A set-theoretic translation method for polymodal logics

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Abstract. The paper presents a *set-theoretic translation method* for polymodal logics that reduces derivability in a large class of propositional polymodal logics to derivability in a very weak first-order set theory Ω. Unlike most existing translation methods, the one we propose applies to any normal complete finitely axiomatizable polymodal logic, regardless of whether it is first-order complete or an explicit semantics is available.
The finite axiomatizability of Ω allows one to implement mechanical proof-search procedures via the deduction theorem. Alternatively, more specialized and efficient techniques can be employed. In the last part of the paper, we briefly discuss the application of *set T-resolution* to support automated derivability in (a suitable extension of) Ω.

Key words: Modal Logic, Translation Methods, Set Theory, Theorem Proving

1. Introduction

In this paper, we propose a novel translation method to support derivability in propositional modal logic, whose basic idea is to map modal formulae into set-theoretic terms. Most inference systems for modal logic are defined in the style of sequent or tableaux calculi, e.g. [10, 24]. As an alternative, a number of *translation* methods for modal logic into classical first-order logic have been proposed in the literature (for an up-to-date survey see [18]). Such methods allow the use of Predicate Calculus mechanical theorem provers to implement modal theorem provers. Compared with the direct approach of finding a proof algorithm for a specific class of modal logics, the translation methods have the advantage of being *independent* of the particular modal logic under consideration: a single theorem prover may be used for any translatable modal logic.

In the standard approach, the first-order language $\mathcal{L}$ into which the translation is carried out contains a constant τ denoting the initial world in the frame, a binary relation $R(x, y)$ denoting the accessibility relation, and a denumerable number of unary predicates $\mathtt{P_i}(x)$. The translation function π is defined by induction on the structural complexity of the modal formula as follows:

– $\pi(P_j, x) \equiv \mathtt{P_j}(x)$;

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– $\pi(-, x)$ commutes with the boolean connectives;

– $\pi(\Box\,\psi, x) \equiv \forall y(xRy \to \pi(\psi, y))$.

Let H be a normal modal logic and ϕ be a modal formula. H is *first-order complete* if there exists a first-order sentence $Axiom_H$, involving only equality and the binary relational symbol $R(x, y)$, such that ϕ is derivable from H if and only if ϕ is true in the initial world τ of all generated frames satisfying $Axiom_H$ [2, 12]. For these logics the following holds:

$$\vdash_H \phi \;\Leftrightarrow \vdash Axiom_H \to \pi(\phi, \tau),$$

where $\vdash$ stands for derivability in classical Predicate Calculus. Hence, as long as we have $Axiom_H$, a classical theorem prover can be used as a theorem prover for H.

Efficiency concerns have motivated further investigations on the above (relational) translation method. Such studies (e.g. [17]) suggested a "functional" semantics for modal logic and resulted in a family of more efficient and general translation methods. From the computational point of view, the functional translation may still cause some problem when using a first-order theorem prover, due to the presence of equalities in $Axiom_H$. A method for limiting the complexity induced by the introduction of equality using a mixed relational/functional translation is proposed in [16].

A common feature of all the methods mentioned above is that, in order to be applied directly, the underlying modal logic must have a first-order semantics: insofar as we are aware, all attempts to deal with logics not having a first-order semantics have required *ad-hoc* techniques. Moreover, if the logic has a first-order semantics, but it is only specified by Hilbert axioms, a preliminary step is necessary to find the corresponding first-order axioms. The question of automatically solving this last problem has been extensively studied and algorithms have been proposed, e.g. [2, 11].

One of the main motivations of the present work was to find a translation applicable to all complete modal logics, regardless of the first-order axiomatizability of their semantics. The set-theoretic translation we propose works for all normal complete finitely axiomatizable modal logics. In particular, our method also works if the modal logic under consideration is only specified by Hilbert axioms.

The basic idea is to represent any Kripke frame as a set, with the accessibility relation modeled using the membership relation $\in$. Given a modal formula $\phi(P_1, \ldots, P_n)$ we define its translation as the *set-theoretic term* $\phi^*(x, x_1, \ldots, x_n)$, with variables $x, x_1, \ldots, x_n$, built using $\cup$, $\setminus$, and Pow. Intuitively, $\phi^*(x, x_1, \ldots, x_n)$ represents the set of those worlds (in the frame x) in which the formula ϕ holds. The inductive definition of $\phi^*(x, x_1, \ldots, x_n)$ is rather straightforward except for the case of $\Box\,\phi$, whose translation is defined as: $(\Box\,\phi)^* \equiv Pow(\phi^*)$ (see Section 2 for details).

In order to achieve a computationally valid result, we want to refer to a *finitely* (first-order) axiomatizable set theory. We succeeded in carrying out our translation in a very weak[1] set theory called Ω.

We prove that, for any normal modal logic $H = K + \psi(\alpha_{j_1}, \ldots, \alpha_{j_m})$, where $\psi(\alpha_{j_1}, \ldots, \alpha_{j_m})$ is an axiom schema, the following holds:

$$\vdash_H \phi \ \Rightarrow\ \Omega \vdash \forall x(Trans(x) \wedge Axiom_H(x) \rightarrow \forall x_1 \ldots \forall x_n(x \subseteq \phi^*(x, x_1, \ldots x_n)))$$

and

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_H(x) \rightarrow \forall x_1 \ldots \forall x_n(x \subseteq \phi^*(x, x_1, \ldots x_n))) \ \Rightarrow\ \psi \models \phi,$$

where $Trans(x)$ and $Axiom_H(x)$ stand for

$$\forall y\ (y \in x \rightarrow y \subseteq x) \text{ and } \forall x_{j_1}, \ldots, \forall x_{j_m}(x \subseteq \psi^*(x, x_{j_1}, \ldots, x_{j_m})),$$

respectively, and $\models$ represents frame logical consequence. In the case of frame-complete theories H, the proposed translation captures exactly the notion of H-derivability.

Instead of translating Hilbert axioms a set-theoretic semantics for H can be used, whenever such a semantics is available. We will study the case of G as an example of this approach.

The proposed set-theoretic translation method is then generalized to polymodal logics. This generalization involves revising the definition of the translation function to cope with a set of distinct modal operators instead of a single one. The technique we employ is similar to the one introduced by Thomason in [22]; the use of a set-theoretic language simplifies Thomason's approach and turns out to be completely symmetric.

The translation method we propose here may also be considered from a more abstract point of view as a means to analyze general deduction for modal formulae. However, this issue is not addressed here, since our focus is on the computational aspects of the technique; an extensive discussion can be found in [3].

In the last part of the paper, we briefly describe the application of set T-resolution techniques to support derivability in Ω. In order to apply such techniques, it is necessary to guarantee the decidability, with respect to Ω, of the class of ground formulae written in any language which extends the one in which the axioms of Ω are written with Skolem functions. We succeeded in providing such a decidability result in a suitable extension of Ω, and the main steps of the proof are outlined in Section 5 (more details can be found in [8], [9]).

The paper is organized as follows. In Section 2, we introduce the set-theoretic translation method and show how to apply it to the modal logic G. In this case, the proofs are simple and a clear description of the main features of the translation method is possible; moreover, G provides an example of how the method

[1] Compare this theory with more classical finite axiomatizations of Set Theory, such as NBG [15].

applies to a logic with a non-first-order semantics. In Section 3, we consider the general case and exploit the possibility of translating the Hilbert axioms of the logic. The proof of soundness of the translation is carried out using a particular universe of non-well-founded sets and applies to a large class of extensions of Ω. In Section 4, we generalize the proposed method to polymodal logics using a set-theoretic counterpart of Thomason's technique for translating polymodal logics into monomodal ones [22, 23]. Finally, in Section 5, we briefly discuss the application of set T-resolution techniques to support derivability in a suitable extension of Ω.

2. A set-theoretic translation of G

We first consider the case of the propositional modal logic G obtained by adding the Löb's axiom schema $\Box(\Box\alpha \rightarrow \alpha) \rightarrow \Box\alpha$ to K. Our goal is to find a translation of G formulae in the language of set theory and a finitely axiomatizable theory Ω such that, for any modal formula ϕ, $\vdash_G \phi$ if and only if Ω proves the translation of ϕ.

We consider the theory Ω specified by the following axioms in the language with relational symbols $\in, \subseteq$, and functional symbols $\cup, \setminus, Pow$:

$$x \in y \cup z \leftrightarrow x \in y \vee x \in z;$$

$$x \in y \setminus z \leftrightarrow x \in y \wedge x \notin z;$$

$$x \subseteq y \leftrightarrow \forall z(z \in x \rightarrow z \in y);$$

$$x \in Pow(y) \leftrightarrow x \subseteq y.$$

Notice that neither the extensionality axiom nor the axiom of foundation are in Ω. In the next section, we will make an essential use of the latter fact: since we will model the accessibility relation by the membership relation, we will be forced to work in universes containing non-well-founded sets. As a matter of fact, it will be convenient to use universes satisfying AFA [1]. However, in the case of G a standard (well-founded) model of set theory is sufficient to carry out the proof of the soundness of the translation.

Given a modal formula $\phi(P_1, \ldots, P_n)$, its translation is the *set-theoretic term* $\phi^*(x, x_1, \ldots, x_n)$, with variables $x, x_1, \ldots, x_n$, inductively defined as follows:

- $P_i^* \equiv x_i$;
- $(\phi \vee \psi)^* \equiv \phi^* \cup \psi^*$;
- $(\phi \wedge \psi)^* \equiv \phi^* \cap \psi^*$;
- $(\neg\phi)^* \equiv x \setminus \phi^*$;

– $(\phi \rightarrow \psi)^* \equiv (x \setminus \phi^*) \cup \psi^*$;

– $(\Box\, \phi)^* \equiv Pow(\phi^*)$,

where x is different from x_i for $i = 1, \ldots, n$, $\phi^* \cap \psi^*$ stands for $\phi^* \setminus (\phi^* \setminus \psi^*)$, and $\diamond$ is translated as $\neg\Box\,\neg$.

We will show that:

$$\vdash_G \phi \Leftrightarrow \Omega \vdash \forall x(Trans(x) \wedge Axiom_G(x) \rightarrow \forall x_1 \ldots \forall x_n(x \subseteq \phi^*(x, x_1, \ldots, x_n))),$$

where $Trans(x)$ stands for $\forall y(y \in x \rightarrow y \subseteq x)$ (x is transitive), and $Axiom_G(x)$ represents the conjunction of $\forall y(y \subseteq x \wedge \exists z(z \in y) \rightarrow \exists s(s \in y \wedge \forall v(v \notin s \cap y)))$ and $\forall z \forall w \forall y(z \in x \wedge w \in x \wedge y \in x \wedge z \in w \wedge w \in y \rightarrow z \in y)$ (x is well-founded and $\in$ restricted to x is transitive, respectively).

We prove that the proposed translation is complete and sound. The proof of completeness is straightforward; the proof of soundness relies on the characterization of G using the class of all finite trees.

THEOREM 1. **(Completeness of the translation method)** *For each modal formula ϕ involving n propositional variables $P_1, \ldots, P_n$,*

$$\vdash_G \phi \;\Rightarrow\; \Omega \vdash \forall x(Trans(x) \wedge Axiom_G(x) \rightarrow \forall x_1 \ldots \forall x_n(x \subseteq \phi^*(x, x_1, \ldots, x_n))).$$

Proof. The proof is by induction on the derivation of $\vdash_G \phi(P_1, \ldots, P_n)$. The cases of tautologies and closure under modus ponens do not present any difficulty, and thus they are left to the reader (a proof can be found in [8]). We explicitly prove the result for K and Löb's axiom schemata, and for closure under necessitation. We first consider the axiom schema K:

$$\Box\,(\alpha \rightarrow \beta) \rightarrow (\Box\,\alpha \rightarrow \Box\,\beta).$$

Without loss of generality, we suppose that α and β involve n propositional variables $P_1, \ldots, P_n$, and show that Ω derives its translation, namely:

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_G(x) \rightarrow \forall x_1 \ldots \forall x_n(x \subseteq (\Box\,(\alpha \rightarrow \beta) \rightarrow (\Box\,\alpha \rightarrow \Box\,\beta))^*)).$$

By definition, $(\Box\,(\alpha \rightarrow \beta) \rightarrow (\Box\,\alpha \rightarrow \Box\,\beta))^* \equiv (x \setminus t) \cup s$, where $t = Pow((x \setminus \alpha^*) \cup \beta^*)$ and $s = (x \setminus Pow(\alpha^*)) \cup Pow(\beta^*)$ are terms on the variables $x, x_1, \ldots, x_n$. We have to prove that $\forall z(z \in x \rightarrow z \in (x \setminus t) \cup s)$, or, equivalently, that $\forall z(z \in x \wedge z \in t \rightarrow z \in s)$. By replacing t and s by their definitions, we may rewrite the last condition as: $z \in x$ and $z \subseteq (x \setminus \alpha^*) \cup \beta^*$ implies that $z \in (x \setminus Pow(\alpha^*)) \cup Pow(\beta^*)$. To prove it, it suffices to show that $z \in x$, $z \subseteq (x \setminus \alpha^*) \cup \beta^*$, and $z \subseteq \alpha^*$ implies that $z \subseteq \beta^*$. Since $z \subseteq \alpha^*$, for each s, if $s \in z$, then $s \in \alpha^*$; from $z \subseteq (x \setminus \alpha^*) \cup \beta^*$, it follows that $s \in \beta^*$. Notice that we never used the hypothesis that x satisfies $Trans(x)$ and $Axiom_G(x)$.

Consider now the closure under necessitation: if $\vdash_G \phi$, then $\vdash_G \Box\,\phi$. In this case, we suppose that:

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_G(x) \rightarrow \forall x_1 \ldots \forall x_n(x \subseteq \phi^*)),$$

and prove that:

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_G(x) \rightarrow \forall x_1 \ldots \forall x_n(x \subseteq Pow(\phi^*))).$$

For each x satisfying $Trans(x)$ and $Axiom_G(x)$, we prove that $\forall x_1, \ldots, \forall x_n(x \subseteq Pow(\phi^*))$, that is, for each z, if $z \in x$, then $z \in Pow(\phi^*)$ or, equivalently, $z \subseteq \phi^*$. Suppose that $z \in x$ and $t \in z$. From the validity of $Trans(x)$, it follows that $z \subseteq x$ and thus $t \in x$. The conclusion $t \in \phi^*$ directly follows from the hypothesis that $x \subseteq \phi^*$.

Finally, let us show that Ω proves the translation of Löb's axiom, that is, if $P_1, \ldots, P_n$ are the n variables occurring in ϕ, then

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_G(x) \rightarrow \forall x_1 \ldots \forall x_n(x \subseteq (\Box\,(\Box\,\phi \rightarrow \phi) \rightarrow \Box\,\phi)^*)).$$

The proof is nothing but the formalization in Ω of the proof of the validity of Löb's axiom schema in any well-founded transitive frame (cf., e.g., [21]).

By definition, $(\Box\,(\Box\,\phi \rightarrow \phi) \rightarrow \Box\,\phi)^* \equiv (x \setminus t) \cup Pow(\phi^*)$, where t stands for the term $Pow((x \setminus Pow(\phi^*)) \cup \phi^*)$. We want to prove that, if x satisfies $Trans(x) \wedge Axiom_G(x)$, then $\forall s(s \in x \wedge s \in t \rightarrow s \in Pow(\phi^*))$. This is equivalent to showing that there exists no set belonging to the subset y of x with $y = x \cap t \setminus Pow(\phi^*)$. We consider the formula:

$$\forall y(\forall s(s \in y \rightarrow \exists v(v \in s \cap y)) \rightarrow (y \subseteq x \rightarrow \forall z(z \notin y))),$$

which can be derived from the axiom stating the well-foundedness of x, and show that for $y = x \cap t \setminus Pow(\phi^*)$ the formula $\forall s(s \in y \rightarrow \exists v(v \in s \cap y))$ holds. Since $y \subseteq x$, this proves the result.

If $s \in y$, then $s \in x$, $s \in t$, with $t = Pow((x \setminus Pow(\phi^*)) \cup \phi^*)$, and $s \notin Pow(\phi^*)$. From the last conjunct, we derive that $\exists v(v \in s \wedge v \notin \phi^*)$. Since x satisfies $Trans(x)$ and $Axiom_G(x)$ (in particular, the transitivity of $\in$ with respect to x holds) and $s \in x$, from $v \in s$, it follows that $v \subseteq s$. Now, from $s \in Pow((x \setminus Pow(\phi^*)) \cup \phi^*)$ and $v \subseteq s$, it follows that $v \in Pow((x \setminus Pow(\phi^*)) \cup \phi^*)$, that is, $v \in t$. Finally, from $v \in s$, $s \subseteq (x \setminus Pow(\phi^*)) \cup \phi^*$, and $v \notin \phi^*$, it follows that $v \in x \setminus Pow(\phi^*)$, and then $v \notin Pow(\phi^*)$. From $v \in x$, $v \in t$, and $v \notin Pow(\phi^*)$, we can conclude that $v \in x \cap t \setminus Pow(\phi^*) = y$, and that proves the result.

(T1) ∎

It is worth noting that all the set-theoretic principles involved in the proof of completeness are those expressed by the (extremely simple) axioms of Ω.

The proof of soundness exploits the (frame) characterization theorem for G stating that $\vdash_G \phi$ if and only if ϕ is valid in every finite tree, where by a finite tree is meant a frame (W, R, r) in which W is a finite set containing the element r (the root), R is transitive and asymmetric, and the set of R−predecessors of any element contains r and is linearly ordered by R (see [21] for details).

THEOREM 2. **(Soundness of the translation method)** *For each modal formula ϕ involving n propositional variables $P_1, \ldots, P_n$,*

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_G(x) \rightarrow \forall x_1 \ldots \forall x_n(x \subseteq \phi^*(x, x_1, \ldots, x_n))) \;\Rightarrow \vdash_G \phi.$$

Proof. Let HF^A be the structure for the language of Ω consisting of all the hereditarily finite sets built from atoms in $A = \{a_0, a_1, \ldots\}$, with the natural set-theoretic interpretation of the relational and functional symbols $\in, \subseteq, \cap, \cup, \setminus$, and Pow. HF^A is a model for Ω [14]. Therefore, for every term $t(x_0, \ldots, x_n)$ and for every $h_0, \ldots, h_n$ in HF^A, we may consider the element $t^{HF^A}(h_0, \ldots, h_n)$ in HF^A. Moreover, if $\phi(P_1, \ldots, P_n)$ is a modal formula, the evaluation of the term $\phi^*(x, x_1, \ldots, x_n)$ over the elements $h_0, \ldots, h_n$ results in an element of HF^A.

Given a finite tree (W, R, r), we determine an element W^* of HF^A such that:

1. $Trans(W^*) \wedge Axiom_G(W^*)$ holds in the model HF^A, and
2. given a modal formula $\phi(P_1, \ldots, P_n)$, if $\forall x_1 \ldots \forall x_n(W^* \subseteq \phi^*(W^*, x_1, \ldots, x_n))$ holds in HF^A, then $\phi(P_1, \ldots, P_n)$ is valid in (W, R, r).

Fix an injection π from the leaves of W (i.e. nodes without any successor) to A. We define W^* in HF^A as follows: for every node $w \in W$, let

$$w^* = \begin{cases} \pi(w) & \text{if } w \text{ is a leaf of } W, \\ \{v^* : wRv\} & \text{otherwise.} \end{cases}$$

Let W^* be r^*. For every $w \in W$, $w^* \in HF^A$; moreover, it is not difficult to see that $Trans(W^*)$ and $Axiom_G(W^*)$ hold in HF^A.

Let $\models$ be a valuation of the propositional variables $P_1, \ldots, P_n$ on W and, for $i = 1, \ldots n$, let $P_i^* = \{w^* \in W^* : w \models P_i\}$. Since W is finite, we have that $P_1^*, \ldots, P_n^*$ belong to HF^A.

If the elements w^* and v^* are equal in HF^A, then $w = v$ (by induction on the height $h(w)$ of the node w in the tree (W, R, r)). This fact will be useful in proving the following lemma.

LEMMA 3. *For all $w \in W$ and for any formula $\phi(P_1, \ldots, P_n)$,*

$$w \models \phi(P_1, \ldots, P_n) \;\Leftrightarrow\; w^* \in \phi^*(W^*, P_1^*, \ldots, P_n^*) \text{ holds in } HF^A.$$

Proof. By induction on the structural complexity of the formula $\phi(P_1, \ldots, P_n)$.

If $\phi(P_1, \ldots, P_n) \equiv P_i$ and $w \models P_i$, then, by definition of P_i^*, $w^* \in P_i^*$. Vice versa, if $w^* \in P_i^*$, then $z \models P_i$ for some $z \in W$ with $w^* = z^*$; hence, as we observed, $w = z$ and therefore $w \models P_i$.

The case of boolean connectives is straightforward.

Now consider the formula $\Box\, \phi(P_1, \ldots, P_n)$:

$$\begin{aligned}
w \models \Box\, \phi(P_1, \ldots, P_n) &\Leftrightarrow \\
\forall z \in W\ (wRz \to z \models \phi(P_1, \ldots, P_n)) &\Leftrightarrow \\
\forall z \in W\ (wRz \to z^* \in \phi^*(W^*, P_1^*, \ldots, P_n^*)) &\Leftrightarrow \\
\{z^* : wRz\} \subseteq \phi^*(W^*, P_1^*, \ldots, P_n^*) &\Leftrightarrow\ w^* \subseteq \phi^*(W^*, P_1^*, \ldots, P_n^*) \Leftrightarrow \\
w^* \in Pow^*(\phi^*(W^*, P_1^*, \ldots, P_n^*)) &\Leftrightarrow\ w^* \in (\Box\, \phi)^*(W^*, P_1^*, \ldots, P_n^*).
\end{aligned}$$

(L3) ∎

From Lemma (3), we have that $\phi(P_1, \ldots, P_n)$ is valid in the model $(W, R, \models)$ if and only if the corresponding set W^* in HF^A is a subset of $\phi^*(W^*, P_1^*, \ldots, P_n^*)$. From this, item 2 above easily follows.

To conclude the proof of Theorem (2), suppose that

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_G(x) \to \forall x_1 \ldots \forall x_n(x \subseteq \phi^*(x, x_1, \ldots, x_n))).$$

If (W, R, r) is a finite tree, the corresponding set W^* in HF^A satisfies $Trans(W^*) \wedge Axiom_G(W^*)$. Hence, from $\Omega \vdash \forall x(Trans(x) \wedge Axiom_G(x) \to \forall x_1 \ldots \forall x_n(x \subseteq \phi^*))$, it follows that, for all elements $h_1, \ldots, h_n$ in HF^A, we have that $W^* \subseteq \phi^*(W^*, h_1, \ldots, h_n)$. In particular, for all valuations $\models$ of the propositional variables $P_1, \ldots, P_n$ on W, the above is true for the sets $P_1^*, \ldots, P_n^*$ defined as in Lemma (3). From the same lemma, one deduces that $\phi(P_1, \ldots, P_n)$ is valid in the model $(W, R, \models)$ and, from the Finite Tree Completeness Theorem [21], it follows that $\vdash_G \phi$.

(T2) ∎

3. The set-theoretic translation method

In this section we generalize the translation method to any normal finitely axiomatizable modal logic, possibly specified by Hilbert axioms only.

Let $\psi(\alpha_{j_1}, \ldots, \alpha_{j_m})$ be an axiom schema and H be the modal logic obtained by adding $\psi(\alpha_{j_1}, \ldots, \alpha_{j_m})$ to K. The completeness of the translation will be shown with respect to derivability in H, while soundness holds with respect to logical consequence. More formally, we will prove that, for any formula ϕ involving n propositional variables $P_1, \ldots, P_n$,

$$\vdash_H \phi\ \Rightarrow\ \Omega \vdash \forall x(Trans(x) \wedge Axiom_H(x) \to \forall x_1 \ldots \forall x_n(x \subseteq \phi^*(x, x_1, \ldots x_n)))$$

and

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_H(x) \rightarrow \forall x_1 \ldots \forall x_n(x \subseteq \phi^*(x, x_1, \ldots x_n))) \;\Rightarrow\; \psi \models \phi,$$

where $Trans(x)$ is the formula $\forall y\ (y \in x \rightarrow y \subseteq x)$ and $Axiom_H(x)$ is the formula $\forall x_{j_1}, \ldots, \forall x_{j_m}(x \subseteq \psi^*(x, x_{j_1}, \ldots, x_{j_m}))$.

In case H is complete, the notions of $\vdash_H$ and $\psi \models$ coincide and modal derivability of a given formula in H is equivalent to first-order derivability of the translated formula in Ω.

THEOREM 4. **(Completeness of the translation method)** *For each modal formula ϕ involving n propositional variables $P_1, \ldots, P_n$,*

$$\vdash_H \phi \;\Rightarrow\; \Omega \vdash \forall x(Trans(x) \wedge Axiom_H(x) \rightarrow \forall x_1 \ldots \forall x_n(x \subseteq \phi^*(x, x_1, \ldots, x_n))).$$

Proof. The proof follows the same path of the proof of Theorem 1, except for the verification of the case in which the formula ϕ is an instance of the axiom schema $\psi(\alpha_{j_1}, \ldots, \alpha_{j_m})$. For this case it is easy to check that the term $(\psi(\alpha_{j_1}, \ldots, \alpha_{j_m}))^*$ is syntactically equal to the term $\psi^*(x_1/\alpha^*_{j_1}, \ldots, x_m/\alpha^*_{j_m})$, and the result follows from $Axiom_H(x)$ and simultaneous substitution in Ω.

(T4) ■

THEOREM 5. **(Soundness of the translation method)** *For each modal formula ϕ involving n propositional variables $P_1, \ldots, P_n$,*

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_H(x) \rightarrow \forall x_1 \ldots \forall x_n(x \subseteq \phi^*(x, x_1, \ldots, x_n))) \Rightarrow \psi \models \phi$$

Proof. Hereafter, let $\mathcal{U}$ denote a universe of hypersets satisfying all the axioms of $ZF - FA$ (ZF except the foundation axiom) and AFA. In $\mathcal{U}$, for any graph (W, R), there is a (unique) function d such that, for every $w \in W$, the following holds (see [1] for details):

$$d(w) = \{d(v) | v \in W \wedge wRv\}.$$

Actually, it can be seen that the use of AFA is not essential for this proof and a model falsifying foundation "whenever needed" could be used in its place. However, as we will see, the use of AFA will simplify our argument making the construction more uniform.

We begin proving the following lemma:

LEMMA 6. *Let α be an ordinal, V_α be the set of all well-founded sets of rank less than α, and $\mathcal{U} \setminus V_\alpha$ be the universe of all hypersets not belonging to V_α.*

The structure for the language of Ω with support (domain) $\mathcal{U} \setminus V_\alpha$ and interpretation function $(.)'$ defined as follows[2]*:*

$$
\begin{array}{rcl}
x \in' y & \textit{iff} & x \in y; \\
x \cup' y & = & x \cup y; \\
x \subseteq' y & \textit{iff} & x \setminus V_\alpha \subseteq y; \\
x \setminus' y & = & \begin{cases} x \setminus y & \textit{if } \ x \setminus y \notin V_\alpha \\ V_\alpha & \textit{otherwise.} \end{cases} ; \\
Pow'(y) & = & \{x : x \setminus V_\alpha \subseteq y\}.
\end{array}
$$

is a model of Ω.

Proof. We first show that $\setminus', Pow'$, and $\cup'$ are well-defined over $\mathcal{U} \setminus V_\alpha$.

The proof for $\setminus'$ follows directly from its definition since either $x \setminus y$ does not belong to V_α and then $\setminus'$ is equal to $x \setminus y$, or it is actually equal to V_α which does not belong to V_α.

The case of Pow' is also straightforward: proceeding by contradiction, suppose that $y \notin V_\alpha$ and $Pow'(y) \in V_\alpha$. By definition, it follows that $y \in Pow'(y)$ and, from the hypothesis, $Pow'(y) \subseteq V_\alpha$ since V_α is transitive. Hence $y \in V_\alpha$, while we assumed $y \notin V_\alpha$.

Finally, for $x \cup' y$ notice that $x \cup' y$ is equal to $x \cup y$ by definition, and $x \cup y \in V_\alpha$ if and only if $x \in V_\alpha$ and $y \in V_\alpha$.

To complete the proof we must show that the proposed interpretation verifies the axioms of Ω.

Since $x \cup' y$ and $\in'$ are defined as $x \cup y$ and $\in$, respectively, the verification of the first axiom is trivial.

Now consider the second axiom. Let x, y, z belong to $\mathcal{U} \setminus V_\alpha$. If $y \setminus z \in V_\alpha$ then $y \setminus' z = V_\alpha$, and thus there are no $x \in \mathcal{U} \setminus V_\alpha$ such that $x \in' y \setminus' z$. Since from $y \setminus z \in V_\alpha$ it follows that $y \setminus z \subseteq V_\alpha$, there are no $x \in \mathcal{U} \setminus V_\alpha$ such that $x \in' y$ and $x \notin' z$. In case $y \setminus z \notin V_\alpha$ we have that $y \setminus' z$ is equal to $y \setminus z$ and therefore the axiom is verified.

For the third axiom, suppose that x, y, z belong to $\mathcal{U} \setminus V_\alpha$. By definition, $x \subseteq' y$ if and only if $x \setminus V_\alpha \subseteq y$, which is equivalent to saying that for all z in $\mathcal{U} \setminus V_\alpha$, if $z \in x$ then $z \in y$, namely that $\forall z(z \in' x \to z \in' y)$ holds in $\mathcal{U} \setminus V_\alpha$.

Finally consider the fourth axiom. Let x belong to $\mathcal{U} \setminus V_\alpha$. From the definition of $Pow'(y)$, it follows that $x \in' Pow'(y)$ if and only if $x \setminus V_\alpha \subseteq y$; but $x \setminus V_\alpha \subseteq y$ if and only if $x \subseteq' y$ and therefore the axiom is verified.

(L6) ■

[2] We denote the defined interpretation of symbols $\in, \cup, \setminus, \subseteq, Pow$ in $\mathcal{U} \setminus V_\alpha$ by $\in', \cup', \setminus', \subseteq'$, Pow', and the *standard* interpretation in $\mathcal{U}$ simply by $\in, \cup, \setminus, \subseteq, Pow$.

Given a frame (W, R), we want to embed it into the universe $\mathcal{U} \setminus V_\alpha$, for some suitable α. Let us associate a set $a\!\downarrow$ in $\mathcal{U}$ with each world $a \in W$. From *AFA* it follows that, for each $a \in W$, there exists a unique labeled decoration * such that $a^* = \{b^* : aRb\} \cup a\!\downarrow$ (cf. [1]). Moreover, it is possible to define $a\!\downarrow$ in such a manner that, for each a, b in W, $a^* \notin b\!\downarrow$ and $a \neq b$ in W implies $a^* \neq b^*$. For this purpose, let us consider a set $\widetilde{W}$ in $\mathcal{U}$, whose elements are well-founded sets of the same rank α, and such that there exists a bijection between $\widetilde{W}$ and W. For each $a \in W$, we denote the image of a in $\widetilde{W}$ by $\widetilde{a}$, and define $a\!\downarrow = \{\widetilde{a}\}$. The following lemma can be easily proved.

LEMMA 7. *For each a, b in W,*

(i) $a^* \notin b\!\downarrow$;

(ii) $a \neq b$ implies $a^* \neq b^*$;

(iii) $a^* \notin V_{\alpha+1}$ and $a^* \setminus V_{\alpha+1} = \{b^* : aRb\}$.

Proof. (i) If $a^* \in b\!\downarrow$, then $b\!\downarrow = \{\widetilde{b}\}$ implies that $a^* = \widetilde{b}$. Since $\widetilde{a} \in a\!\downarrow$ and $a\!\downarrow \subseteq a^*$, it follows that $\widetilde{a} \in \widetilde{b}$, which is impossible because $\widetilde{a}$ and $\widetilde{b}$ have the same rank α. (ii) If $a^* = b^*$, then $\widetilde{a} \in b^* = \{c^* : bRc\} \cup b\!\downarrow$. If $\widetilde{a} \in b\!\downarrow$, then $\widetilde{a} = \widetilde{b}$, contradicting $a \neq b$. If $\widetilde{a} = c^*$, for a given c such that bRc, then $\widetilde{c} \in \widetilde{a}$ (contradiction). (iii) Immediate.

(L7) ∎

Now consider a valuation $\models$ of the propositional variables $P_1, \ldots, P_n$ over the frame (W, R), and let W^* be equal to $\{a^* : a \in W\}$, where * is the labeled decoration previously introduced. W^* does not belong to $V_{\alpha+1}$ because $V_{\alpha+1}$ is transitive and, for each $a \in W$, $a^* \notin V_{\alpha+1}$. Furthermore, let P_i^* be equal to $\{a^* \in W^* : a \models P_i\}$ if this set is not empty, and to $V_{\alpha+1}$ otherwise. The following lemma holds:

LEMMA 8. *For each $a \in W$ and each formula $\phi(P_1, \ldots, P_n)$*

$$a \models \phi(P_1, \ldots, P_n) \;\Leftrightarrow\; a^* \in \phi^*(W^*, P_1^*, \ldots, P_n^*) \text{ in the universe } \mathcal{U} \setminus V_{\alpha+1}.$$

Proof. By induction on the structural complexity of the formula $\phi(P_1, \ldots, P_n)$.

The cases of propositional variables and boolean combinations of formulae are left to the reader.

In the case of a formula of the form $\Box\, \phi(P_1, \ldots, P_n)$, we have that:

$$
\begin{array}{rl}
a \models \Box\,\phi(P_1,\ldots,P_n) & \Leftrightarrow \\
\forall b \in W\ (aRb \to b \models \phi(P_1,\ldots,P_n)) & \Leftrightarrow \\
\forall b \in W\ (aRb \to b^* \in' \phi^*(W^*,P_1^*,\ldots,P_n^*)) & \Leftrightarrow \\
a^* \setminus V_{\alpha+1} \subseteq \phi^*(W^*,P_1^*,\ldots,P_n^*) & \Leftrightarrow \\
a^* \in' Pow'(\phi^*(W^*,P_1^*,\ldots,P_n^*)) & \Leftrightarrow \\
a^* \in' (\Box\,\phi)^*(W^*,P_1^*,\ldots,P_n^*). &
\end{array}
$$

(L8) ∎

From Lemma (8), it follows that a formula $\phi(P_1,\ldots,P_n)$ is valid in a model $(W,R,\models)$ if and only if W^* is a subset of $\phi^*(W^*,P_1^*,\ldots,P_n^*)$ in the model $\mathcal{U}\setminus V_{\alpha+1}$. This result can be generalized to frames.

LEMMA 9. *A formula $\phi(P_1,\ldots,P_n)$ is valid in the frame (W,R) if and only if, for the corresponding hyperset W^*,*

$$\forall x_1,\ldots,x_n(W^* \subseteq' \phi^*(W^*,x_1,\ldots,x_n))$$

holds in $\mathcal{U}\setminus V_{\alpha+1}$.

Proof. First of all, we show that, for each $a \in W$ and $x_1,\ldots,x_n \in \mathcal{U}\setminus V_{\alpha+1}$,

$$a^* \in' \phi^*(W^*,x_1,\ldots,x_n) \;\Leftrightarrow\; a^* \in' \phi^*(W^*,x_1 \cap' W^*,\ldots,x_n \cap' W^*).$$

The proof is by induction on the structural complexity of the formula ϕ. We only report the proof of the inductive step for $\phi \equiv \Box\,\beta$, leaving the remaining cases to the reader (complete details can be found in [8]):

$$
\begin{array}{rl}
a^* \in' (\Box\,\beta)^*(W^*,x_1,\ldots,x_n) & \Leftrightarrow \\
a^* \in' Pow'(\beta^*(W^*,x_1,\ldots,x_n)) & \Leftrightarrow \\
a^* \setminus V_{\alpha+1} \subseteq \beta^*(W^*,x_1,\ldots,x_n) & \Leftrightarrow \\
\forall b \in W\ (aRb \to b^* \in' \beta^*(W^*,x_1,\ldots,x_n)) & \Leftrightarrow \\
\forall b \in W\ (aRb \to b^* \in' \beta^*(W^*,x_1 \cap' W^*,\ldots,x_n \cap' W^*)) & \Leftrightarrow \\
a^* \setminus V_{\alpha+1} \subseteq \beta^*(W^*,x_1 \cap' W^*,\ldots,x_n \cap' W^* & \Leftrightarrow \\
a^* \in' (\Box\,\beta)^*(W^*,x_1 \cap' W^*,\ldots,x_n \cap' W^*). &
\end{array}
$$

Given n hypersets $x_1,\ldots,x_n$ in $\mathcal{U}\setminus V_{\alpha+1}$, let $\models$ be a valuation of $P_1,\ldots,P_n$ such that, for each $a \in W$, $a \models P_i$ if and only if $a^* \in' x_i \cap' W^*$.

It is straightforward to see that, if P_1^*, …, P_n^* are the hypersets defined in Lemma (8) on the basis of the valuation $\models$, then P_i^* and $x_i \cap' W^*$ have the same elements in the model $\mathcal{U}\setminus V_{\alpha+1}$. From this, it is easy to verify by induction on ϕ that $\phi^*(W^*,P_1^*,\ldots,P_n^*)$ and $\phi^*(W^*,x_1 \cap' W^*,\ldots,x_n \cap' W^*)$ have the same elements.

If a formula ϕ is valid in the frame (W,R), then it is also valid in the model $(W,R,\models)$, and from Lemma (8) it follows that, for all $a \in W$, $a^* \in' \phi^*(W^*,x_1 \cap'$

$W^*, \ldots, x_n \cap' W^*)$. This allows us to conclude that $a^* \in' \phi^*(W^*, x_1, \ldots, x_n)$, and, therefore, for all hypersets $x_1, \ldots, x_n$ in $\mathcal{U} \setminus V_{\alpha+1}$, $W^* \subseteq' \phi^*(W^*, x_1, \ldots, x_n)$.

The converse can easily be proved by associating the hypersets $P_1^*, \ldots, P_n^*$ (where P_i^* is equal to $\{a^* \in' W^* : a \models P_i\}$ if this set is not empty, and to $V_{\alpha+1}$ otherwise) with each valuation $\models$ of $P_1, \ldots, P_n$.

(L9) ∎

To conclude the proof of Theorem (5), let us suppose that

$$\Omega \vdash \forall x(Trans(x) \wedge Axiom_H(x) \rightarrow \forall x_1, \ldots, \forall x_n(x \subseteq \phi^*(x, x_1, \ldots, x_n))).$$

Let (W, R) be a frame in which the formula $\psi(P_{j_1}, \ldots, P_{j_m})$ is valid; from Lemma (9), it follows that the formula $\forall x_{j_1}, \ldots, \forall x_{j_m}(W^* \subseteq' \psi^*(W^*, x_{j_1}, \ldots, x_{j_m}))$ is true in the universe $\mathcal{U} \setminus V_{\alpha+1}$. Furthermore, it is easy to prove that $Trans(W^*)$ holds as well. Since $\mathcal{U} \setminus V_{\alpha+1}$ is an Ω-model, from the hypotheses it follows that the formula $\forall x_1, \ldots, \forall x_n\ (W^* \subseteq' \phi^*(W^*, x_1, \ldots, x_n))$ is true in $\mathcal{U} \setminus V_{\alpha+1}$, and thus, again by Lemma (9), that the formula ϕ is valid in (W, R).

(T5) ∎

Remark 10. From the preceding proof, it should be clear that the proposed translation method works for any theory Ω^* extending Ω, provided that the model $\mathcal{U} \setminus V_\alpha$ of Lemma (6) is a model of Ω^*. This fact will play an essential role in Section 5, where we will discuss the decidability results needed to apply the machinery of T-theorem proving to a theory Ω', somehow stronger than Ω, having $\mathcal{U} \setminus V_\alpha$ as a model. One could observe that this remark does not apply to theories containing the extensionality and/or the foundation axioms. As far as theories with extensionality are concerned, it is possible to show that we can deal with such theories by a minor technical change in the definition of the translation function $(\cdot)^*$. The status of the axiom of foundation is more delicate, in the sense that, at least as long as one wants to represent the accessibility relation using the membership relation, some form of *anti-foundation* does seem to be the best possible choice.

4. The generalization to polymodal logics

In this section we generalize the proposed set-theoretic translation method to polymodal logics. Our approach can be seen as a (completely symmetric) set-theoretic version of Thomason's technique [22, 23].

The main problem is to map a polymodal frame, consisting of a set U endowed with k accessibility relations $\triangleleft_1, \ldots, \triangleleft_k$, with $k > 1$, into a set provided with the membership relation only. We solved this problem by first providing polymodal logics with an alternative semantics that transforms the plurality of accessibility relations $\lhd_1, \ldots, \lhd_k$ into a single accessibility relation R together with k subsets $U_1, \ldots, U_k$ of U.

4.1. An alternative semantics for polymodal logics

Let us introduce an alternative semantics for polymodal logics, called *p-semantics*, and the relevant notions of frame, valuation, and validity. To distinguish such notions from the standard ones, we add the prefix p to the usual terms (e.g. p-valuation, p-model, p-frame).

DEFINITION 11. A *p-frame* $\mathcal{F}$ is a $(k+2)$-tuple $(U, U_1, \ldots, U_k, R)$, where $U, U_1, \ldots, U_k$ are sets and R is a binary relation on $U \cup U_1 \cup \ldots \cup U_k$, such that, for all u, v, t in $U \cup U_1 \cup \ldots \cup U_k$, if $u \in U$, uRv and vRt, then $t \in U$ (we will denote this property by $Trans^2(U)$).

A p-valuation assigns a truth value to propositional variables only at worlds belonging to U. Formally

DEFINITION 12. A *p-valuation* $\models_p$ is a subset of $U \times \Phi$, where Φ is the set of propositional variables.

In the case of boolean combinations, the p-valuation $\models_p$ may be lifted to the set of all polymodal formulae in the canonical fashion. In the case of $\Box_i$, with $i = 1, \ldots, k$, for all $u \in U$ we put

$$u \models_p \Box_i \phi \Leftrightarrow \forall v(uRv \wedge v \in U_i \rightarrow \forall t(vRt \rightarrow t \models_p \phi)).$$

DEFINITION 13. A polymodal formula ϕ is p-valid in a p-frame $(U, U_1, \ldots, U_k, R)$ if and only if for all p-valuations $\models_p$ and all worlds $u \in U$, $u \models_p \phi$ holds.

On the basis of the above definitions, the following lemma holds:

LEMMA 14. *Given a p-frame $(U, U_1, \ldots, U_k, R)$, there exists a classical polymodal frame $(U, \lhd_1, \ldots, \lhd_k)$, based on the set U, that validates all and only the formulae ϕ which are p-valid in $(U, U_1, \ldots, U_k, R)$.*

Proof. Let $\lhd_1, \ldots, \lhd_k$ be defined as follows:

$$u \lhd_i v \;\Leftrightarrow\; \exists t(t \in U_i \wedge uRt \wedge tRv).$$

Any p-valuation $\models_p$ on the p-frame $(U, U_1, \ldots, U_k, R)$ may be interpreted as a valuation on $(U, \lhd_1, \ldots, \lhd_k)$, and vice versa.

For any $u \in U$ and any polymodal formula ϕ, we show that:

$$u \models_p \phi \Leftrightarrow u \models \phi.$$

The proof is by induction on ϕ. We confine ourselves to the case of $\Box_i$ operators (the proof in the other cases is straightforward).

Suppose that $u \models_p \Box_i\psi$. We want to prove that $u \models \Box_i\psi$, that is, $\forall w(u \lhd_i w \rightarrow w \models \psi)$. Consider a world w such that $u \lhd_i w$. By definition of $\lhd_i$, we have that $\exists t(t \in U_i \wedge uRt \wedge tRw)$. Since $u \models_p \Box_i\psi$ is defined as $\forall v(uRv \wedge v \in U_i \rightarrow \forall t(vRt \rightarrow t \models_p \psi))$, it follows that $w \models_p \psi$ and hence $w \models \psi$ by induction.

Suppose now that $u \models \Box_i\psi$. If $v \in U_i$ is such that uRv, then, for all t such that vRt, it follows that $u \lhd_i t$. From the hypothesis, we have that $t \models \psi$ and, by induction, $t \models_p \psi$.

If the formula ϕ is p-valid in $(U, U_1, \ldots, U_k, R)$, then, given any classical valuation $\models$ on $(U, \lhd_1, \ldots, \lhd_k)$, it follows that, for all $u \in U$, $u \models_p \phi$ holds in the corresponding p-model $(U, U_1, \ldots, U_k, R, \models_p)$, and thus $u \models \phi$ in $(U, \lhd_1, \ldots, \lhd_k)$. Since this is true for all $u \in U$ and all valuations $\models$, it follows that ϕ is classically valid in the frame $(U, \lhd_1, \ldots, \lhd_k)$. Symmetrically, it is possible to prove that if ϕ is valid in $(U, \lhd_1, \ldots, \lhd_k)$, then it is p-valid in the p-frame $(U, U_1, \ldots, U_k, R)$.

(L14) ■

LEMMA 15. *For every classical polymodal frame* $(U, \lhd_1, \ldots, \lhd_k)$ *there exists a p-frame* $(U, U_1, \ldots, U_k, R)$ *that p-validates exactly the formulae which are valid in* $(U, \lhd_1, \ldots, \lhd_k)$.

Proof. Let $U_1, \ldots, U_k, R$ be defined as follows:

- let $U_1, \ldots, U_k$ be pairwise disjoint sets isomorphic to U, each one disjoint from U (let us denote by $u \mapsto u_i$ a fixed correspondence between U and U_i);
- for $i = 1, \ldots, k$ and $u, v \in U$, let u_iRv if and only if $u \lhd_i v$;
- for all $u \in U$ and $i = 1, \ldots, k$, let uRu_i.

It is easy to show that $Trans^2(U)$ holds in $(U, U_1, \ldots, U_k, R)$. Moreover, any valuation $\models$ on $(U, \lhd_1, \ldots, \lhd_k)$ can be seen as a p-valuation $\models_p$ on $(U, U_1, \ldots, U_k, R)$.

For any $u \in U$ and any polymodal formula ϕ, the following holds:

$$u \models \phi \Leftrightarrow u \models_p \phi$$

The verification for boolean combinations is left to the reader.

Let us consider the case in which ϕ is of the form $\Box_i\psi$. If $u \models \Box_i\psi$, then, to prove that $u \models_p \Box_i\psi$, take v, t in $U \cup U_1 \cup \ldots \cup U_k$ such that $v \in U_i$, uRv, and vRt. By definition of R, there exists $u_i \in U_i$ such that $v = u_i$ (uRv and u is different from any u_i), and thus vRt can be rewritten as u_iRt. By definition of R, u_iRt if and only if $u \lhd_i t$. Therefore, by the hypothesis, it follows that $t \models \psi$, and, by induction, $t \models_p \psi$.

Now suppose that $u \models_p \Box\, \psi$, and let v be $\lhd_i$-related to u, that is, $u \lhd_i v$. By definition of R, it follows that uRu_i and u_iRv; thus, by definition of $\models_p$, it follows that $v \models_p \psi$, and, by induction, $v \models \psi$.

The result easily follows as in Lemma (14).

(L15) ∎

Lemmas (14) and (15) together show that any p-frame $\mathcal{F} = (U, U_1, \ldots, U_k, R)$ can be reduced to a p-frame $\mathcal{F}' = (U, U'_1, \ldots, U'_k, R')$ such that $U, U'_1, \ldots, U'_k$ are pairwise disjoint.

From the previous two lemmas we have:

THEOREM 16. *If ψ, ϕ are polymodal formulae, then*

$$\psi \models \ \phi \ \Leftrightarrow \phi \textit{ is p-valid in all p-frames in which } \psi \textit{ is p-valid.}$$

Proof. Apply Lemma (15) and Lemma (14).

(T16) ∎

4.2. A set-theoretic translation method for polymodal logics

As in the soundness proof for the monomodal case, we interpret any p-frame $(U, U_1, \ldots, U_k, R)$ as a $(k+1)$-tuple $U^*, U^*_1, \ldots, U^*_k$ of "sets" in a particular Ω-model such that, for all elements t^* of the model which are $\in$-related to $U^* \cup U^*_1 \cup \ldots \cup U^*_k$, we have:

$$t^* = \{s^* : tRs\}.$$

As in the monomodal case, every p-valuation of $P_1, \ldots, P_n$ on the p-frame is interpreted in terms of n subsets $P^*_1, \ldots, P^*_n$ of U^*. Moreover, for each polymodal formula ϕ, we define its translation as a term $\phi^*(x, y_1, \ldots, y_k, x_1, \ldots, x_n)$ such that, for all $u \in U$,

$$u \models_p \phi \ \Leftrightarrow u^* \in \phi^*(U^*, U^*_1, \ldots, U^*_k, P^*_1, \ldots P^*_n).$$

Under this constraint, the definition of the translation of $\Box_i\phi$ directly follows from the definition of $\models_p$ (and induction):

$u \models_p \Box_i\phi$ iff $\forall v(uRv \wedge v \in U_i \rightarrow \forall t(vRt \rightarrow t \models_p \phi))$ iff $\forall v(v^* \in u^* \wedge v^* \in U^*_i \rightarrow \forall t(t^* \in v^* \rightarrow t^* \in \phi^*))$ iff $u^* \cap U^*_i \subseteq Pow(\phi^*)$ iff $u^* \subseteq ((U^* \cup U^*_1 \cup \ldots \cup U^*_k) \setminus U^*_i) \cup Pow(\phi^*)$ iff $u^* \in Pow(((U^* \cup U^*_1 \cup \ldots \cup U^*_k) \setminus U^*_i) \cup P(\phi^*))$.

Thus, the translation of the term $\Box_i \phi(P_1, \ldots, P_n)$ is defined as follows:

$$(\Box_i \phi)^* \equiv Pow(((x \cup y_1 \cup \ldots \cup y_k) \setminus y_i) \cup Pow(\phi^*)).$$

Now let us prove the soundness and completeness of the translation method for polymodal logics.

THEOREM 17. (**Soundness of the translation method**) *Let H be a k-dimensional polymodal logic extending $K \otimes \ldots \otimes K$ with the axiom schema $\psi(\alpha_{j_1}, \ldots, \alpha_{j_m})$. For any polymodal formula ϕ involving n propositional variables $P_1, \ldots, P_n$,*

$$\begin{aligned}\Omega \vdash\ & \forall x \forall y_1 \ldots \forall y_k (Trans^2(x) \wedge Axiom_H(x, y_1, \ldots, y_k) \rightarrow \\ & \forall x_1 \ldots \forall x_n (x \subseteq \phi^*(x, y_1, \ldots, y_k, x_1, \ldots, x_n)))\end{aligned}$$

$$\Rightarrow \psi \models \phi$$

where $Axiom_H(x, y_1, \ldots, y_k)$ is $\forall x_1 \ldots \forall x_m (x \subseteq \psi^*(x, y_1, \ldots, y_k, x_1, \ldots, x_m)))$, and $Trans^2(x)$ stands for $\forall y \forall z\ (y \in z \wedge z \in x \rightarrow y \subseteq x)$, that is, $x \subseteq Pow(Pow(x))$.

Proof. To show that $\psi \models \phi$ it is sufficient to prove that ϕ is p-valid in all p-frames in which ψ is p-valid (Theorem (16)).

Let $(U, U_1, \ldots, U_k, R)$ be a p-frame in which ψ is p-valid. Then, we proceed as in the monomodal case to prove that in a model of Ω there are $k+1$ sets $U^*, U_1^*, \ldots, U_k^*$ such that $Trans^2(U^*)$ holds and, for any polymodal formula $\alpha(P_1, \ldots, P_n)$,

$$\forall x_1, \ldots, x_n (U^* \subseteq \alpha^*(U^*, U_1^*, \ldots, U_k^*, x_1, \ldots, x_n))$$

holds in the model if and only if $\alpha(P_1, \ldots, P_n)$ is p-valid in $(U, U_1, \ldots, U_k, R)$. Hence, $Axiom_H(U^*, U_1^*, \ldots, U_k^*)$ holds in the model, and, by the hypothesis, it follows that $\forall x_1, \ldots, x_n (U^* \subseteq \phi^*(U^*, U_1^*, \ldots, U_k^*, x_1, \ldots, x_n))$. This allows us to conclude that ϕ is p-valid in $(U, U_1, \ldots, U_k, R)$.

(T17) ∎

THEOREM 18. (**Completeness of the translation method**) *Let H be a k-dimensional polymodal logic extending $K \otimes \ldots \otimes K$ by means of the axiom schema $\psi(\alpha_{j_1}, \ldots, \alpha_{j_m})$. For each polymodal formula ϕ involving n propositional variables $P_1, \ldots, P_n$,*

$$\vdash_H \phi \Rightarrow$$

$$\begin{aligned}\Omega \vdash\ & \forall x \forall y_1 \ldots \forall y_k (Trans^2(x) \wedge Axiom_H(x, y_1, \ldots, y_k) \rightarrow \\ & \forall x_1 \ldots \forall x_n (x \subseteq \phi^*(x, y_1, \ldots, y_k, x_1, \ldots, x_n)))\end{aligned}$$

Proof. The proof is by induction on the length of a derivation of ϕ in H.

The cases of tautologies, closure under the substitution rule and modus ponens, and closure under the axiom H are as in the monomodal case.

We prove the closure under the axiom K and necessitation rule of the modalities $\Box_i$. For the axiom K ($\Box_i(\alpha \to \beta) \to (\Box_i\alpha \to \Box_i\beta)$), we show that Ω proves the formula

$$\forall x \forall y_1 \ldots \forall y_k (Trans^2(x) \wedge Axiom_H(x, y_1, \ldots, y_k) \to$$
$$\forall x_1 \ldots \forall x_n (x \subseteq (x \setminus u) \cup (x \setminus v) \cup Pow(((x \cup y_1 \cup \ldots \cup y_k) \setminus y_i) \cup P(\beta^*)))$$

where u and v stand for $Pow(((x \cup y_1 \cup \ldots \cup y_k) \setminus y_i) \cup Pow((x \setminus \alpha^*) \cup \beta^*))$ and $Pow(((x \cup y_1 \cup \ldots \cup y_k) \setminus y_i) \cup Pow(\alpha^*))$, respectively.

Let $x_1, \ldots, x_n$ be fixed and consider $x, y_1 \ldots, y_k$ such that both $Trans^2(x)$ and $Axiom_H(x, y_1, \ldots, y_k)$ hold. Suppose that $t \in x$, $t \in u$, and $t \in v$. We prove that $t \subseteq ((x \cup y_1 \cup \ldots \cup y_k) \setminus y_i) \cup Pow(\beta^*)$, that is, if $s \in t$, then either $s \in (x \cup y_1 \cup \ldots \cup y_k) \setminus y_i$ or $s \in P(\beta^*)$. If $s \notin (x \cup y_1 \cup \ldots \cup y_k) \setminus y_i$, then $t \in u$ implies $s \subseteq (x \setminus \alpha^*) \cup \beta^*$, and $t \in v$ implies $s \subseteq \alpha^*$.

In order to prove the closure under necessitation rule, we show that from

$$\begin{aligned}\Omega \vdash\ & \forall x \forall y_1 \ldots \forall y_k (Trans^2(x) \wedge Axiom_H(x, y_1, \ldots, y_k) \to \\ & \forall x_1 \ldots \forall x_n (x \subseteq \phi^*(x, y_1, \ldots, y_k, x_1, \ldots, x_n)))\end{aligned}$$

it follows that

$$\begin{aligned}\Omega \vdash\ & \forall x \forall y_1 \ldots \forall y_k (Trans^2(x) \wedge Axiom_H(x, y_1, \ldots, y_k) \to \\ & \forall x_1 \ldots \forall x_n (x \subseteq (\Box_i\phi)^*(x, y_1, \ldots, y_k, x_1, \ldots, x_n))).\end{aligned}$$

Let $x_1, \ldots, x_n$ be fixed and consider $x, y_1 \ldots, y_k$ such that both $Trans^2(x)$ and $Axiom_H(x, y_1, \ldots, y_k)$ hold. By the hypothesis, $x \subseteq \phi^*(x, y_1, \ldots, y_k, x_1, \ldots, x_n)$. Hence, $Pow(x) \subseteq Pow(\phi^*) \subseteq ((x \cup y_1 \cup \ldots \cup y_k) \setminus y_i) \cup Pow(\phi^*)$ and therefore $Pow(Pow(x)) \subseteq Pow(((x \cup y_1 \cup \ldots \cup y_k) \setminus y_i) \cup Pow(\phi^*))$. From $Trans^2(x)$, it follows that $x \subseteq Pow(Pow(x))$, and, therefore, $x \subseteq Pow(((x \cup y_1 \cup \ldots \cup y_k) \setminus y_i) \cup Pow(\phi^*)) = (\Box_i\phi)^*$.

(T18) ∎

Remark 19. As in the case of monomodal logics, if H is complete then by Theorem (17) and Theorem (18) modal derivability of a given formula in H is equivalent to first-order derivability of the translated formula in Ω.

5. On the application of set T-resolution

As we said in the introduction, on the basis of the results presented in the preceding sections it is possible to automatically test modal derivability–from modal theories in the specified class–using a classical first-order theorem prover.

Recently a more specialized technique (called T-theorem proving) for automated theorem proving in first-order theories has been proposed (see [20]). Based on the translation method introduced above, a suitable application of T-theorem proving in which the underlying theory T is Ω (or one similar to it) can now be considered as an alternative for automatically testing modal derivability. In this section we briefly discuss the problem of applying *set* T-resolution together with our translation method.

A prerequisite to employing T-resolution in the context of a given theory T, is the decidability, with respect to T, of the class of ground formulae written in any language which extends the one in which the axioms of T are written with Skolem (uninterpreted) function symbols. In [20], it was shown that the satisfiability problem with respect to any theory T of ground formulae on a given language $\mathcal{L}^*$ obtained from $\mathcal{L}(T)$ by adding an arbitrary number of functional and constant symbols, is equivalent to the T-satisfiability of the class of purely existential formulae written in $\mathcal{L}(T)$. Therefore we are interested in this last problem in the case of Ω, whose language ($\mathcal{L}(\Omega)$ from now on) consists of the symbols $\emptyset, \cup, \setminus, \subseteq, \in$, and *Pow*.

Before commenting on the above mentioned problem, notice that the decidability of classes very similar to the one we want to deal with has already been proved by Cantone, Schwartz, and Ferro [4, 5, 6]. Unfortunately, the results mentioned–among the most complex in the field of computable set theory–cannot be applied to our context, the problem being the underlying set theory on which they rest. Our theory Ω is very weak; in fact it can easily be verified that the proofs in [4, 6] make an essential use of assumptions such as regularity, existence of the transitive closure of sets, extensionality, etc., which are certainly not derivable in Ω.

We succeeded in providing a proof of the decidability result we need for a theory Ω' slightly stronger than Ω (but having essentially the same language). The main difference between Ω and Ω' is that Ω' contains as axioms some simple consequences–not derivable in Ω–of Cantor's theorem on the number of subsets of a given set [8].

The proof is based on a technique first introduced in [7, 19]. The main idea is the following: in order to establish whether there exists a model of Ω' satisfying a formula $\varphi(x_1, \dots, x_n)$ (an unquantified formula written in $\mathcal{L}(\Omega')$), we assume that there exists a model M of Ω' such that $M \models \varphi(x_1, \dots, x_n)$, and we concentrate our attention on n elements $a_1, \dots, a_n$ in the support of M satisfying φ. The goal is to show that under this hypothesis we can build another (*simpler*) n-tuple $a_1^*, \dots, a_n^*$ of elements in the support of a model M' of Ω' still satisfying φ. The elements $a_1^*, \dots, a_n^*$ are completely described by a graph G whose size is bounded by a function of n, in the sense that, in order to test the existence of (M' and) $a_1^*, \dots, a_n^*$, it is sufficient to test the existence of G, and this result guarantees the decidability.

The problem of determining $a_1^*, \dots, a_n^*$ is combinatorially non-trivial. First of all, notice that if in the formula $\varphi(x_1, \dots, x_n)$ we had no conjuncts of the form

$Pow(x_i) = x_j$, then we could define $a_1^*, \ldots, a_n^*$ simply as a n-tuple satisfying $a_i^* = \{a_j^* \mid a_j \in^M a_i\}$, and it would be easy to check that all our requirements are satisfied (recall that we do not have to deal with the extensionality axiom). As a matter of fact we can think of the map * as a way of *marking* some of the elements in each of the a_i (the marked elements being those of the form a_j) and then take a_i^* as the set of (images with respect to $*$ of) marked elements in a_i. In order to deal with a literal of the form $Pow(x_i) = x_j$, we need to mark more elements: at least all those elements which are subsets of the set of marked elements in a_i. Notice that if one simply does so and marks all such elements (subsets of a_i) without "care", new elements can turn out marked in a_i, and the marking process may not terminate. We solved the above problem processing the a_i's in an order compatible with their size and applying the simple consequences of Cantor's theorem that were forced to hold in Ω' precisely for this purpose (the details of the proof are given in [8]).

It may be interesting to note that it is still an open problem whether the class of purely existential formulae of $\mathcal{L}(\Omega)$ is decidable with respect to Ω. In other words it is not known whether T-theorem proving can be applied directly to Ω; hence, up to this point, despite its simplicity, Ω seems to be a less suitable theory for computational purposes, than a more complex one (i.e. Ω').

Conclusions

In this paper we proposed a new translation method mapping polymodal formulae into set-theoretic terms of the very week set theory Ω. The method can be used for any normal complete finitely axiomatizable polymodal logic, possibly specified with Hilbert axioms only, and applies to a large class of theories extending Ω. An important and interesting line of investigation at this point is the generalization of the proposed method to first-order (poly)modal logics.

As another line of development of the work presented here, we mention a systematic comparison of the approach introduced in this paper (and possibly suitable variations of it) with standard translations into first-order and monadic second-order logics [2, 18]. From a theoretical perspective, the aspects that can be considered are, for example, the ability to translate specific (classes of) logics and modal operators [3], the problems related to the correspondence between proofs in modal and set-theoretic systems, and the relationships with tableaux-like methods for modal logics [10]. From a more practical point of view, it could be of interest to investigate the computational costs of the set-theoretic reasoning mechanisms sketched in Section 5.

We are also investigating the possibility of exploiting our translation method to reduce undecidable decision problems for particular propositional polymodal logics, e.g. [13], to the derivability problem with respect to Ω of formulae of type $\forall^*\exists$, thereby showing the undecidability of the latter problem.

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