# Extending Kamp's Theorem to Model Time Granularity

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## Abstract

In this paper, a generalization of Kamp's theorem relative to the functional completeness of the *until* operator is proved. Such a generalization consists in showing the functional completeness of more expressive temporal operators with respect to the extension of the first-order theory of linear orders $MFO[<]$ with an extra binary relational symbol. The result is motivated by the search of a modal language capable of expressing properties and operators suitable to model time granularity in $\omega$-layered temporal structures.

*Keywords*: Kamp's theorem, linear temporal logic, functional completeness, time granularity

## 1 Introduction

One of the most important results in temporal logic is Kamp's theorem on the functional completeness of the so-called (strict) *until* operator [8]. Kamp's theorem guarantees that any formula of the monadic first-order theory of linear orders $MFO[<]$ can be rewritten as a temporal formula using only the until operator. The result has been re-proved and generalized in many different ways (cf. [2, 6, 7]), e.g., by adding properties to the class of models for the underling first-order language of linear orders. A particularly elegant approach is that proposed by Immerman and Kozen [7], which uses pebble games and Gabbay's result on the first-order completeness of theories enjoying the $k$-variable property [5]. In this paper, we present a new kind of extension based on the addition of relational operators to $MFO[<]$. In order to prove our result we need to introduce more expressive temporal operators, that we called *stereo* operators, capable of expressing properties of *pairs* of segments of the underlying linear model (*stereo-intervals*).

Our results are motivated by the search of a formalism suitable to represent time granularity. The ability of providing and relating temporal representations at different 'grain levels' of the same reality is widely recognized as an important research

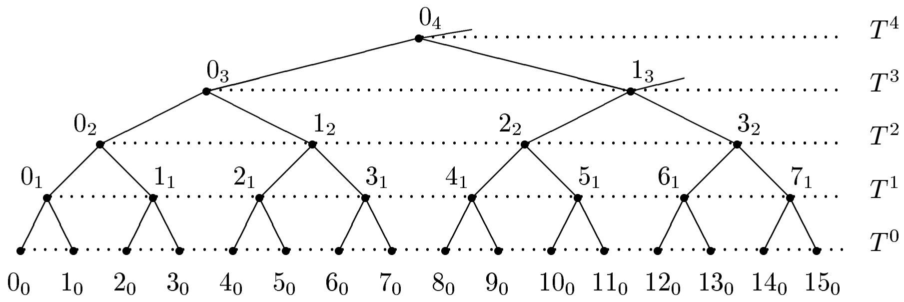


FIG. 1. A 2-refinable upward unbounded layered structure.

theme for temporal logic and a major requirement for many applications, including formal specifications, artificial intelligence, temporal databases, and data mining, e.g. [1, 3, 4, 13, 14]. As for the area of formal specifications, the explicit representation of time granularity makes it possible to specify in a concise way reactive systems whose behaviour can be naturally modeled with respect to a (possibly infinite) set of differently-grained temporal domains. This is the case, for instance, of the wide-ranging class of real-time reactive systems whose components have dynamic behaviours regulated by very different—even by orders of magnitude—time constants, e.g., days, hours, and seconds (called *granular reactive systems* in [9]). To explicitly represent time granularity we put forward a number of results whose common denominator is the introduction of a layered temporal structure capable to characterize the different *grains* we wish to model [9, 10, 11]. A systematic framework for *time granularity* can be found in [9]. It is based on a many-level view of temporal structures that replaces the flat temporal domain of Propositional Linear Time Logic ($PLTL$ hereafter) by a temporal universe consisting of a (possibly infinite) set of differently-grained temporal domains. The theory of $n$-layered (there exist exactly $n$ temporal domains) $k$-refinable (each time point can be refined into $k$ time points of the immediately finer temporal domain, if any) temporal structures for time granularity, with matching decidability results, has been investigated in [11]. The theory of *($k$-refinable) upward unbounded layered structures*, that is, $\omega$-layered structures consisting of a finest temporal domain together with an infinite number of coarser and coarser domains, and the theory of *($k$-refinable) downward unbounded layered structures*, that is, $\omega$-layered structures consisting of a coarsest domain together with an infinite number of finer and finer domains, have been studied in [10].

In this paper, we focus on the problem of providing the first-order fragment of the theory of 2-refinable upward unbounded layered structures with a temporal logic counterpart (the generalization to $k > 2$ is straightforward). A (finite portion of a) 2-refinable upward unbounded layered structure is depicted in Figure 1. Each layer $T^i$ of the structure is isomorphic to $\langle \mathbb{N}, < \rangle$ and each time point–except for the ones belonging to the finest layer–can be decomposed into two time points of the immediately finer one, thus connecting the elements of the temporal structure into a rootless, perfectly-balanced binary tree. A total order of time points is induced by the in-order visit of the tree-like structure. The theory of 2-refinable upward unbounded layered structures, denoted by $MSO[<, \downarrow_0, \downarrow_1]$ (the notation is borrowed from [16]), is a monadic second-order theory with equality endowed with the binary relation symbols $<, \downarrow_0$, and $\downarrow_1$. In [10], we prove that $MSO[<, \downarrow_0, \downarrow_1]$ is (non-elementarily) decidable by encoding its layered models into standard linear models. Such linear models, however, need to be equipped with a language including some extra predicate (besides $<$) that provides them with the ability of switching across layers of the original model. Such an extra relation symbol is called $flip$, since its second argument can be obtained by flipping the least significant digit of the binary representation of its first argument. In [10], we show that the monadic second-order theory $MSO[<, flip]$, interpreted over standard linear orders, is exactly as expressive as $MSO[<, \downarrow_0, \downarrow_1]$. In this paper, we prove the functional completeness of stereo operators with respect to the predicate $flip$. More precisely, we prove that the temporal logic of stereo operators is expressively complete with respect to the first-order fragment $MFO[<, flip]$ of $MSO[<, flip]$. Such a result also provides a temporal logic counterpart to the first-order fragment $MFO[<, \downarrow_0, \downarrow_1]$ of $MSO[<, \downarrow_0, \downarrow_1]$, since $MFO[<, flip]$ is at least as expressive as $MFO[<, \downarrow_0, \downarrow_1]$, as it will be shown in the next section. As in the case of PLTL, the functional completeness of stereo operators is intimately related to a separation property (with respect to stereo-intervals). Such a separation property is expressed by a suitable normal form for $MFO[<, flip]$-formulas which generalizes the normal form for $MFO[<]$-formulas outlined in [2]. Since the functional completeness result we are going to establish seems to be ultimately based on simple basic properties of the $flip$ relation (parenthesis-like properties), we believe that it is worth to be investigated the expressive completeness of the general class of relational operators sharing these properties.

The paper is organized as follows. We start by giving some basic definitions and results in Section 2. Then, we introduce the notion of stereo-interval (Section 3), together with the corresponding stereo operators (Section 4). The expressive completeness of stereo operators is proved in Section 6, on the basis of the normal form theorem given in Section 5. At the end, we provide an assessment of the work done and outline future research directions. In the Appendix, we report the proofs of the most technical results.

## 2 On the relationships between linear and layered structures

The temporal structure underlying $PLTL$ is the set of natural numbers $\mathbb{N}$ with their usual ordering $<$, denoted $\langle \mathbb{N}, < \rangle$. Given a set of (atomic) proposition symbols $AP$, a *linear time model* is a tuple $\alpha = \langle \mathbb{N}, <, (Q_p)_{p \in AP} \rangle$, where, for every $p \in AP$, $Q_p \subseteq \mathbb{N}$. For each $p \in AP$, $Q_p$ identifies the positions at which $p$ holds. A linear time model $\alpha$ can be alternatively expressed as an infinite word over the alphabet $PowerSet(AP)$. Accordingly, for all $i \in \mathbb{N}$, we will write $\alpha(i)$ to denote the set $\{p : i \in Q_p\}$.

The theory of upward unbounded layered structures $MSO[<, \downarrow_0, \downarrow_1]$ is a monadic second-order theory with equality, endowed with the binary relation symbols $<, \downarrow_0$, and $\downarrow_1$. The second-order language of $MSO[<, \downarrow_0, \downarrow_1]$ is defined as follows. The *vocabulary* consists of individual variable symbols $x, y, z, \ldots$, proposition (set variable) symbols $p_1, \ldots, p_n$, and the binary relation symbols $<$, $\downarrow_0$, and $\downarrow_1$. *Atomic formulas* are of the forms $x = y$, $p_i(x)$, $x < y$, and $\downarrow_i (x, y)$, where $x$ and $y$ are individual variables and $i = 1, 2$. *Formulas* are obtained from atomic formulas by using the

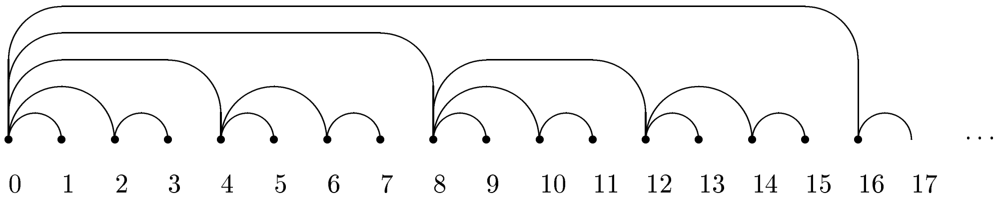


FIG. 2. The structure of the function *flip*.

Boolean connectives $\wedge$, $\vee$, $\neg$, and $\rightarrow$, and the quantifiers $\exists$ and $\forall$ ranging over both individual and set variables. The relation symbol $<$ is interpreted as the total ordering over the upward unbounded layered structure induced by the in-order visit of the tree-like structure, while the relation symbols $\downarrow_0$ and $\downarrow_1$ are interpreted as the left and right child functions, respectively. Since both $\downarrow_0$ and $\downarrow_1$ are functional in nature, in the following we will often write $\downarrow_0 (x) = y$ (resp. $\downarrow_1 (x) = y$) for $\downarrow_0 (x, y)$ (resp. $\downarrow_1 (x, y)$). Formulas of $MSO[<, \downarrow_0, \downarrow_1]$ are interpreted in the usual way.

In [10], we describe in detail how the basic temporal operators for time granularity, namely, the displacement, contextualization, and projection operators, can be defined in $MSO[<, \downarrow_0, \downarrow_1]$. As an example, we report the definition of the unary predicate $\Delta_0$ holding at the origin of each layer. The predicate $\Delta_0$ is interpreted as the set of all and only the elements belonging to the left-edge of an upward unbounded layered structure, which is defined as the least set containing the element $0_0$ and all its ancestors $0_1$, $0_2$, ... (cf. Figure 1). This predicate can be defined in $MSO[<, \downarrow_0, \downarrow_1]$ as follows. Given a second-order formula $\phi(\mathtt{p})$, with free set variable $\mathtt{p}$, let $\mu(\phi(\mathtt{p}))(\mathtt{x})$ be the following second-order formula, with free individual variable $\mathtt{x}$:

$$\exists \mathtt{p}(\mathtt{p}(\mathtt{x}) \wedge \phi(\mathtt{p}) \wedge \forall \mathtt{q}(\phi(\mathtt{q}) \rightarrow \forall \mathtt{y}(\mathtt{p}(\mathtt{y}) \rightarrow \mathtt{q}(\mathtt{y})))).$$

$\mu(\phi(\mathtt{p}))(\mathtt{x})$ evaluates to true if and only if the valuation for $\mathtt{x}$ belongs to the smallest valuation for $\mathtt{p}$ for which $\phi(\mathtt{p})$ holds true. Using the operator $\mu$, $\Delta_0(\mathtt{x})$ can be expressed as follows:

$$\mu\,(\mathtt{p}(0_0) \wedge \forall \mathtt{y}, \mathtt{z}((\mathtt{p}(\mathtt{z}) \wedge \downarrow_0 (\mathtt{y}) = \mathtt{z}) \rightarrow \mathtt{p}(\mathtt{y})))\,(\mathtt{x}),$$

where $\mathtt{p}(0_0)$ is a shorthand for $\exists \mathtt{y}(\mathtt{p}(\mathtt{y}) \wedge \forall \mathtt{z}(\mathtt{y} \leq \mathtt{z}))$. It is easy to verify that such a formula captures the smallest interpretation for $\mathtt{p}$ which contains $0_0$ and it is closed parent-wise.

Further, we prove that $MSO[<, \downarrow_0, \downarrow_1]$ is strictly more expressive than $MSO[<]$, and we show how to embed it into $MSO[<, flip]$, a proper extension of $MSO[<]$ with the binary relation symbol $flip$. The language of $MSO[<, flip]$ is defined in the standard way. The domain of interpretation is the set of natural numbers $\mathbb{N}$. The relational symbol $<$ is interpreted as the usual ordering over $\mathbb{N}$, while the relational symbol $flip$ is interpreted as a unary function *flip*, which, for any natural number $x > 0$, returns the natural number $x - x'$, where $x'$ is the least power of 2, with a non-null coefficient, that occurs in the binary representation of $x$.

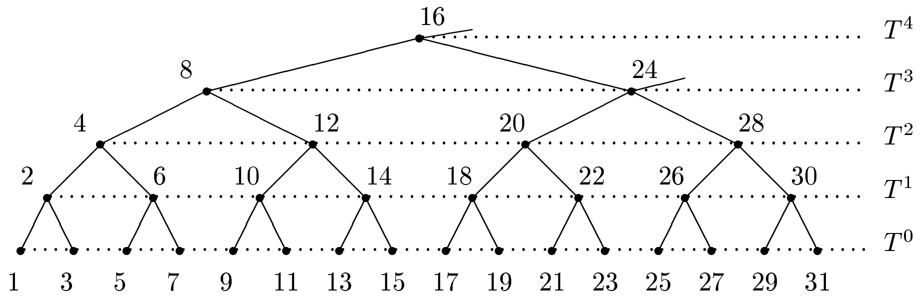


FIG. 3. The concrete 2-refinable upward unbounded layered structure.

**Definition 2.1** (THE FUNCTION *flip*)
*The function* $flip : \mathbb{N}^+ \to \mathbb{N}$ *is such that, for all* $x \in \mathbb{N}^+$,

$$flip(x) = y \quad \text{iff} \quad x = \sum_{j=0}^{n} 2^{i_j}, \text{ with } i_n > i_{n-1} > \ldots > i_0 \geq 0, \text{ and } y = x - 2^{i_0}.$$

The function $flip$ is not defined for $x = 0$; however, totality can be recovered by extending it with $flip(0) = 0$[1]. Furthermore, it is useful to add a maximum element $\infty$ to $\mathbb{N}$, with $flip(\infty) = 0$. A graphical representation of the function $flip$ is given in Figure 2.

An embedding of $MSO[<, \downarrow_0, \downarrow_1]$ into $MSO[<, flip]$ can be obtained as follows [10]. First, it is possible to show that any two (2-refinable) upward unbounded layered structures are isomorphic, by defining a simple correspondence $f$ that maps each element of the first structure into an element of the second one, preserving projection and ordering relations. As an easy consequence, we have that any given formula $\phi(\mathtt{p}_1, \ldots, \mathtt{p}_m)$, where $p_1, \ldots, p_m$ are the set variables occurring free in $\phi$, is satisfiable under an interpretation $\mathtt{p}_1^{\mathcal{I}}, \ldots \mathtt{p}_m^{\mathcal{I}}$ if and only if it is satisfiable under the interpretation $f(\mathtt{p}_1^{\mathcal{I}}), \ldots, f(\mathtt{p}_m^{\mathcal{I}})$. This allows one to replace the class of upward unbounded layered structures by a (unique) suitable *concrete* structure which can be easily encoded into $\mathbb{N}$. Such a *concrete* upward unbounded layered structure is defined as follows:

- for all $i \geq 0$, the $i$-th layer $T^i$ is the set $\{2^i + n2^{i+1} : n \geq 0\} \subseteq \mathbb{N}$;
- for every element $x = 2^i + n2^{i+1}$ belonging to $T^i$, with $i \geq 1$, $\downarrow_0 (x) = 2^i + n2^{i+1} - 2^{i-1} = 2^{i-1} + 2n2^i$ and $\downarrow_1 (x) = 2^i + n2^{i+1} + 2^{i-1} = 2^{i-1} + (2n+1)2^i$;
- $<$ is the usual ordering over $\mathbb{N}$.

A fragment of this concrete structure is depicted in Figure 3. Notice that all odd numbers are associated with layer $T^0$, while even numbers are distributed over the remaining layers. Notice also that the labeling of the concrete structure does not include the number 0. In the following, we will see that it is convenient to consider 0 as the label of the first node of an imaginary additional finest layer, whose remaining nodes are not labeled (notice that the node with label 0 turns out to be the left son of the node with label 1).

[1] Notice that $flip(x) < x$, for all $x \in \mathbb{N}^+$, and $flip(x) \leq x$, for all $x \in \mathbb{N}$. Later, we will often use these properties of $flip$ to simplify definitions.

$$\begin{array}{ccccc}
MSO[<] & \supset & MFO[<] & \equiv & PLTL \\
\cap & & \cap & & \\
MSO[<,\downarrow_0,\downarrow_1] & \supset & MFO[<,\downarrow_0,\downarrow_1] & & \cap \\
\cap\!\mid & & \cap\!\mid & & \\
MSO[<,flip] & \supset & MFO[<,flip] & \equiv & UUTL
\end{array}$$

FIG. 4. The relationships among temporal theories and logics.

Both relations $\downarrow_0$ and $\downarrow_1$ can be easily encoded in terms of the function $flip$ as follows. For any given even number $x$,

$$\downarrow_0(x) = max\{y : y < x, flip(y) = flip(x)\}, \text{ and} \tag{2.1}$$

$$\downarrow_1(x) = max\{y : flip(y) = x\}. \tag{2.2}$$

By exploiting such a correspondence, it is possible to define a translation $\tau$ of $MSO[<,\downarrow_0,\downarrow_1]$-formulas into $MSO[<,flip]$-formulas such that, for any formula $\phi \in MSO[<,\downarrow_0,\downarrow_1]$, $\phi$ is satisfiable if and only if $\tau(\phi) \in MSO[<,flip]$ is satisfiable.

In [12], Monti and Peron prove that $MSO[<,flip]$ is strictly more expressive than $MSO[<]$ and that it is (non-elementarily) decidable. By exploiting the proposed embedding of $MSO[<,\downarrow_0,\downarrow_1]$ into $MSO[<,flip]$), the decidability of $MSO[<,\downarrow_0,\downarrow_1]$ immediately follows from the decidability of $MSO[<,flip]$. The same expressiveness relationships hold among the first-order fragments of the considered theories. First, it can be easily shown that both $MFO[<,flip]$ (the first-order fragment of $MSO[<,flip]$ and $MFO[<,\downarrow_0,\downarrow_1]$ (the first-order fragment of $MSO[<,\downarrow_0,\downarrow_1]$) are strictly more expressive than $MFO[<]$ (the first-order fragment of $MSO[<]$). As an example, it is well known that the property which states that a given predicate $p$ holds true at all odd time points is not expressible in $MFO[<]$ [17]. On the contrary, such a property can be easily captured in $MFO[<,flip]$ (as well as in $MFO[<,\downarrow_0,\downarrow_1]$, as shown in [10]) by means of the following formula:

$$\forall x((\neg\exists y\, flip(y) = x) \to p(x)).$$

As another example, unlike $MFO[<]$, $MFO[<,flip]$ allows one to constrain a predicate $p$ to be satisfied at a time point $i$ only if $i$ is a power of two. This condition can be captured by means of the following $MFO[<,flip]$-formula:

$$\forall x(p(x) \to \exists y(flip(x) = y \wedge y = 0)),$$

where $y = 0$ is a shorthand for $\neg\exists z(z < y)$.

Finally, it is not difficult to show that $MFO[<,flip]$ is at least as expressive as $MFO[<,\downarrow_0,\downarrow_1]$. Both expressions (2.1) and (2.2) are indeed first-order definable:

$$\begin{array}{lll}
\downarrow_0(x) = y & \text{iff} & y < x \wedge flip(y) = flip(x) \wedge \\
& & \neg\exists z(y < z \wedge z < x \wedge flip(z) = flip(x)); \\
\downarrow_1(x) = y & \text{iff} & flip(y) = x \wedge \neg\exists z(y < z \wedge flip(z) = x).
\end{array}$$

The relationships among the various temporal theories and logics are summarized in Figure 4, where $UUTL$ denotes the proposed temporal logic for time granularity.

## 3 Replacing intervals by stereo-intervals

In order to prove the expressive completeness of the $UUTL$ operators with respect to $MFO[<, flip]$, we firstly need to introduce a generalized notion of interval, called *stereo-interval.* Later on, we will show that the relationships between stereo-intervals and $UUTL$ operators is analogous to the relationships between intervals and the classical until operator $U$.

**Definition 3.1** (Stereo-interval)
*Given $x_1, x_2, x_3$, and $x_4$ belonging to $\mathbb{N}$, the* stereo-interval $[x_1, x_2] - [x_3, x_4]$ *is the set $\{x : x_1 \leq x \leq x_2 \vee x_3 \leq x \leq x_4\}$, with $x_1 \leq x_2$, $x_2 \leq x_3$, and $x_3 \leq x_4$.*

For the sake of simplicity, we assume, and unless otherwise said we will always do, that $x_1 \neq x_4$. Moreover, we will write $[x_1, x_4]$ for $[x_1, x_2] - [x_3, x_4]$, whenever $x_2 = x_3$ (degenerate stereo-intervals or *mono-intervals*). In the following, we will often use the term stereo-interval to designate both proper and degenerate stereo-intervals. Finally, we will write $[x_1, x_2] - [x_3, x_4)$ (resp. for $[x_1, x_2) - [x_3, x_4]$) for the set $\{x : x_1 \leq x \leq x_2 \vee x_3 \leq x < x_4\}$ (resp. $\{x : x_1 \leq x < x_2 \vee x_3 \leq x \leq x_4\}$). In particular, we will write $[x_1, x_2] - [x_3, x_3)$ (resp. $[x_1, x_1) - [x_3, x_4]$) for $[x_1, x_2]$ (resp. $[x_3, x_4]$).

The addition to the domain of a new element $\infty$, such that, for all $x \in \mathbb{N}$, $x < \infty$, allows us to define right unbounded intervals. Hereafter, we write $[x_1, x_2] - [x_3, \infty]$ to denote the stereo-interval whose elements $x$ satisfy either the condition $x_1 \leq x \leq x_2$ or the condition $x_3 \leq x$.

**Definition 3.2** (Composition of stereo-intervals)
*The* composition *of two stereo-intervals $[x_1, x_2] - [x_3, x_4]$ and $[x'_1, x'_2] - [x'_3, x'_4]$, where $x'_1 = x_2$ and either $x'_4 = x_3$ or $x'_4 + 1 = x_3$, denoted by $[x_1, x_2] - [x_3, x_4] \uplus [x'_1, x'_2] - [x'_3, x'_4]$, is the stereo interval $[x_1, x'_2] - [x'_3, x_4]$.*

Notice that the composition of two stereo-intervals is just the set-theoretic union of the two suitable pairs of convex intervals.

**Definition 3.3** (Weakly flip-closed stereo-interval)
*A stereo-interval $[x_1, x_2] - [x_3, x_4]$ is* weakly flip-closed *if and only if it satisfies the following two conditions:*

*(i) for each $x \in (x_1, x_2] - [x_3, x_4]$, $flip(x) \in [x_1, x_2] - [x_3, x_4]$;*
*(ii) for each $x \notin (x_1, x_2] - [x_3, x_4]$, $flip(x) \notin (x_1, x_2] - [x_3, x_4)$.*

If $[x_1, x_2]$ is not empty, then the least element of $[x_1, x_2] - [x_3, x_4]$ is $x_1$; otherwise, it is $x_3$. From the properties of $flip(\cdot)$, it follows that a stereo-interval $[x_1, x_2] - [x_3, x_4]$ is weakly flip-closed only if, for all $x_1 < x \leq x_2$, it holds that $x_1 \leq flip(x) < x_2$ and, for all $x_3 < x \leq x_4$, it holds that either $x_1 \leq flip(x) \leq x_2$ or $x_3 \leq flip(x) < x_4$. Moreover, if $x_1 \neq x_3$, then $x_1 \leq flip(x_3) \leq x_2$. In particular, given a weakly flip-closed stereo-interval $[x_1, x_2] - [x_3, x_4]$, $x_4 = \infty$ implies $x_1 = 0$.

The following flip-closure properties of stereo-intervals can be easily proved (for the sake of brevity, we will omit the term weakly).

**Proposition 3.4** (Flip-Closure Properties I)

*a) if $[x_1, x_4]$ is flip-closed, then $[x_1, x_4']$ is flip-closed, for all $x_1 \leq x_4' \leq x_4$;*
*b) if $[x_1, x_2] - [x_3, x_4]$ is flip-closed, then $[x_1, x_2'] - [x_3, x_4]$ is flip-closed, for all $x_2 \leq x_2' \leq x_3$;*
*c) if $[x_1, x_2] - [x_3, x_4]$ is flip-closed, $v \in [x_1, x_2]$, $w \in [x_3, x_4]$, and $flip(w) = v$, then both $[x_1, v] - [w, x_4]$ and $[v, x_2] - [x_3, w]$ are flip-closed.*

Below, we will rely on the following properties to build flip-closed mono- and stereo-intervals.

**Proposition 3.5** (Flip-Closure Properties II)

*d) if $flip(x_4) = x_1$, then $[x_1, x_4]$ is a flip-closed mono-interval;*
*e) if $flip(x_4) = x_1$ and $flip(x_3) = x_2$, then $[x_1, x_2] - [x_3, x_4]$ is a flip-closed stereo-interval.*

In order to prove our main result, we will focus on a proper subset of the set of weakly flip-closed stereo-intervals, which is based on the following notion of *break point*. For any given point $x$, we say that $y > x$ is a *jump* for $x$ if and only if $flip(y) < x (< y)$. Given a pair $(x, y)$ of flip-related points, that is, $flip(y) = x$, a *break point* $z$, with $x < z < y$, is a point for which there exists no jump $w$, with $x < w < y$. We will denote any break point $z$ of a flip-related pair of points $(x, y)$ by $BreakP_{x,y}(z)$. For instance, the break points generated by the interval $(0, 16)$ are $8, 12, 14$, and $15$ (cf. Figure 2). In general, given $x$ and $y$ such that $flip(y) = x$ and $y = 2^{n_k} + \ldots + 2^{n_0}$, with $n_k > .. > n_0$, the break points of $(x, y)$ are the points $x_i = x - 2^i$, with $0 \leq i < n_0$. In the following, we will often write $BreakP(x, y)$ as a shorthand for $\exists z, w(z < x < y < w \wedge flip(w) = z \wedge BreakP_{z,w}(x) \wedge BreakP_{z,w}(y))$.

**Definition 3.6** (Strongly Flip-closed stereo-interval)
*A (non-degenerate) stereo-interval $[x_1, x_2] - [x_3, x_4]$ is* strongly flip-closed *if and only if it satisfies the following two conditions:*

1. *$flip(x_3) = x_2$ and*
2. *either $x_4 = \infty$ or $BreakP(x_1, x_4)$ or there exists $w$ such that $BreakP_{x_1,w}(x_4)$.*

*The definition of strongly flip-closed mono-interval $[x_1, x_4]$ is obtained by removing condition 1.*

From Definition 3.6, it follows that there exists a chain of flip-related points from $x_4$ to $x_1$, that is, there exist $y_1 \ldots y_n$ such that $y_1 = x_1$, $y_n = x_4$, and $flip(x_i) = x_{i-1}$, for all $2 \leq i \leq n$. It is not difficult to prove that Definition 3.6 implies Definition 3.3, but not vice versa. As an example, the stereo interval $[0, 13] - [14, 17]$ is flip-closed according to Definition 3.3, but not according to Definition 3.6. In the following, we will always use the expression flip-closed intervals to refer to strongly flip-closed intervals.

Flip-closed mono- and stereo-intervals can be interpreted in terms of the concrete upward unbounded layered structure (cf. Figure 3) as follows. If $BreakP(x_1 x_2)$, the mono-interval $[x_1, x_2]$ consists of the element of the tree rooted in $x_1$, devoid of the elements of the left subtree of $x_1$ and of the elements of the right subtree of $x_2$. Notice that $BreakP(x_1, x_2)$ holds whenever $x_1$ and $x_2$ belong to the same right path, that is, to a path involving only right son edges. This is the case, for instance, of mono-intervals $[10, 11]$, with $z = 8$ and $w = 12$, and $[8, 14]$, with $z = 0$ and $w = 16$. If there exists $w$ such that $BreakP_{x_1,w}(x_2)$, the set of elements of the mono-interval $[x_1, x_2]$ is the union of the singleton $\{x_1\}$ and the set of elements of the left subtree of $w$, devoid of the elements of the right subtree of $x_2$. This is the case, for instance, of mono-intervals $[8, 11]$, with $w = 12$, and $[8, 10]$, with $w = 12$. Finally, the elements of a stereo-interval $[x_1, x_2] - [x_3, x_4]$ are the elements of the mono-interval $[x_1, x_4]$, determined according to the above rules, devoid of the elements of the left subtree of $x_3$. This is the case, for instance, of the stereo-intervals $[8, 12] - [14, 14]$, with $z = 0$ and $w = 16$, and $[0, 8] - [12, 16]$, with $w = 32$.

A deeper understanding of the relationships between the structure of the function $flip$ (cf. Figura 2) and the concrete 2-refinable upward unbounded layered structure (cf. Figure 3) can be achieved by assigning a meaning to (suitable) paths over the layered structures in terms of the $flip$. Formally, let us define $y$ a *left* (resp. *right*) *ancestor* of $x$ if $y(\neq x)$ is an ancestor of $x$ and the path from $y$ to $x$ starts with a left (resp. right) edge, that is, $x$ belongs to the left (resp. right) subtree of $y$. Let $y(\neq x)$ be a left (resp. right) ancestor of $x$. The path connecting $y$ to $x$ is called a left (resp. right) path if all its edges are left (resp. right) edges. The elements $x_1 < x_2 < \ldots < x_n$ of any given (descending) left path of the concrete layered structure have the same $flip$, that is, $flip(x_1) = flip(x_2) = \ldots = flip(x_n)$, and the mono-intervals $[flip(x_1), x_1], \ldots, [flip(x_n), x_n]$ are ordered by inclusion, that is, $[flip(x_1), x_1] \subset \ldots \subset [flip(x_n), x_n]$. The elements $x_1 < x_2 < \ldots < x_n$ of any given (ascending) right path of the concrete layered structure are such that all consecutive pairs are flip-related, that is, $flip(x_n) = x_{n-1}, \ldots, flip(x_2) = x_1$, and, for all $1 \leq i, j \leq n$, $BreakP(x_i, x_j)$.

## 4 A temporal logic for time granularity

The temporal logic counterpart of $MFO[<, flip]$, that we called $UUTL$ (double Until Temporal Logic), is obtained by replacing the classical until operator of $PLTL$ with suitable stereo operators.

**Definition 4.1** (TEMPORAL LOGIC - SYNTAX)
*Let $AP$ be a set of atomic proposition symbols. The propositional (modal) language of $UUTL$ consists of the set of formulas generated by the following rules:*

1. *each symbol in $AP$ is in $UUTL$;*
2. *if $\phi_1$ and $\phi_2$ are in $UUTL$, then $\phi_1 \wedge \phi_2$ and $\neg\phi_1$ are in $UUTL$;*
3. *if $\phi_1, \phi_2, \phi_3, \psi_1, \psi_2$, and $\psi_3$ are in $UUTL$, then $\psi_1 \cdot \phi_1 U \psi_2 - \psi_3 U \phi_2 \cdot \phi_3$, $\psi_1 \diamond \psi_2$, and $\psi_1 \cdot \psi_2 \uparrow \psi_3$ are in $UUTL$.*

*We will use the standard abbreviations $\phi_1 \vee \phi_2$ and $\phi_1 \rightarrow \phi_2$ for $\neg(\neg\phi_1 \wedge \neg\phi_2)$ and $\neg\phi_1 \vee \phi_2$, respectively.*

*A formula is called* general *if it is a boolean combination of formulas of the forms $\psi_1 \cdot \phi_1 U \psi_2 - \psi_3 U \phi_2 \cdot \phi_3$ and $\psi_1 \cdot \psi_2 \uparrow \psi_3$.*

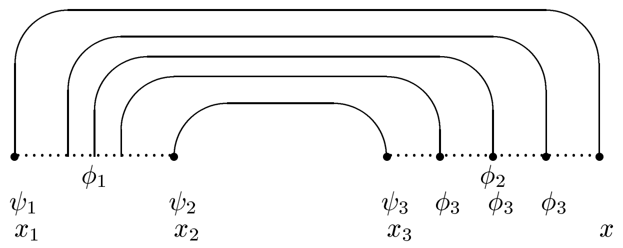


FIG. 5. The semantics of the formula $\psi_1 \cdot \phi_1 \ U \ \psi_2 {-} \psi_3 \ U \ \phi_2 \cdot \phi_3$.

Any $UUTL$-formula is interpreted over initial segments of $\mathbb{N}$. In addition, general formulas are also interpreted over $\mathbb{N}$. Let $\alpha$ be a linear time model. For all $x \in \mathbb{N}$, we denote the prefix of $\alpha$ over the interval $[0, x]$ by $\langle \alpha, x \rangle$. Accordingly, we alternatively denote the whole model $\alpha$ by $\langle \alpha, \infty \rangle$.

**Definition 4.2** (TEMPORAL LOGIC - SEMANTICS)
*The semantics of $UUTL$-formulas is defined as follows:*

1. $\langle \alpha, x \rangle \models p$ *iff* $p \in \alpha(x)$, *for* $p \in AP$;
2. $\langle \alpha, x \rangle \models \phi_1 \wedge \phi_2$ *iff* $\langle \alpha, x \rangle \models \phi_1$ *and* $\langle \alpha, x \rangle \models \phi_2$;
3. $\langle \alpha, \infty \rangle \models \phi_1 \wedge \phi_2$ *iff* $\langle \alpha, \infty \rangle \models \phi_1$ *and* $\langle \alpha, \infty \rangle \models \phi_2$, *where* $\phi_1$ *and* $\phi_2$ *are general formulas;*
4. $\langle \alpha, x \rangle \models \neg \phi_1$ *iff it is not the case that* $\langle \alpha, x \rangle \models \phi_1$;
5. $\langle \alpha, \infty \rangle \models \neg \phi_1$ *iff it is not the case that* $\langle \alpha, \infty \rangle \models \phi_1$, *where* $\phi_1$ *is a general formula;*
6. $\langle \alpha, x \rangle \models \psi_1 \cdot \phi_1 U \psi_2 {-} \psi_3 U \phi_2 \cdot \phi_3$ *iff*
$\exists x_1, x_2, x_3 (x_3 < x \ \wedge flip(x) = x_1 \wedge flip(x_3) = x_2 \wedge \bigwedge_{i=1}^{3} \langle \alpha, x_i \rangle \models \psi_i \wedge$
$\forall y ((x_1 < y < x_2 \rightarrow \langle \alpha, y \rangle \models \phi_1) \wedge (x_3 < y < x \rightarrow (\langle \alpha, y \rangle \models \phi_2 \wedge$
$(\exists z (x_1 \leq z \leq x_2 \wedge flip(y) = z) \rightarrow \langle \alpha, y \rangle \models \phi_3)))))$;
7. $\langle \alpha, \infty \rangle \models \psi_1 \cdot \phi_1 U \psi_2 {-} \psi_3 U \phi_2 \cdot \phi_3$ *iff*
$\langle \alpha, 0 \rangle \models \psi_1 \wedge \exists x_2, x_3 (flip(x_3) = x_2 \wedge \langle \alpha, x_2 \rangle \models \psi_2 \wedge \langle \alpha, x_3 \rangle \models \psi_3 \wedge$
$\forall y ((0 < y < x_2 \rightarrow \langle \alpha, y \rangle \models \phi_1) \wedge (y > x_3 \rightarrow (\langle \alpha, y \rangle \models \phi_2 \wedge$
$(\exists z (0 \leq z \leq x_2 \wedge flip(y) = z) \rightarrow \langle \alpha, y \rangle \models \phi_3)))))$;
8. $\langle \alpha, x \rangle \models \psi_1 \diamond \psi_2$ *iff*
$\exists x_1 (x_1 < x \wedge BreakP(x_1, x) \wedge \langle \alpha, x_1 \rangle \models \psi_1 \wedge$
$\forall y ((x_1 < y < x \wedge BreakP(y, x)) \rightarrow \langle \alpha, y \rangle \models \psi_2))$;
9. $\langle \alpha, x \rangle \models \psi_1 \cdot \psi_2 \uparrow \psi_3$ *iff*
$\exists x_1 (x_1 < x \wedge flip(x) = x_1 \wedge \langle \alpha, x_1 \rangle \models \psi_1 \wedge \forall y (x_1 < y < x \rightarrow (\langle \alpha, y \rangle \models \psi_2 \vee$
$\exists z, w (x_1 \leq z < y < w < x \wedge flip(w) = z \wedge \langle \alpha, w \rangle \models \psi_3))))$;
10. $\langle \alpha, \infty \rangle \models \psi_1 \cdot \psi_2 \uparrow \psi_3$ *iff*
$\langle \alpha, 0 \rangle \models \psi_1 \wedge \forall y (y > 0 \rightarrow (\langle \alpha, y \rangle \models \psi_2 \vee$
$\exists z, w (0 \leq z < y < w \wedge flip(w) = z \wedge \langle \alpha, w \rangle \models \psi_3)))$.

Notice that, from the pair of conditions $x_1 < x$ and $flip(x) = x_1$ of clause (9), it follows that any formula of the form $\psi_1 \cdot \psi_2 \uparrow \psi_3$ can never be satisfied in 0. Since the

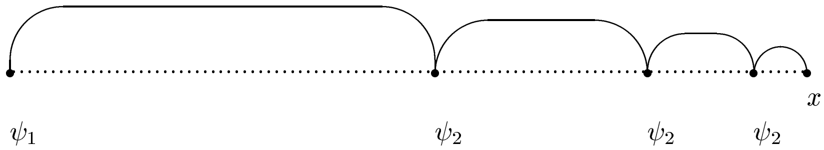


FIG. 6. The semantics of the formula $\psi_1 \diamond \psi_2$.

initial position 0 is the only position that does not satisfy $True \cdot True \uparrow True$, it can be characterized by means of the formula $\neg(True \cdot True \uparrow True)$. In the following, we will use 0 as a shorthand for $\neg(True \cdot True \uparrow True)$.

A common feature of the stereo operators $UU$, $\diamond$, and $\uparrow$ is that, when evaluated at a given point $x$, they look *backwards* in time. This reflects the fact that the relation $flip$ is functional only towards the past. A graphical account of the semantics of the formula $\psi_1 \cdot \phi_1 U \psi_2 - \psi_3 U \phi_2 \cdot \phi_3$ is given in Figure 5. The operator $UU$ generalizes the classical until operator $U$ to cope with stereo-intervals (the name of the logic, $UUTL$, is borrowed from its most significant operator $UU$). Point $x_1$, at which $\psi_1$ holds, represents the starting point of an interval over which a classical $\phi_1 U \psi_2$-formula holds. Point $x$ is the starting point of an interval satisfying (backwards) $\phi_2 U \psi_3$, that is, of an interval satisfying $\phi_2 S \psi_3$. Moreover, the $flip$ operator forces $x_1$ and $x$, as well as $x_2$ and $x_3$, to be flip-related. Finally, $\phi_3$ holds at all time points in between $x_3$ and $x$ which are flip-related to time points located between $x_1$ and $x_2$. The semantics of the formula $\psi_1 \diamond \psi_2$ is depicted in Figure 6. The operator $\diamond$ is a sort of since operator $S$ over break points. Points $x_1$ and $x$, with $x_1 < x$, are both break points of a pair of flip-related points $z$ and $w$, with $z < x_1$ and $x < w$, $\psi_1$ holds at $x_1$, and $\psi_2$ holds at all the break points of $(z, w)$ which lie between $x_1$ and $x$. Finally, the operator $\uparrow$ imposes that (i) $x$ is flip-related to $x_1$, that is, $flip(x) = x_1$, and $\psi_1$ holds at $x_1$, (ii) $\psi_2$ holds at every position $z$ such that $BreakP_{x_1,x}(z)$, and (iii) at all other positions $z$ in between $x_1$ and $x$, either $\psi_2$ holds or there exists a jump $y(< x)$ for $z$ such that $\psi_3$ holds at $y$.

The following examples show how meaningful timing properties can be expressed in $UUTL$. The first example gives a characterization of odd positions.

**Example 4.3** (ODD POSITIONS)
*For any given position $x$, the following equivalence holds:*

$$\langle \alpha, x \rangle \models True \cdot False \uparrow True \quad \textit{iff} \quad x \textit{ is an odd number.}$$

*In the following, we will use $Odd$ as a shorthand for $True \cdot False \uparrow True$.*

In the next example, we show that the operator $UU$ can be used to express non-regular properties, which cannot be captured in $(Q)PLTL$.

**Example 4.4** (NON-REGULAR PROPERTIES)
*A well-known non-regular property is the one stating that a predicate p is satisfied at a point $x$ only if $x$ is a power of two. The set of positions $x$ such that $x = 2^n$, for*

*some $n \geq 0$, is captured by the following UUTL-formula:*

$$True \cdot True\ U\ 0{-}True\ U\ True \cdot True.$$

*Let us use $2^n$ as a shorthand for such a formula. It holds that:*

$$\langle \alpha, \infty \rangle \models \neg p \cdot True\ U\ 0{-}0\ U\ p \rightarrow 2^n \cdot True \quad \textit{iff} \quad p \textit{ holds at } x \textit{ only if } x = 2^n.$$

The last example shows that the relations $\downarrow_0$ and $\downarrow_1$ can be expressed in $UUTL$. At the end of Section 2, we proposed an embedding of both $\downarrow_0$ and $\downarrow_1$ in $MFO[<, flip]$. An alternative characterization of these relations can be given in terms of break points. It is not difficult to prove that $\downarrow_0 (x) = y$ if and only if $BreakP_{flip(x),x}(y) \wedge \neg\exists z(z < y \wedge BreakP_{flip(x),x}(z))$. As for $\downarrow_1$, it is possible to show that $\downarrow_1 (x) = y$ if and only if $x < y \wedge \exists z, w(BreakP_{z,w}(x) \wedge BreakP_{z,w}(y) \wedge \neg\exists u(x < u < y \wedge BreakP_{z,w}(u)))$. We will exploit such a characterization to capture $\downarrow_0$ and $\downarrow_1$ in $UUTL$.

**Example 4.5** (The relations $\downarrow_0$ and $\downarrow_1$)
*Let Break be a shorthand for the formula $True \diamond True$. According to Definition 4.2, clause (8), Break holds at a given point $x$ if and only if there exists $x_1$ such that $x_1 < x$ and $BreakP(x_1, x)$. We also introduce $EvenPred(\varphi)$ as a shorthand for the formula $True \cdot True\ U\ True{-}\varphi\ U\ False \cdot True$. It is easy to check that*

$$\langle \alpha, x \rangle \models EvenPred(\varphi) \quad \textit{iff} \quad \begin{array}{l} \textit{either } x \textit{ is even or } x = 1, \textit{ and} \\ \varphi \textit{ holds at the predecessor of } x. \end{array}$$

*As for $\downarrow_0$, for any given point $x$ and UUTL-formulas $\phi_1$ and $\phi_2$, it is possible to show that $\langle \alpha, x \rangle \models \phi_1$ and $\langle \alpha, \downarrow_0 (x) \rangle \models \phi_2$ (denoted by $\langle \alpha, x \rangle \models \phi_1 \downarrow_0 \phi_2$) is equivalent to:*

$$\langle \alpha, x \rangle \models \phi_1 \wedge (EvenPred(\phi_2 \wedge \neg Break) \vee EvenPred((\phi_2 \wedge \neg Break) \diamond\ True)).$$

*The (derived) operator $\downarrow_0$ allows us to introduce a notion of temporal context, or layer, in UUTL. For all $i \geq 0$, let $Level_i$ be a propositional variable such that $\langle \alpha, x \rangle \models Level_i$ if and only if $x$ belongs to the $i$-th layer of the upward unbounded layered structure. $Level_i$ can be inductively defined as follows: $Level_0$ is the formula Odd; for all $i > 0$, $Level_i$ can be defined as $True \downarrow_0 Level_{i-1}$.*

*As for $\downarrow_1$, for any given point $x$ and UUTL-formulas $\phi_1$ and $\phi_2$, it is not difficult to show that $\langle \alpha, x \rangle \models \phi_1$ and $\langle \alpha, y \rangle \models \phi_2$, with $\downarrow_1 (y) = x$ (denoted by $\langle \alpha, x \rangle \models \phi_1 \uparrow_1 \phi_2$), is equivalent to:*

$$\langle \alpha, x \rangle \models \phi_1 \wedge (\phi_2 \diamond False).$$

*It is possible to show that $Level_i$ cannot be defined using the (derived) operator $\uparrow_1$. Such an operator can only be used to select the subset of elements belonging to $Level_i$ which are right sons of elements belonging to $Level_{i+1}$. For all layers $Level_i$, these elements are indeed all and only the elements that satisfy the formula $Level_i \uparrow_1 True$.*

We conclude the section by providing $UUTL$ operators with an interpretation in terms of the concrete upward unbounded layered structure (cf. Figure 3).

A node $x$ of the concrete upward unbounded layered structure satisfies the formula $\psi_1 \cdot \phi_1 U \psi_2 {-} \psi_3 U \phi_2 \cdot \phi_3$ if and only if the following conditions hold. First, $\psi_1$ holds at

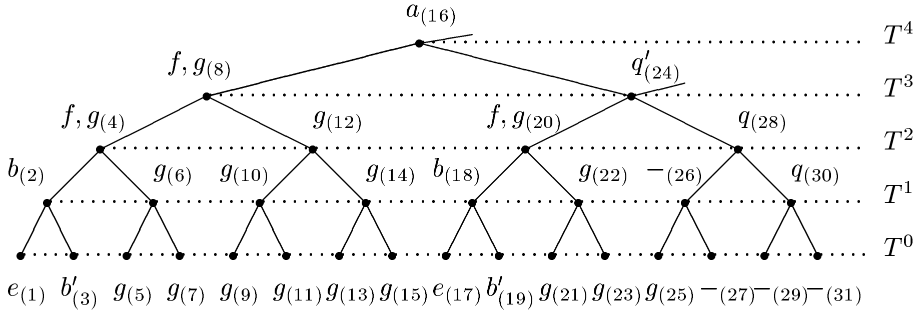


FIG. 7. A labeled concrete upward unbounded layered structure.

the closest right ancestor $x_1$ of $x$, if any. If $x$ has not a right ancestor, then it belongs to the left-edge of the structure; in such a case, $\psi_1$ holds at $x_1 = 0$. Second, there exists a node $x_3$, belonging to the left subtree of $x$, such that $\psi_3$ holds at $x_3$ and $\psi_2$ holds at the closest right ancestor $x_2$ of $x_3$. Third, $\phi_3$ holds at all left ancestors of $x_3$, belonging to the left subtree of $x$. Fourth, and last, $\phi_1$ (resp. $\phi_2$) holds at each node less than $x_2$ (resp. greater than $x_3$) belonging to the left subtree of $x$. As an example, consider the labeled concrete upward unbounded layered structure depicted in Figure 7, assuming that labels belong to $AP$ and that the label of node 0 is $a$. It is easy to check that the formula $a \cdot e\ U\ b - b'\ U\ g \cdot f$ holds at nodes 16 and 24.

A node $x$ satisfies the formula $\psi_1 \diamond \psi_2$ if there exists $y (\neq x)$ such that $y$ is an ancestor of $x$, $\psi_1$ holds at $y$, the path connecting $y$ to $x$ is a right path consisting of at least two nodes, and $\psi_2$ holds at every node $z$ belonging to such a path (with $z \neq y$ and $z \neq x$). With reference to Figure 7, the formula $q' \diamond q$ holds at nodes 28, 30, and 31, while the formula $True \diamond True$ holds at nodes 3, 6, 7, 14, and 15, but it does not hold at nodes 1 and 5. Furthermore, it is easy to check that the formula $(\varphi \wedge q') \diamond q$ (where $\varphi$ is the formula $a \cdot e\ U\ b - b'\ U\ g \cdot f$ given above) holds at 28, 30, and 31.

It is worth noting that the $UU$ operator allows one to express until-like conditions over (descending) left paths , while the $\diamond$ operator allows one to express until-like conditions over (ascending) right paths. As an example, the condition stating that a given formula $\phi$ holds at all intermediate nodes of a left path from a node $x$ to a node $y$ at which $\psi$ holds is expressed by the following $UUTL$-formula:

$$True \cdot False\ U\ True - \psi\ U\ True \cdot \phi.$$

With reference to Figure 7, if we assume $\phi$ and $\psi$ to be equal to $f$ and $b$, respectively, the above formula holds both at node 16 and at node 24.

Finally, a node $x$ satisfies the formula $\psi_1 \cdot \psi_2 \uparrow \psi_3$ if $\psi_1$ holds at the closest right ancestor $x_1$ of $x$ (at 0, if $x$ has not a right ancestor) and, for each node $y$ belonging to the left subtree of $x$, either $\psi_2$ holds at $y$ or there exists a node $z$, belonging to the left subtree of $x$, such that $z$ is a left ancestor of $y$ and $\psi_3$ holds at $z$. As an example, the formula $a \cdot g \uparrow f$ holds at node 16 (cf. Figure 7).

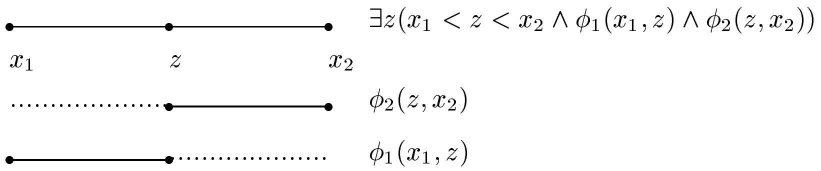


FIG. 8. The separation property for $MFO[<]$.

## 5 A separation result for $MFO[<, flip]$-formulas

As in the case of PLTL, the functional completeness of the stereo operators with respect to $MFO[<, flip]$ is based on a separation property relative to stereo-intervals.

We briefly recall the classical separation result for $MFO[<]$ [2]. Such a result basically states that any $MFO[<]$-formula $\phi(x_1, x_2)$, which refers to the elements of an interval $[x_1, x_2]$ only, can be replaced by a logically equivalent pair of formulas $\phi(x_1, z)$ and $\phi(z, x_2)$ (in fact, a set of pairs of this form), for some $z$ in between $x_1$ and $x_2$, which respectively refer to the elements of the subintervals $[x_1, z)$ (the past) and $[z, x_2]$ (the present and the future) only. The situation is graphically depicted in Figure 8. More precisely, each $MFO[<]$-formula $\phi(x_1, x_2)$, whose interpretation is restricted to an interval $[x_1, x_2]$, is equivalent to a finite disjunction of formulas belonging to the class of *Between formulas* over $[x_1, x_2]$, denoted by $BET(x_1, x_2)$, which is inductively defined as follows:

- $True$, $p(x_i)$, with $i = 1, 2$ and $p \in AP$, $x_1 \leq x_2$, and $x_1 = x_2$ are in $BET(x_1, x_2)$;
- $BET(x_1, x_2)$ is closed under Boolean connectives $\vee$, $\wedge$, and $\neg$;
- $\exists z(x_1 < z < x_2 \wedge \phi_1(x_1, z) \wedge \phi_2(z, x_2))$, where $\phi_1(x_1, z)$ belongs to $BET(x_1, z)$ and $\phi_2(z, x_2)$ belongs to $BET(z, x_2)$.

In the following, we will generalize the separation result for $MFO[<]$-formulas to deal with $MFO[<, flip]$-formulas. The normal form for $MFO[<]$-formulas is based on the notion of interval and the operation of linear split/concatenation of intervals [2]; the normal form for $MFO[<, flip]$-formulas is based on the notion of stereo-interval (cf. Definition 3.1) and the operation of composition of stereo-intervals (cf. Definition 3.2).

We start by introducing the notions of interval formula, that is, a formula whose interpretation is restricted to a stereo-interval, and of flip-closed (interval) formula. From now on, we will use a variable as endpoint of an interval meaning that the endpoint is actually the interpretation of the variable.

**Definition 5.1** (INTERVAL FORMULA)
*An $MFO[<, flip]$-formula $\phi(x_1, \ldots, x_i\text{-}x_{i+1}, \ldots, x_n)$ is an* interval formula *over the stereo-interval $[x_1, x_i] - [x_{i+1}, x_n]$ if and only if it is equivalent to an $MFO[<, flip]$-formula $\bigwedge_{j=1}^{n-1} x_j \leq x_{j+1} \wedge \theta$, where $\theta$ is a $\theta$-formula in $x_1, \ldots, x_n$. $\theta$-formulas are inductively defined as follows:*

- $p(x_k)$, $flip(x_k) = x_h$, *and* $x_h = x_k$ *are* $\theta$*-formulas;*
- *Boolean combinations of* $\theta$*-formulas are* $\theta$*-formulas;*
- *if* $\upsilon(z, x_1, \ldots, x_n)$ *is a* $\theta$*-formula, then* $\exists z(x_1 \leq z \leq x_i \wedge \upsilon(z, x_1, \ldots, x_i, x_{i+1}, \ldots, x_n))$ *and* $\exists z(x_{i+1} \leq z \leq x_n \wedge \upsilon(z, x_1, \ldots, x_i, x_{i+1}, \ldots, x_n))$ *are* $\theta$*-formulas.*

Intuitively, an interval formula $\psi(x_1, \ldots, x_i\text{-}x_{i+1}, \ldots, x_n)$ constrains (the interpretation of) each free or bound occurrence of a variable (to be ordered and) to belong to the stereo-interval $[x_1, x_i]$-$[x_{i+1}, x_n]$.

For instance, the formula $\phi$ equal to $x_1 \leq z \wedge z \leq x_2 \wedge x_1 \leq w \wedge w \leq x_2 \wedge p(z) \wedge q(z)$ is equivalent to the disjunction of two interval formulas $\phi_1(x_1, z, w, x_2)$ and $\phi_2(x_1, w, z, x_2)$ over the mono-interval $[x_1, x_2]$, with $\phi_1(x_1, z, w, x_2)$ equal to $\phi \wedge z \leq w$ and $\phi_2(x_1, w, z, x_2)$ equal to $\phi \wedge w < z$. On the contrary, the formula $\phi$ equal to $x_1 \leq x_2 \wedge x_2 \leq x_3 \wedge x_3 \leq x_4 \wedge \exists z(flip(x_4) = z)$ is neither an interval formula $\phi(x_1, x_2, x_3, x_4)$ over the mono-interval $[x_1, x_4]$ nor an interval formula $\phi(x_1, x_2\text{-}x_3, x_4)$ over the stereo-interval $[x_1, x_2\text{-}x_3, x_4]$ since in both cases $z$ is not forced to belong to the respective intervals. Finally, the formula $\phi$ equal to $x_1 \leq x_2 \wedge x_2 \leq x_3 \wedge x_3 \leq x_4 \wedge flip(x_3) = x_2 \wedge flip(x_4) = x_1 \wedge \exists z(flip(x_4) = z)$ is both an interval formula $\phi(x_1, x_2, x_3, x_4)$ over the mono-interval $[x_1, x_4]$ and an interval formula $\phi(x_1, x_2\text{-}x_3, x_4)$ over the stereo-interval $[x_1, x_2\text{-}x_3, x_4]$ since it is equivalent to the formula $\phi'$ equal to $x_1 \leq x_2 \wedge x_2 \leq x_3 \wedge x_3 \leq x_4 \wedge flip(x_3) = x_2 \wedge flip(x_4) = x_1 \wedge (\exists z(x_3 \leq z \leq x_4 \wedge flip(x_4) = z) \vee \exists z(x_1 \leq z \leq x_2 \wedge flip(x_4) = z))$.

It is immediate to see that any interval formula $\psi(x_1, \ldots, x_i\text{-}x_{i+1}, \ldots, x_n)$ is also an interval formula $\psi(x_1, \ldots, x_n)$, while the converse does not hold. Whenever the stereo-interval underlying an interval formula is flip-closed, the interpretation of the interval formula depends only on elements belonging to the stereo-interval. This situation is captured by the following definition.

**Definition 5.2** (Flip-closed formula)
*An interval formula* $\phi(x_1, \ldots, x_i\text{-}x_{i+1}, \ldots, x_n)$ *is* flip-closed *if and only if it is equivalent to an* $MFO[<, flip]$*-formula of the form*

$$flip(x_{i+1}) = x_i \wedge (BreakP(x_1, x_n) \vee \exists w(flip(w) = x_1 \wedge \\ \wedge BreakP_{x_1,w}(x_n))) \wedge \psi(x_1, \ldots, x_i\text{-}x_{i+1}, \ldots, x_n),$$

*where* $\psi(x_1, \ldots, x_i\text{-}x_{i+1}, \ldots, x_n)$ *is an interval formula.*

Any formula $\phi(x_1, \ldots, x_n)$ can be (trivially) viewed as a flip-closed formula $\phi'(x_0, x_1, \ldots, x_n, x_{n+1})$ equal to $\phi(x_1, \ldots, x_n) \wedge x_0 = 0 \wedge \forall z(z < x_{n+1})$ over the stereo-interval $[x_0, x_{n+1}] = [0, \infty]$.

Our separation result consists in showing that any (flip-closed) interval formula can be rewritten as a flip-closed formula whose subformulas are flip-closed formulas as well. The key notions are those of *Mono-Between* formulas (over mono-intervals) and *Stereo-Between* formulas (over stereo-intervals), *M-BET* and *S-BET* for short, respectively. *M-BET* and *S-BET* formulas allow us to localize temporal constraints over mono- and stereo-intervals, thus playing the same role of *Between* formulas in the proof of the separation result for $MFO[<]$. A graphical account of Mono-Between and Stereo-Between formulas is given in Figure 9. They are formally defined as follows.

**Definition 5.3** ($M$-$BET$ AND $S$-$BET$ FORMULAS)
*The class of Mono-Between formulas over the mono-interval* $[x_1, x_2]$, *denoted by* $M$-$BET(x_1, x_2)$, *is inductively defined as follows:*

1. *any formula of the form True,* $p(x_1)$, *with* $p \in AP$, *and* $flip(x_2) = x_1$ *is an* $M$-$BET(x_1, x_2)$*-formula;*
2. *Boolean combination of* $M$-$BET(x_1, x_2)$*-formulas are* $M$-$BET(x_1, x_2)$*-formulas;*
3. $\exists z(x_1 < z < x_2 \wedge flip(x_2) = z \wedge \phi_1 \wedge \phi_2)$, *where* $\phi_1$ *is an* $M$-$BET(x_1, z)$*-formula and* $\phi_2$ *is an* $M$-$BET(z, x_2)$*-formula, is an* $M$-$BET(x_1, x_2)$*-formula (Figure 9.a);*
4. $\exists w(x_1 < w < x_2 \wedge flip(w) = x_1 \wedge \phi_1 \wedge \phi_2)$, *where* $\phi_1$ *is an* $S$-$BET(x_1, x_1$-$w, x_2)$*-formula and* $\phi_2$ *is an* $M$-$BET(x_1, w)$*-formula, is an* $M$-$BET(x_1, x_2)$*-formula (Figure 9.b);*
5. $\exists z, w(x_1 < z < w < x_2 \wedge flip(w) = z \wedge \phi_1 \wedge \phi_2)$, *where* $\phi_1$ *is an* $S$-$BET(x_1, z$-$w, x_2)$*-formula and* $\phi_2$ *is an* $M$-$BET(z, w)$*-formula, is an* $M$-$BET(x_1, x_2)$*-formula (Figure 9.c).*

*The class of Stereo-Between formulas over the stereo-interval* $[x_1, x_2$-$x_3, x_4]$, *denoted by* $S$-$BET(x_1, x_2$-$x_3, x_4)$, *is inductively defined as follows:*

1. *any formula of the form True,* $p(x_i)$, *with* $p \in AP$ *and* $1 \leq i \leq 3$, $x_i = x_j$, *with* $1 \leq i, j \leq 4$, *and* $flip(x_j) = x_i$, *with* $1 \leq i < j \leq 4$, *is an* $S$-$BET(x_1, x_2$-$x_3, x_4)$*-formula;*
2. *Boolean combinations of* $S$-$BET(x_1, x_2$-$x_3, x_4)$*-formulas are* $S$-$BET(x_1, x_2$-$x_3, x_4)$*-formulas;*
3. $flip(x_3) = x_2 \wedge \phi_1 \wedge \phi_2$, *where* $\phi_1$ *is an* $M$-$BET(x_1, x_2)$*-formula and* $\phi_2$ *is an* $M$-$BET(x_3, x_4)$*-formula, is an* $S$-$BET(x_1, x_2$-$x_3, x_4)$*-formula (Figure 9.d);*
4. $\exists z(x_1 < z < x_2 \wedge flip(x_i) = z \wedge \phi_1 \wedge \phi_2)$, *where* $\phi_1$ *is an* $S$-$BET(x_1, z$-$x_i, x_4)$*-formula,* $\phi_2$ *is an* $S$-$BET(z, x_2$-$x_3, x_i)$*-formula, and* $i \in \{3, 4\}$, *is an* $S$-$BET(x_1, x_2$-$x_3, x_4)$*-formula (Figure 9.e, with* $i = 4$*);*
5. $\exists w(x_3 < w < x_4 \wedge flip(w) = x_i \wedge \phi_1 \wedge \phi_2)$, *where* $\phi_1$ *is an* $S$-$BET(x_1, x_i$-$w, x_4)$*-formula,* $\phi_2$ *is an* $S$-$BET(x_i, x_2$-$x_3, w)$*-formula, and* $i \in \{1, 2\}$, *is an* $S$-$BET(x_1, x_2$-$x_3, x_4)$*-formula (Figure 9.f, with* $i = 1$*);*
6. $\exists z(x_1 < z < x_2 \wedge \exists w(x_3 < w < x_4 \wedge flip(w) = z \wedge \phi_1 \wedge \phi_2))$, *where* $\phi_1$ *is an* $S$-$BET(x_1, z$-$w, x_4)$*-formula and* $\phi_2$ *is an* $S$-$BET(z, x_2$-$x_3, w)$*-formula, is an* $S$-$BET(x_1, x_2$-$x_3, x_4)$*-formula (Figure 9.g) .*

We now outline the main steps which we will follow to prove that any $MFO[<, flip]$-formula can be transformed into an equivalent combination of $(M$-$/S$-$)BET$-formulas. We will make use of the following notion of *internal* (free) variable: given an interval formula $\phi(x_1, \ldots, x_i$-$x_{i+1}, \ldots, x_n)$, we say that the free variable $x_1 \leq x_j \leq x_n$ is an *internal* (free) variable if and only if $x_j \notin \{x_1, x_i, x_{i+1}, x_n\}$.

We apply the following sequence of (pairs of) steps.

1. In the first step, we show that each interval formula $\phi(x_1, \ldots, x_i$-$x_{i+1}, \ldots, x_n)$ can be split into a set of interval formulas devoid of internal variables (the basic *structured formulas*, basic **s**-formulas for short, of Definition 5.4), that is, formulas in at most four free variables. Moreover, $x_1, \ldots, x_n$ turn out to be endpoints of the stereo-intervals associated with the resulting basic **s**-formulas and, possibly, further endpoints can be introduced as bound variables.

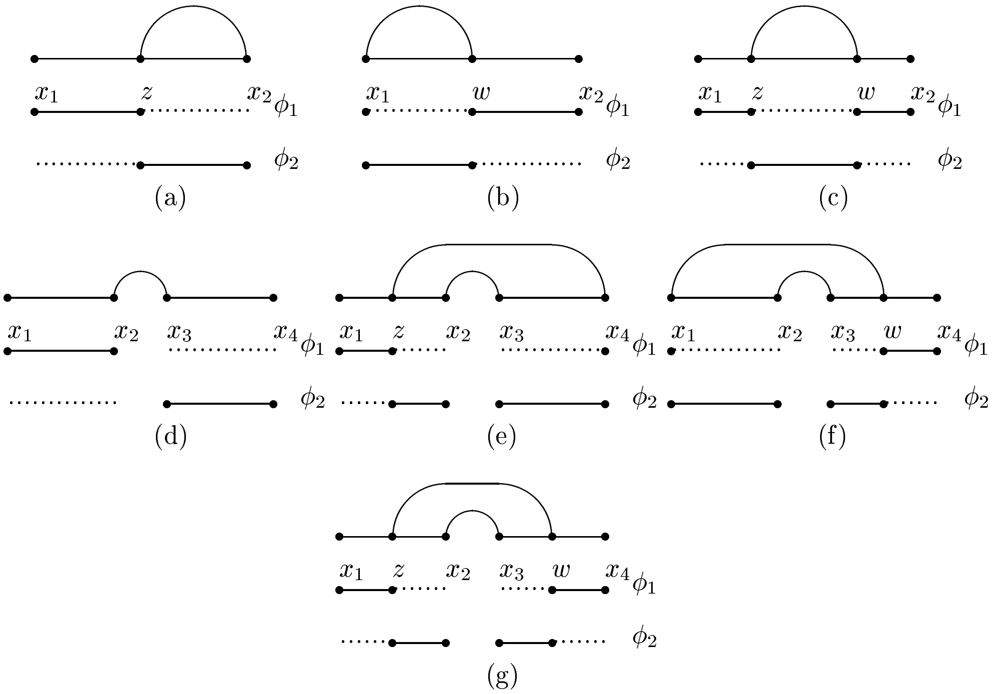


FIG. 9. Mono-Between and Stereo-Between formulas.

2. In the second step, each resulting $\mathbf{s}$-formula is singled out, its outermost quantifiers are dropped, and a formula suitable for a further (recursive) application of step 1 is produced.

A simple argument can be used to prove that the process terminates.

**Definition 5.4** ($\mathbf{s}$-FORMULAS)
*The class of* $\mathbf{s}$*-formulas is inductively defined as follows:*

1. *each interval formula* $\phi(x_1, x_2\text{-}x_3, x_4)$ *is a* basic $\mathbf{s}$*-formula;*
2. $flip(x_l) = x_i \wedge \phi_1(x_1, .., x_i\text{-}x_l, .., x_n) \wedge \phi_2(x_i, .., x_j\text{-}x_k, \dots, x_l)$*, where* $\phi_1$ *and* $\phi_2$ *are* $\mathbf{s}$*-formulas, is an* $\mathbf{s}$*-formula (over the stereo-interval* $[x_1, x_j]\text{-}[x_k, x_n]$*);*
3. $\exists z(x_i < z < x_{i+1} \wedge flip(x_l) = z \wedge \phi_1(x_1, .., z\text{-}x_l, .., x_n) \wedge \phi_2(z, .., x_j\text{-}x_k, .., x_l))$*, as well as* $\exists z(x_l < z < x_{l+1} \wedge flip(z) = x_i \wedge \phi_1(x_1, .., x_i\text{-}z, .., x_n) \wedge \phi_2(x_i, .., x_j\text{-}x_k, .., z))$*, where both* $\phi_1$ *and* $\phi_2$ *are* $\mathbf{s}$*-formulas, is an* $\mathbf{s}$*-formula (over the stereo-interval* $[x_1, x_j]\text{-}[x_k, x_n]$*);*
4. $\exists z(x_i < z < x_{i+1} \wedge \exists w(x_l < w < x_{l+1} \wedge flip(w) = z \wedge \phi_1(x_1, .., z\text{-}w, .., x_n) \wedge \phi_2(z, .., x_j\text{-}x_k, .., w)))$*, where both* $\phi_1$ *and* $\phi_2$ *are* $\mathbf{s}$*-formulas, is an* $\mathbf{s}$*-formula (over the stereo-interval* $[x_1, x_j]\text{-}[x_k, x_n]$*).*

Since interval formulas force the (non strict) ordering of variables, cases 2, 3, and 4 of the above definition can actually be collapsed into the following case:

$$\exists z(\exists w(\phi_1(x_1, \ldots, z\text{-}w, \ldots, x_n) \wedge \phi_2(z, \ldots, x_j\text{-}x_k, \ldots, w))).$$

The first step towards the normal form for $MFO[<, flip]$ is the following lemma.

**Lemma 5.5** *Each flip-closed formula $\phi(x_1,..,x_k)$ (resp. $\phi(x_1,..,x_m\text{-}x_{m+1},..,x_k)$) of quantification nesting $n$ is equivalent to a finite disjunction of flip-closed* **s***-formulas $\bigvee_i \phi_i(x_1,..,x_k)$ (resp. $\bigvee_i \phi_i(x_1,..,x_m\text{-}x_{m+1},..,x_k)$) such that each basic* **s***-formula occurring in $\phi_i$ has quantification nesting at most $max\{2, n\}$.*

In fact, it is also possible to show that each $\phi_i(x_1, \ldots, x_k)$ (resp. $\phi_i(x_1, \ldots, x_m\text{-}x_{m+1}, \ldots, x_k)$) has quantification nesting at most $5(k-2) + max\{2, n\}$ (resp. $5(k-4) + max\{2, n\}$).

Lemma 5.5 is proved by induction on the number of internal (free) variables of $\phi(x_1, \ldots, x_k)$ (resp. $\phi(x_1, \ldots, x_m\text{-}x_{m+1}, \ldots, x_k)$) by applying the Ehrenfeucht-Fraïssé theorem. In order to manage the inductive step, we need to apply a decomposition result for stereo-intervals and a decomposition result for the Ehrenfeucht-Fraïssé games played over stereo-models (roughly speaking, a stereo-model is a model over a stereo-interval). The decomposition result for stereo-intervals is the following one.

**Proposition 5.6** *Given a flip-closed interval $[x_1, x_2]$ (resp. $[x_1, x_2]$-$[x_3, x_4]$), $m$ elements $y_1 < \ldots < y_m$ such that $x_1 < y_1$ and $y_m < x_2$, (resp. $m$ elements $y_1 < \ldots < y_m$ as above and $n$ elements $z_1 < \ldots < z_n$ such that $x_3 < z_1$ and $z_n < x_4$), the given interval can be decomposed (with respect to the operation $\uplus$) in at most 6 flip-closed intervals, each one properly including a proper subset of $\{y_1, \ldots, y_m\}$ (resp. $\{y_1, \ldots, y_m, z_1, \ldots, z_n\}$).*

PROOF. We first consider the case of mono-intervals, and then that of stereo-intervals.

1. Any mono-interval $[x_1, x_2]$ can be decomposed in at most 2 flip-closed stereo-intervals: take the pair of stereo-intervals $[x_1, v]$-$[y_i, x_2]$ and $[v, y_i]$, where $v = flip(y_i)$ for any given $y_1 \leq y_i \leq y_n$. It is immediate to see that the two stereo-intervals are flip-closed and properly contain a proper subset of $\{y_1, \ldots, y_m\}$.
2. Consider now the case of the stereo-interval $[x_1, x_2]$-$[x_3, x_4]$.
   (a) If there exist $v$ and $w$ such that $x_1 \leq v \leq x_2$, $x_3 \leq w \leq x_4$, $flip(w) = v$, and $y_1 \leq v \leq y_m$ or $z_1 \leq w \leq z_n$, then $[x_1, x_2]$-$[x_3, x_4] = [x_1, v]$-$[w, x_4] \uplus [v, x_2]$-$[x_3, w]$, where both $[x_1, v]$-$[w, x_4]$ and $[v, x_2]$-$[x_3, w]$ are flip-closed, by property $(c)$ of Proposition 3.4, and properly contain a proper subset of $\{y_1, \ldots, y_m, z_1, \ldots, z_n\}$.
   (b) Otherwise, if there exist $v_1$ and $w_1$ such that $flip(w_1) = v_1$, $x_1 \leq v_1 < y_1$, and $z_n < w_1 \leq x_4$, and there exist $v_2$ and $w_2$ such that $flip(w_2) = v_2$, $y_m < v_2 \leq x_2$, and $x_3 \leq w_2 < z_1$, we first decompose the stereo-interval $[x_1, x_2]$-$[x_3, x_4]$ into the three flip-closed stereo-intervals $[x_1, v_1']$-$[w_1', x_4]$, $[v_1', v_2']$-$[w_2', w_1']$, and $[v_2', x_2]$-$[x_3, w_2']$, where
   $v_1' = max(\{v : \exists w\ flip(w) = v \wedge x_1 \leq v < y_1 \wedge z_n < w \leq x_4\})$,
   $v_2' = min(\{v : \exists w\ flip(w) = v \wedge y_m < v \leq x_2 \wedge x_3 \leq w < z_1\})$,
   $w_1' = min(\{w : flip(w) = v_1' \wedge z_n < w \leq x_4\})$, and
   $w_2' = max(\{w : flip(w) = v_2' \wedge x_3 \leq w < z_1\})$.
   It is immediate to see that $\{y_1, \ldots, y_m, z_1, \ldots, z_n\}$ is a (proper) subset of $[v_1', v_2']$-$[w_2', w_1']$. Given the definitions of $v_1', v_2', w_2'$, and $w_1'$, it is possible to decompose $[v_1', v_2']$-$[w_2', w_1']$ into the two flip-closed stereo-intervals $[v_1', v_2']$-$[w_1', w_1']$ and

$[v_2', v_2']$-$[w_2', w_1')$. If both $m$ and $n$ are greater than 0, we are done. Otherwise, suppose that $n = 0$. In such a case, the interval $[v_1', v_2']$-$[w_1', w_1']$ can be decomposed into the intervals $[v_1', v_1']$-$[w_1', w_1']$ and $[v_1', v_2']$-$[w_1', w_1')$. Since $[v_1', v_2']$-$[w_1', w_1')$ is equal to $[v_1', v_2']$, we can proceed as in case (1). The case in which $m = 0$ is perfectly symmetric.

(c) Finally, if there exist $v_2$ and $w_2$ such that $flip(w_2) = v_2$, $y_m < v_2 \leq x_2$, and $x_3 \leq w_2 < z_1$, but there exist no $v_1$ and $w_1$ such that $flip(w_1) = v_1$, $x_1 \leq v_1 < y_1$, and $z_n < w_1 \leq x_4$, we can decompose the stereo-interval $[x_1, x_2]$-$[x_3, x_4]$ into the flip-closed stereo-intervals $[x_1, v_2']$-$[w_2', x_4]$ and $[v_2', x_2]$-$[x_3, w_2']$, where $v_2'$ and $w_2'$ are defined as in case $(b)$. The stereo-interval $[x_1, v_2']$-$[w_2', x_4]$ can be further decomposed into the stereo-intervals $[x_1, v_2']$-$[x_4, x_4)$ and $[v_2', v_2']$-$[w_2', x_4]$, and then we can proceed as in case $(b)$. (Notice that it cannot happen that there exist no $v_2$ and $w_2$ such that $flip(w_2) = v_2$, $y_m < v_2 \leq x_2$, and $x_3 \leq w_2 < z_1$, because $[x_1, x_2]$-$[x_3, x_4]$ is flip-closed and thus $flip(x_3)$ belongs to $[x_1, x_2]$.)

∎

It is worth noting that 6 flip-closed stereo-intervals, involving at most 5 new endpoints, that is, endpoints which differ from $x_1, x_2, x_3, x_4$, $y_1, \ldots, y_m$, $z_1, \ldots, z_n$, are needed to deal with case $(2b)$, when either $m$ or $n$ are equal to 0. In cases (1), $(2a)$, and $(2c)$, we add at most 1, 2, and 3 new endpoints, respectively.

We introduce now the notion of stereo-model and the decomposition result for the Ehrenfeucht-Fraïssé games played over stereo-models. A *stereo-model* $\mathbf{w}$ is a pair $\langle I, f\rangle$, where $I$ is a stereo-interval and $f : I \to 2^{AP}$. The notation $\mathbf{w}_{x_1,\ldots,x_k}$ will be used to denote a stereo-model with $k$ distinguished positions $x_1, \ldots, x_k$, with $x_i \in I$ for all $i$. Stereo-models correspond to stereo-intervals and can be thought as (possibly finite) models for $MFO[<, flip]$. Hence, by referring to the usual notion of quantifier depth, we can define the equivalence relation $\equiv_n$ over stereo-models as follows:

$$\langle I, f\rangle \equiv_n \langle I', f'\rangle \quad \text{iff} \quad \langle I, f\rangle, \langle I', f'\rangle \text{ satisfy the same sentences of quantifier depth } n.$$

On the basis of $\equiv_n$, we introduce the relation $\equiv_{n,k}$ over stereo-models with $k$ distinguished positions in order to deal with formulas with $k$ free variables (cf. [15]). For any stereo-model $\mathbf{w}_{x_1,\ldots,x_k}$ with $k$ distinguished positions, let $[\mathbf{w}_{x_1,\ldots,x_k}]_{n,k}$ be the equivalence class of $\equiv_{n,k}$ $\mathbf{w}_{x_1,\ldots,x_k}$ belongs to. Since our signature is finite, by the Ehrenfeucht-Fraïssé theorem, we can conclude that $\equiv_{n,k}$ is of finite index for any $n$ and $k$. Moreover, any equivalence class can be characterized by a single $MFO[<, flip]$-formula of quantifier depth $n$ and any $MFO[<, flip]$-formula of quantifier depth $n$ is equivalent to a finite disjunction of formulas associated with $\equiv_{n,k}$-classes. Finally, we denote by $\approx_{n,k}$ the relation that holds between any two stereo-models $\langle I, f\rangle, \langle I', f'\rangle$ if and only if Duplicator has a winning strategy for any Ehrenfeucht-Fraïssé game of $n$ rounds over $\langle I, f\rangle, \langle I', f'\rangle$ with $k$ initial associations.

We also introduce a suitable notion of stereo-model composition. Given two stereo-models $\mathbf{w} = \langle I, f\rangle$ and $\mathbf{w}' = \langle I', f'\rangle$ such that (i) there exists $I \uplus I'$ and (ii) $f(x) = f'(x)$, for all $x \in I \cap I'$, we define the composition $\mathbf{w} \uplus \mathbf{w}'$ of $\mathbf{w}$ and $\mathbf{w}'$ as the pair $\langle I \uplus I', f \cup f'\rangle$. As shown by the following proposition, the notion of composition we have introduced enjoys the congruence property, with respect to $\approx_{n,k}$, at the ground of McNaughton and Papert theorem [15, 16]. Such a result generalizes to stereo-models the well-known results on linear (and partial) orders. To this end, we need to introduce a constraint on stereo-models (to be *flip-closed*), which is related to the *binary* flip operator, and that does not have a counterpart in standard models.

**Proposition 5.7** *Let* $\mathbf{w}^{I}_{x^I_1,\ldots,x^I_k} = \langle I, f^I \rangle$, $\mathbf{w}^{J}_{x^J_1,\ldots,x^J_k} = \langle J, f^J \rangle$, $\mathbf{w}^{I'}_{x^{I'}_1,\ldots,x^{I'}_h} = \langle I', f^{I'} \rangle$, *and* $\mathbf{w}^{J'}_{x^{J'}_1,\ldots,x^{J'}_h} \langle J', f^{J'} \rangle$ *be four stereo-models such that each of them is flip-closed and the set of its distinguished positions includes the endpoints, and there exist both* $\mathbf{w}^{I}_{x^I_1,\ldots,x^I_k} \uplus \mathbf{w}^{I'}_{x^{I'}_1,\ldots,x^{I'}_h}$ *and* $\mathbf{w}^{J}_{x^J_1,\ldots,x^J_k} \uplus \mathbf{w}^{J'}_{x^{J'}_1,\ldots,x^{J'}_h}$. *It holds that*

$$\begin{gathered} \mathbf{w}^{I}_{x^I_1,\ldots,x^I_k} \approx_{n,k} \mathbf{w}^{J}_{x^J_1,\ldots,x^J_k} \text{ and } \mathbf{w}^{I'}_{x^{I'}_1,\ldots,x^{I'}_h} \approx_{n,h} \mathbf{w}^{J'}_{x^{J'}_1,\ldots,x^{J'}_h} \\ \text{if and only if} \\ \mathbf{w}^{I}_{x^I_1,\ldots,x^I_k} \uplus \mathbf{w}^{I'}_{x^{I'}_1,\ldots,x^{I'}_h} \approx_{n,l} \mathbf{w}^{J}_{x^J_1,\ldots,x^J_k} \uplus \mathbf{w}^{J'}_{x^{J'}_1,\ldots,x^{J'}_h}, \end{gathered} \tag{5.1}$$

*where* $k + h - 2 \leq l \leq k + h - 1$ *(*$I$ *and* $I'$ *share at most two positions and at least one position; the same for* $J$ *and* $J'$*).*

Proof. ($\Rightarrow$) By hypothesis, Duplicator has a winning strategy $s_1$ for a game over $I$ and $J$ as well as a winning strategy $s_2$ for a game over $I'$ and $J'$. A winning strategy of Duplicator over $I \uplus I'$ and $J \uplus J'$ can be obtained by exploiting strategy $s_1$ whenever Spoiler picks up an element in $I$ or $J$, and by exploiting strategy $s_2$ whenever Spoiler picks up an element in $I'$ or $J'$. To prove that such a strategy is a winning strategy for Duplicator we have to show that, for any pair of elements $x, y$ picked up from $I \uplus I'$ and the corresponding pair of elements $x', y'$ in $J \uplus J'$, it holds that (i) $x < y$ if and only if $x' < y'$ and (ii) $flip(y) = x$ if and only if $flip(y') = x'$. The only nontrivial case is the case in which $x \in I - I'$ and $y \in I' - I$, or vice versa. Let $I^L$ and $I^R$ be the left and right components of the stereo-interval $I$, respectively. Since the set of distinguished positions associated with the stereo-models includes the endpoints of the stereo-intervals, it holds that $x \in I^L$ (resp. $x \in I^R$) if and only if $x' \in J^L$ (resp. $x' \in J^R$), and the same for $y, y'$. This allows us to conclude that $x < y$ if and only if $x' < y'$. Furthermore, since the stereo-intervals are flip-closed, $flip(y) \neq x$ and $flip(y') \neq x'$.

($\Leftarrow$) By hypothesis, Duplicator has a winning strategy $s$ for a game over $I \uplus I'$ and $J \uplus J'$. Since the set of distinguished positions associated with the stereo-models includes the endpoints of the stereo-intervals, if the Spoiler makes his choices only in $I$ and $J$ (resp. $I'$ and $J'$), then Duplicator is obliged to make his choices in $I$ and $J$ too (resp. $I'$ and $J'$). ∎

We are now able to prove Lemma 5.5. The proof is given in the Appendix.

Some further steps are needed to guarantee that an **s**-formula satisfies the separation property. First, **s**-formulas may differ from $M$-/$S$-$BET$-formulas on the presence of internal (free) variables. Furthermore, their basic **s**-formulas may fail to satisfy the separation property. Such weaknesses can be recovered as follows.

Let a *preM-BET* (resp. *preS-BET*) formula be an **s**-formula $\phi$ over a mono-interval (resp. stereo-interval) such that each basic **s**-formula occurring in it is an $M$-/$S$-$BET$-formula.

**Proposition 5.8** *Let $\chi(x_1, z, x_2)$ (resp. $\phi(x_1, z, x_2\text{-}x_3, x_4)$, $\psi(x_1, x_2\text{-}x_3, z, x_4)$) be a preM-BET-formula (resp. a preS-BET-formula). The formula $\exists z\ \chi(x_1, z, x_2)$ (resp. $\exists z\ \phi(x_1, z, x_2\text{-}x_3, x_4)$, $\exists z\ \psi(x_1, x_2\text{-}x_3, z, x_4)$) is equivalent to an M-BET (resp. S-BET) formula.*

The proof is given in the Appendix.

The following theorem proves the separation result for flip-closed formulas, devoid of internal (free) variables, over mono- and stereo-intervals. The subsequent corollary generalizes such a result to any flip-closed formula.

**Theorem 5.9** *Any flip-closed formula $\phi(x_1, x_2)$ (resp. $\phi(x_1, x_2\text{-}x_3, x_4)$) is equivalent to an M-BET$(x_1, x_2)$ (resp. S-BET$(x_1, x_2\text{-}x_3, x_4)$) formula.*

PROOF. The proof is by induction on the quantification nesting $n$ of $\phi$ and it is explicitly given only for flip-closed formulas over stereo-intervals. The easier case of flip-closed formulas over mono-intervals can be proved in a similar way.

Case $n = 0$. Obvious. In particular, notice that a Boolean combination of basic *S-BET*$(x_1, x_2\text{-}x_3, x_4)$ formulas is an *S-BET*$(x_1, x_2\text{-}x_3, x_4)$ formula (basic *S-BET* formulas are *S-BET* formulas defined by means of rules 1 and 3 of Definition 5.3).

Case $n = 1$. Since *S-BET* formulas are closed under the application of Boolean connectives, it suffices to prove the thesis for $\phi(x_1, x_2\text{-}x_3, x_4)$ of the forms $\exists z(x_1 < z < x_2 \wedge \psi(x_1, z, x_2\text{-}x_3, x_4))$ or $\exists z(x_3 < z < x_4 \wedge \psi(x_1, x_2\text{-}x_3, z, x_4))$, where $\psi(x_1, z, x_2\text{-}x_3, x_4)$ and $\psi(x_1, x_2\text{-}x_3, z, x_4)$ are conjunctions of (negated) atomic formulas. We consider formulas of the first form (formulas of the second form can be dealt with analogously). Without loss of generality, we also assume that if either $x_i < x_j$ or $x_i = x_j$ occur in $\psi$, then $j = i + 1$, and that if either $x_i < z$ or $x_i = z$ or $z < x_j$ or $z = x_j$ occur in $\psi$, then $i = 1$ and $j = 2$. This means that (i) $\leq$ is rewritten in terms of $<$ and $=$, (ii) any application of $<$ (resp. $=$) to non-consecutive pairs of points is replaced by a suitable set of applications of $<$ (resp. $=$) to consecutive pairs of points (in the obvious way), and (iii) any contradiction with respect to the given ordering of points $x_1, z, x_2, x_3$, and $x_4$ is preliminarily dealt with, that is, replaced by $False$.

According to Proposition 5.6, given a flip-closed stereo-interval $[x_1, x_2]$-$[x_3, x_4]$ and an element $z$ in between $x_1$ and $x_2$, $[x_1, x_2]$-$[x_3, x_4]$ can be decomposed into a set of flip-closed intervals such that at least one of them has $z$ as one of its endpoints. We exploit such a decomposition to replace the formula $x_1 < z < x_2 \wedge \psi(x_1, z, x_2\text{-}x_3, x_4)$ by the equivalent formula $x_1 < z < x_2 \wedge \psi'(x_1, z, x_2\text{-}x_3, x_4)$, where $\psi'(x_1, z, x_2\text{-}x_3, x_4)$ is defined as follows (for the sake of readability, we isolate the six disjuncts 5.2-5.7 of $\psi'(x_1, z, x_2\text{-}x_3, x_4)$):

$$flip(x_4) = z \wedge \psi_{1,1}(x_1, z\text{-}x_4, x_4) \wedge \psi_{1,2}(z, x_2\text{-}x_3, x_4) \vee \tag{5.2}$$

$$\exists w(x_3 < w < x_4 \wedge flip(w) = z \wedge \psi_{2,1}(x_1, z\text{-}w, x_4) \wedge \psi_{2,2}(z, x_2\text{-}x_3, w)) \vee \tag{5.3}$$

$$\begin{gathered} \neg(\exists k \exists w(x_1 \leq k \leq x_2 \wedge x_3 < w < x_4 \wedge flip(w) = k)) \wedge \\ \psi_{3,1}(x_2, x_2\text{-}x_3, x_3) \wedge \psi_{3,2}(x_1, x_1\text{-}x_4, x_4) \wedge \exists w(x_1 \leq w < x_2 \wedge \\ flip(z) = w \wedge \psi_{3,3}(x_1, w\text{-}z, x_2) \wedge \psi_{3,4}(w, z)) \vee \end{gathered} \tag{5.4}$$

$$\begin{array}{c}\exists v_1 \exists w_1 (x_1 \le v_1 < z \wedge x_3 < w_1 \le x_4 \wedge flip(w_1) = v_1 \wedge \\ \neg(\exists k \exists w (v_1 \le k \le x_2 \wedge x_3 < w < w_1 \wedge flip(w) = k)) \wedge \\ \psi_{4,1}(x_1, v_1\text{-}w_1, x_4) \wedge \psi_{4,2}(x_2, x_2\text{-}x_3, x_3) \wedge \exists w(v_1 \le w < x_2 \wedge \\ flip(z) = w \wedge \psi_{4,3}(v_1, w\text{-}z, x_2) \wedge \psi_{4,4}(w, z))) \vee\end{array} \tag{5.5}$$

$$\begin{array}{c}\exists v_1 \exists w_1 (z < v_1 \le x_2 \wedge x_3 < w_1 \le x_4 \wedge flip(w_1) = v_1 \wedge \\ \psi_{5,1}(x_1, x_1\text{-}x_4, x_4) \wedge \psi_{5,2}(v_1, x_2\text{-}x_3, w_1) \wedge \\ \neg(\exists k \exists w (x_1 \le k \le v_1 \wedge w_1 < w < x_4 \wedge flip(w) = k)) \wedge \\ \exists w(x_1 \le w < v_1 \wedge flip(z) = w \wedge \psi_{5,3}(x_1, w\text{-}z, v_1) \wedge \psi_{5,4}(w, z)) \vee\end{array} \tag{5.6}$$

$$\begin{array}{c}\exists v_1 \exists w_1 (x_1 \le v_1 < z \wedge x_3 < w_1 \le x_4 \wedge flip(w_1) = v_1 \wedge \psi_{6,1}(x_1, v_1\text{-}w_1, x_4) \wedge \\ \exists v_2 \exists w_2 (z < v_2 \le x_2 \wedge x_3 < w_2 \le w_1 \wedge flip(w_2) = v_2 \wedge \psi_{6,2}(v_2, x_2\text{-}x_3, w_2) \wedge \\ \neg(\exists k \exists w (v_1 \le k \le v_2 \wedge w_2 < w < w_1 \wedge flip(w) = k)) \wedge \\ \exists w(v_1 \le w < v_2 \wedge flip(z) = w \wedge \psi_{6,3}(v_1, w\text{-}z, v_2) \wedge \psi_{6,4}(w, z)))).\end{array} \tag{5.7}$$

All subformulas $\psi_{i,j}(y_1, y_2\text{-}y_3, y_4)$ are obtained from the original formula $\psi(x_1, z, x_2\text{-}x_3, z, x_4)$ by replacing each (negated) atom $(\neg)\phi$ occurring in it by the formula $\gamma((\neg)\phi, y_1, y_2, y_3, y_4)$, which is defined as follows:

$$\begin{array}{ll}\gamma((\neg)\phi, y_1, y_2, y_3, y_4) = (\neg)\phi & \text{if } var(\phi) \subseteq \{y_1, y_2, y_3, y_4\}; \\[2ex] \gamma((\neg)\phi, y_1, y_2, y_3, y_4) = True & \text{if } var(\phi) \cap \{y_1, y_2, y_3, y_4\} = \emptyset; \\[2ex] \gamma(\phi, y_1, y_2, y_3, y_4) = False & \text{if } var(\phi) \cap \{y_1, y_2, y_3, y_4\} \neq \emptyset, \\ & var(\phi) \not\subseteq \{y_1, y_2, y_3, y_4\}, \text{ and} \\ & \phi \text{ of the forms } flip, =; \\[2ex] \gamma(\phi, y_1, y_2, y_3, y_4) = True & \text{if } var(\phi) \cap \{y_1, y_2, y_3, y_4\} \neq \emptyset, \\ & var(\phi) \not\subseteq \{y_1, y_2, y_3, y_4\}, \text{ and} \\ & \phi \text{ of the form } <; \\[2ex] \gamma(\neg\phi, y_1, y_2, y_3, y_4) = True & \text{if } var(\phi) \cap \{y_1, y_2, y_3, y_4\} \neq \emptyset, \\ & var(\phi) \not\subseteq \{y_1, y_2, y_3, y_4\}, \text{ and} \\ & \phi \text{ of the forms } flip, =; \\[2ex] \gamma(\neg\phi, y_1, y_2, y_3, y_4) = False & \text{if } var(\phi) \cap \{y_1, y_2, y_3, y_4\} \neq \emptyset, \\ & var(\phi) \not\subseteq \{y_1, y_2, y_3, y_4\}, \text{ and} \\ & \phi \text{ of the form } <,\end{array} \tag{5.8}$$

where $var(\phi)$ denotes the set of variables in $\phi$. Notice that $\gamma((\neg)True, y_1, y_2, y_3, y_4) = (\neg)True$, since $var((\neg)True) = \emptyset$ and $\emptyset \subseteq \{y_1, y_2, y_3, y_4\}$.

Taking advantage of the decomposition of flip-closed stereo-intervals given in Proposition 5.6, it is easy to check that $x_1 < z < x_2 \wedge \psi'(x_1, z, x_2\text{-}x_3, x_4)$ is a *preS-BET* formula which is equivalent to $x_1 < z < x_2 \wedge \psi(x_1, z, x_2\text{-}x_3, x_4)$. From Proposition 5.8, it follows that the formula $\exists z(x_1 < z < x_2 \wedge \psi'(x_1, z, x_2\text{-}x_3, x_4))$ is equivalent to an *S-BET*-formula.

Case $n = 2$. Since *S-BET*-formulas are closed under the application of Boolean connectives, it suffices to prove the thesis for a formula of the form $\exists z(x_1 < z < x_2 \wedge \psi(x_1, z, x_2\text{-}x_3, x_4))$, where $\psi(x_1, z, x_2\text{-}x_3, x_4)$ abbreviates $\theta(x_1, z, x_2\text{-}x_3, x_4) \wedge (\neg)\exists w(x_3$

$< w < x_4 \wedge \nu(z,w) \wedge \chi(x_1, x_2\text{-}x_3, w, x_4))$. Formulas $\theta(x_1, z, x_2\text{-}x_3, x_4)$, $\nu(z,w)$, and $\chi(x_1, x_2\text{-}x_3, w, x_4)$ are conjunction of (negated) atoms involving the variable $z$ (and $x_1, x_2, x_3, x_4$), the variables $z$ and $w$ (but not $x_1, x_2, x_3, x_4$), and the variable $w$ (and $x_1, x_2, x_3, x_4$), respectively. In the following, we restrict our analysis to the most difficult case in which $\psi(x_1, z, x_2\text{-}x_3, x_4)$ has the form $\theta(x_1, z, x_2\text{-}x_3, x_4) \wedge \neg\exists w(x_3 < w < x_4 \wedge \nu(z,w) \wedge \chi(x_1, x_2\text{-}x_3, w, x_4))$.

As in the case $n = 1$ (disjuncts 5.2-5.7), given a flip-closed stereo-interval $[x_1, x_2]$-$[x_3, x_4]$ and an element $z$ in between $x_1$ and $x_2$, $[x_1, x_2]$-$[x_3, x_4]$ can be decomposed into a set of flip-closed intervals such that at least one of them has $z$ as one of its endpoints. As an example, the analogous of formula 5.3 is the formula:

$$\begin{array}{l}\exists h(x_3 < h < x_4 \wedge flip(h) = z \wedge \theta_1(x_1, z\text{-}h, x_4) \wedge \theta_2(z, x_2\text{-}x_3, h) \wedge \\ \quad \neg\exists w(w = h \wedge \nu(z,w) \wedge \chi_1(x_1, z\text{-}h, x_4) \wedge \chi_2(z, x_2\text{-}x_3, h)) \wedge \\ \neg\exists w(h < w < x_4 \wedge \nu(z,w) \wedge \chi_3(x_1, z\text{-}h, w, x_4) \wedge \chi_4(z, x_2\text{-}x_3, h)) \wedge \\ \neg\exists w(x_3 < w < h \wedge \nu(z,w) \wedge \chi_5(z, x_2\text{-}x_3, w, h) \wedge \chi_6(x_1, z\text{-}h, x_4))),\end{array} \tag{5.9}$$

where all $\theta_i$ (resp. $\chi_i$) are obtained from $\theta$ (resp. $\chi$) by replacing each occurrence of a (negated) atom by its transformation through $\gamma$ (cf. definition 5.8 of $\gamma$). Formulas $\neg\exists w(w = h \wedge \nu(z,w) \wedge \chi_1(x_1, z\text{-}h, x_4) \wedge \chi_2(z, x_2\text{-}x_3, h))$, $\neg\exists w(h < w < x_4 \wedge \nu(z,w) \wedge \chi_3(x_1, z\text{-}h, w, x_4) \wedge \chi_4(z, x_2\text{-}x_3, h))$, and $\neg\exists w(x_3 < w < h \wedge \nu(z,w) \wedge \chi_5(z, x_2\text{-}x_3, w, h) \wedge \chi_6(x_1, z\text{-}h, x_4))$ are flip-closed formulas of quantification nesting 1. As in the case $n = 1$, this implies that they can be replaced by equivalent *S-BET* formulas, thus transforming formula 5.9 into a *preS-BET* formula. The analogous of formulas 5.2 and 5.4-5.7 can be obtained by using the same technique, thus proving that the formula $x_1 \leq z \leq x_2 \wedge \psi(x_1, z, x_2\text{-}x_3, x_4)$ is equivalent to a *preS-BET* formula $x_1 < z < x_2 \wedge \psi'(x_1, z, x_2\text{-}x_3, x_4)$. By Proposition 5.8, the formula $\exists z(x_1 < z < x_2 \wedge \psi'(x_1, z, x_2\text{-}x_3, x_4))$ is equivalent to an *S-BET*-formula.

Case $n > 2$. By standard manipulation of Boolean connectives, the formula $\phi(x_1, x_2\text{-}x_3, x_4)$ can be rewritten as $\bigvee_i(\phi_{i,1}(x_1, x_2\text{-}x_3, x_4) \wedge \ldots \wedge \phi_{i,n_i}(x_1, x_2\text{-}x_3, x_4))$, where every $\phi_{i,j}(x_1, x_2\text{-}x_3, x_4)$ is a flip-closed formula of quantification nesting at most $n$. Each formula $\phi_{i,j}(x_1, x_2\text{-}x_3, x_4)$ is either (a negation of) an atomic formula or a formula of the form $(\neg)\exists z(x_1 < z < x_2 \wedge \psi(x_1, z, x_2\text{-}x_3, x_4))$ or $(\neg)\exists z(x_3 < z < x_4 \wedge \psi(x_1, x_2\text{-}x_3, z, x_4))$.

Let us assume that $\phi_{i,j}(x_1, x_2\text{-}x_3, x_4)$ is of the form $\exists z(x_1 < z < x_2 \wedge \psi(x_1, z, x_2\text{-}x_3, x_4))$ (the case $\exists z(x_3 < z < x_4 \wedge \psi(x_1, x_2\text{-}x_3, z, x_4))$ can be treated analogously). By Lemma 5.5, it holds that the flip-closed formula $x_1 < z < x_2 \wedge \psi(x_1, z, x_2\text{-}x_3, x_4)$ is equivalent to a finite disjunction $\bigvee_j \chi_j(x_1, z, x_2\text{-}x_3, x_4)$, where each $\chi_j(x_1, z, x_2\text{-}x_3, x_4)$ is an **s**-formula such that every basic **s**-formula occurring in it has quantification nesting at most $n - 1(\geq 2)$. Let $\gamma$ be a basic **s**-formula occurring in $\chi_j$. By the inductive hypothesis, $\gamma$ is equivalent to an *S-BET*-formula $\gamma'$. Now, let $\chi'_j(x_1, z, x_2\text{-}x_3, x_4)$ be the formula obtained by replacing each occurrence of a basic **s**-formula $\gamma$ in $\chi_j(x_1, z, x_2\text{-}x_3, x_4)$ by the equivalent *S-BET*-formula $\gamma'$. Since $\chi'_j(x_1, z, x_2\text{-}x_3, x_4)$ is a *preS-BET*-formula, by Proposition 5.8, it holds that $\exists z\ \chi'_j(x_1, z, x_2\text{-}x_3, x_4)$ is equivalent to an *S-BET*-formula $\chi''_j(x_1, x_2\text{-}x_3, x_4)$. Therefore, $\exists z(x_1 < z < x_2 \wedge \psi(x_1, x_2\text{-}x_3, z, x_4))$ is equivalent to the *S-BET*-formula $\bigvee_j \chi''_j(x_1, x_2\text{-}x_3, x_4)$.

If $\phi_{i,j}(x_1, x_2\text{-}x_3, x_4)$ has the form $\neg\exists z(x_1 < z < x_2 \wedge \psi(x_1, z, x_2\text{-}x_3, x_4))$, then it is equivalent to an *S-BET*-formula $\neg\psi'(x_1, x_2\text{-}x_3, x_4)$, where $\psi'(x_1, x_2\text{-}x_3, x_4)$ is an *S-*

$MFO[<, flip]$**-formulas**

$\downarrow$

**(**$M$**-/**$S$**-)**$BET$**-formulas** — Normal form guaranteeing a separation property

$\downarrow$

**(**$M$**-/**$S$**-)**$DEC$**-formulas** — Formulas which are the semantics of $UUTL$-formulas

$\downarrow$

$UUTL$**-formulas**

FIG. 10. The main steps of the proof of expressive completeness: a summary.

$BET$-formula equivalent to $\exists z(x_1 < z < x_2 \wedge \psi(x_1, z, x_2\text{-}x_3, x_4))$. It is now easy to see that $\phi(x_1, x_2\text{-}x_3, x_4) = \bigvee_i (\phi_{i,1}(x_1, x_2\text{-}x_3, x_4) \wedge \ldots \wedge \phi_{i,n_i}(x_1, x_2\text{-}x_3, x_4))$ is equivalent to an $S$-$BET$-formula, since each $\phi_{i,j}$ is equivalent to an $S$-$BET$-formula. ∎

**Corollary 5.10** *Each flip-closed $MFO[<, flip]$-formula $\phi(x_1, \ldots, x_k)$ over a mono-interval (resp. $\phi(x_1, \ldots, x_m\text{-}x_{m+1}, \ldots, x_k)$ over a stereo-interval) is equivalent to a flip-closed pre$M$-$BET$ (resp. pre$S$-$BET$) formula.*

Corollary 5.10 easily follows from Lemma 5.5 and Theorem 5.9.

## 6 Expressive completeness of $UUTL$ with respect to $MFO[<, flip]$

In the previous section, we have shown that each $MFO[<, flip]$-formula can be turned into an equivalent *(pre)*$M$-/$S$-$BET$-formula satisfying the separation property. In this section, in order to prove the expressive completeness of $UUTL$ with respect to (the class of sentences of) $MFO[<, flip]$, we translate *(pre)*$M$-/$S$-$BET$-formulas into a more restrictive, and somewhat technical, normal form. Intuitively, the formulas which enjoy such a new normal form, called *Decomposition Formulas* ($M$-/$S$-$DEC$-formulas for short), are $MFO[<, flip]$-formulas which correspond to (the semantics of) $UUTL$-formulas. We will show that *(pre)*$M$-/$S$-$BET$ formulas can be translated into an equivalent combination of $M$-/$S$-$DEC$-formulas and that $M$-/$S$-$DEC$-formulas can be easily mapped into $UUTL$-formulas. The steps of the whole transformation which leads to the expressive completeness result are summarized in Figure 10.

We first define an operator $T(\gamma, x)$ which provides any $UUTL$-formula $\gamma$ with an $MFO[<, flip]$-formula $\psi(x)$ which expresses the semantics of $\gamma$ in the case in which $\gamma$ is evaluated at the point $x$ of the mono-interval $[0, x]$. In order to deal with formulas over stereo-intervals (this is the case of $S$-$BET$ formulas), we have to show how to interpret an $UUTL$-formula $\gamma$ over stereo-intervals. To this end, we define a generalized version $T(\gamma, y_1\text{-}y_2, x)$ of the operator $T(\gamma, x)$ which provides any $UUTL$-formula

$\gamma$ with an $MFO[<, flip]$-formula $\psi(y_1, y_2, x)$ which expresses the semantics of $\gamma$ in the case in which $\gamma$ is evaluated at the point $x$ of the stereo-interval $[0, y_1]$-$[y_2, x]$.

**Definition 6.1** *For any given UUTL-formula $\phi$, we denote by $T(\phi, y_2\text{-}y_3, x)$ the $MFO[<, flip]$-formula recursively defined as follows (the definition of $T(\phi, x)$ is obtained from that of $T(\phi, y_2\text{-}y_3, x)$ by removing any constraint of the form $x_i \leq y_j$ or $y_j \leq x_i$, with $i = 1, 2, 3$ and $j = 2, 3$):*

1. $T(Q_a, y_2\text{-}y_3, x) = Q_a(x)$;
2. $T(\phi_1 \wedge \phi_2, y_2\text{-}y_3, x) = T(\phi_1, y_2\text{-}y_3, x) \wedge T(\phi_2, y_2\text{-}y_3, x)$ *and* $T(\phi_1 \wedge \phi_2, y_2\text{-}y_3, \infty) = T(\phi_1, y_2\text{-}y_3, \infty) \wedge T(\phi_2, y_2\text{-}y_3, \infty)$;
3. $T(\neg\phi, y_2\text{-}y_3, x) = \neg T(\phi, y_2\text{-}y_3, x)$ *and* $T(\neg\phi, y_2\text{-}y_3, \infty) = \neg T(\phi, y_2\text{-}y_3, \infty)$;
4. $T(\psi_1 \cdot \phi_1 U \psi_2 - \psi_3 U \phi_2 \cdot \phi_3, y_2\text{-}y_3, x) =$
$\exists x_1, x_2, x_3 (x_2 \leq y_2 \leq y_3 \leq x_3 < x \wedge flip(x) = x_1 \wedge flip(x_3) = x_2 \wedge T(\psi_1, x_1) \wedge T(\psi_2, x_2) \wedge T(\psi_3, y_2\text{-}y_3, x_3) \wedge \forall y((x_1 < y < x_2 \to T(\phi_1, y)) \wedge (x_3 < y < x \to (T(\phi_2, y) \wedge (\exists z(x_1 \leq z \leq x_2 \wedge flip(y) = z) \to T(\phi_3, y_2\text{-}y_3, y))))))$;
5. $T(\psi_1 \cdot \phi_1 U \psi_2 - \psi_3 U \phi_2 \cdot \phi_3, y_2\text{-}y_3, \infty) =$
$\exists x_2, x_3 (x_2 \leq y_2 \leq y_3 \leq x_3 \wedge flip(x_3) = x_2 \wedge T(\psi_1, 0) \wedge T(\psi_2, x_2) \wedge T(\psi_3, y_2\text{-}y_3, x_3) \wedge \forall y((0 < y < x_2 \to T(\phi_1, y)) \wedge (y > x_3 \to (T(\phi_2, y) \wedge (\exists z(0 \leq z \leq x_2 \wedge flip(y) = z) \to T(\phi_3, y_2\text{-}y_3, y))))))$;
6. $T(\psi \diamond \phi, y_2\text{-}y_3, x) =$
$\exists x_1 (y_3 \leq x_1 < x \wedge BreakP(x_1, x) \wedge T(\psi, y_2\text{-}y_3, x_1) \wedge \forall y((x_1 < y < x \wedge BreakP(y, x)) \to T(\phi, y)))$;
7. $T(\psi_1 \cdot \psi_2 \uparrow \psi_3, y_2\text{-}y_3, x) =$
$\exists x_1 (x_1 < x \wedge flip(x) = x_1 \wedge T(\psi_1, x_1) \wedge \forall y(x_1 < y < x \to (T(\psi_2, y) \vee \exists z, w(x_1 \leq z < y < w < x \wedge flip(w) = z \wedge T(\psi_3, w)))))$;
8. $T(\psi_1 \cdot \psi_2 \uparrow \psi_3, y_2\text{-}y_3, \infty) =$
$T(\psi_1, 0) \wedge \forall y(y > 0 \to (T(\psi_2, y) \vee \exists z, w(0 \leq z < y < w \wedge flip(w) = z \wedge T(\psi_3, w))))$.

From Definition 6.1, it immediately follows the proposition below.

**Proposition 6.2** *For any given UUTL-formula $\phi$ and model $\alpha$, $\langle \alpha, x \rangle \models \phi$ if and only if $\langle \alpha, x \rangle$ satisfies $T(\phi, x)$ and $\langle \alpha, \infty \rangle \models \phi$ if and only if $\alpha$ satisfies $T(\phi, \infty)$.*

Now, we inductively define *Mono-Decomposition* formulas, which are closely related to the semantics of $UUTL$-operators (interpreted) over mono-intervals.

**Definition 6.3** *The class of* Mono-Decomposition *formulas over a mono-interval $[x_1, x_2]$, denoted by $M\text{-}DEC(x_1, x_2)$, is inductively defined as follows:*

1. $flip(x_2) = x_1 \wedge T(\phi, x_1) \wedge \forall y(x_1 < y < x_2 \to (T(\phi', y) \vee \exists z, w(x_1 \leq z < y < w < x_2 \wedge flip(w) = z \wedge T(\phi'', w))))$;
2. $flip(x_2) = x_1 \wedge \exists z(x_1 < z < x_2 \wedge BreakP_{x_1, x_2}(z) \wedge \delta'(x_1, z) \wedge T(\psi, z) \wedge \forall y((z < y < x_2 \wedge BreakP_{x_1, x_2}(y)) \to T(\phi, y)))$,
*where $\delta'(x_1, z)$ is an $M\text{-}DEC(x_1, z)$ formula;*
3. $\neg flip(x_2) = x_1 \wedge \exists z(x_1 < z < x_2 \wedge BreakP(z, x_2) \wedge \delta'(x_1, z) \wedge T(\psi, z) \wedge \forall y((z < y < x_2 \wedge BreakP(y, x_2)) \to T(\phi, y)))$,
*where $\delta'(x_1, z)$ is an $M\text{-}DEC(x_1, z)$ formula;*

4. $flip(x_2) = x_1 \wedge \exists z, w(x_1 \leq z \leq w \leq x_2 \wedge flip(w) = z \wedge \delta'(z,w) \wedge T(\psi, x_1) \wedge$ $T(\psi', z) \wedge T(\psi'', w) \wedge \forall y(x_1 < y < z \rightarrow T(\phi, y)) \wedge \forall y(w < y < x_2 \rightarrow T(\phi', y)) \wedge$ $\forall y((w < y < x_2 \wedge \exists y'(x_1 \leq y' \leq z \wedge flip(y) = y')) \rightarrow T(\phi'', y)))$, *where* $\delta'(z,w)$ *is an* $M\text{-}DEC(z,w)$ *formula;*
5. $\neg flip(x_2) = x_1 \wedge \exists k(x_1 < k < x_2 \wedge flip(x_2) = k \wedge \delta'(x_1, k) \wedge \delta''(k, x_2))$, *where* $\delta'(x_1, z)$ *and* $\delta''(k, x_2)$ *are* $M\text{-}DEC(x_1, z)$ *and* $M\text{-}DEC(k, x_2)$ *formulas, respectively.*

Mono-decomposition formulas of the first form ($M\text{-}DEC.1$ for short) can be associated with the operator $\uparrow$, $M\text{-}DEC.2$ and $M\text{-}DEC.3$ can be associated with the operator $\diamond$, and $M\text{-}DEC.4$ can be associated with the operator $UU$. $M\text{-}DEC.5$ allows one to recursively combine the decomposition formulas. It is worth noting that the recursive nesting of decomposition formulas is performed both in a top-down fashion, with respect to the nesting of flip-related pairs (cf. $M\text{-}DEC.4$ and $M\text{-}DEC.5$), and left to right, with respect to the ordering of natural numbers (cf. $M\text{-}DEC.2$, $M\text{-}DEC.3$, and $M\text{-}DEC.5$).

The following definition extends the notion of decomposition formula to deal with formulas over stereo-intervals. *Stereo-Decomposition* formulas are closely related to the semantics of $UUTL$-operators (interpreted) over stereo-intervals (cf. Definition 4.2). From a syntactical point of view, $S\text{-}DEC(x_1, x_2\text{-}x_3, x_4)$ formulas can be viewed as $M\text{-}DEC(x_1, x_4)$ formulas which place a hole (bounded by $x_2$ and $x_3$) between $x_1$ and $x_4$ in a suitable way. In this respect, $S\text{-}DEC.3$ generalizes $M\text{-}DEC.2$; $S\text{-}DEC.4$ generalizes $M\text{-}DEC.3$; $S\text{-}DEC.1$, $S\text{-}DEC.6$, and $S\text{-}DEC.8$ generalizes $M\text{-}DEC.4$; $S\text{-}DEC.5$ and $S\text{-}DEC.7$ generalize $M\text{-}DEC.5$.

**Definition 6.4** *The class of* Stereo-Decomposition *formulas over a stereo-interval* $[x_1, x_2]\text{-}[x_3, x_4]$, *written* $S\text{-}DEC(x_1, x_2\text{-}x_3, x_4)$ *is inductively defined as follows:*

1. $flip(x_4) = x_1 \wedge flip(x_3) = x_2 \wedge T(\psi, x_1) \wedge T(\psi', x_2) \wedge T(\psi'', x_2\text{-}x_3, x_3) \wedge$ $T(\psi''', x_2\text{-}x_3, x_4) \wedge \forall y(x_1 < y < x_2 \rightarrow T(\phi, y)) \wedge \forall y(x_3 < y < x_4 \rightarrow T(\phi', y)) \wedge$ $\forall y((x_3 < y < x_4 \wedge \exists y'(x_1 \leq y' \leq x_2 \wedge flip(y) = y')) \rightarrow T(\phi'', x_2 - x_3, y))$;
2. $\neg flip(x_4) = x_1 \wedge x_4 = x_3 \wedge flip(x_3) = x_2 \wedge T(\psi, x_2) \wedge T(\psi', x_3) \wedge \delta(x_1, x_2)$, *where* $\delta(x_1, x_2)$ *is an* $M\text{-}DEC(x_1, x_2)$ *formula;*
3. $flip(x_4) = x_1 \wedge \exists z(x_3 \leq z < x_4 \wedge BreakP_{x_1, x_4}(z) \wedge \delta'(x_1, x_2\text{-}x_3, z) \wedge$ $T(\psi, x_1\text{-}x_2, z) \wedge \forall y((z < y < x_4 \wedge BreakP_{x_1, x_4}(y)) \rightarrow T(\phi, x_2\text{-}x_3, y)))$, *where* $\delta'(x_1, x_2\text{-}x_3, z)$ *is an* $S\text{-}DEC(x_1, x_2\text{-}x_3, z)$ *formula;*
4. $\neg flip(x_4) = x_1 \wedge \exists z(x_3 \leq z < x_4 \wedge BreakP(z, x_4) \wedge \delta'(x_1, x_2\text{-}x_3, z) \wedge$ $T(\psi, x_1\text{-}x_2, z) \wedge \forall y((z < y < x_4 \wedge BreakP(y, x_4)) \rightarrow T(\phi, x_2\text{-}x_3, y)))$, *where* $\delta'(x_1, x_2\text{-}x_3, z)$ *is an* $S\text{-}DEC(x_1, x_2\text{-}x_3, z)$ *formula;*
5. $\neg flip(x_4) = x_1 \wedge \exists k(x_3 \leq k < x_4 \wedge flip(x_4) = k \wedge \delta'(x_1, x_2\text{-}x_3, k) \wedge \delta''(k, x_4))$, *where* $\delta'(x_1, x_2\text{-}x_3, k)$ *and* $\delta''(k, x_4)$ *are an* $S\text{-}DEC(x_1, x_2\text{-}x_3, k)$ *and* $M\text{-}DEC(k, x_4)$ *formula, respectively;*
6. $flip(x_4) = x_1 \wedge \exists z, w(x_1 \leq z \leq x_2 \leq x_3 \leq w \leq x_4 \wedge flip(w) = z \wedge \delta'(z, x_2\text{-}x_3, w) \wedge T(\psi, x_1) \wedge T(\psi', z) \wedge T(\psi'', x_2\text{-}x_3, w) \wedge \forall y(x_1 < y < z \rightarrow T(\phi, y)) \wedge \forall y(w < y < x_4 \rightarrow T(\phi', y)) \wedge \forall y((w < y < x_4 \wedge \exists y'(x_1 \leq y' \leq z \wedge flip(y) = y')) \rightarrow T(\phi'', x_2\text{-}x_3, y)))$, *where* $\delta'(z, x_2\text{-}x_3, w)$ *is an* $S\text{-}DEC(z, x_2\text{-}x_3, w)$*;*

7. $\neg flip(x_4) = x_1 \wedge \exists k(x_1 < k \leq x_2 \wedge flip(x_4) = k \wedge \delta'(x_1, k) \wedge \delta''(k, x_2 - x_3, x_4))$, *where* $\delta'(x_1, z)$ *is an* $M\text{-}DEC(x_1, z)$ *formula and* $\delta''(k, x_2\text{-}x_3, x_4)$ *is an* $S\text{-}DEC(k, x_2\text{-}x_3, x_4)$ *formula;*
8. $flip(x_4) = x_1 \wedge \exists w(x_3 < w < x_4 \wedge flip(w) = x_1 \wedge T(\psi, x_2\text{-}x_3, w) \wedge \delta'(x_1, x_2\text{-}x_3, w) \wedge \forall y(w < y < x_4 \rightarrow T(\phi, y)) \wedge \forall y((w < y < x_4 \wedge flip(y) = x_1) \rightarrow T(\phi'', x_2\text{-}x_3, y))$, *where* $\delta'(x_1, x_2\text{-}x_3, w)$ *is an* $S\text{-}DEC(x_1, x_2\text{-}x_3, w)$ *formula.*

The following lemma proves that any Boolean combination of Mono-Decomposition (resp. Stereo-Decomposition) formulas is equivalent to a finite disjunction of Mono-Decomposition (resp. Stereo-Decomposition) formulas. The proof is given in the Appendix.

**Lemma 6.5** *Any Boolean combination of Mono- (resp. Stereo-) Decomposition formulas is equivalent to a finite disjunction of Mono- (resp. Stereo-) Decomposition formulas.*

**Proposition 6.6** *Let* $\delta_1(x_1, x_2\text{-}x_3, x_4)$ *be an* $S\text{-}DEC(x_1, x_2\text{-}x_3, x_4)$ *formula and let* $\delta_2(x_2, x_3)$ *(resp.* $\delta_2(x_2, x_2'\text{-}x_3', x_3)$*) be an* $M\text{-}DEC(x_2, x_3)$ *(resp. an* $S\text{-}DEC(x_2, x_2'\text{-}x_3', x_3)$ *formula). The formula* $\exists x_2, x_3(flip(x_3) = x_2 \wedge \delta_1(x_1, x_2\text{-}x_3, x_4) \wedge \delta_2(x_2, x_3))$ *(resp.* $\exists x_2, x_3(flip(x_3) = x_2 \wedge \delta_1(x_1, x_2\text{-}x_3, x_4) \wedge \delta_2(x_2, x_2'\text{-}x_3', x_3))$*) is equivalent to a disjunction of* $M\text{-}DEC(x_1, x_4)$ *formulas (resp.* $S\text{-}DEC(x_1, x_2'\text{-}x_3', x_4)$ *formulas).*

PROOF. The proof is obtained by an easy structural induction on the definition of Mono- and Stereo-Decomposition formulas. ■

The next step is the definition of a mapping of $M\text{-}BET(x_1, x_2)$ (resp. $S\text{-}BET(x_1, x_2\text{-}x_3, x_4)$) formulas into $M\text{-}DEC(x_1, x_2)$ (resp. $S\text{-}DEC(x_1, x_2\text{-}x_3, x_4)$) formulas. To this end, we firstly state the following lemma, whose proof is given in the Appendix.

**Lemma 6.7** *Any flip-closed* $M\text{-}BET(x_1, x_2)$ *(resp.* $S\text{-}BET(x_1, x_2\text{-}x_3, x_4)$*) formula is equivalent to a finite disjunction of* $M\text{-}DEC(x_1, x_2)$ *(resp.* $S\text{-}DEC(x_1, x_2\text{-}x_3, x_4)$ *formulas, respectively.*

To complete the proof we only need to show that Mono- and Stereo-Decomposition formulas can be easily translated into $UUTL$-formulas, as stated by the following lemma. The proof is given in the Appendix.

**Lemma 6.8** *For each* $M\text{-}DEC$ *formula* $\delta(x_1, x_2)$*, with* $x_1 < x_2$ *(resp.* $S\text{-}DEC$ *formula* $\delta(x_1, x_2\text{-}x_3, x_4)$*, with* $x_1 < x_4$*), there exists a formula* $\gamma \in UUTL$ *such that* $\exists x_1\ \delta(x_1, x_2)$ *is equivalent to* $T(\gamma, x_2)$ *(resp.* $\exists x_1\ \delta(x_1, x_2\text{-}x_3, x_4)$ *is equivalent to* $T(\gamma, x_2\text{-}x_3, x_4)$*).*

On the ground of the above results, we are now ready to show the expressive completeness of $UUTL$ with respect to $MFO[<, flip]$.

**Theorem 6.9** *For any* $MFO[<, flip]$*-sentence* $\phi$*, there exists an* $UUTL$*-formula* $\gamma$ *such that, for any model* $\alpha$*,* $\alpha \models \phi$ *if and only if* $\langle \alpha, \infty \rangle \models \gamma$*.*

PROOF. By Theorem 5.9 and Lemma 6.7, it holds that any $MFO[<, flip]$-sentence $\phi$ is equivalent to a finite disjunction $\bigvee_i \delta_i(0,\infty)$ of $M$-$DEC(0,\infty)$-formulas. By Lemma 6.8, for every $M$-$DEC(0,\infty)$-formula $\delta_i(0,\infty)$, there exists an $UUTL$-formula $\gamma_i$ such that $\langle \alpha, \infty \rangle \models \gamma_i$ if and only if $\alpha$ satisfies $\delta_i(0,\infty)$. This implies that $\langle \alpha, \infty \rangle \models \bigvee_i \gamma_i$ if and only if $\alpha$ satisfies $\bigvee_i \delta_i(0,\infty)$, and thus the thesis. ∎

As a corollary of Proposition 6.2 and Theorem 6.9, we obtain the following equivalence between $UUTL$ and $MFO[<, flip]$.

**Corollary 6.10** *$MFO[<, flip]$ is as expressive as $UUTL$.*

## 7 Conclusions and further work

In this paper, we proposed a temporal logic for (a certain form of) time granularity and showed that it is expressively complete with respect to the theory $MFO[<, flip]$. To this end, we provided an extension of Kamp's theorem to linear orders with an additional binary predicate. To the best of our knowledge, this is the first generalization of Kamp's result to the case in which a binary relation symbol, different from $<$, is included in the language. As a matter of fact, we restricted our attention to natural numbers.

We are currently investigating the minimality of the proposed set of temporal operators as well as the possibility of identifying alternative operators which are both expressively complete and natural to use in expressing properties of layered structures. In particular, we are considering a combined temporal logic approach that embeds a logic for branching within (discrete) time points into a linear temporal logic. We are also analyzing the computational properties of $UUTL$ to establish whether or not the elementary decidability obtainable using the until operator in case of $PLTL$ generalizes to $UUTL$.

What we consider the most important goal of our research at this point, however, is a complete axiomatic specification of the properties enjoyed by the $flip$ operator which guarantee the validity of the functional completeness result. We believe that the ideas underlying our result can be adapted to deal with a variety of structures that enjoy a separation property which can be characterized in a general (axiomatic) way. We also believe that such a generalization will contribute to a simplification of the argument presented in this paper.

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# 8 Appendix: Proofs

## *8.1 Proofs of Section 5*

**Lemma 5.5** *Each flip-closed formula* $\phi(x_1,..,x_k)$ *(resp.* $\phi(x_1,..,x_m\text{-}x_{m+1},\dots,x_k)$*) of quantification nesting* $n$ *is equivalent to a finite disjunction of flip-closed* $\mathbf{s}$*-formulas* $\bigvee_i \phi_i(x_1,\dots,x_k)$ *(resp.* $\bigvee_i \phi_i(x_1,\ \dots,x_m\text{-}x_{m+1},\dots,x_k)$*) such that each each basic* $\mathbf{s}$*-formula occurring in* $\phi_i$ *has quantification nesting at most* $max\{2,n\}$*.*

PROOF. The proof is by induction on the number $s$ of internal variables of the formula $\phi(x_1, \ldots, x_k)$ (resp. $\phi(x_1, \ldots, x_m\text{-}x_{m+1}, \ldots, x_k))$[2]. In particular, $s = |\{x_2, \ldots x_{m-1}, x_{m+2}, \ldots, x_{k-1}\}|$ in the case of formulas over stereo-intervals. The inductive case rests on a decomposition of formulas based on the interval decomposition given in Proposition 5.6.

Case $s = 0$. In both the cases of a mono- and a stereo-interval, $\phi$ is a basic $\mathbf{s}$-formula of quantification nesting $n$.

Case $s > 0$ (inductive case). If $\phi$ is not satisfiable, then it is equivalent to false.

Suppose that $\phi$ is satisfiable. We first consider the case of a mono-interval $[x_1, x_k]$. Let $\mathbf{w}_{x_1,\ldots,x_k}$ be a model over the flip-closed interval $[x_1, x_k]$ of the formula $\phi(x_1, \ldots, x_k)$. By Proposition 5.6 (case (1)), there exists $z$ such that $flip(x_{k-1}) = z$ and $x_i \leq z < x_{i+1}$, for some $i = 1, 2, \ldots, k-2$. Hence, the formula $\phi(x_1, \ldots, x_k)$ is equivalent to the formula

$$\exists z \ \bigvee\nolimits_{1 \leq i < k-1} \phi_i(x_1, \ldots, x_i, z, x_{i+1}, \ldots, x_k),$$

where $\phi_i(x_1, \ldots, x_i, z, x_{i+1}, \ldots, x_k)$ stands for

$$x_i \leq z < x_{i+1} \wedge flip(x_{k-1}) = z \wedge \phi(x_1, \ldots, x_k).$$

From the Ehrenfeucht-Fraïssé theorem, it holds that the set of models of

$$\bigvee_{1 \leq i < k-1} \phi_i(x_1, \ldots, x_i, z, x_{i+1}, \ldots, x_k)$$

is a finite union $\bigcup_l [\mathbf{w}^l_{x_1,\ldots,z,\ldots,x_k}]_{n,k+1}$. Consider a specific model $\mathbf{w}^l_{x_1,\ldots,z,\ldots,x_k}$ belonging to $[\mathbf{w}^l_{x_1,\ldots,z,\ldots,x_k}]_{n,k+1}$. Since the interval $[x_1, x_k]$ is equal to $[x_1, z]\text{-}[x_{k-1}, x_k] \uplus [z, x_{k-1}]$, by Proposition 5.7, $[\mathbf{w}^l_{x_1,\ldots,z,\ldots,x_k}]_{n,k+1}$ is equal to $[\mathbf{w}^l_{x_1,\ldots,z,x_{k-1},x_k}]_{n,k_1} \uplus [\mathbf{w}^l_{z,\ldots,x_{k-1}}]_{n,k_2}$, with $k_1 + k_2 = k + 3$. Since each equivalence class $[\mathbf{w}]_{n,k+1}$ can be described by a formula $\psi_{[\mathbf{w}]_{n,k+1}}$ of quantification degree $n$, with $k+1$ free variables, we have that

$$\psi_{[\mathbf{w}^l_{x_1,\ldots,z,\ldots,x_k}]_{n,k+1}} = \psi_{[\mathbf{w}^l_{x_1,\ldots,z,x_{k-1},x_k}]_{n,k_1}} \wedge \psi_{[\mathbf{w}^l_{z,\ldots,x_{k-1}}]_{n,k_2}}.$$

As a consequence, we have that $\exists z \bigvee_{1 \leq i < k-1} \phi_i(x_1, \ldots, x_i, z, x_{i+1}, \ldots, x_k)$ is equivalent to $\exists z \bigvee_l \psi_{[\mathbf{w}^l_{x_1,\ldots,z,\ldots,x_k}]_{n,k+1}}$, which, in its turn, is equivalent to:

$$\exists z \bigvee\nolimits_l (\psi_{[\mathbf{w}^l_{x_1,\ldots,z,x_{k-1},x_k}]_{n,k_1}} \wedge \psi_{[\mathbf{w}^l_{z,\ldots,x_{k-1}}]_{n,k_2}}).$$

By construction, $\psi_{[\mathbf{w}^l_{x_1,\ldots,z,x_{k-1},x_k}]_{n,k_1}}$ and $\psi_{[\mathbf{w}^l_{z,\ldots,x_{k-1}}]_{n,k_2}}$ are flip-closed formulas, each one properly including at most $s-1$ internal variables, and thus, by the inductive hypothesis, each of them can be replaced by a suitable disjunction of flip-closed interval $\mathbf{s}$-formulas. For instance, the formula $\psi_{[\mathbf{w}^l_{x_1,\ldots,z,x_{k-1},x_k}]_{n,k_1}}$ is equivalent to a finite

[2] The proof, as given here, is not constructive, but it could be made so using a suitable satisfiability algorithm, e.g. [10], as a subroutine.

disjunction $\bigvee_{h_1} \delta^l_{h_1}(x_1, \ldots, z, x_{k-1}, x_k)$, where, for each $h_1$, $\delta^l_{h_1}(x_1, \ldots, z, x_{k-1}, x_k)$ is an **s**-formula of quantification nesting at most $n$. Hence,

$$\exists z \bigvee_{1 \leq i < k-1} \phi_i(x_1, \ldots, x_i, z, x_{i+1}, \ldots, x_k)$$

is equivalent to:

$$\exists z \bigvee_l \bigvee_{h_1} \bigvee_{h_2} (\delta^l_{h_1}(x_1, \ldots, z\text{-}x_{k-1}, x_k) \wedge \delta^l_{h_2}(z, \ldots, x_{k-1})),$$

which is equivalent to:

$$\bigvee_l \bigvee_{h_1} \bigvee_{h_2} \exists z (\delta^l_{h_1}(x_1, \ldots, z\text{-}x_{k-1}, x_k) \wedge \delta^l_{h_2}(z, \ldots, x_{k-1})).$$

It is immediate to see that, for all $l, h_1, h_2$, the disjunct $\exists z(\delta^l_{h_1}(x_1, \ldots, z\text{-}x_{k-1}, x_k) \wedge \delta^l_{h_2}(z, \ldots, x_{k-1}))$ is (equivalent to) an **s**-formula, and thus the thesis.

As for the case of a stereo-interval $[x_1, x_m]$-$[x_{m+1}, x_k]$, let $\mathbf{w}_{x_1, \ldots, x_m, x_{m+1}, \ldots, x_k}$ be a stereo-model over the flip-closed stereo-interval $[x_1, x_m]$-$[x_{m+1}, x_k]$ of the formula $\phi(x_1, \ldots, x_m\text{-}x_{m+1}, \ldots, x_k)$. By Proposition 5.6, either (i) there exist $v$ and $w$ such that $x_1 \leq v \leq x_m$, $x_{m+1} \leq w \leq x_k$, $flip(w) = v$, and $x_2 \leq v \leq x_{m-1}$ or $x_{m+2} \leq w \leq x_{k-1}$ (case (2a)), or (ii) there exist $v_1$ and $w_1$ such that $flip(w_1) = v_1$, $x_1 \leq v_1 < x_2$, and $x_{k-1} < w_1 \leq x_k$, and there exist $v_2$ and $w_2$ such that $flip(w_2) = v_2$, $x_{m-1} < v_2 \leq x_m$, and $x_{m+1} \leq w_2 < x_{m+2}$ (case (2b)), or (iii) there exist $v_2$ and $w_2$ such that $flip(w_2) = v_2$, $x_{m-1} < v_2 \leq x_m$, and $x_{m+1} \leq w_2 < x_{m+2}$, but there exist no $v_1$ and $w_1$ such that $flip(w_1) = v_1$, $x_1 \leq v_1 < x_2$, and $x_{k-1} < w_1 \leq x_k$ (case (2c)).

Accordingly, the formula

$$\phi(x_1, \ldots, x_m\text{-}x_{m+1}, \ldots, x_k)$$

is equivalent to the formula

$$\bigvee_{i=1}^{3} \phi_i(x_1, \ldots, x_m\text{-}x_{m+1}, \ldots, x_k),$$

where $\phi_1(x_1, \ldots, x_m\text{-}x_{m+1}, \ldots, x_k)$, which captures case (2a), stands for

$$\begin{gathered}\bigvee_{(1<i<m-1 \wedge m+1 \leq j \leq k-1) \vee ((i=1 \vee i=m-1) \wedge m+1<j<k-1)} \\ \exists z(x_i \leq z \leq x_{i+1} \wedge \exists w(x_j \leq w \leq x_{j+1} \wedge \\ flip(w) = z \wedge \phi(x_1, \ldots, x_m\text{-}x_{m+1}, \ldots, x_k))),\end{gathered}$$

$\phi_2(x_1, \ldots, x_m\text{-}x_{m+1}, \ldots, x_k)$, which captures case (2b), stands for

$$\begin{gathered}\exists z_1(x_1 \leq z_1 < x_2 \wedge \exists w_1(x_{k-1} < w_1 \leq x_k \wedge flip(w_1) = z_1 \wedge \\ \exists z_2(x_{m-1} < z_2 \leq x_m \wedge \exists w_2(x_{m+1} \leq w_2 < x_{m+2} \wedge flip(w_2) = z_2 \wedge \\ \psi^m(z_1, x_2, \ldots, x_{m-1}, z_2\text{-}w_2, x_{m+2}, \ldots, x_{k-1}, w_1) \wedge \\ close(z_1, z_2, w_2, w_1) \wedge \phi(x_1, \ldots, x_m\text{-}x_{m+1}, \ldots, x_k))))),\end{gathered}$$

and $\phi_3(x_1, \ldots, x_m\text{-}x_{m+1}, \ldots, x_k)$, which captures case (2c), stands for

$$\begin{gathered}\exists z(x_{m-1} < z \leq x_m \wedge \exists w(x_{m+1} \leq w < x_{m+2} \wedge flip(w) = z \wedge \\ \psi^m(x_1, \ldots, x_{m-1}, z\text{-}w, x_{m+2}, \ldots, x_k) \wedge \\ close(x_1, z, w, x_k) \wedge \phi(x_1, \ldots, x_m\text{-}x_{m+1}, \ldots, x_k)))),\end{gathered}$$

where $\psi^m(y_1, .., y_m\text{-}y_{m+1}, .., y_k)$ stands for

$$\psi^m(y_1, .., y_m\text{-}y_{m+1}, .., y_k) = \begin{cases} \mathit{true} \ \text{ if } m > 2 \text{ and } k > m+2; \\ \exists z \bigvee_{1 \leq i < m-2} (y_i \leq z < y_{i+1} \wedge \mathit{flip}(y_{m-1}) = z) \\ \quad \text{if } (m > 2 \text{ and}) \ k = m+2;, \\ \exists z \bigvee_{m+1 \leq i < k-2} (y_i \leq z < y_{i+1} \wedge \mathit{flip}(y_{k-1}) = z) \\ \quad \text{if } m = 2 \ (\text{and } k > m+2), \end{cases}$$

and $\mathit{close}(y_1, y_2, y_3, y_4)$ stands for

$$\neg(\exists z \exists w (y_1 \leq z \leq y_2 \wedge y_3 \leq w < y_4 \wedge \mathit{flip}(w) = z)).$$

To complete the proof, it suffices to show that $\phi_i(x_1, \ldots, x_m\text{-}x_{m+1}, \ldots, x_k)$, for $1 \leq i \leq 3$, is equivalent to a finite disjunction of flip-closed $\mathbf{s}$-formulas.

Let us first consider the formula $\phi_1(x_1, \ldots, x_m\text{-}x_{m+1}, \ldots, x_k)$, which is equivalent to:

$$\exists z \exists w \bigvee_{(1<i<m-1 \wedge m+1 \leq j \leq k-1) \vee ((i=1 \vee i=m-1) \wedge m+1<j<k-1)} \phi_{i,j}(x_1, \ldots, x_i, z, x_{i+1}, \ldots, x_j, w, x_{j+1}, \ldots, x_k),$$

where, for all $i, j$, $\phi_{i,j}(x_1, \ldots, x_i, z, x_{i+1}, \ldots, x_j, w, x_{j+1}, \ldots, x_k)$ stands for:

$$x_i \leq z \leq x_{i+1} \wedge x_j \leq w \leq x_{j+1} \wedge \mathit{flip}(w) = z \wedge \phi(x_1, \ldots, x_m\text{-}x_{m+1}, \ldots, x_k).$$

From the Ehrenfeucht-Fraïssé theorem, it holds that the set of models of

$$\bigvee_{(1<i<m-1 \wedge m+1 \leq j \leq k-1) \vee ((i=1 \vee i=m-1) \wedge m+1<j<k-1)} \phi_{i,j}(x_1, \ldots, x_i, z, x_{i+1}, \ldots, x_j, w, x_{j+1}, \ldots, x_k)$$

is a finite union $\bigcup_l [\mathbf{w}^l_{x_1, \ldots, z, \ldots, w, \ldots, x_k}]_{n,k+2}$. Since the stereo-interval $[x_1, x_m]$-$[x_{m+1}, x_k]$ is equal to $[x_1, z]$-$[w, x_k] \uplus [z, x_m]$-$[x_{m+1}, w]$, we can proceed as in the case of mono-intervals to prove that $\phi_1(x_1, \ldots, x_m\text{-}x_{m+1}, \ldots, x_k)$ is equivalent to:

$$\bigvee_l \bigvee_{h_1} \bigvee_{h_2} \exists z \exists w (\delta^l_{h_1}(x_1, \ldots, z\text{-}w, \ldots, x_k) \wedge \delta^l_{h_2}(z, \ldots, x_m\text{-}x_{m+1}, \ldots, w)),$$

where, for all $l, h_1$, and $h_2$, both $\delta^l_{h_1}(x_1, \ldots, z\text{-}w, \ldots, x_k)$ and $\delta^l_{h_2}(z, \ldots, x_m\text{-}x_{m+1}, \ldots, w)$ are flip-closed $\mathbf{s}$-formulas of quantification nesting $n$. The thesis immediately follows.

As for the formula $\phi_2(x_1, \ldots, x_m\text{-}x_{m+1}, \ldots, x_k)$, we must distinguish three different cases, depending on the value of $\psi^m(z_1, x_2, \ldots, x_{m-1}, z_2\text{-}w_2,\ x_{m+2}, \ldots, x_{k-1},\ w_1)$.

If it is equal to *true*, then $\phi_2(x_1, \ldots, x_m\text{-}x_{m+1}, \ldots, x_k)$ can be rewritten as follows:

$$\exists z_1 \exists w_1 \exists z_2 \exists w_2 (x_1 \leq z_1 < x_2 \wedge x_{k-1} < w_1 \leq x_k \wedge \mathit{flip}(w_1) = z_1 \wedge \\ x_{m-1} < z_2 \leq x_m \wedge x_{m+1} \leq w_2 < x_{m+2} \wedge \mathit{flip}(w_2) = z_2 \wedge \\ \mathit{close}(z_1, z_2, w_2, w_1) \wedge \phi(x_1, \ldots, x_m\text{-}x_{m+1}, \ldots, x_k)).$$

By exploiting the decomposition of the stereo-interval $[x_1, x_m]$-$[x_{m+1}, x_k]$ into $[x_1, z_1]$-$[w_1, x_k] \uplus (([z_1, z_2]\text{-}[w_1, w_1] \uplus [z_2, z_2]\text{-}[w_2, w_1]) \uplus [z_2, x_m]\text{-}[x_{m+1}, w_2])$, we can proceed as in the previous case to prove that $\phi_2(x_1, \ldots, x_m\text{-}x_{m+1}, \ldots, x_k)$ is equivalent to:

$$\bigvee_l \bigvee_{h_1} \bigvee_{h_2} \bigvee_{h_3} \bigvee_{h_4} \exists z_1 \exists w_1 \exists z_2 \exists w_2 (\delta^l_{h_1}(x_1, z_1\text{-}w_1, x_k) \wedge \\ \delta^l_{h_2}(z_1, \ldots, z_2\text{-}w_1, w_1) \wedge \delta^l_{h_2}(z_2, z_2\text{-}w_2, \ldots, w_1) \wedge \delta^l_{h_4}(z_2, x_m\text{-}\ x_{m+1} w_2)),$$

where, for all $l, h_i$, with $1 \leq i \leq 4$, $\delta^l_{h_i}$ are flip-closed **s**-formulas of quantification nesting equal to $max\{2, n\}$. The thesis immediately follows.

If $\psi^m(z_1, x_2, \ldots, x_{m-1}, z_2\text{-}w_2, x_{m+2}, \ldots, x_{k-1}, w_1)$ is equal to

$$\exists z(z_1 \leq z < x_2 \land flip(x_{m-1}) = z) \lor \bigvee_{2 \leq i < m-2} (x_i \leq z < x_{i+1} \land flip(x_{m-1}) = z),$$

then $\phi_2(x_1, \ldots, x_m\text{-}x_{m+1}, \ldots, x_k)$ can be rewritten as follows:

$$\begin{gathered}\exists z_1 \exists w_1 \exists z_2 \exists w_2 \exists z (x_1 \leq z_1 < x_2 \land x_{k-1} < w_1 \leq x_k \land flip(w_1) = z_1 \land \\ x_{m-1} < z_2 \leq x_m \land x_{m+1} \leq w_2 < x_{m+2} \land flip(w_2) = z_2 \land \\ ((z_1 \leq z \leq x_2 \land flip(x_{m-1}) = z) \lor \textstyle\bigvee_{2 \leq i < m-2} (x_i \leq z < x_{i+1} \land flip(x_{m-1}) = z)) \land \\ \phi(x_1, \ldots, x_m\text{-}x_{m+1}, \ldots, x_k)).\end{gathered}$$

By exploiting the decomposition of the stereo-interval $[x_1, x_m]\text{-}[x_{m+1}, x_k]$ into $[x_1, z_1]\text{-}[w_1, x_k] \uplus ((([z_1, z_1]\text{-}[w_1, w_1] \uplus ([z_1, z]\text{-}[x_{m-1}, z_2] \uplus [z, x_{m-1}])) \uplus [z_2, z_2]\text{-}[w_2, w_1]) \uplus [z_2, x_m]\text{-}[x_{m+1}, w_2])$, we can proceed as in the previous case to prove that $\phi_2(x_1, \ldots, x_m\text{-}x_{m+1}, \ldots, x_k)$ is equivalent to:

$$\begin{gathered}\textstyle\bigvee_l \bigvee_{h_1} \bigvee_{h_2} \bigvee_{h_3} \bigvee_{h_4} \bigvee_{h_5} \bigvee_{h_6} \exists z_1 \exists w_1 \exists z_2 \exists w_2 \exists z (\delta^l_{h_1}(x_1, z_1\text{-}w_1, x_k) \land \\ \delta^l_{h_2}(z_1, z_1\text{-}w_1, w_1) \land \delta^l_{h_3}(z_1, \ldots, z\text{-}x_{m-1}, \ldots, z_2) \land \delta^l_{h_4}(z, \ldots, x_{m+1}) \land \\ \delta^l_{h_5}(z_2, z_2\text{-}w_2, \ldots, w_1) \land \delta^l_{h_6}(z_2, x_m\text{-}x_{m+1}, w_2)),\end{gathered}$$

where, for all $l, h_i$, $1 \leq i \leq 6$, $\delta^l_{h_i}$ are flip-closed **s**-formulas of quantification nesting equal to $max\{2, n\}$. The thesis immediately follows.

A very similar sequence of steps can be applied to manage the last case in which $\psi^m(z_1, x_2, \ldots, x_{m-1}, z_2\text{-}w_2, x_{m+2}, \ldots, x_{k-1}, w_1)$ is equal to

$$\exists w(w_2 \leq w < x_{m+2} \land flip(x_{k-1}) = w) \lor \bigvee_{x_{m+2} \leq i < k-2} (x_i \leq z < x_{i+1} \land flip(x_{k-1}) = w).$$

As for the formula $\phi_3(x_1, \ldots, x_m\text{-}x_{m+1}, \ldots, x_k)$, it is immediate to see that it is a subformula of $\phi_2$, and thus this case can be dealt with as the previous one. ∎

**Proposition 5.8** *Let* $\chi(x_1, z, x_2)$ *(resp.* $\phi(x_1, z, x_2\text{-}x_3, x_4)$, $\psi(x_1, x_2\text{-}x_3, z, x_4)$*) be a preM-BET-formula (resp. a preS-BET-formula). The formula* $\exists z\ \chi(x_1, z, x_2)$ *(resp.* $\exists z\ \phi(x_1, z, x_2\text{-}x_3, x_4)$, $\exists z\ \psi(x_1, x_2\text{-}x_3, z, x_4)$*) is equivalent to an M-BET (resp. S-BET) formula.*

PROOF. We prove the thesis for the **s**-formula $\phi(x_1, z, x_2\text{-}x_3, x_4)$. The proofs for $\chi(x_1, z, x_2)$ and $\psi(x_1, x_2\text{-}x_3, z, x_4)$ are analogous, and thus omitted. The proof is by induction on the structure of the formula.

Basic case. The thesis is obvious, since there exist at most two free variables $x_1$ and $x_2$ and $z$ must be equal to one of them.

Inductive case. Assume that $\phi$ has the form of Definition 5.4, case 2. If $z$ occurs both in $\phi_1$ and $\phi_2$, then (the interpretation of) $z$ must be an endpoint of both the stereo-intervals associated with $\phi_1$ and $\phi_2$, and thus $\phi_1$ and $\phi_2$ are basic **s**-formulas. Hence $\phi_1$ and $\phi_2$ are *S-BET*-formulas, and, from Definition 5.3 (*S-BET*-formulas, case 4, S-BET.4 for short), $\exists z\ \phi$ is an *S-BET*-formula too. If $z$ occurs in $\phi_1$, but not in $\phi_2$, then $\exists z(flip(x_l) = x_i \wedge \phi_1 \wedge \phi_2)$, with $x_i \in \{1,2\}$ and $x_l \in \{3,4\}$, is equivalent to $flip(x_l) = x_i \wedge \exists z(\phi_1) \wedge \phi_2$. Since $\phi_2$ is a basic **s**-formula, it is also an $S\text{-}BET$-formula. By the inductive hypothesis, $\exists z\phi_1$ is equivalent to an $S\text{-}BET$-formula $\phi_1'$, and thus the thesis, since $flip(x_l) = x_i \wedge \phi_1' \wedge \phi_2$ is an $S\text{-}BET$-formula. The symmetric case is dealt with analogously.

Assume that $\phi$ has the form $\exists w(x_3 < w < x_4 \wedge flip(w) = z \wedge \phi_1(x_1, z\text{-}w, x_4) \wedge \phi_2(z, x_2\text{-}x_3, w))$. Since both $\phi_1(x_1, z\text{-}w, x_4)$ and $\phi_2(z, x_2\text{-}x_3, w)$ are $S\text{-}BET$ formulas, $\exists z\phi$ is an $S\text{-}BET$ formula (S-BET.4). Consider the case in which $\phi$ has the form $\exists h(x_1 < h < z \wedge flip(x_i) = h \wedge \phi_1(x_1, h\text{-}x_i, x_4) \wedge \phi_2(h, z, x_2\text{-}x_3, x_i))$, with $i \in \{3,4\}$. The formula $\exists z\phi$ is equivalent to the formula $\exists h(x_1 < h < x_i \wedge flip(x_i) = h \wedge \phi_1(x_1, h\text{-}x_i, x_4) \wedge \exists z\phi_2(h, z, x_2\text{-}x_3, x_i)$, where $\phi_1(x_1, h\text{-}x_i, x_4)$ is an $S\text{-}BET$-formula. By the inductive hypothesis, $\exists z\phi_2(h, z, x_2\text{-}x_3, x_i)$ is equivalent to an $S\text{-}BET$- formula $\phi_2'(h, x_2\text{-}x_3, x_i)$ and thus the formula $\exists z\phi$ is equivalent to the $S\text{-}BET$-formula $\exists h(x_1 < h < x_i \wedge flip(x_i) = h \wedge \phi_1(x_1, h\text{-}\ x_i, x_4) \wedge \phi_2'(h, x_2\text{-}x_3, x_i))$. In all the remaining cases in which $\phi$ has the form of Definition 5.4, case 3, we can proceed exactly in the same way.

Assume that $\phi$ has the form $\exists h\exists w(x_1 < h < z \wedge x_3 < w < x_4 \wedge flip(w) = h \wedge \phi_1(x_1, h\text{-}w, x_4) \wedge \phi_2(h, z, x_2\text{-}x_3, w))$. The formula $\exists z\phi$ is equivalent to the formula $\exists h\exists w(x_1 < h < z \wedge x_3 < w < x_4 \wedge flip(w) = h \wedge \phi_1(x_1, h\text{-}w, x_4) \wedge \exists z\phi_2(h, z, x_2\text{-}x_3, w))$ and by inductive hypothesis $\exists z\phi_2(h, z, x_2\text{-}x_3, w)$ is equivalent to an $S\text{-}BET$-formula $\phi'(h, x_2\text{-}x_3, w)$. The resulting formula $\exists h\exists w(x_1 < h < z \wedge x_3 < w < x_4 \wedge flip(w) = h \wedge \phi_1(x_1, h\text{-}w, x_4) \wedge \phi'(h, z, x_2\text{-}x_3, w))$ is an $S\text{-}BET$ formula (S-BET.6). The other case in which $z < h < x_2$ can be handled analogously. ∎

## 8.2 *Proofs of Section 6*

**Lemma 6.5** *Any Boolean combination of Mono- (resp. Stereo-) Decomposition formulas is equivalent to a finite disjunction of Mono- (resp. Stereo-) Decomposition formulas.*

Proof. Since the assertion trivially holds for disjunction of decomposition formulas, and conjunctions can be equivalently expressed in terms of disjunction and negations, we only give the proof for negation of decomposition formulas.

We start proving that $\neg\delta(x_1, x_2)$ is equivalent to a disjunction of $M\text{-}DEC$ formulas. The proof is by induction on the structure of the $M\text{-}DEC$ formula $\delta(x_1, x_2)$.

Base. $\delta(x_1, x_2)$ has the form $flip(x_2) = x_1 \wedge T(\overline{\phi}, x_1) \wedge \forall y(x_1 < y < x_2 \to (T(\overline{\phi}', y) \vee \exists z, w(x_1 \leq z < y < w < x_2 \wedge flip(w) = z \wedge T(\overline{\phi}'', w)))$. In this case $\neg\delta(x_1, x_2)$ is equivalent to $\bigvee_{i=1}^{4} \delta_i(x_1, x_2)$, where $\delta_1(x_1, x_2)$ is a $M - DEC.3$-formula (i.e. it holds $\neg flip(x_2) = x_1$); $\delta_2$ is a $M - DEC.1$ formula where $\phi$ is set to $\neg\overline{\phi}$; $\delta_3$ is a $M - DEC.2$ formula where $\psi$ is set to $\neg\overline{\phi}'$ (i.e. there is a break point where $\neg\overline{\phi}'$ holds); $\delta_4$ is a $M - DEC.4$ formula where $\psi''$ is set to $\neg\overline{\phi}'' \wedge \neg always(\overline{\phi}')$ (with $always(\gamma) = True \cdot \gamma \uparrow \gamma$) and $\phi''$ is set to $\neg\overline{\phi}''$.

Inductive case.

(**M-DEC.2**) Let us consider the case in which $\delta(x_1, x_2)$ has the form $flip(x_2) = x_1 \wedge \exists z(x_1 < z < x_2 \wedge BreakP_{x_1,x_2}(z) \wedge \overline{\delta}'(x_1, z) \wedge T(\overline{\psi}, z) \wedge \forall y((z < y < x_2 \wedge BreakP_{x_1,x_2}(y)) \to T(\overline{\phi}, y)))$. By inductive hypothesis $\neg\overline{\delta}'(x_1, z)$ is equivalent to a

disjunction of $M-DEC$-formulas $\bigvee_i \overline{\delta}_i(x_1,z)$. By Lemma 6.8, for each $i$, there is a $UUTL$ formula $\gamma_i$ such that $\exists x_1\overline{\delta}_i(x_1,z)$ is equivalent to $T(\gamma_i,z)$. So, $\neg\delta(x_1,x_2)$ is equivalent to $\bigvee_{i=1}^{4}\delta_i$ where $\delta_1$ is a $M-DEC.3$-formula (i.e. $\neg flip(x_2)=x_1$); $\delta_2$ is a $M-DEC.1$-formula where $\phi'$ is set to $False$ (i.e. there is no break point, namely $x_2$ is odd);
$\delta_3$ is a $M-DEC.2$-formula with $\psi$ set to $\neg\overline{\phi}\wedge(\bigvee_i\gamma_i\vee\neg\overline{\psi})$ and $\phi$ set to $\bigvee_i\gamma_i\vee\neg\overline{\psi}$ (i.e. $T(\neg\overline{\phi},z)$ holds in some break point $z$);
$\delta_4$ is a $M-DEC.2$-formula with $\psi$ and $\phi$ set to $\bigvee_i\gamma_i\vee\neg\overline{\psi}$, and $\delta''(x_1,z)$ a $M-DEC.1$ formula ($T(\overline{\phi},z)$ holds in each break point $z$).
(**M-DEC.3**) Let us consider the case where $\delta(x_1,x_2)$ has the form $\neg flip(x_2)=x_1\wedge\exists z(x_1<z<x_2\wedge BreakP(z,x_2)\wedge\overline{\delta}'(x_1,z)\wedge T(\overline{\psi},z)\wedge\forall y((z<y<x_2\wedge BreakP(y,x_2))\rightarrow T(\overline{\phi},y)))$. By inductive hypothesis $\neg\overline{\delta}'(x_1,z)$ is equivalent to a disjunction of $M-DEC$-formulas $\bigvee_i\overline{\delta}_i(x_1,z)$. By Lemma 6.8, for each $i$, there is a $UUTL$ formula $\gamma_i$ such that $(\exists x_1)\overline{\delta}_i(x_1,z)$ is equivalent to $T(\gamma_i,z)$. In this case $\neg\delta(x_1,x_2)$ is equivalent to $\bigvee_{i=1}^{3}\delta_i$ with
$\delta_1$ a $M-DEC.1$-formula (i.e. $flip(x_2)=x_1$); $\delta_2$ a $M-DEC.3$-formula with $\psi$ set to $\neg\overline{\phi}\wedge(\bigvee_i\gamma_i\vee\neg\overline{\psi})$, and $\phi$ set to $\bigvee_i\gamma_i\vee\neg\overline{\psi}$ ($T(\neg\phi,z)$ holds in some break point $z$); $\delta_3$ a $M-DEC.3$-formula with $\psi$ and $\phi$ are set to $\bigvee_i\gamma_i\vee\neg\overline{\psi}$, and $\delta''(x_1,z)$ a $M-DEC.1$ formula ($T(\phi,z)$ holds in each break point $z$).
(**M-DEC.4**) Let us consider the case where $\delta(x_1,x_2)$ has the form $flip(x_2)=x_1\wedge\exists z\exists w(x_1\leq z\leq w\leq x_2\wedge flip(w)=z\wedge\overline{\delta}'(z,w)\wedge T(\overline{\psi},x_1)\wedge T(\overline{\psi}',z)\wedge T(\overline{\psi}'',w)\wedge\forall y(x_1<y<z\rightarrow T(\overline{\phi},y))\wedge\forall y(w<y<x_2\rightarrow T(\overline{\phi}',y))\wedge\forall y((w<y<x_2\wedge\exists y'(x_1\leq y'\leq z\wedge flip(y)=y'))\rightarrow T(\overline{\phi}'',y)))$.
By inductive hypothesis $\neg\overline{\delta}'(z,w)$ is equivalent to a disjunction of $M$-$DEC$-formulas $\bigvee_i\overline{\delta}_i(z,w)$. By Lemma 6.8, for each $i$, there is a $UUTL$ formula $\gamma_i$ such that $\exists z\overline{\delta}_i(z,w)$ is equivalent to $T(\gamma_i,w)$ (and in this case such a $z$ is $flip(w)$).
The cases in which $\neg flip(x_2)=x_1$ or $T(\neg\overline{\psi},x_1)$ holds can be expressed by a $M-DEC.3$ or a $M-DEC.1$, respectively.
We consider all the possible ways in which $T(\overline{\phi},x)$ and $T(\overline{\phi}',x)$ are satisfied, for $x_1<x<x_2$.
(Case A) The case in which there is no point $x_1<x<x_2$ such that either $\neg T(\overline{\phi},x)$ or $\neg T(\overline{\phi}',x)$ hold is captured by a $M-DEC.1$ formula where $\phi'$ is set to $\Gamma=\bigvee_i\gamma_i\vee\neg\overline{\psi}''\vee(\neg\overline{\psi}'\cdot True\uparrow True)$ and $\phi''$ is set to $\neg\overline{\phi}''$.
(Case B) Let us consider the case in which $T(\overline{\phi}',x)$ holds for any $x_1<x<x_2$ and $\neg T(\overline{\phi},x)$ holds in some $x_1<x<x_2$. We focus our attention on the least point $\overline{x}$ where $\neg T(\overline{\phi},\overline{x})$ holds. Let us assume that $\overline{x}$ is a break point of $(x_1,x_2)$. If it is the first break point we have a $M-DEC.2$ formula where $\psi$ is set to $Odd\wedge\neg\overline{\phi}\wedge\Delta$ with $\Delta=\Gamma\wedge(True\cdot\Gamma\uparrow\neg\overline{\phi}'')$; $\delta'(x_1,z)$ is either an arbitrary $M-DEC.1$ formula or it is a $M-DEC.3$ formula with $\delta'$ forced to be a $M-DEC.1$ formula, and with $\psi$ and $\phi$ set to $\Delta$.
We consider now the case where $\overline{x}$ is a break point but not the first. In this case we have a $M-DEC.2$ formula where $\delta'(x_1,z)$ is a $M-DEC.5$ formula. The $M-DEC.5$ formula is as follows: $\delta''$ is a $M-DEC.4$ formula with $\psi$ set to $\neg\overline{\phi}\wedge\Delta$, $\phi$ set to $False$, $\psi''$ set to $(Odd\vee\neg\overline{\phi}'')\wedge\Gamma$ and $\phi''$ set to $\Gamma$; $\delta'$ is either an arbitrary $M-DEC.1$ formula or it is a $M-DEC.3$ formula with $\delta'$ forced to be a $M-DEC.1$ formula,

and with $\psi$ and $\phi$ set to $\Delta$.
The case in which $\overline{x}$ is not a break point is similar and it is as follows. We have a $M-DEC.2$ formula where $\psi$ is set to $\neg always(\overline{\phi}) \wedge \Gamma$ and $\delta'(x_1, z)$ is either a $M-DEC.4$ formula ($\overline{x}$ belongs to the last jump) or a $M-DEC.5$ formula. Let us consider the former case. The idea is that in the considered $M-DEC.4$ formula $z$ and $w$ are chosen in such a way that $z < \overline{x} < w$ and $\overline{x}$ is a break point of $(z, w)$. In such a situation, $\delta'(z, w)$ can be defined exactly as in the already examined case. We have also to consider the case whether there is a jump point $y$ for $w$ which satisfies $\neg T(\overline{\phi}'', y)$ or not. Assume that such a jump point does not exists. With this assumption the formulas $\phi''$ and $\psi''$ of the $M-DEC.4$ formula are both set to $(\neg Break \wedge \Gamma) \vee (Break \wedge \Gamma \wedge (Fill(\Delta).True \uparrow True))$ with $Fill(\Delta)$ the formula $(\neg Break \wedge \Delta) \vee ((\neg Break \wedge \Delta) \diamond \Delta)$ (notice that $Fill(\Delta)$ imposes $\Delta$ in each break point).
The case in which there is a jump point $y$ for $\overline{x}$ which satisfies $\neg T(\overline{\phi}'', y)$ can be handled similarly.
Now, let us consider the case in which $\delta'(x_1, z)$ is a $M-DEC.5$ formula. In this case, $\delta''$ is defined exactly as in the former case and $\delta'$ is the (disjunction of) $M-DEC$ formulas which requires the formula $\Delta$ in each break point of the interval $(x_1, z)$.
(Case C) Let us consider the case in which $T(\overline{\phi}, x)$ holds for any $x_1 < x < x_2$ and $\neg T(\overline{\phi}', x)$ holds for some $x_1 < x < x_2$. In this case we focus our attention on the greatest point $\overline{x}$ such that $\neg T(\overline{\phi}', \overline{x})$ holds. If $\overline{x}$ is a break point we have a $M-DEC.2$ formula where $\phi$ is set to $(\overline{\phi}' \vee always(\overline{\phi}')) \wedge \Delta$; $\psi$ is set to $\neg\overline{\phi}' \wedge \Gamma$ and $\delta'(x_1, z)$ is an arbitrary $M$-$DEC$ formula.
The case in which $\overline{x}$ is not a break point is similar and it is as follows. We have a $M-DEC.2$ formula where $\phi$ is set as in the previous case and $\psi$ is set to $\neg always(\overline{\phi}') \wedge \Gamma$. Now, $\delta'(x_1, z)$ is either a $M-DEC.4$ formula ($\overline{x}$ belongs to the last jump) or a $M-DEC.5$ formula. (We treat explicitly only the former case from which also the latter case can be easily derived.) We have also to consider the case whether there is a jump point $y$ for $\overline{x}$ which satisfies $\neg T(\overline{\phi}'', y)$ or not. Assume that such a jump point does not exists. With this assumption the $M-DEC.4$ formula for $\delta'(x_1, z)$ is as follows: $\psi''$ is set to $\neg\overline{\phi}' \wedge \Gamma$; $\phi''$ is set to
$EvenPred([(\neg\overline{\phi}' \vee \neg always(\overline{\phi}')) \wedge \Gamma] \diamond (always(\overline{\phi}') \wedge \Delta))$. The case in which there is a jump point $y$ for $\overline{x}$ which satisfies $\neg T(\overline{\phi}'', y)$ can be handled similarly.
(Case D) Finally, we consider the case in which for some $x_1 < x < x_2$ $\neg T(\overline{\phi}, x)$ holds (let $\overline{x}$ be the least of these points) and for some $x_1 < y < x_2$ $\neg\overline{\phi}'$ holds (let $\overline{y}$ be the greatest of these points). Let us assume that $\overline{x} \leq \overline{y}$. The case in which there is a break point $z$ such that $\overline{x} < z < \overline{y}$ is given by a $M-DEC.2$ formula where $\psi$ is set to

$$(\neg\overline{\phi}' \vee \neg always(\overline{\phi}')) \wedge \overline{\phi} \wedge always(\overline{\phi}) \wedge (somewhere(\neg\overline{\phi}) \cdot True \uparrow True)$$

and $\phi$ is set to $\overline{\phi} \wedge always(\overline{\phi})$, with $somewhere(\gamma)$ the formula $\neg always(\neg\gamma) \vee ((\gamma \vee \neg always(\neg\gamma)) \diamond True)$. The other cases in which either $\overline{x} = z < \overline{y}$ or $\overline{x} < z = \overline{y}$ can be treated analogously.
If $flip(\overline{y}) = \overline{x}$, then we have to consider the two cases whether there exists or not a jump point $k$ where $\neg T(\overline{\phi}'', k)$ holds. We consider only the latter case which is given

by a $M-DEC.4$ formula with $\psi'$ set to $\neg\overline{\phi}$; $\psi''$ set to $\Gamma \wedge \neg\overline{\phi}'$ and $\phi''$ set to $\Gamma$.
The other case in which $\overline{x} = \overline{y}$ can be dealt with analogously.
(Case E) If all the previously examined cases are excluded there are three points $z$, $w$ and $k$ such that $flip(w) = z$, and $k$ is a break point (with respect to $[z, w]$) with $z < \overline{x} \leq k \leq \overline{y} < w$. In this case if we assume the simplifying assumption that $T(\overline{\phi}'', k)$ holds for each jump point $k$ of $w$, we have a disjunction of $M-DEC.4$ formulas where $\psi''$ and $\phi''$ are set to $\Gamma$ and $\delta'(z, w)$ is a disjunct of the formulas defined in the Case D.
The case in which $\overline{y} < \overline{x}$ can be handled in a similar way.
(**M-DEC.5**) Let us consider the case where $\delta(x_1, x_2)$ has the form $\neg flip(x_2) = x_1 \wedge \exists k(x_1 < k < x_2 \wedge flip(x_2) = k \wedge \overline{\delta}'(x_1, k) \wedge \overline{\delta}''(k, x_2))$. By inductive hypothesis $\neg\overline{\delta}'(x_1, k)$ and $\neg\overline{\delta}''(k, x_2)$ are equivalent to a disjunction of $M$-$DEC$-formulas $\bigvee_i \delta_i'(x_1, k)$ and $\bigvee_j \delta_j''(k, x_2)$, respectively. Therefore, $\neg\delta(x_1, x_2)$ is equivalent to $\delta_1 \vee \bigvee_i \delta_{2,i} \vee \bigvee_j \delta_{3,j}$, where $\delta_1$ is an arbitrary $M-DEC.1$ formula (i.e. $flip(x_2) = x_1$); $\delta_{2,i}$ is a $M-DEC.5$ with $\delta'(x_1, k)$ set to $\delta_i'(x_1, k)$ and $\delta_{3,j}$ is a $M-DEC.5$ with $\delta''(k, x_2)$ set to $\delta_i''(k, x_2)$.
Let us consider now $S$-$DEC$ formulas.
(**S-DEC.1**) Let us consider the case where $\delta(x_1, x_2, x_3, x_4)$ has the form $flip(x_4) = x_1 \wedge flip(x_3) = x_2 \wedge T(\overline{\psi}, x_1) \wedge T(\overline{\psi}', x_2) \wedge T(\overline{\psi}'', x_2\text{-}\ x_3, x_3) \wedge T(\overline{\psi}''', x_2\text{-}x_3, x_4) \wedge \forall y(x_1 < y < x_2 \rightarrow T(\overline{\phi}, y)) \wedge \forall y(x_3 < y < x_4 \rightarrow T(\overline{\phi}', y)) \wedge \forall y((x_3 < y < x_4 \wedge \exists y'(x_1 \leq y' \leq x_2 \wedge flip(y) = y')) \rightarrow T(\phi'', x_2\text{-}x_3, y))$. The situation in which $\neg flip(x_4) = x_1$ holds can be represented by disjuncts of the form $S-DEC.2, 4, 5, 7$. The cases where $\neg T(\overline{\psi}, x_1)$ (resp. $\neg T(\overline{\psi}', x_2)$, $\neg T(\overline{\psi}'', x_2\text{-}x_3, x_3)$, $\neg T(\overline{\psi}''', x_2\text{-}x_3, x_4)$) does not hold is expressed by a $S-DEC.1$ formula with $\psi$ (resp.: $\psi', \psi'', \psi'''$) set to $\neg\overline{\psi}$ (resp.: $\neg\overline{\psi}'$, $\neg\overline{\psi}''$, $\neg\overline{\psi}'''$). In order to handle in a simple way the case where there is a jump $k$ for $x_3$ where $T(\overline{\phi}'', x_2\text{-}x_3, k)$ does not hold, one can previously show that a format which differs from $S-DEC.6$ only for the requirement that $x_3 < w < x_4$ allows to define formulas which are equivalent to disjunction of $S-DEC$ formulas. We refer to this format as $S-DEC.6^*$. Therefore, the case we are considering is given by a $S-DEC.6^*$ formula with $\psi''$ set to $\neg\overline{\phi}''$.
We consider now the case in which there is a point $x_1 < \overline{x} < x_2$ where $T(\overline{\phi}, \overline{x})$ does not hold. The basic case we consider first is the one in which there is a break point $k$ for $(x_1, x_4)$ such that either $\overline{x} < k \leq x_2$ or $\overline{x} = k < x_2$.
In the former case, we have a $S-DEC.3$ having the recursive parameter $\delta'(x_1, x_2 - x_3, z)$ either of the form $S-DEC.2$ or $S-DEC.7$. If we consider a $S-DEC.2$ formula, then $\psi$ is set to $somewhere(\neg\overline{\phi})$. If we consider a $S-DEC.7$ formula, then $\delta''$ is a $S-DEC.1$ formula with $\psi$ set to $somewhere(\neg\overline{\phi})$. In the latter case (i.e. $\overline{x} = k < x_2$), we have a $S-DEC.3$ having the recursive parameter of the form $S-DEC.7$. We have two possibilities for $S-DEC.7$: either $\delta''$ is a $S-DEC.1$ formula and $\psi$ is set to $\neg\overline{\phi} \diamond True$ or $\psi$ is set to $\neg\overline{\phi}$ and $\delta''$ is $S$-$DEC$ formulas guaranteeing that $k < x_2$ (it can be easily show that such a formula can be defined).
If such a break point $k$ does not exists, there are $z$, $w$ and $k$ such that $flip(w) = z$, with $x_3 \leq w$, $k$ is a break point for $[z, w]$ and either $\overline{x} < k \leq x_2$ or $\overline{x} = k < x_2$. In this case we have a formula $S-DEC.6$ where $\delta'(z, x_2\text{-}x_3, w)$ is an element of the disjunction of $S$-$DEC$ formulas obtained by handling the above considered basic case.
The symmetrical case in which there is a point $x_3 < \overline{x} < x_4$ where $T(\overline{\phi}', \overline{x})$ does not

hold can be handled analogously.

(**S-DEC.2**) Let us consider the case where $\delta(x_1, x_2, x_3, x_4)$ has the form $\neg flip(x_4) = x_1 \wedge x_4 = x_3 \wedge flip(x_3) = x_2 \wedge T(\overline{\psi}, x_2) \wedge T(\overline{\psi}', x_3) \wedge \overline{\delta}(x_1, x_2)$. The situation in which $flip(x_4) = x_1$ holds can be represented by a $S - DEC.1$ formula. The case in which $x_3 < x_4$ is given either by a $S - DEC.4$ formula, or by a $S - DEC.7$ formula having as $\delta''$ a formula of the form $S - DEC.3$.

The case in which $T(\overline{\psi}, x_2)$ (resp. $T(\overline{\psi}', x_3)$) does not hold is treated by a $S.DEC.2$ formula with $\psi$ (resp. $\psi'$) set to $\neg\overline{\psi}$ (resp. $\neg\overline{\psi}'$).

Finally, as already proved, $\neg\overline{\delta}$ is equivalent to a disjunction $\bigvee_j \delta_j(x_1, x_2)$. Therefore the case in which $\overline{\delta}(x_1, x_2)$ does not hold true is given by a disjunction (indexed by $j$) of $S.DEC.2$ formulas with $\delta(x_1, x_2)$ set to $\delta_j(x_1, x_2)$.

(**S-DEC.3**) Let us consider the case where $\delta(x_1, x_2, x_3, x_4)$ has the form $flip(x_4) = x_1 \wedge \exists z(x_3 \leq z < x_4 \wedge BreakP_{x_1,x_4}(z) \wedge \overline{\delta}'(x_1, x_2\text{-}x_3, z) \wedge T(\overline{\psi}, x_2\text{-}x_3, z) \wedge \forall y((z < y < x_4 \wedge BreakP_{x_1,x_4}(y)) \rightarrow T(\overline{\phi}, x_2\text{-}x_3, y)))$.

By inductive hypothesis $\neg\overline{\delta}'(x_1, x_2\text{-}x_3, z)$ is equivalent to a disjunction of $S\text{-}DEC$ formulas $\bigvee_i \delta_i(x_1, x_2\text{-}x_3, z)$ and by Lemma 6.8, for each $i$, there is a $UUTL$ formula $\gamma_i$ such that $\exists x_1\ \delta_j(x_1, x_2\text{-}x_3, z)$ is equivalent to $T(\gamma_i, x_2\text{-}x_3, z)$.

The case in which $flip(x_4) = x_1$ does not hold has been already considered. The case in which there is no break point, namely $x_4$ is odd, is handled by a $S{-}DEC.1$-formula where $\psi'''$ is set to $Odd$.

We consider now the case in which $T(\overline{\phi}, x_2\text{-}x_3, z)$ holds in each break point $z$ greater than (or equal to) $x_3$. In such a case we have a $S - DEC.3$-formula with $\psi$ and $\phi$ set to $\bigvee_i \gamma_i \vee \neg\overline{\psi}$, and with $\delta'(x_1, x_2\text{-}x_3, z)$ a $S - DEC.7$ or a $S - DEC.1$ formula. The case in which $T(\neg\overline{\phi}, z)$ holds in some break point $z$ preceding $x_3$ is given by a $S - DEC.3$-formula where $\phi$ is set to $\bigvee_i \gamma_i \vee \neg\overline{\psi}$; $\psi$ is set to $\neg\overline{\phi} \wedge \bigvee_i \gamma_i \vee \neg\overline{\psi}$ and $\delta'$ an arbitrary $S$- $DEC$ formula.

(**S-DEC.4**) This case can be treated, with obvious changes, exactly as the case $S - DEC.3$, and it is therefore omitted.

(**S-DEC.5**) Let us consider the case where $\delta(x_1, x_2, x_3, x_4)$ has the form $\neg flip(x_4) = x_1 \wedge \exists k(x_3 \leq k < x_4 \wedge flip(x_4) = z \wedge \overline{\delta}'(x_1, x_2\text{-}x_3, k) \wedge \overline{\delta}''(k, x_4))$. As previously proved, $\neg\overline{\delta}''(k, x_4)$ is equivalent to a disjunction of $M\text{-}DEC$ formulas $\bigvee_i \delta_i''(k, x_4)$ and, by inductive hypothesis, $\neg\overline{\delta}'(x_1, x_2\text{-}\ x_3, k)$ is equivalent to a disjunction of $S$-$DEC$ formulas $\bigvee_j \delta_j'(x_1, x_2\text{-}x_3, k)$.

The case in which $flip(x_4) = x_1$ holds is handled as in the previous cases. The other cases are given by a disjunction $\bigvee_i \delta_i \vee \bigvee_j \delta_j$ where each $\delta_i$ is a $S - DEC.5$ formula with $\delta''(k, x_4)$ set to $\delta_i''(k, x_4)$, and each $\delta_j$ is a $S - DEC.5$ formula with $\delta'(x_1, x_2\text{-}x_3, k)$ set to $\delta_j'(x_1, x_2\text{-}\ x_3, k)$.

(**S-DEC.6**) Let us consider the case where $\delta(x_1, x_2, x_3, x_4)$ has the form $flip(x_4) = x_1 \wedge \exists z \exists w(x_1 \leq z \leq x_2 \wedge x_3 \leq w \leq x_4 \wedge flip(w) = z \wedge \overline{\delta}'(z, x_2\text{-}x_3, w) \wedge T(\overline{\psi}, x_1) \wedge T(\overline{\psi}', z) \wedge T(\overline{\psi}'', x_2\text{-}x_3, w) \wedge \forall y(x_1 < y < z \rightarrow T(\overline{\phi}, y)) \wedge \forall y(w < y < x_4 \rightarrow T(\overline{\phi}', y)) \wedge \forall y((w < y < x_4 \rightarrow \exists y'(x_1 \leq y' \leq z \wedge flip(y) = y')) \rightarrow T(\overline{\phi}'', x_2\text{-}\ x_3, y)))$.

By inductive hypothesis, $\neg\overline{\delta}'(x_1, x_2\text{-}x_3, k)$ is equivalent to a disjunction of $S\text{-}DEC$ formulas $\bigvee_j \delta_j'(z, x_2\text{-}x_3, w)$, and by Lemma 6.8, for each $j$ there is a $UUTL$ formula $\gamma_j$ such that $\exists z\ \delta_i'(z, x_2\text{-}x_3, w)$ is equivalent to $T(\gamma_i, x_2\text{-}x_3, x_4)$.

(Case A) Assume that $T(\overline{\phi}, x)$ holds true for any $x_1 < x < x_2$, that $T(\overline{\phi}', y)$ holds

true for any $x_3 < y < x_4$ and that $T(\overline{\phi}'', x_2\text{-}\ x_3, y)$ holds true for each jump point $y$ of $x_3$. In this case, we have a $S - DEC.6$ formula as follows: $\phi''$ is set to $\Gamma = \bigvee_i \gamma_i \vee \neg\overline{\psi}'' \vee (\neg\overline{\psi}' \cdot True \uparrow True)$; $z$ and $w$ are forced to coincide with $x_2$ and $x_3$, respectively, by choosing a $\delta'(z, x_2\text{-}x_3, w)$ of the form $S - DEC.1$ with $\phi$ and $\phi''$ set to $False$;
Otherwise, if there exists a jump point $y$ of $x_3$ where $T(\neg\overline{\phi}'', x_2\text{-}x_3, y)$ holds true we have a $S - DEC.6$ formula as follows: $\psi''$ is set to $\neg\overline{\phi}'' \wedge \Gamma$ and $\phi''$ is set to $\Gamma$.
(Case B) Assume the basic case in which there is a point $x_1 < \overline{x} < x_2$ where $T(\overline{\phi}, \overline{x})$ does not hold and assume that $\overline{x}$ is the least of these points. Assume also that $T(\overline{\phi}', y)$ holds true in any $x_3 < y < x_4$. The basic case we consider first is the one in which there is a break point $k$ for $(x_1, x_4)$ such that either $\overline{x} < k \leq x_2$ or $\overline{x} = k < x_2$.
For instance, the case in which $\overline{x} < k \leq x_2$ is given by a $S - DEC.3$ formula where the recursive argument $\delta'(x_1, x_2\text{-}x_3, z)$ is a $S - DEC.7$ formula having as $\delta''$ a $S - DEC.1$ formula with $\psi$ set to $somewhere(\neg\overline{\phi})$. The case $\overline{x} = k < x_2$ can be given in a similar way.
If such a break point $k$ does not exists, there are $z$ and $w$, $k$ such that $flip(w) = z$, $k$ is a break point for $[z, w]$ and either $\overline{x} < k \leq x_2$ or $\overline{x} = k < x_2$. Assume also for simplicity, that $T(\overline{\phi}'', x_2\text{-}x_3, y)$ holds true in each jump point $y$ of $w$ (the opposite case can be dealt with analogously). In this case we have a disjunction of $S - DEC.6$ formula where $\psi''$ and $\phi''$ are set to $\Gamma$ and where $\delta'(z, x_2\text{-}x_3, w)$ is an element of the disjunction of $S\text{-}DEC$ formulas obtained by handling the basic case.
(Case C) The symmetric case where there is a point $x_3 < \overline{y} < x_4$ such that $T(\neg\overline{\phi}', \overline{y})$ and $T(\overline{\phi}, x)$ for any $x_1 < x < x_2$, can be treated with obvious changes, exactly as case $B$.
(Case D) Let us assume that there are a least point $x_1 < \overline{x} < x_2$, where $T(\overline{\phi}, \overline{x})$ does not hold, and a greatest point $x_3 < \overline{y} < x_4$, where $T(\overline{\phi}', \overline{y})$ does not hold. Also in this case we have to deal with a basic case where there exists a break point $k$ for $[x_1, x_4]$ such that either $\overline{x} < k \leq x_2$, or $\overline{x} = k < x_2$, or $x_3 \leq k < \overline{y}$ or $x_3 < k = \overline{y}$. (These cases are dealt with as in case $B$ and $C$.) Then we have to consider the general case in which there are $z$, $w$ with $flip(w) = z$, and a break point $k$ for $[z, w]$ such that $z < x_2 \leq x_3 < w$ and one of the four conditions of the basic case is satisfied. In this case we can proceed in the standard way by exploiting a disjunction of $S - DEC.6$ formulas with $\phi''$ set to $\Gamma$ (if any jump point $y$ for $w$ satisfies $T(\overline{\phi}'', x_2\text{-}\ x_3, y)$ and with $\delta'(z, x_2\text{-}x_3, w)$ a $S\text{-}DEC$ formula obtained from the analysis of the basic case.
(**S-DEC.7**) This case can be treated, with obvious changes, exactly as the case $S - DEC.5$, and it is therefore omitted.
(**S-DEC.8**) This case can be treated, with obvious changes, exactly as the case $S - DEC.6$, and it is therefore omitted. ∎

**Lemma 6.7** *Any flip-closed $M\text{-}BET(x_1, x_2)$ and $S\text{-}BET(x_1, x_2\text{-}x_3, x_4)$ formula is equivalent to a finite disjunction of $M\text{-}DEC(x_1, x_2)$ formulas and $S\text{-}DEC(x_1, x_2\text{-}x_3, x_4)$ formulas, respectively.*

PROOF. The proof is by induction on the structure of $M\text{-}BET$ and $S\text{-}BET$ formulas.
**Base.** We consider first $M\text{-}BET$ formulas.
The formula $True$ is equivalent to $\delta_1 \vee \delta_2$, where $\delta_1$ is a $M - DEC.1$ formula and $\delta_2$ is a $M - DEC.3$ formula.

The formula $p(x_1)$ is equivalent to $\delta_1 \vee \delta_2$, where $\delta_1$ is a $M-DEC.1$ formula with $\psi$ set to $p$, and $\delta_2$ is a $M-DEC.3$ formula where the recursive parameter $\delta'$ is a $M-DEC.1$ formula with $\psi$ set to $p$. The formula $flip(x_2) = x_1$ is a $M-DEC.1$ formula with $\psi = True$.
We consider now the base case for *S-BET* formulas.
The formula $True$ is equivalent to the a formula $\bigvee_{i=1}^{3} \delta_i$ which describes all of the possible flip-closed stereo intervals. In particular, $\delta_1$ is a $S-DEC.1$ formula; $\delta_2$ is a $S-DEC.7$ formula (a multiple-sequence stereo interval with $x_2$ and $x_3$ in the first sequence element); $\delta_3$ is a $S-DEC.4$ formula (a multiple-sequence stereo interval with $x_2$ and $x_3$ not placed in the first sequence element).
The cases for $p(x_i)$, with $i = 1, 2, 3$, exploits a translation analogous to that we have introduced for $True$ taking care to suitably place in such a translation a sub-formula $T(p, x_i)$. For instance the formula $p(x_2)$ is equivalent to the a formula $\bigvee_{i=1}^{3} \delta_i$, where $\delta_1$ is a $S-DEC.1$ formula with $\psi'$ set to $p$; $\delta_2$ is a $S-DEC.7$ formula where the recursive parameter $\delta''$ is a $S-DEC.1$ formula with $\psi'$ set to $Q_a$; $\delta_3$ is a $S-DEC.4$ where the recursive parameter has either the form of $\delta_1$ or $\delta_2$.
The formula $flip(x_4) = x_1$ is a $S-DEC.1$ formula.
The formula $flip(x_4) = x_2$ is equivalent to a disjunction $\delta_1 \vee \delta_2$, where
$\delta_1$ is a $S-DEC.1$ with $\phi = False$ (notice that $flip(x_4) = x_1$ implies $x_1 = x_2$) and $\delta_2$ is a $S-DEC.7$ where the recursive parameter $\delta''$ has the same form of $\delta_1$.
The formula $flip(x_4) = x_3$ is equivalent to a disjunction $\delta_1 \vee \delta_2 \vee \delta_3$, where $\delta_1$ is a $S-DEC.1$ with $\psi''$ set to 0 (notice that $flip(x_4) = x_1$ implies $x_1 = x_2 = x_3 = 0$); $\delta_2$ is a $S-DEC.4$ where $\delta'(x_1, x_2\text{-}x_3, k)$ is a $S-DEC.1$ with $\phi$ and $\phi'$ set to $False$ and $\psi''$ set to $Even$ (i.e $x_1 = x_2$ and $x_3 = z$); $\delta_3$ is a $S-DEC.4$ where $\delta'(x_1, x_2\text{-}x_3, k)$ is a $S-DEC.2$.
The formula $flip(x_3) = x_1$ is equivalent to a disjunction $\delta_1 \vee \delta_2$, where
$\delta_1$ is a $S-DEC.1$ formula with $\phi$ set to $False$ (notice that $flip(x_3) = x_1$ implies $x_1 = x_2$ since $flip(x_3) = x_2$ by hypothesis); $\delta_2$ is a $S-DEC.4$ with the recursive parameter $\delta'$ defined as $\delta_1$.
The formula $flip(x_2) = x_1$ is equivalent to a disjunction $\bigvee_{i=1}^{4} \delta_i$, where $\delta_1$ is a $S-DEC.2$ with $\delta(x_1, x_2)$ a $M-DEC.1$;
$\delta_2$ is a $S-DEC.4$ with $\delta(x_1, x_2\text{-}x_3, z)$ having the same form of $\delta_1$;
$\delta_3$ is a $S-DEC.8$ formula with $\delta(x_1, x_2\text{-}\ x_3, w)$ having either the form of $\delta_1$ or the form of $\delta_2$;
$\delta_4$ is a $S-DEC.4$ with $\delta(x_1, x_2\text{-}x_3, z)$ having the same form of $\delta_3$.
The formulas $x_4 = x_1$ and $x_4 = x_2$ are equivalent to $False$;
The formula $x_4 = x_3$ is equivalent to a formula $\delta_1 \vee \delta_2$ where $\delta_1$ is a $S-DEC.2$ formula and $\delta_2$ is a $S-DEC.1$ formula with $\phi$ and $\phi'$ set to $False$ and either both $\psi''$ and $\psi'''$ set to *odd* or $\psi''$ set to $Even$ (notice that in this case $flip(x_4) = x_1$ implies $x_1 = x_2$).
The formula $x_3 = x_2$ and $x_1 = x_3$ are equivalent to a disjunction $\delta_1 \vee \delta_2$ where $\delta_1$ is a $S-DEC.1$ with $\psi'' = 0$ and $\delta_2$ is a $S-DEC.4$ with $\delta'(x_1, x_2\text{-}x_3, z)$ a formula defined exactly as $\delta_1$.
The formula $x_2 = x_1$ is equivalent to a disjunction $\delta_1 \vee \delta_2$ where $\delta_1$ is a $S-DEC.1$ with $\phi = False$ and $\delta_2$ is a $S-DEC.4$ with $\delta'(x_1, x_2\text{-}\ x_3, z)$ having the same form as $\delta_1$;
**Inductive case.** We consider first *M-BET* formulas.

(**M-BET.2**). By Lemma 6.5.
(**M-BET.3**) Let us consider the case of $\phi(x_1, x_2)$ having the form $\exists z(x_1 < z < x_2 \wedge flip(x_2) = z \wedge \phi_1(x_1, z) \wedge \phi_2(z, x_2))$. By inductive hypothesis $\phi_1(x_1, z)$ is equivalent to a disjunction of *M-DEC* formulas $\bigvee_i \delta_i'(x_1, z)$ and $\phi_2(z, x_2)$ is equivalent to a disjunction of *M-DEC* formulas$\bigvee_j \delta_j''(z, x_2)$. If suffices to prove the thesis for a disjunct $\phi_{i,j}$ of the form $\exists z(x_1 < z < x_2 \wedge flip(x_2) = z \wedge \delta_i'(x_1, z) \wedge \delta_j''(z, x_2))$.
The formula $\phi_{i,j}$ is equivalent to a $M - DEC.5$ formula, where the recursive parameter $\delta'$ is $\delta_i'(x_1, z)$ and the recursive parameter $\delta''$ is $\delta_j''(z, x_2)$.
(**M-BET.4** and **M-BET.5**) Both the cases directly follow from inductive hypothesis and Proposition 6.6.
We consider now *S-BET* formulas.
(**S-BET.2**). The thesis follows from Lemma 6.5.
(**S-BET.3**). Let us consider the case of $\phi(x_1, x_2\text{-}x_3, x_4)$ having the form $flip(x_3) = x_2 \wedge \phi_1(x_1, x_2) \wedge \phi_2(x_3, x_4)$. Notice that in this case, the hypothesis of flip-closure implies that $\neg flip(x_4) = x_1$.
By inductive hypothesis $\phi_1(x_1, x_2)$ is equivalent to a disjunction of *M-DEC* formulas $\bigvee_i \delta_i'(x_1, x_2)$ and $\phi_2(x_3, x_4)$ is equivalent to a disjunction of *M-DEC* formulas $\bigvee_j \delta_j''(x_3, x_4)$. If suffices to prove the thesis for a disjunct $\phi_{i,j}$ of the form $flip(x_3) = x_2 \wedge \delta_i'(x_1, x_2) \wedge \delta_j''(x_3, x_4)$.
If $\delta_j''(x_3, x_4)$ has the form $M - DEC.1$, $M - DEC.2$ or $M - DEC.4$ then $\phi_{i,j}$ has the form $S - DEC.5$, where $\delta''(k, x_4)$ is the formula obtained from $\delta_j''(x_3, x_4)$ by renaming $x_3$ with $k$, and $\delta'(x_1, x_2\text{-}x_3, k)$ is the $S - DEC.2$ formula $\neg flip(k) = x_1 \wedge k = x_3 \wedge flip(x_3) = x_2 \wedge \delta_i'(x_1, x_2)$.
If $\delta_j''(x_3, x_4)$ has the form $M - DEC.3$, then $\phi_{i,j}$ is a $S - DEC.4$ formula, where $\delta'(x_1, x_2\text{-}x_3, k)$ is the $S - DEC.2$ formula defined exactly as in the previous case.
Finally, let us consider the case where $\delta_j''(x_3, x_4)$ has the form $M - DEC.5$, namely it has the form $\neg flip(x_4) = x_3 \wedge \exists k(x_3 < k < x_4 \wedge flip(x_4) = k \wedge \overline{\delta}'(x_3, k) \wedge \overline{\delta}''(k, x_4))$.
In this case, $\phi_{i,j}$ has the form $S - DEC.5$ where $\delta''(k, x_4)$ is $\overline{\delta}''(k, x_4)$, and $\delta'(x_1, x_2\text{-}x_3, k)$ is recursively defined as follows:
if $\overline{\delta}'(x_3, k)$ has not the form $M - DEC.5$, then $\delta'(x_1, x_2\text{-}x_3, k)$ is obtained by transforming $\overline{\delta}'(x_3, k)$ as in the previous already examined cases; otherwise, if $\overline{\delta}'(x_3, k)$ has the form $M - DEC.5$ $\delta'(x_1, x_2\text{-}x_3, k)$ is obtained by recursively applying the transformation to $\overline{\delta}'(x_3, k)$.
(**S-BET.4, S.5** and **S.6**) The cases directly follow from inductive hypothesis and Proposition 6.6. ∎

**Lemma 6.8** *For each M-DEC formula $\delta(x_1, x_2)$ (with $x_1 < x_2$) (resp. S-DEC formula $\delta(x_1, x_2\text{-}x_3, x_4)$, with $x_1 < x_4$) there exists a formula $\gamma \in UUTL$ such that $(\exists x_1)\delta(x_1, x_2)$ is equivalent to $T(\gamma, x_2)$ (resp. $(\exists x_1)\delta(x_1, x_2\text{-}x_3, x_4)$ is equivalent to $T(\gamma, x_2\text{-}x_3, x_4)$).*

PROOF. Let us consider first *M-DEC*-formulas. The proof is by induction on their structure.
Base. If $\delta(x_1, x_2)$ has the form $M - DEC.1$, then $\gamma \in UUTL$ is $\phi \cdot \phi' \uparrow \phi''$.
Inductive case.
($M - DEC.2$) By inductive hypothesis, there is $\gamma'$ such that $(\exists x_1)\delta'(x_1, z)$ is equivalent to $T(\gamma', z)$. The formula $\gamma \in UUTL$ is
$(True \cdot True\ U\ True -\ \psi \wedge \gamma' U\ False \cdot True) \vee (True \cdot True\ U\ True -\ \phi \wedge (\psi \wedge \gamma') \diamond$

$\phi U\ False \cdot True)$.
$(M-DEC.3)$ By inductive hypothesis, there is $\gamma'$ such that $(\exists x_1)\delta'(x_1, z)$ is equivalent to $T(\gamma', z)$. The formula $\gamma \in UUTL$ is $(\gamma' \wedge \psi) \diamond \phi$.
$(M-DEC.4)$ By inductive hypothesis, there is $\gamma'$ such that $(\exists z)\delta'(z, w)$ is equivalent to $T(\gamma', w)$. The formula $\gamma \in UUTL$ is
$((\psi \wedge \psi' \cdot True \uparrow True) \wedge \psi'' \wedge \gamma') \vee\ (\psi \cdot \phi\ U\ \psi' {-} \psi'' \wedge \gamma'\ U\ \phi' \cdot \phi'')$.
Notice that the former disjunct is for the case $w = x_4$ and the latter for the case $w < x_4$.
$(M-DEC.5)$ By inductive hypothesis, there are $\gamma'$ and $\gamma''$ such that $(\exists x_1)\delta'(x_1, z)$ is equivalent to $T(\gamma', z)$ and $(\exists k)\delta''(k, x_2)$ is equivalent to $T(\gamma'', x_2)$. The formula $\gamma \in UUTL$ is $(\gamma' \cdot True \uparrow True) \wedge \gamma''$.
We consider now $S$-$DEC$ formulas.
Base. Let $\delta(x_1, x_2\text{-}x_3, x_4)$ have the form $S-DEC.1$. In this case the formula $\gamma \in UUTL$ is $(\Delta \wedge \psi'' \wedge \psi''' \wedge ((\psi \wedge \psi') \cdot True \uparrow True)) \vee\ (\psi \cdot \phi\ U\ \psi' {-} \psi'' \wedge \Delta\ U\ \phi' \cdot \phi'')$, where $\Delta$ is the formula $\neg(True \cdot True\ U\ True\text{-}True\ U\ True \cdot True)$. Notice that the former disjunct is for the case $x_3 = x_4$ and the latter for the case $x_3 < x_4$. Notice also that the conjunct $\Delta$ is used to ensure, in the first disjunct, that $x_3 = x_4$, and in the second that the internal *flip*-related points chosen by the $UU$ operator are precisely $x_2$ and $x_3$.
$(S-DEC.2)$ As already proved, there exists $\gamma'$ such that $T(\gamma', x_2)$ is equivalent to $(\exists x_1)\delta(x_1, x_2)$. The formula $\gamma$ is $\psi' \wedge ((\gamma' \wedge \psi) \cdot True \uparrow True)$.
Inductive case.
$(S-DEC.3)$ By inductive hypothesis there is $\gamma'$ such that $(\exists x_1)\delta(x_1, x_2\text{-}x_3, z)$ is equivalent to $T(\gamma', x_2\text{-}x_3, z)$. The formula $\gamma$ is $(True \cdot True\ U\ True {-}\ \psi \wedge \gamma'\ U\ False \cdot True) \vee\ (True \cdot True\ U\ True {-} \phi \wedge ((\psi \wedge \gamma') \diamond \phi) U\ False \cdot True)$
$(S-DEC.4)$ By inductive hypothesis there is $\gamma'$ such that $(\exists x_1)\delta(x_1, x_2\text{-}x_3, z)$ is equivalent to $T(\gamma', x_2\text{-}x_3, z)$. The formula $\gamma$ is $(\gamma' \wedge \psi) \diamond \phi$.
$(S-DEC.5)$ By inductive hypothesis there are $\gamma'$ and $\gamma''$ such that $T(\gamma', x_2\text{-}x_3, k)$ and $T(\gamma'', x_4)$ are equivalent to $(\exists x_1)\delta'(x_1, x_2\text{-}x_3, k)$ and $(\exists k)\delta''(k, x_4)$, respectively. The formula $\gamma$ is $\gamma'' \wedge (\gamma' \cdot True \uparrow True)$.
$(S-DEC.6)$ By inductive hypothesis there is $\gamma'$ such that $T(\gamma', x_2\text{-}x_3, w)$ is equivalent to $(\exists z)\delta'(z, x_2\text{-}x_3, w)$. The formula $\gamma$ is $((\psi \wedge \psi' \cdot True \uparrow True) \wedge \psi'' \wedge \gamma') \vee\ (\psi \cdot \phi\ U\ \psi' {-} \psi'' \wedge \gamma'\ U\ \phi' \cdot \phi'')$.
$(S-DEC.7)$ is quite similar to $S-DEC.5$.
$(S-DEC.8)$ By inductive hypothesis there is $\gamma'$ such that $T(\gamma', w)$ is equivalent to $(\exists x_1)\delta'(x_1, x_2\text{-}x_3, w)$. The formula $\gamma$ is $((\psi \wedge \psi' \cdot True \uparrow True) \wedge \psi'' \wedge \gamma') \vee\ (\psi \cdot False\ U\ \psi' {-} \psi'' \wedge \gamma'\ U\ \phi' \cdot \phi'')$. ∎

Received July 2, 2002