Alternative Translation Techniques for Propositional and First-Order Modal Logics

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Abstract. We describe and analyze techniques, other than the standard relational/functional methods, for translating validity problems of modal logics into first-order languages. For propositional modal logics we summarize the □-as-Pow method, a complete and automatic translation into a weak set theory, and then describe an alternative method, which we call *algebraic*, that achieves the same full generality of □-as-Pow, but is simpler and computationally more attractive. We also discuss the relationships between the two methods, showing that □-as-Pow generalizes to the first-order case. For first-order modal logics, we describe two extensions, of different degrees of generality, of □-as-Pow to logics of rigid designators and constant domains.

Keywords: Translation methods, (first-order) modal logics, computable set theories.

1. Introduction

Translation techniques have offered valuable aid for the study and development of theorem provers for non-classical logics, and their naïve use can be ascribed to tradition. The first systematic study of the *relational* translation was carried out within Correspondence Theory (van Benthem, 1983). In later years further studies of the relational and similar techniques for computational purposes were initiated ((Ohlbach, 1991; Ohlbach, 1993; Nonnengart, 1993); see also (Ohlbach et al., 2001) for a comprehensive survey). The field has revealed itself to be fruitful and interesting.

The *relational translation* of a modal formula $\varphi = \varphi(p_1, \ldots, p_n)$ is the first-order formula $\mathrm{RT}(\varphi, x)$ (it has also been called ST, for "standard translation", but we will reserve this designation for "set-theoretic translation"), inductively defined by

$$\begin{aligned} \mathrm{RT}(p_i, x) &= P_i(x) \ , \\ \mathrm{RT}(\neg\varphi, x) &= \neg\mathrm{RT}(\varphi, x) \ , \\ \mathrm{RT}(\varphi \vee \psi, x) &= \mathrm{RT}(\varphi, x) \vee \mathrm{RT}(\psi, x) \ , \end{aligned}$$

$$\mathrm{RT}(\Box\varphi, x) \;=\; \forall y\,(R(x,y) \rightarrow \mathrm{RT}(\varphi, y)) \ .$$

Here, each propositional variable p_i is mapped one-to-one to a unary predicate symbol P_i. Let $\overline{\mathrm{RT}}(\varphi)$ be

$$\forall P_1, \ldots, P_n\, \forall x\ \mathrm{RT}(\varphi, x) \ .$$

If γ axiomatizes a frame-complete logic, then

$$\gamma \vDash_f \varphi \Leftrightarrow\ \vDash \overline{\mathrm{RT}}(\gamma) \rightarrow \overline{\mathrm{RT}}(\varphi) \ .$$

Here $\vDash_f$ denotes validity of φ in the logic axiomatized by γ, and $\vDash$ denotes classical (in this case second-order) validity. The second-order quantifiers in $\overline{\mathrm{RT}}(\varphi)$ can be removed, since they are universal and appear only as prefix to the consequent of an implication. However, the second-order character of $\overline{\mathrm{RT}}(\gamma)$ is, in general, not eliminable, since, in this case, the second-order (universal) quantification appears in the antecedent of an implication. This makes the technique inapplicable for computational purposes, since the target language is highly undecidable (non-arithmetic, cf. (van Benthem, 1983)). Whenever $\overline{\mathrm{RT}}(\gamma)$ can be proved equivalent to a first-order formula, the logic axiomatized by γ is said to be first-order definable and the modal validity problem can be solved by a first-order theorem prover. Determining when $\overline{\mathrm{RT}}(\gamma)$ has a first-order equivalent is the main topic of correspondence theory. In some cases it can even be done automatically (e.g. Sahlqvist formulae, the SCAN procedure (Gabbay and Ohlbach, 1992)). Unfortunately, not all modal logics have a first-order (frame) correspondent, and, even when they do, there is no complete algorithm to find one. A lot of work has been done to avoid these difficulties, resulting in some extensions of the class of tractable logics and, especially, in improved performance (Nonnengart, 1993; Ohlbach, 1998). In particular, (Ohlbach, 1998) proposed a way to incorporate axioms and rules of Hilbert calculi into relational translations, thus creating a hybrid system able to deal with the axioms that cannot directly be captured by the translation.

A different perspective on the problem is taken in the approach proposed in (D'Agostino et al., 1995), where the $\Box$-as-Pow translation was first introduced. The novelty of $\Box$-as-Pow lies in the fact that, instead of determining which modal theories can be mapped into a first-order language, a *fragment* of second-order logic is used as a framework for the translation. It turns out that the fragment required is very small and can be specified as a weak first-order axiomatic set theory. Under this theory, the resulting translation is sound and complete for all frame-complete (not only with respect to a first-order class of frames) logics. More specifically, given a frame-complete logic with axiom γ and a formula φ, the $\Box$-as-Pow translation $(\gamma, \varphi)^*$ is such that

$$\gamma \models_f \varphi \Leftrightarrow \Omega \models (\gamma, \varphi)^* \ ,$$

where Ω is the weak axiomatic set theory that will be defined in Section 2. Several extensions of this method were discussed in subsequent papers (Montanari and Policriti, 1997a; Montanari and Policriti, 1997b; van Benthem et al., 1998), including a variation (van Benthem et al., 1997) that allows for completeness for any logic (not necessarily frame-complete) and another with strong decidability results (D'Agostino et al., 1996).

In this paper we systematically present set-theoretic and algebraic translation techniques that are less restrictive than RT and its variations, and whose range includes propositional modal logics as well as an important class of first-order modal logics. After a short review of the original $\Box$-as-Pow translation (Section 2), we introduce a new, very natural, translation method, that we call *algebraic*. This algebraic translation, suggested by the algebraic semantics for modal logics, achieves completeness for all logics in a simpler way than $\Box$-as-Pow. We point out strong relationships between $\Box$-as-Pow and the algebraic method, showing a new way to obtain the former from an algebraic point of view (Section 3). We show that, in a certain sense to be made precise later, $\Box$-as-Pow and the algebraic translation coincide at the propositional level. However, no such strong relationship holds when we try to extend these methods to handle first-order modal logics: $\Box$-as-Pow can be so extended, but for the algebraic method there seems to be a significant stumbling block (unless we allow the target language of the translation to be second-order, which is unacceptable). The relational translation and its derivations can be extended, in their restricted setting, to first-order modal logics in a fairly straightforward manner. In Section 4, we collect and organize some results (Montanari et al., 2000a; Montanari et al., 2000b) on the extension of $\Box$-as-Pow to the important class of first-order modal logics of rigid designators and constant domain, which is significant given the aforementioned observations. Section 5 contains some lengthy proofs that would have broken the continuity of the exposition if put in the main text. We conclude with some discussion on open problems and possible future work.

We have tried to use a commonly understood symbolism. Some less common notations are the ones for logical derivation/truth:

$\vdash$ syntactic derivation in classical and modal logic;

$\vdash_{K_s}$	syntactic derivation in the modal calculus K_s, that is, K plus the Instantiation Rule;
$\models$	truth in a model, semantic consequence in classical logic;
$\models_f$	truth in a world or model, frame semantic consequence in modal logic.

2. $\Box$-as-Pow Translation (Propositional Case)

The $\Box$-as-Pow translation, at the propositional level, can be described as an extension to the $\Box$ operator of the set-theoretic interpretation of the propositional connectives $\wedge$, $\vee$, and $\neg$. The approach briefly described here was first presented in (D'Agostino et al., 1995) and subsequently developed in (van Benthem et al., 1997) and (van Benthem et al., 1998). The key idea is simply to replace the accessibility relation R of the possible world semantics by the membership relation $\in$. Accordingly, a world y accessible from x becomes an *element* of x, and taking a further step from y via R amounts to moving *into* y to inspect one of its elements. This viewpoint allows us to identify a frame $\mathfrak{F}$ with its support w. Moreover, since we clearly want all worlds y accessible from a given world x in a frame w to be themselves elements of w, it is natural to require that w be a *transitive* set: $w \subseteq \mathrm{P}(w)$.

Valuations for propositional variables, i.e. sets of worlds, can now be identified with worlds themselves. $\Box$-as-Pow is designed in such a way that the translation φ^* of a compound formula φ turns out to be the set

$$\{x \in W \mid x \models \varphi\}$$

of all worlds where φ is true.

For propositional connectives we simply follow the Boolean tradition and say that

$$\begin{aligned} (\neg\varphi)^* &= w \setminus \varphi^* \ , \\ (\varphi \wedge \psi)^* &= \varphi^* \cap \psi^* \ , \\ (\varphi \vee \psi)^* &= \varphi^* \cup \psi^* \ . \end{aligned}$$

In order to complete the definition it is sufficient to *calculate* the Kripke semantics of the $\Box$ operator replacing R by $\in$:

$$x \models \Box\varphi \Leftrightarrow \forall y \in x \, (y \in \varphi^*) \ ,$$

which is to say: $\Box\varphi$ holds true at x if and only if x is a subset of φ^*. This allows us to complete our definition by putting

$$(\Box\varphi)^* = \mathrm{P}(\varphi^*) \ .$$

It is possible to show that the collection Ω of the following set-theoretic principles suffices for proving the soundness and completeness of the translation:

$$
\begin{aligned}
z \in x \cup y &\leftrightarrow z \in x \vee z \in y \ , \\
z \in x \setminus y &\leftrightarrow z \in x \wedge \neg z \in y \ , \\
x \subseteq y &\leftrightarrow \forall z \,(z \in x \rightarrow z \in y) \ , \\
x \in \mathrm{P}(y) &\leftrightarrow x \subseteq y \ .
\end{aligned}
$$

Defining $\mathit{Trans}(w)$ as $w \subseteq \mathrm{P}(w)$ it can be shown that

$$
\begin{aligned}
\vdash_{K_s} \varphi &\Rightarrow \Omega \vdash \vec{\forall}(\mathit{Trans}(w) \rightarrow w \subseteq \varphi^*) \ , \\
\models_f \varphi &\Leftarrow \Omega \models \vec{\forall}(\mathit{Trans}(w) \rightarrow w \subseteq \varphi^*) \ .
\end{aligned}
$$

When a modal logic is characterized by a finite collection of Hilbert axioms, whose conjunction is γ, these results generalize to

$$
\begin{aligned}
\gamma \vdash_{K_s} \varphi &\Rightarrow \Omega \vdash (\gamma, \varphi)^* \ , \\
\gamma \models_f \varphi &\Leftarrow \Omega \vdash (\gamma, \varphi)^* \ .
\end{aligned}
$$

where

$$(\gamma, \varphi)^* = \vec{\forall}(\mathit{Trans}(w) \wedge \mathit{Axiom}_\gamma(w) \rightarrow w \subseteq \varphi^*) \ .$$

$\mathit{Axiom}_\gamma(w)$ stands for $\forall \vec{x}\,(w \subseteq \gamma^*)$, where $\vec{x}$ is the list of all the variables in γ^* distinct from w.

Thus when α is frame-complete an exact correspondence between modal and set-theoretic derivability is guaranteed[1]:

$$\gamma \vdash_{K_s} \varphi \Leftrightarrow \Omega \vdash (\gamma, \varphi)^* \ .$$

It turns out that the theory Ω is *almost* minimal in providing a formal counterpart to the idea underlying the $\Box$-as-Pow translation. Let us call MM (an acronym for "Minimal Modal") the theory whose axioms result from those of Ω by replacing $x \in \mathrm{P}(y) \leftrightarrow x \subseteq y$ with the following two theorems of Ω:

$$x \subseteq y \rightarrow \mathrm{P}(x) \subseteq \mathrm{P}(y) \ ,$$

$$\mathrm{P}(x) \cap \mathrm{P}(y) \subseteq \mathrm{P}(x \cap y) \ ,$$

[1] As a technical note, if we were to deal with a logic with an infinite axiom set Γ, we could reformulate the result as follows (use skolemization on w/c and the Deduction Theorem for first-order logic to obtain the equivalence):

$$\Gamma \vdash_{K_s} \varphi \Leftrightarrow \Omega \cup \{\mathit{Trans}(c)\} \cup \{\vec{\forall}(c \subseteq \gamma^*) \mid \gamma \in \Gamma\} \vdash \vec{\forall}(c \subseteq \varphi^*) \ .$$

where $u \cap v$ is an abbreviation for $u \setminus (u \setminus v)$. Although MM is even weaker than Ω, it can still drive the $\Box$-as-Pow translation. A further possible simplification would be to eliminate the binary union symbol, observing that, by exploiting complementation, it can be defined in terms of set difference.

Theorem 1 (Completeness of $\Box$-as-Pow). *If $\gamma \vdash_{K_s} \varphi$, then $MM \vdash (\gamma, \varphi)^*$.*

Theorem 2 (Soundness of $\Box$-as-Pow). *If $\Omega \vdash (\gamma, \varphi)^*$, then $\gamma \vDash_f \varphi$.*

3. Algebraic Translation (Propositional Case)

The same mental shift that proceeds from relational semantics to the relational translation enables one to discover the algebraic translation as a straightforward rewriting of algebraic semantics in syntactic terms. The basic objects playing the role of Kripke frames are Boolean algebras with one extra operator (Bull and Segerberg, 1984; Blackburn et al., 2001).

Definition 3. Consider the first-order language of Boolean algebras (with meet $\sqcap$, join $\sqcup$, complementation $'$, bottom element 0, and top element 1) and an additional operator ℓ. Define the theory *MA* (for *Modal Algebra*) to be the first-order theory of Boolean algebras with the two additional axioms:

$$\begin{aligned} \ell\, 1 &= 1 \ , \\ \forall x\, \forall y \ \ \ell\,(x \sqcap y) &= (\ell\, x) \sqcap (\ell\, y) \ . \end{aligned}$$

Definition 4. Let us map each propositional symbol p to a distinct individual variable x_p of the target first-order language and define the algebraic translation $\varphi^{\#}$ of a modal formula φ by the following structural induction:

$$\begin{aligned} p^{\#} &= x_p \ , \\ \bot^{\#} &= 0 \ , \\ (\neg\varphi)^{\#} &= (\varphi^{\#})' \ , \\ (\varphi \wedge \psi)^{\#} &= \varphi^{\#} \sqcap \psi^{\#} \ , \\ (\varphi \vee \psi)^{\#} &= \varphi^{\#} \sqcup \psi^{\#} \ , \\ (\Box\varphi)^{\#} &= \ell\, \varphi^{\#} \ . \end{aligned}$$

Based on this definition and following the truth definition of algebraic semantics, we put

$$(\gamma, \varphi)^{\#} \ = \ \vec{\forall}(\gamma^{\#} = 1) \rightarrow \vec{\forall}(\varphi^{\#} = 1) \ .$$

It is straightforward to prove that this translation is sound and complete for all (finitely axiomatizable) modal logics, since the algebraic semantics is complete for all logics (in fact, it is equivalent to the general frame semantics).

Theorem 5. *Given a modal logic with axiom γ and a formula φ,*

$$\gamma \vdash_{K_s} \varphi \Leftrightarrow MA \vdash (\gamma, \varphi)^{\#} \ .$$

The algebraic translation converts a validity problem in any modal logic to a validity problem of first-order logic, thus permitting the use of any first-order theorem prover. As a validity checking method, this is slower than the use of the relational or functional translation, when the latter is applicable, but it has the advantage of being a completely syntactic method and it can turn out to be extremely useful for experimenting with new logics.[2] A further advantage of this technique is that it also works, with no modification, for multi-modal logics: just introduce one ℓ operator for each of the modalities involved and augment the underlying theory with a pair of axioms, of the form of (1), for each of them.

What we are left with after the algebraic translation is a deduction problem in *equational* first-order logic. Tools that deal with such problems, such as EQP, are available and can be used for modal theorem proving based on the algebraic translation. Equational problems can be solved particularly efficiently when rewriting techniques can be applied. In the case of modal logic, Foret (Foret, 1992) discovered confluent (modulo associativity and commutativity of the Boolean operators) and terminating rewriting systems for the logics K, D, T, and S5. The rewriting systems are oriented versions of the equational axioms of modal algebras and can be seen as a specialized solver for the algebraic translation of the formulae. For example, Foret's system for K adds the oriented versions of (1) to Hsiang's system for propositional logic—an oriented version of the equational theory of Boolean rings with multiplicative identity (Hsiang, 1982; Hsiang, 1985):

$$\begin{aligned} \ell\, 1 \ &\longrightarrow \ 1 \ , \\ (\ell\, x) \sqcap (\ell\, y) \ &\longrightarrow \ \ell\, (x \sqcap y) \ . \end{aligned}$$

[2] We did some simple experimentation with a raw algebraic translation given to EQP (McCune, 1997), and were able to prove 31 out of the 72 problems of (Goré et al., 1997) in seconds (at most a few minutes in a few cases) on a Pentium class computer.

3.1. Relations with □-as-Pow.

If one looks at the original formulation of the □-as-Pow translation, there is one peculiar feature: the membership symbol $\in$ does not appear explicitly. In fact, the translation seems to be expressed in an almost equational language and *looks* very much like the algebraic translation. Moreover, the axioms of MM are strikingly similar to those of modal algebras.

We will now prove that these remarks have indeed a deep underpinning: there is an equational theory of sets, capturing the Boolean properties of Ω and MM, that proves the propositional □-as-Pow translation sound and complete. In other words, at the propositional level, the algebraic and the □-as-Pow translation are essentially equivalent. However, as will be shown in the next section, there is no simple way to lift this equivalence to the first-order level.

One would be inclined to use, as an equational set theory, a theory of Boolean algebras. However, we cannot hope to have a complementation function, since there is no constant for a top element. Hence, we add a relativized complementation which mimics the behavior of set difference. This turns out to be all we need and the following axioms are sufficient.

Definition 6. We define the theory Ω_E ("E" for "equational") by the universal closure of the following axioms:

$$\begin{array}{rcll} x \cup y &=& y \cup x \ , & x \cap y = y \cap x \ , \\ x \cup (y \cup z) &=& (x \cup y) \cup z \ , & x \cap (y \cap z) = (x \cap y) \cap z \ , \\ x \cup (x \cap y) &=& x \ , & x \cap (x \cup y) = x \ , \\ x \cup (y \cap z) &=& (x \cup y) \cap (x \cup z) \ , & x \cap (y \cup z) = (x \cap y) \cup (x \cap z) \ , \\ x \cup (z \setminus x) &=& z \cup x \ , & x \cap (z \setminus x) = \emptyset \ , \end{array}$$

$$\mathrm{P}(x) \cap \mathrm{P}(y) \subseteq \mathrm{P}(x \cap y) \ ,$$

$$x \subseteq y \to \mathrm{P}(x) \subseteq \mathrm{P}(y) \ .$$

Intuitively, the first ten axioms describe a theory of "Boolean algebras without maximum element". It can be shown that the models of these axioms are such that the cone of elements below any given element is a Boolean algebra. This kind of structure is exactly a Boolean ring in the sense of, e.g., (Jacobson, 1951), that is, an idempotent ring with no multiplicative null element. This allows us to establish a two-way correspondence between models of Ω_E and models of MA. Using this correspondence and the □-as-Pow translation, which can be applied directly to the language of Ω_E, we can prove the following equivalence:

Theorem 7. *Consider a finitely axiomatizable modal logic, with axiom γ, and a modal formula φ. Then*

$$MA \vdash (\gamma, \varphi)^{\#} \Leftrightarrow \Omega_E \vdash (\gamma, \varphi)^* \ .$$

Proof. See Section 5.

Corollary 8. *If γ axiomatizes a frame-complete modal logic, then*

$$\Omega \vdash (\gamma, \varphi)^* \Leftrightarrow \Omega_E \vdash (\gamma, \varphi)^* \ .$$

Proof. $\Omega \vdash (\gamma, \varphi)^* \Leftrightarrow \gamma \vdash_{K_s} \varphi$ by soundness and completeness of $\Box$-as-Pow. By soundness and completeness of the algebraic translation (Theorem 5), $\gamma \vdash_{K_s} \varphi \Leftrightarrow MA \vdash (\gamma, \varphi)^{\#}$. To conclude apply Theorem 7. $\Box$

4. $\Box$-as-Pow Translation (First-Order Case)

The analogy between the $\Box$-as-Pow and the algebraic translation does not lift from the propositional to the first-order level. An extension of the algebraic method to first-order modal logics would necessitate employing a theory of complete Boolean algebras, which is second-order, therefore leading again to problems of the same sort as those mentioned in Section 1. However, we have successfully extended $\Box$-as-Pow to an important class of first-order modal logics maintaining the target language of the translation at the first-order level. Hence, $\Box$-as-Pow and the semantics it is based on can be seen as genuine generalizations of the algebraic models; only at the lowest (propositional) level do the two happen to be equivalent.

Here we describe two extensions of the $\Box$-as-Pow translation method to the class of first-order modal logics with *rigid designators* and *fixed domains*, known as $Q1$ systems (see (Garson, 1984; Fitting, 1993; Fitting and Mendelsohn, 1998)). Informally, the language of such logics is a first-order language together with a $\Box$ modality. A model M consists of a frame $\langle W, R\rangle$ (possibly with restrictions defined by an underlying propositional modal logic) plus a first-order logic interpretation with a single domain D, an interpretation $\mathcal{F}_C$ for constants and one $\mathcal{F}_F$ for function symbols, and—the only part that varies with worlds—the interpretation $\mathcal{F}_P$ of predicate symbols:

$$M = \langle W, R, D, \mathcal{F}_C, \mathcal{F}_F, \mathcal{F}_P\rangle \ .$$

There is a sound and complete calculus for any such logic, obtained by adding to the propositional calculus for the underlying modal logic the axioms and rules of first-order logic—the Universal Quantifier Axiom schema and the Universal Generalization Rule—and the Barcan Formula schema:

$$\forall x\, \Box\varphi(x) \rightarrow \Box\forall x\, \varphi(x) \tag{1}$$

for every formula $\varphi(x)$.

Both translations will be targeted to a two-sorted first-order logic. This is done essentially for better understanding and ease of implementation: in principle everything could be done in a one-sorted language (see (Slanina, 2001) for details; for a general discussion of the reducibility of many-sorted to one-sorted logic see also (Davis, 1993)). If L is the first-order modal language, we define L' to be a two-sorted language where the sort `term` denotes L-terms and the sort `set` denotes objects of the translation (possible worlds and sets thereof). The underlying theory Ω is modified accordingly, with the third axiom now becoming

$$\forall x\, \forall y\, (x \subseteq y \leftrightarrow \forall z\colon \mathtt{set}\, (z \in x \rightarrow z \in y)) \enspace .$$

We call this modified theory Ω_2.

The first translation (Montanari et al., 2000a) works for the fragment of the $Q1$ language where modal operators cannot appear in the scope of quantifiers (below: "locally quantified formulae"). This case appears often in practice and can be treated using the observation that sentences can be decomposed into two disjoint parts: one first-order, but classical, and the other modal, but propositional.[3] More precisely, a sentence of this fragment can be written as $\psi(\varphi_1, \ldots, \varphi_n)$, where $\varphi_1, \ldots, \varphi_n$ ($\vec{\varphi}$ for short) are classical first-order sentences and the formula we obtain by replacing each φ_i by a propositional variable p_i, namely, $\psi = \psi(p_1, \ldots, p_n)$ ($\psi(\vec{p})$ for short), is a propositional modal formula. The simple, but crucial, observation at this point is that derivability can be split accordingly, as stated in the following lemma.

Lemma 9. *There is a finite set Σ of classical propositional formulae (over the variables $p_1, \ldots, p_n$) such that $\vDash \sigma\{\vec{p} \mapsto \vec{\varphi}\}$ for all $\sigma \in \Sigma$, and, for all ψ,*

$$\vDash_{Q1-L} \psi(\varphi_1, \ldots, \varphi_n) \quad \Leftrightarrow \quad \Sigma \vDash_L \psi(p_1, \ldots, p_n) \enspace .$$

Proof (sketch). Clearly,

$$\vDash_{Q1-L} \psi(\varphi_1, \ldots, \varphi_n) \Leftrightarrow \{p_1 \leftrightarrow \varphi_1, \ldots, p_n \leftrightarrow \varphi_n\} \vDash_L \psi(p_1, \ldots, \varphi_n). \tag{2}$$

[3] This is the case, for instance, of Manna and Pnueli's temporal framework (Manna and Pnueli, 1995), where most specifications can be confined to this restricted form of quantification.

Let Σ be the set of propositional formulae on $p_1, \ldots, p_n$ that are valid under the substitution $\{p_1 \mapsto \varphi_1, \ldots, p_n \mapsto \varphi_n\}$. Then, it is easy to see that the right-hand side of (2) is equivalent to $\Sigma \models \psi(p_1, \ldots, p_n)$. Σ is finite modulo logical equivalence. □

Using the above results, the translation is defined as follows.

Definition 10. As an auxiliary notation, we introduce a transformation $\tilde{}$ that adds an extra variable (representing a world in a Kripke frame) to the arguments of each predicate symbol ($\vec{t}$ are terms):

$$\begin{aligned}
(P(\vec{t}))^{\sim}(z) &= P(\vec{t}, z) \ , \\
(\theta \vee \eta)^{\sim}(z) &= \theta^{\sim}(z) \vee \eta^{\sim}(z) \ , \\
(\neg\theta)^{\sim}(z) &= \neg\theta^{\sim}(z) \ , \\
(\forall u\, \theta)^{\sim}(z) &= \forall u{:}\, \mathtt{term}\ \theta^{\sim}(z) \ .
\end{aligned}$$

Now fix a variable w of sort $\mathtt{set}$. The translation $(\gamma, F)^{\dagger}$ of a locally quantified formula $F = \psi(\varphi_1, \ldots, \varphi_n)$, with respect to the logic axiomatized by γ, is expressed as follows:

$$\begin{aligned}
(\gamma, F)^{\dagger} = \forall w\, , \vec{x}{:}\ \mathtt{set}\ \ (&\mathit{Trans}(w) \wedge \mathit{Axiom}^2_{\gamma}(w) \wedge \\
&\bigwedge_{i=1}^{n} ST^2(\varphi_i)(\vec{x}) \rightarrow w \subseteq \psi^*(w, \vec{x})) \ ,
\end{aligned}$$

where

$$ST^2(\varphi_i)(\vec{x}) = \forall y{:}\, \mathtt{set}\, (y \in w \rightarrow (y \in x_i \leftrightarrow \varphi_i^{\sim}(y))) \ .$$

Here $\mathit{Axiom}^2_{\gamma}(w)$ and ψ^* are the same as in the propositional case, with all bound variables of sort $\mathtt{set}$.

The adequateness theorems are similar to the ones for the propositional case:

Theorem 11 (Completeness). $\gamma \vdash_{Q1} F \Rightarrow \Omega_2 \vdash (\gamma, F)^{\dagger}$.

Theorem 12 (Soundness). $\Omega_2 \vdash (\gamma, F)^{\dagger} \Rightarrow \gamma \models_{Q1} F$.

The proofs of these theorems can be found in (Montanari et al., 2000a). We omit them here, since our next translation is stronger.

The second translation works for the full $Q1$ language and is developed along a different line.

Definition 13. The set-theoretic translation $(\gamma,\varphi)^{**}$ of a formula φ with respect to the logic with axiom γ (a propositional modal axiom) is the formula

$$(\gamma,\varphi)^{**} = \vec{\forall}(\mathit{Trans}(w) \wedge \mathit{Axiom}_\gamma(w) \wedge \beta_\varphi(w) \rightarrow w \subseteq t_\varphi(w,\vec{x})) \ .$$

$\mathit{Trans}(w)$ and $\mathit{Axiom}_\gamma(w)$ are the same as in the propositional case, while $\beta_\varphi(w)$ encodes the semantics of quantifiers. The notation is as follows: variables occurring—free or not—in φ are denoted by $\vec{x}$, which stands for $x_1,\ldots,x_n$; there are fresh function symbols f_ψ for some subformulae ψ of φ (to be specified below), with signature $\texttt{set} \times \texttt{term}^n \rightarrow \texttt{set}$. Each term t_ψ, with ψ subformula of φ, represents the set of worlds where ψ is true, in the frame w and under variable assignment $\vec{x}$. Formally, let $\vec{x} = x_1,\ldots,x_n$ be a superset of the set of the variables appearing in φ. Then

$$t_\varphi(w,\vec{x}) = \begin{cases} f_Q(\vec{s}) & \text{if } \varphi \text{ is } Q(\vec{s}), \\ f_\varphi(w,x_{j_1},\ldots,x_{j_m}) & \text{if } \varphi \text{ is } \forall x_i\, \eta \text{ and } x_{j_1},\ldots,x_{j_m} \text{ are} \\ \qquad \text{the free variables in } \varphi, \text{ with } j_1 < j_2 < \cdots < j_m, & \\ t_\eta(w,\vec{x}) \cup t_\theta(w,\vec{x}) & \text{if } \varphi \text{ is } \eta \vee \theta, \\ w \setminus t_\eta(w,\vec{x}) & \text{if } \varphi = \neg\eta, \\ \mathrm{P}(t_\eta(w,\vec{x})) & \text{if } \varphi = \Box\eta. \end{cases}$$

$\beta_\varphi(w)$ is simply the conjunction, over all subformulae ψ of φ, of the formulae $\alpha_\psi(w)$, where α_ψ is defined as

$$\alpha_\psi(w) = \begin{cases} \forall x_{j_1},\ldots,x_{j_m}\colon \texttt{term}\, \forall y\colon \texttt{set}\, (y \in w \rightarrow \\ \quad (y \in f_\psi(w,x_{j_1},\ldots,x_{j_m}) \leftrightarrow \forall x_i\colon \texttt{term}\ y \in t_\eta(w,\vec{x}))) \\ \quad \text{if } \psi \text{ is } \forall x_i\, \eta \text{ and } x_{j_1},\ldots,x_{j_m} \text{ are the free variables in} \\ \quad \psi, \text{ with } j_1 < j_2 < \cdots < j_m, \\ \top \text{ otherwise.} \end{cases}$$

In analogy to the propositional case, we now have the following theorems, whose proofs are given in Section 5.

Theorem 14 (Completeness). $\gamma \vdash_{Q1} \varphi \Rightarrow \Omega_2 \vdash (\gamma,\varphi)^{**}$.

Theorem 15 (Soundness). $\Omega_2 \models (\gamma,\varphi)^{**} \Rightarrow \gamma \models_{Q1} \varphi$.

Example. Let φ be the formula

$$\forall x\, {}^8\Diamond^7(\exists y\, {}^2P^1(x,y) \vee^6 \forall z\, {}^5\Box^4 Q^3(z,x)) \ .$$

The superscripts mark the main operator of each of the eight subformulae of φ. Let γ be the axiom of seriality, $D = \Diamond\top$. Then, with some freedom of notation (e.g., we use $\exists$, whose direct translation is obtained patching the one for $\forall$ in the obvious way), the translation is the following:

$$\begin{aligned}
(\gamma,\varphi)^{**} &= \forall w{:}\,\texttt{set}\,(\mathit{Trans}(w) \wedge \mathit{Axiom}_\gamma(w) \wedge \beta_\varphi(w) \rightarrow \\
&\quad w \subseteq f_8(w)) \ , \\
\mathit{Axiom}_\gamma(w) &= w \subseteq w \setminus \mathrm{P}(w \setminus w) \ , \\
\beta_\varphi(w) &= \forall v{:}\,\texttt{set}\,(v \in w \rightarrow (v \in f_8(w) \\
&\qquad \leftrightarrow \forall x{:}\,\texttt{term}\; v \in w \setminus \mathrm{P}(w \setminus (f_2(w,x) \cup f_5(w,x))))) \\
&\quad \wedge\ \forall x{:}\,\texttt{term}\,\forall v{:}\,\texttt{set}\,(v \in w \rightarrow (v \in f_2(w,x) \\
&\qquad \leftrightarrow \exists y{:}\,\texttt{term}\; v \in f_P(w,x,y))) \\
&\quad \wedge\ \forall x{:}\,\texttt{term}\,\forall v{:}\,\texttt{set}\,(v \in w \rightarrow (v \in f_5(w,x) \\
&\qquad \leftrightarrow \forall z{:}\,\texttt{term}\; v \in f_Q(w,z,x))) \ .
\end{aligned}$$

5. Proofs

5.1. Proof of Theorem 7.

We only give a sketch of the proof. The interested reader is referred to (Slanina, 2001) for the details.

"Only if". The proof is by contradiction: assume that $\Omega_E \nvdash (\gamma,\varphi)^*$, i.e., by completeness of first-order logic, $\Omega_E \not\models (\gamma,\varphi)^*$. This means there exists a model $\mathfrak{A}$ of Ω_E and an assignment σ on $\mathfrak{A}$ such that, putting $a = \sigma(w)$,

$$\begin{aligned}
a &\subseteq^{\mathfrak{A}} \mathrm{P}^{\mathfrak{A}}(a) \ , \\
\mathfrak{A}, [w/a] &\models \mathit{Axiom}_\gamma(w) \ , \\
\mathfrak{A}, \sigma &\models w \not\subseteq \varphi^* \ .
\end{aligned}$$

Let $\mathfrak{B}$ be the model for MA defined in the obvious way taking as domain the lower cone of $\mathfrak{A}$ with supremum a (see (Slanina, 2001) for details). Each term t in the language of Ω_E such that any occurrence of $\setminus$ has w as its left argument can be mapped to a term t' in the language of MA, and the assignment σ can be mapped into the assignment σ' on $\mathfrak{B}$ defined by $\sigma'(x) = a \cap^{\mathfrak{A}} \sigma(x)$, in such a way that

$$\sigma'(t') = a \cap^{\mathfrak{A}} \sigma(t)$$

and

$$\begin{aligned}
\varphi^{\#} &= (\varphi^*)' \ , \\
\gamma^{\#} &= (\gamma^*)' \ .
\end{aligned}$$

It is then straightforward to prove that $\mathfrak{B} \not\models (\gamma, \varphi)^{\#}$.

"If". The argument is analogous to the "only if" case, proceeding by contradiction from a model $\mathfrak{A}$ of MA such that $\mathfrak{A} \models \vec{\forall}(\gamma^{\#} = 1)$, $\mathfrak{A} \not\models \vec{\forall}(\varphi^{\#} = 1)$, and building a model $\mathfrak{B}$ of Ω_E such that $\mathfrak{B} \not\models (\gamma, \varphi)^{*}$. □

5.2. Proof of Theorem 14.

We need a preliminary lemma, whose omitted proof directly follows from Definition 13.

Lemma 16. *If there is no free occurrence of x_i in φ, then there is no occurrence of x_i in $t_\varphi(w, \vec{x})$.*

The proof of the theorem is by induction on the derivation of φ.

Propositional Axioms. The proof is exactly the same as the one for the propositional $\Box$-as-Pow translation (cf. (D'Agostino et al., 1995)).

Universal Quantifier Axioms. Let us assume, without loss of generality, that we are instanciating the variable x_1 and that the other free variables in $\psi(x_1)$ are $x_2, \ldots, x_m$, $m \leq n$. Thus φ is

$$\forall x_1 \; \psi(x_1) \rightarrow \psi(s) \enspace ,$$

where the substitution $\{x_1 \mapsto s\}$ is free for x_1 in $\psi(x_1)$. Let η be $\forall x_1 \, \psi(x_1)$. Then, by Def. 13,

$$t_\varphi(w, \vec{x}) = (w \setminus f_\eta(w, x_2, \ldots, x_m)) \cup t_{\psi(s)}(w, \vec{x})$$

and

$$\begin{aligned}
\alpha_\eta(w) &= \forall x_2, \ldots, x_m \colon \mathtt{term} \; \forall y \colon \mathtt{set} \; (y \in w \rightarrow \\
&\qquad (y \in f_\eta(w, x_2, \ldots, x_m) \leftrightarrow \forall x_1 \colon \mathtt{term} \; y \in t_{\psi(x_1)}(w, \vec{x}))) \enspace , \\
\beta_\varphi(w) &= \alpha_\eta(w) \wedge \beta_{\psi(x_1)}(w) \enspace .
\end{aligned}$$

Assume $\beta_\varphi(w)$ ($\mathit{Trans}(w)$ and $\mathit{Axiom}_\gamma(w)$ are not needed in the proof). We shall prove that $w \subseteq t_\varphi(w, \vec{x})$ by showing that, whenever $y \in w$ and $y \notin t_{\psi(s)}(w, \vec{x})$, then $y \notin f_\eta(w, x_2, \ldots, x_m)$. Assume $y \in w$ and $y \notin t_{\psi(s)}(w, \vec{x})$. From Def. 13, we have that $t_{\psi(s)}(w, \vec{x}) = t_{\psi(x_1)}(w, \vec{x})\{x_1 \mapsto s\}$; hence $\exists x_1 \colon \mathtt{term} \; y \notin t_{\psi(x_1)}(w, \vec{x})$, that is, taking the negation out of the quantifier, $\neg \forall x_1 \colon \mathtt{term} \; y \in t_{\psi(x_1)}(w, \vec{x})$; together with $\beta_\varphi(w)$ (the $\alpha_\eta(w)$ conjunct), this implies $y \notin f_\eta(w, x_2, \ldots, x_m)$.

Modus Ponens and Necessitation Rule. Both proofs are basically the same as those for the propositional case. Refer, again, to (D'Agostino et al., 1995).

Universal Generalization Rule. Assume the same notation as with the Universal Quantifier Axiom schema. The rule has then the form

$$\frac{\varphi \to \psi}{\varphi \to \forall x_1\ \psi} \ ,$$

where x_1 does not occur free in φ. Let η be the subformula $\forall x_1\ \psi$. From Def. 13, we get

$$\begin{aligned} t_{\varphi\to\psi}(w,\vec{x}) &= (w \setminus t_\varphi(w,\vec{x})) \cup t_\psi(w,\vec{x}) \ , \\ t_{\varphi\to\forall x_1\ \psi}(w,\vec{x}) &= (w \setminus t_\varphi(w,\vec{x})) \cup f_\eta(w,x_2,\ldots,x_m) \ , \end{aligned}$$

$$\begin{aligned} \alpha_\eta(w) &= \forall x_2,\ldots,x_m\colon \mathtt{term}\ \forall y\colon \mathtt{set}\ (y \in w \to \\ &\qquad (y \in f_\eta(w,x_2,\ldots,x_m) \leftrightarrow \forall x_1\colon \mathtt{term}\ y \in t_\psi(w,\vec{x}))) \ , \\ \beta_{\varphi\to\forall x_1\ \psi}(w) &= \alpha_\eta(w) \wedge \beta_\varphi(w) \wedge \beta_\psi(w) \ . \end{aligned}$$

Assume $\beta_{\varphi\to\forall x_1\ \psi}(w)$ (*Trans*(w) and *Axiom*$_\gamma(w)$ are not needed in the proof). We shall prove that $w \subseteq t_{\varphi\to\forall x_1\ \psi}(w,\vec{x})$ by showing that, whenever $y \in w$ and $y \in t_\varphi(w,\vec{x})$, necessarily $y \in f_\eta(w,x_2,\ldots,x_m)$. Assume $y \in w$ and $y \in t_\varphi(w,\vec{x})$. By induction hypothesis, $w \subseteq (w \setminus t_\varphi(w,\vec{x})) \cup t_\psi(w,\vec{x})$, hence $y \in t_\psi(w,\vec{x})$. By Lemma 16, the formula $y \in t_\varphi(w,\vec{x})$ contains no occurrence of x_1; hence, we can apply classical Universal Generalization to conclude that $\forall x_1\colon \mathtt{term}\ y \in t_\psi(w,\vec{x})$ and, thus, from the conjunct $\alpha_\eta(w)$ of the assumption $\beta_{\varphi\to\forall x_1\ \psi}(w)$, that $y \in f_\eta(w,x_2,\ldots,x_m)$.

Barcan Formulae. Let $\varphi = \forall x_1\, \Box\psi(x_1) \to \Box\forall x_1\, \psi(x_1)$. Let us call η and θ the formulae $\forall x_1\, \Box\psi(x_1)$ and $\forall x_1\, \psi(x_1)$, respectively, and let $x_2,\ldots,x_m$ be the free variables in η (or, equivalently, in θ). Then

$$\begin{aligned} t_\varphi(w,\vec{x}) &= (w \setminus f_\eta(w,x_2,\ldots,x_m)) \cup \mathrm{P}(f_\theta(w,x_2,\ldots,x_m)) \ , \\ \alpha_\eta(w) &= \forall x_2,\ldots,x_m\colon \mathtt{term} \forall y\colon \mathtt{set}\ (y \in w \to \\ &\qquad (y \in f_\eta(w,x_2,\ldots,x_m) \leftrightarrow \forall x_1\colon \mathtt{term} y \in \mathrm{P}(t_{\psi(x_1)}(w,\vec{x})))) \ , \\ \alpha_\theta(w) &= \forall x_2,\ldots,x_m\colon \mathtt{term} \forall y\colon \mathtt{set}\ (y \in w \to \\ &\qquad (y \in f_\theta(w,x_2,\ldots,x_m) \leftrightarrow \forall x_1\colon \mathtt{term} y \in t_{\psi(x_1)}(w,\vec{x}))) \ , \\ \beta_\varphi(w) &= \alpha_\eta(w) \wedge \alpha_\theta(w) \wedge \beta_{\psi(x_1)}(w) \ . \end{aligned}$$

Assume $\beta_\varphi(w)$ and $y \in w$. Then

$$\begin{aligned} & y \in t_\varphi(w,\vec{x}) \\ &\leftrightarrow \quad (y \in w \wedge y \notin f_\eta(w,x_2,\ldots,x_m)) \vee y \subseteq f_\theta(w,x_2,\ldots,x_m) \\ &\leftrightarrow \quad \exists x_1\colon \mathtt{term} y \not\subseteq t_{\psi(x_1)}(w,\vec{x}) \\ &\qquad\quad \vee \forall z\colon \mathtt{set}\ (z \in y \to z \in f_\theta(w,x_2,\ldots,x_m)) \\ &\leftrightarrow \quad \exists x_1\colon \mathtt{term} \exists z\colon \mathtt{set}\ (z \in y \wedge z \notin t_{\psi(x_1)}(w,\vec{x})) \\ &\qquad\quad \vee \forall z\colon \mathtt{set}\ (z \in y \to \forall x_1\colon \mathtt{term} z \in t_{\psi(x_1)}(w,\vec{x})) \ . \end{aligned}$$

The first equivalence is just the unfolding of the definition of $t_\varphi(w,\vec{x})$ by the axioms of Ω_2. In the second we dropped the subformula $y \in w$ from the conjunct because it is true by assumption, and then we rewrote the first disjunct using $\alpha_\eta(w)$ and the second using the Ω_2 axiom for $\subseteq$. In the last, we used the definition of $\subseteq$ in the left disjunct and $\alpha_\theta(w)$ in the right one.

Now, to show that $w \subseteq t_\varphi(w,\vec{x})$, we take, as above, $y \in w$ and assume that the second disjunct of the last line is false, which is equivalent to

$$\exists z\colon \mathtt{set}\,(z \in y \wedge \exists x_1\colon \mathtt{term}\, z \notin t_{\psi(x_1)}(w,\vec{x})) \enspace .$$

This obviously implies the first disjunct, thus proving the claim. □

5.3. Proof of Theorem 15.

The proof is an adaptation of the technique introduced in the soundness proof of the propositional case in (D'Agostino et al., 1995). More precisely, we are going to follow the proof of Theorem 5 of that paper. The first part is almost the same and we will state some lemmas without proof.

Let $\mathcal{U}$ be a universe of hypersets satisfying all the axioms of $ZF-FA$ (ZF except the Foundation Axiom) and AFA (see (Aczel, 1988) for details).

Lemma 17. *Let α be an ordinal, V_α be the set of all well-founded sets of rank less than α, and $\mathcal{U} \setminus V_\alpha$ be the universe of all hypersets not belonging to V_α. Then any model for the language of Ω_2 with domain $\mathcal{U} \setminus V_\alpha$ for sort* set *and interpretation function $(\cdot)'$ satisfying*

$$\begin{aligned} x \in' y &\Leftrightarrow x \in y \enspace , \\ x \cup' y &= x \cup y \enspace , \\ x \subseteq' y &\Leftrightarrow x \setminus V_\alpha \subseteq y \enspace , \\ x \setminus' y &= \begin{cases} x \setminus y & \text{if } x \setminus y \notin V_\alpha \enspace , \\ V_\alpha & \text{otherwise} \enspace , \end{cases} \\ \mathrm{P}'(x) &= \{y \mid y \setminus V_\alpha \subseteq x\} \end{aligned}$$

is a model of Ω_2.

Proof. See Lemma 6 of (D'Agostino et al., 1995). □

Given a frame (W,R,D), we want to embed it into the universe $\mathcal{U} \setminus V_\alpha$, for some suitable α. In (D'Agostino et al., 1995), the authors prove the existence of a labeled decoration * such that the following lemma holds.

Lemma 18. *For each* $a, b \in W$,

1. $a \neq b \Rightarrow a^* \neq b^*$,
2. $a^* \notin V_{\alpha+1}$ *and* $a^* \setminus V_{\alpha+1} = \{b^* \mid aRb\}$.

Proof. See (ii) and (iii) in Lemma 7 of (D'Agostino et al., 1995). □

Now consider an interpretation $(\mathcal{F}_C, \mathcal{F}_F, \mathcal{F}_R)$ over (W, R, D) and an assignment $\sigma : \{\vec{x}\} \to D$. Moreover, let $\mathcal{M}$ be the $Q1$ model $(W, R, D, \mathcal{F}_C, \mathcal{F}_F, \mathcal{F}_R)$.

A model for the language of Ω_2 can be obtained as follows. First, take the interpretation for `set` symbols as defined by the previous lemmas and let D be the domain for the sort `term`. Then, interpret `term` constant and function symbols as in $\mathcal{F}_C$ and $\mathcal{F}_F$, respectively, and, for each k-ary relational symbol Q, let $f^*_Q(d_1, \ldots, d_k)$ be equal to

$$\{a^* \mid a \in W \text{ and } a, [x_1/d_1, \ldots, x_k/d_k] \models_{Q1-L} Q(x_1, \ldots, x_k)\}$$

if this set is not empty, and to $V_{\alpha+1}$ otherwise.

Finally, for each universally quantified formula $\varphi = \forall x_i\, \eta$, with free variables $x_{j_1}, \ldots, x_{j_m}$, let $f^*_\varphi(w, d_1, \ldots, d_m)$ be equal to

$$\{a^* \mid a \in W \text{ and } a, [x_{j_1}/d_1, \ldots, x_{j_m}/d_m] \models_{Q1-L} \varphi\}$$

if this set is not empty, and to $V_{\alpha+1}$ otherwise.

We call this model $\mathfrak{A}_\mathcal{M}$. Furthermore, let W^* be equal to $\{a^* \mid a \in W\}$. $W^* \notin V_{\alpha+1}$ because $V_{\alpha+1}$ is transitive and, for each $a \in W$, $a^* \notin V_{\alpha+1}$ (Lemma 18).

Lemma 19. *We have:*

1. *for each* $a \in W$, $d_1, \ldots, d_n \in D$ *and formula* φ,
$$a, [x_1/d_1, \ldots, x_n/d_n] \models_{Q1-L} \varphi$$
if and only if
$$a^* \in' t^*_\varphi(W^*, d_1, \ldots, d_n) \text{ in } \mathfrak{A}_\mathcal{M} \quad ,$$
2. $\mathfrak{A}_\mathcal{M} \models \beta_\varphi(W^*)$.

Proof. 1. Let $\sigma = [x_1/d_1, \ldots, x_n/d_n]$. The proof is by induction on φ.

$\varphi = Q(s_1, \ldots, s_k)$: immediate from the definition of f^*_Q.

$\varphi = \eta \vee \theta$: $a, \sigma \models_{Q1-L} \varphi \Leftrightarrow a, \sigma \models_{Q1-L} \eta$ or $a, \sigma \models_{Q1-L} \theta \Leftrightarrow$ (by induction hypothesis) $a^* \in' t^*_\eta(W^*, \vec{d})$ or $a^* \in' t^*_\theta(W^*, \vec{d}) \Leftrightarrow$ (for $\mathfrak{A}_\mathcal{M}$ is a model of Ω_2) $a^* \in' t^*_\eta(W^*, \vec{d}) \cup' t^*_\theta(W^*, \vec{d}) \Leftrightarrow$ (by definition of t_φ) $a^* \in' t^*_\varphi(W^*, \vec{d})$.

$\varphi = \neg\eta$: $a, \sigma \models_{Q1-L} \varphi \Leftrightarrow a, \sigma \not\models_{Q1-L} \eta \Leftrightarrow$ (by i.h.) $a^* \not\in' t^*_\eta(W^*, \vec{d})$ $\Leftrightarrow$ (for $a^* \in' W^*$) $a^* \in' W^*$ and $a^* \not\in' t^*_\eta(W^*, \vec{d}) \Leftrightarrow$ (for $\mathfrak{A}_\mathcal{M}$ is a model of Ω_2) $a^* \in' W^* \setminus t^*_\eta(W^*, \vec{d}) \Leftrightarrow$ (by definition of t_φ) $a^* \in' t^*_\varphi(W^*, \vec{d})$.

$\varphi = \Box\eta$: $a, \sigma \models_{Q1-L} \varphi \Leftrightarrow b, \sigma \models_{Q1-L} \eta$ for every $b \in W$ such that $aRb \Leftrightarrow$ (by i.h.) for every $b \in W$ such that aRb, $b^* \in' t^*_\eta(W^*, \vec{d})$ $\Leftrightarrow$ (by Lemma 18) $a^* \setminus V_{\alpha+1} \subseteq t^*_\eta(W^*, \vec{d}) \Leftrightarrow$ (by definition of P' in Lemma 17) $a^* \in' \mathrm{P}(t^*_\eta(W^*, \vec{d})) \Leftrightarrow$ (by definition of t_φ) $a^* \in' t^*_\varphi(W^*, \vec{d})$.

$\varphi = \forall x_i\, \eta$: immediate from the definitions of f^*_φ and t_φ.

2. Assume, without loss of generality, that $\varphi = \forall x_1\, \eta$. Let $x_2, \ldots, x_m$ be the free variables in φ. We are going to prove $\alpha_\varphi(W^*)$. Let $a \in W$ and $d_1, \ldots, d_n$ be arbitrary elements of D. Then $a^* \in' f^*_\varphi(d_2, \ldots, d_m) \Leftrightarrow$ (by definition of f^*_φ) $a, [x_2/d_2, \ldots, x_m/d_m] \models_{Q1-L} \varphi \Leftrightarrow$ for every $d_1 \in D$, $a, [x_1/d_1, \ldots, x_m/d_m] \models_{Q1-L} \eta \Leftrightarrow$ (for the first part of this lemma) for every $d_1 \in D$, $a^* \in' t^*_\eta(W^*, d_1, \ldots, d_n)$[4] $\Leftrightarrow \forall x_1 \colon \mathtt{term}\ a^* \in' t^*_\eta(W^*, x_1, d_2, \ldots, d_n)$.

□

Corollary 20.

$$\mathcal{M} \models_{Q1-L} \varphi$$
if and only if
$$\forall \vec{x} \colon \mathtt{term}\, W^* \subseteq' t^*_\varphi(W^*, \vec{x}) \text{ in } \mathfrak{A}_\mathcal{M} \ .$$

Proof. Follows directly from Lemma 19 recalling that $\mathcal{M} \models_{Q1-L} \varphi$ means that $\mathcal{M} \models_{Q1-L} \forall \vec{x}\, \varphi$. □

Lemma 21. *A modal propositional formula γ, over the propositional variables $P_1, \ldots, P_r$, is valid in the frame (W, R) if and only if, for the corresponding hyperset W^*,*

$$\forall x_1, \ldots, x_r \colon \mathtt{set}\, W^* \subseteq' t^*_\gamma(W^*, x_1, \ldots, x_r)$$

holds in $\mathcal{U} \setminus V_{\alpha+1}$.

Proof. See Lemma 9 of (D'Agostino et al., 1995). □

To conclude the proof of Theorem 15, let us suppose that

$$\Omega_2 \ \models \ \forall w \colon \mathtt{set}\, (\mathit{Trans}(w) \wedge \mathit{Axiom}^2_\gamma(w) \wedge \beta_\varphi(w) \\ \rightarrow \forall \vec{x} \colon \mathtt{term}\, w \subseteq t_\varphi(w, \vec{x})) \ .$$

[4] By Lemma 16, all the d's beyond the m-th do not count in the interpretation.

Let $\mathcal{M} = (W, R, D, \mathcal{F}_C, \mathcal{F}_F, \mathcal{F}_R)$ be a model based on a frame in which γ is valid (a $Q1$-L model). From Lemma 21 it follows that $Axiom^2_\gamma(W^*)$ is true in $\mathfrak{A}_\mathcal{M}$, and from the second part of Lemma 19 the same follows for $\beta_\varphi(W)$. Furthermore, it is easy to prove that $Trans(W^*)$ holds as well. Since $\mathfrak{A}_\mathcal{M}$ is a model of Ω_2 and, by hypothesis, the formula $\forall \vec{x}\colon \mathtt{term}\ W^* \subseteq' t^*_\varphi(W^*, \vec{x})$ is true in $\mathfrak{A}_\mathcal{M}$, it follows from Corollary 20 that $\mathcal{M} \models_{Q1-L} \varphi$. $\square$

6. Conclusions and Open Questions

All the results presented in this paper uniformly rewrite the deduction problem for various families of modal logics into deduction problems with some form of set-theoretic flavor. The main contributions are the proposal of an equational theory capable of driving the translation at the propositional level and the extension of the $\Box$-as-Pow translation technique to a significant class of first-order modal logics.

A systematic study of set-theoretic translation techniques for different kinds of first-order modal logics could be carried out starting from the ideas presented here. In this paper, we deliberately confined ourselves to one of the most significant classes. Moreover, we did not consider the extension of the algebraic method to the first-order case: although we did not see any particular obstacle in principle, its development would almost certainly have called for the introduction of a very *ad-hoc* (two-sorted, non-equational) extension of the theory of Boolean rings, thus stripping the method of its elegance (which was the main motivation for its use in the propositional case).

All of the results mentioned call for specialized techniques explicitly designed for deduction in such theories as Ω or, better yet, able to exploit the syntactic features of the formulae obtained from the translation. Preliminary work in this direction is presented in (Piazza and Policriti, 2000), where a tableaux-based decision procedure for a class of formulae in which the resulting formulae can be embedded is presented. Other related results can be found in (Cantone and Zarba, 2000; Cantone et al., 2001).

A further step toward the applicability of the results presented could be obtained via a higher level of integration of the set-theoretic translation with more classical techniques. For example, it would be interesting to study the possibility of separating the first-order definable portions of a logic (for which more efficient techniques are often available) from contexts in which the $\Box$-as-Pow is the only available translation. In this context, we mention as an open problem, the question of precisely characterizing the expressive power of locally quantified first-order modal logics (cf. Lemma 9).

Finally, another research task, related to the quest for specialized techniques designed to be coupled with the translations presented here, is the search for reduction orders and rewriting systems to be used in automated engines for testing derivability in the theory Ω_E introduced in Section 3.

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