Theories of ω-Layered Metric Temporal Structures: Expressiveness and Decidability

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Abstract

This paper focuses on decidability problems for metric and layered temporal logics. The considered logics—suitable to model time granularity in various contexts—allow one to build granular temporal models by referring to the natural scale in any component of the model and properly constraining the interactions between differently grained components. We prove the decidability of both the theory of metric temporal structures provided with an infinite number of arbitrarily coarse temporal layers and the theory of metric temporal structures provided with an infinite number of arbitrarily fine temporal layers. The proof for the first theory is obtained by a reduction to the decidability problem of an extension of the standard monadic second-order theory of one successor $MSO[<]$, while the proof for the second one is done through a reduction to the monadic second-order decidable theory of k successors $MSO[<_k, succ_0, \ldots, succ_{k-1}]$.

Keywords: Time Granularity, Temporal Logic, Monadic Second-Order Theories, Tree Automata

1 Introduction

This paper focuses on decidability problems for metric and layered temporal logics [7, 20, 24]. The considered logics—suitable to model time granularity in various contexts—allow one to build granular temporal models by referring to the natural scale in any component of the model and properly constraining the interactions between differently grained components.

As pointed out in [3], the ability of providing and relating temporal representations at different 'grain levels' of the same reality is widely recognized as an important research theme for temporal logic and a major requirement for many applications, including formal specifications of real-time systems [8, 11, 16], temporal databases [5, 9, 32, 33, 34], and data mining [4]. The use of temporal and real-time logics for representing and reasoning about timing properties of reactive and real-time systems

has proved to be highly beneficial in the area of formal specifications [18, 19]. However, there are real-time systems whose temporal specification is far from being simple even with these logics. This is the case of a large class of real-time systems whose components have dynamic behaviours regulated by very different—even by orders of magnitude—time constants (granular real-time systems): it is indeed quite unsatisfactory to compel the specifier of these systems to use a flat temporal logic, provided with a unique time granularity, to describe the behaviour of all the components. With regard to temporal databases, most applications in the areas of heterogeneous and federated temporal databases must be able to manage differently-grained time-stamps associated with data collected from different sources which are not under the same control. Finally, time granularity has been successfully exploited to prune the search space in the data mining process.
The decidability of the satisfiability and validity problems for some simple finitely-layered metric temporal logics has already been shown in [24] (since a formula is valid if and only if its negation is not satisfiable, satisfiability and validity are actually equivalent problems). As most decidability results for temporal logics, it has been proved by reducing, through coding into the well-known theory $MSO[<]$, the given problem to the decidable problem of emptiness for regular ω-languages. In this paper, we deal with the satisfiability problem for metric temporal logics provided with an infinite number of layers (ω-layered), in which each time point belonging to a given layer can be decomposed into k time points of the immediately finer one (k-refinable). Finitely-layered metric temporal logics can be used to specify timing properties of granular real-time systems composed by a finite number of differently-grained temporal components, which have been fixed once and for all (closed/rigid systems). Furthermore, if provided with a rich enough layered structure, they suffice to deal with conditions like "p holds at all even times of a given temporal domain" that cannot be expressed using flat propositional temporal logics [10]. ω-layered metric temporal logics are needed to specify granular real-time systems that can dynamically change their structure, e.g., by adding temporal components of possibly different grain sizes (open/flexible systems). As a matter of fact, ω-layered metric temporal logics also allow one to express relevant properties of infinite sequences of states over a single temporal domain that cannot be captured by using flat or finitely-layered temporal logics. This is the case, for instance, of conditions like "p holds at all times 2^i, with $i \in \mathbb{N}$, of a given temporal domain". In order to investigate the decidability problem for ω-layered (k-refinable) metric temporal structures, we introduce a suitable second-order language $MSO[<, \downarrow_0, \ldots, \downarrow_{k-1}]$ and show how to interpret it over different classes of ω-layered structures. We first consider the case of temporal structures in which there is a finest temporal domain together with an infinite number of coarser and coarser domains (*upward unbounded layered structures*); then, we switch to the class of infinitely refinable structures consisting of a coarsest domain together with an infinite number of finer and finer domains (*downward unbounded layered structures*). We prove that the theories of upward unbounded layered structures and of downward unbounded ones are decidable by reducing each of them to a (different) proper extension of $MSO[<]$. Furthermore, as a by-product of our work, we define a uniform framework for time and states, within which the theory of timed state sequences, underlying real-time logics [1], can be recovered as one possible abstraction of the proposed theories of ω-layered metric temporal structures. More precisely, we will show that both the theory of up-

ward unbounded metric temporal structures and the theory of downward unbounded ones allow one to characterize the relevant properties of timed state sequences.

The paper is organized as follows. In Section 2, we recall the definitions of the theories $MSO[<]$ and $MSO[<_k, succ_0, \ldots, succ_{k-1}]$, and define the theory $MSO[<, pow_k, hp_k]$. In Section 3, we formally define the theories of upward and downward unbounded layered structures. In Section 4, we show that the theory of timed state sequences can be embedded into both theories of ω-layered metric temporal structures. In Section 5, we prove the decidability of the two theories by reducing the former to the theory $MSO[<, pow_k, hp_k]$ and the latter to the theory $MSO[<_k, succ_0, \ldots, succ_{k-1}]$. In Section 6, we show how the basic functionalities of metric and layered temporal logics can be expressed in $MSO[<, \downarrow_0, \ldots, \downarrow_{k-1}]$.

2 Preliminaries

In this section, we introduce syntax and semantics of the (monadic) second-order languages used in this paper. From the syntactic point of view, we first define the notion of second-order language over a (parametric) set of embedded predicates; then, the set of predicates is instantiated to obtain the concrete languages we are interested in. (The notation we use is borrowed from [31].)

Definition 2.1
Let $MSO[u_1, \ldots, u_s, b_1, \ldots, b_t]$ be the second-order language with equality over a set $U = \{u_1, \ldots, u_s\}$ of unary relational symbols and a set $B = \{b_1, \ldots, b_t\}$ of binary relational symbols, which is built up as follows:

- *atomic formulas* are of the forms $x = y$, $u_i(x)$, with $1 \leq i \leq s$, $b_j(x, y)$, with $1 \leq j \leq t$, and $p(x)$, where x, y are individual variables and p is a set variable;
- *formulas* are built up from atomic formulas by means of the boolean connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$ and the quantifiers $\forall, \exists$ ranging over both individual and set variables.

Furthermore, let us denote by $MFO[u_1, \ldots, u_s, b_1, \ldots, b_t]$ the first-order fragment of $MSO[u_1, \ldots, u_s, b_1, \ldots, b_t]$.
As usual, if $MSO[u_1, \ldots, u_s, b_1, \ldots, b_t]$ is interpreted over a domain $\mathcal{D}$, individual variables are interpreted as elements of $\mathcal{D}$, set variables and unary relational symbols in U are interpreted as subsets of $\mathcal{D}$, binary relational symbols in B are interpreted as subsets of $\mathcal{D} \times \mathcal{D}$, $x = y$ is interpreted as equality between x and y, $u_i(x)$ (resp.: $b_j(x, y)$, $p(x)$) are interpreted as membership of x (resp.: $\langle x, y\rangle$, x) to u_i (resp.: b_j, p).
Let $\phi(p_1, \ldots p_n)$ be a formula, where at most the n set variables $p_1, \ldots, p_n$ occur free, and let $P_1, \ldots, P_n$ be subsets of $\mathcal{D}$. We write $\mathcal{D}, P_1, \ldots P_n \models \phi(p_1, \ldots p_n)$ if ϕ is satisfied in $\mathcal{D}$, with P_i as interpretation for p_i (for $1 \leq i \leq n$).
Both the classical theories of k successors, with $k \geq 1$, and the proposed extension of the theory of one successor can be defined as suitable instantiation of $MSO[u_1, \ldots, u_s, b_1, \ldots, b_t]$.

The theory of one successor. If we consider the case where $\mathcal{D}$ is the set of natural numbers $\mathbb{N}$, U is the empty set, and $B = \{<\}$, with $<$ interpreted as the standard ordering of natural numbers, then $MSO[<]$ is the well-known monadic second-order theory of one successor (denoted also by $S1S$). For $P_1, \ldots, P_n$ (fixed) subsets of $\mathbb{N}$, a model-theoretic structure for $MSO[<]$ can be viewed as an ω-word α over the alphabet $\Sigma = \{0,1\}^n$ such that $j \in P_i$ iff the i-th element of $\alpha(j)$ is set to 1, with $\alpha(j)$ denoting the symbol in the j-th position of α. The language defined by a $MSO[<]$-formula is the set of ω-words satisfying it. The correspondence between ω-languages and $MSO[<]$-formulas is given by Büchi Theorem which states that the class of regular ω-languages is precisely the class of languages defined by $MSO[<]$-formulas. The theory $MSO[<]$ is decidable (cf. [30]).

The theory of k successors. If we consider the case where $\mathcal{D}$ is the set of words over the alphabet $\{0,\ldots,k-1\}$, U is the empty set, and $B = \{<_k, succ_0, \ldots, succ_{k-1}\}$, with $<_k$ interpreted as the standard strict prefix relation over words and $succ_i$ as the function over words such that $succ_i(w, w \cdot i)$ (where $\cdot$ denotes concatenation of words), written in the following as $succ_i(w) = w \cdot i$, then $MSO[<_k, succ_0, \ldots, succ_{k-1}]$ is the well-known monadic second-order theory of k successor (denoted also by SkS). For $P_1, \ldots, P_n$ (fixed) subsets of $\{0,\ldots,k-1\}^*$, a model-theoretic structure for $MSO[<_k, succ_0, \ldots, succ_{k-1}]$ can be viewed as a leafless, perfectly balanced k-ary tree labelled over the alphabet $\Sigma = \{0,1\}^n$ The theory $MSO[<_k, succ_0, \ldots, succ_{k-1}]$ is decidable (cf. [30]).

The theory $MSO[<, pow_k, hp_k]$. In [22, 23, 26], we defined a (decidable) extension of $MSO[<]$, called $MSO[<, pow_k, hp_k]$, which we proved to be the logical counterpart of the operational definition of systolic ω-languages. The theory $MSO[<, pow_k, hp_k]$ extends $MSO[<]$ with a binary relational symbol pow_k (actually a partial function), called *power function*, and a unary relational symbol hp_k. For any natural number $x > 0$, the power function computes the natural number $x - x'$, where x' is the least power of k (with non-null coefficient) in the k-ary representation of x. In the following, we will use the functional notation $pow_k(x) = y$ instead of $pow_k(x, y)$. The predicate hp_k holds for a natural number x if and only if the least power of k (with non-null coefficient) in the k-ary representation of x has coefficient $k-1$.

DEFINITION 2.2
The function $pow_k : \mathbb{N}^+ \to \mathbb{N}$ is such that $pow_k(x) = y$ iff

$$\begin{aligned} x &= a_n k^n + a_{n-1}k^{n-1} + \ldots + a_m k^m,\ 0 \le a_i \le k-1, \text{ and } a_m \neq 0 \\ y &= a_n k^n + a_{n-1}k^{n-1} + \ldots + (a_m - 1)k^m. \end{aligned}$$

The predicate hp_k holds at x iff

$$x = a_n k^n + a_{n-1}k^{n-1} + \ldots + a_m k^m,\ 0 \le a_i \le k-1, \text{ and } a_m = k-1.$$

Notice that the predicate "*x is a power of k*" can be easily expressed as $pow_k(x) = 0$; hence, $MSO[<, pow_k, hp_k]$ is at least as expressive as the well-known (decidable) extension of $MSO[<]$ with the predicate "*is a power of k*".

In [22], we proved that the theory $MSO[<, pow_k, hp_k]$ is a decidable extension of the theory $MSO[<]$. First, notice that, as in the case of $MSO[<]$, the model-theoretic structures for the theory $MSO[<, pow_k, hp_k]$ are ω-words. Then, recall that the set of model-theoretic structures satisfying a formula ϕ of $MSO[<]$ is a regular ω-language and the decidability of $MSO[<]$ can be reduced to the (decidable) problem of emptiness for regular ω-languages (Büchi Theorem). The decidability of $MSO[<, pow_k, hp_k]$ has been proved in a similar way. In [25], Monti and Peron

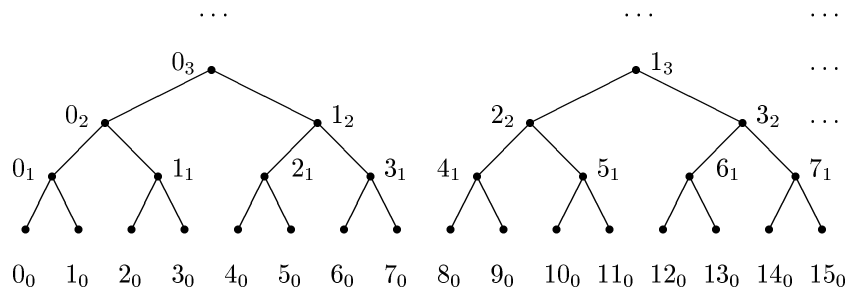

FIG. 1. An upward unbounded (2-refinable) structure (i_j stands for $(+_j 1)^i(0_j)$).

defined the class of ω-languages accepted by systolic tree automata, and showed that it properly extends the class of regular ω-languages, preserving all its closure (with respect to operations) and decidability properties. Furthermore, in [22, 23, 26], it has been proved that $MSO[<, pow_k, hp_k]$ is the logical counterpart of systolic ω-languages, namely, the set of ω-words satisfying a formula ϕ in $MSO[<, pow_k, hp_k]$ is a systolic ω-language (an extension of Büchi Theorem). This allowed as to reduce the decidability problem for $MSO[<, pow_k, hp_k]$ to the (decidable) emptiness problem for systolic ω-languages. The proof of the decidability of $MSO[<, pow_2, hp_2]$ can be found in [26]. The (non-trivial) extension of the proof to any theory $MSO[<, pow_k, hp_k]$, with $k \geq 2$, is given in [22].

3 Theories of ω-layered temporal structures

In this section, we introduce the theory of ω-layered structures consisting of an infinite number of *arbitrarily coarse* infinite temporal domains, called upward unbounded layered structures, and the theory of ω-layered structures consisting of an infinite number of *arbitrarily fine* infinite temporal domains, called downward unbounded layered structures. From a syntactical point of view, we consider a second-order language $MSO[<, \downarrow_0, \ldots, \downarrow_{k-1}]$, where $\downarrow_i$, with $0 \leq i \leq k-1$, is a binary relational symbol (which is actually a function). In the following, we will use the functional notation $\downarrow_i (x) = y$ instead of $\downarrow_i (x, y)$. The intuitive meanning is that $\downarrow_0 (x), \ldots, \downarrow_{k-1} (x)$ are the k elements of the domain into which the element x is refined. The same language is shared by both the above mentioned theories. In the remaining part of this section, we show how $MSO[<, \downarrow_0, \ldots, \downarrow_{k-1}]$ is interpreted over the class of upward unbounded layered structures as well as over the class of downward unbounded ones.

3.1 The theory of upward unbounded layered structures

We first define upward unbounded layered structures, and then show how to interpret $MSO[<, \downarrow_0, \ldots, \downarrow_{k-1}]$ over them. Intuitively, an upward unbounded ω-layered k-refinable metric temporal structure is an infinite k-branching tree generated starting from the leaves (cf. Figure 1). The total ordering $<$ is provided by the inorder visit of its nodes.

DEFINITION 3.1
An upward unbounded ω-layered k-refinable metric temporal structure is a triplet $\langle \bigcup_{i\geq 0} T^i, \downarrow, < \rangle$, where

- $\{T^i\}_{i\geq 0}$ are pairwise disjoint enumerable sets;
- $\downarrow$: $\{0,\ldots,k-1\} \times \bigcup_{i\geq 1} T^i \to \bigcup_{i\geq 0} T^i$ is a bijection such that for $i > 0$, $\downarrow|_{T^i}$: $\{0,\ldots,k-1\} \times T^i \to T^{i-1}$ is a bijection;
- $<$ is the transitive closure of the relation $\tilde{<}$ on $\bigcup_{i\geq 0} T^i$ such that
  0. $\langle T^0, <|_{T^0\times T^0} \rangle$ is isomorphic to the set of natural numbers $\mathbb{N}$ with the standard ordering;
  1. $\downarrow(0,t) \tilde{<} t$ and $t \tilde{<} \downarrow(j,t)$, for $1 \leq j \leq k-1$;
  2. $\downarrow(j,t) \tilde{<} \downarrow(j+1,t)$, for $1 \leq j \leq k-2$;
  3. if $t \tilde{<} t'$ and $t' \not\in \downarrow^n(t)$ (for $n \geq 0$), then $\downarrow(k-1,t) \tilde{<} t'$;
  4. if $t \tilde{<} t'$ and $t \not\in \downarrow^n(t')$ (for $n \geq 0$), then $t \tilde{<} \downarrow(0,t')$,

  where, for $t \in \bigcup_{i\geq 0} T^i$ and $n \in \mathbb{N}$, $\downarrow^n(t) \subseteq \bigcup_{i\geq 0} T^i$ is the set such that $\downarrow^0(t) = \{t\}$ and $\downarrow^n(t) = \{\downarrow(j,\bar{t}) : \bar{t} \in \downarrow^{n-1}(t), j \in \{0,\ldots,k-1\}$, for $n \geq 1$.

Notice that, for each $t \in T^i$, with $i \geq 1$, and $0 \leq j \leq k-1$, $\downarrow(j,t)$ associates with t the $(j+1)$-th element of its k-decomposition with respect to T^{i-1}.

It can be shown that for each $i \geq 0$, T^i together with the restriction $<|_{T^i\times T^i}$ of $<$ to $T^i \times T^i$ is isomorphic to the set of natural numbers $\mathbb{N}$ with the standard ordering. For all $i \geq 0$ and $t \in T^i$, we denote by $+_i 1(t)$ the immediate successor on T^1 of t, that is, the element $t' \in T^i$ such that $t < t'$ and $\neg\exists t''(t'' \in T^i \wedge t < t'' < t')$.

An example of upward unbounded structure is given in Figure 1. It is easy to see that the function $\downarrow$ gives a root-less, tree-like k-ary structure to the domain $\bigcup_{i\geq 0} T^i$ and that $<$ is the total ordering of the domain induced by the inorder visit of the tree-like structure.

The domain of the interpretation of $MSO[<,\downarrow_0,\ldots,\downarrow_{k-1}]$ over an upward unbounded layered structure $\langle \bigcup_{i\geq 0} T^i, \downarrow, < \rangle$ is $\bigcup_{i\geq 0} T^i$; the relational symbol $<$ of $MSO[<,\downarrow_0,\ldots,\downarrow_{k-1}]$ is interpreted as the total ordering of the domain; the interpretation of $\downarrow_j(x)$ is $\downarrow(j,x)$ if x is interpreted over $\bigcup_{i>0} T^i$ and it is undefined if x is interpreted over T^0. (Atomic formulas are assumed to evaluate to false whenever at least one of their arguments is undefined.)

In Section 6, we will show how the basic relations of metric and layered temporal logics, as well as other natural temporal relations, can be encoded in $MSO[<,\downarrow_0,\ldots,\downarrow_{k-1}]$. As an example, let us show how to define a unary predicate Δ_0 holding at the origin of each layer. The interpretation of Δ_0 is exactly the set of all the elements belonging to the left-edge of the structure (cf. Figure 1), which can be defined as the least set containing the element 0_0 and all its ancestors 0_1, 0_2, ... This definition can be given in the language $MSO[<,\downarrow_0,\ldots,\downarrow_{k-1}]$ as follows. Given a second-order formula $\phi(\mathtt{p})$, with free set variable $\mathtt{p}$, let $\mu(\phi(\mathtt{p}))(\mathtt{x})$ be the following second-order formula, with free individual variable $\mathtt{x}$:

$$\exists \mathtt{p}(\mathtt{p}(\mathtt{x}) \wedge \phi(\mathtt{p}) \wedge \forall \mathtt{q}(\phi(\mathtt{q}) \to \forall \mathtt{y}(\mathtt{p}(\mathtt{y}) \to \mathtt{q}(\mathtt{y})))).$$

$\mu(\phi(\mathtt{p}))(\mathtt{x})$ evaluates to true if and only if the valuation for $\mathtt{x}$ belongs to the smallest valuation for $\mathtt{p}$ for which $\phi(\mathtt{p})$ holds true. Using the operator μ and introducing $\mathtt{p}(0_0)$ as a shorthand for $\exists \mathtt{y}(\mathtt{p}(\mathtt{y}) \wedge \forall \mathtt{z}(\mathtt{y} \leq \mathtt{z}))$, $\Delta_0(\mathtt{x})$ can be expressed as:

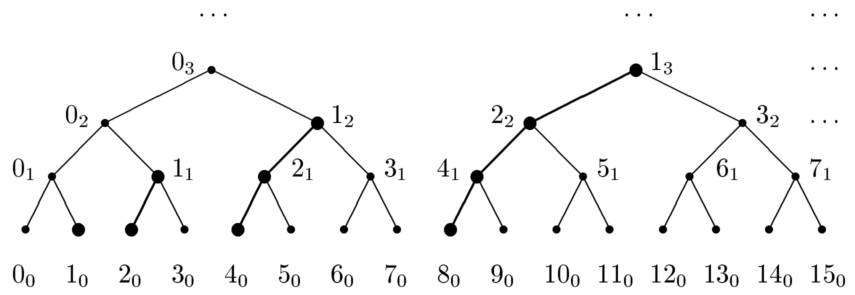

FIG. 2. The set of powers of two (big circles and thick lines)

$$\mu\,(\mathtt{p}(0_0) \wedge \forall \mathtt{y},\mathtt{z}((\mathtt{p}(\mathtt{z}) \wedge \downarrow_0 (\mathtt{y}) = \mathtt{z}) \rightarrow \mathtt{p}(\mathtt{y})))\,(\mathtt{x}), \tag{3.1}$$

which captures the smallest interpretation for $\mathtt{p}$ containing 0_0 and closed parent-wise.

In Section 6, we will define a family of unary predicates of horizontal contextualization $\Delta_\mathtt{i}(\mathtt{x})$, with $i \geq 0$, holding at all time points at distance i from the origin of the layer they belong to. All of them can be defined in terms of the predicate $\Delta_0(\mathtt{x})$. For instance, the predicate $\Delta_1(\mathtt{x})$ can be defined as follows:

$$\exists \mathtt{y}(\Delta_0(\mathtt{y}) \wedge \downarrow_1 (\mathtt{y}) = \mathtt{x}).$$

EXAMPLE 3.2
Consider the condition: "for every layer (of an upward unbounded layered structure), p holds at all its 2^i-th time points, with $i \geq 0$". By looking at Figure 2, it is easy to see that such points are all and only the points in the left downward closure of the right children of points belonging to (the interpretation of) Δ_0.

Therefore, the condition can be expressed by the following formula:

$$\forall \mathtt{x}(\mathtt{power}(\mathtt{x}) \rightarrow \mathtt{p}(\mathtt{x})),$$

where $\mathtt{power}(\mathtt{x})$ stands for the second-order formula:

$$\mu\,((\Delta_1(\mathtt{x}) \rightarrow \mathtt{p}(\mathtt{x})) \wedge \forall \mathtt{y},\mathtt{z}((\mathtt{p}(\mathtt{z}) \wedge \downarrow_0 (\mathtt{z}) = \mathtt{y}) \rightarrow \mathtt{p}(\mathtt{y})))(\mathtt{x})\,.$$

The above condition can be easily restricted to a single temporal domain, e.g., the finest layer T^0, as follows:

$$\forall \mathtt{x}((\mathtt{T}^0(\mathtt{x}) \wedge \mathtt{power}(\mathtt{x})) \rightarrow \mathtt{p}(\mathtt{x})),$$

where $\mathtt{T}^0(\mathtt{x})$ is a shorthand for $\neg\exists \mathtt{y}(\downarrow_0 (\mathtt{x}) = \mathtt{y} \vee \downarrow_1 (\mathtt{x}) = \mathtt{y})$.

Remark. It is worth mentioning the fact that conditions like "p holds at all even times of a given temporal domain" can be expressed using (sufficiently rich) finitely-layered metric temporal structures. Indeed, given a 2-layered (2-refinable) metric temporal structure consisting of two temporal domains T^0 and T^1, with T^0 finer than T^1, the above condition can be expressed by the following first-order formula:

$$\forall \mathtt{x}((\mathtt{even}(\mathtt{x}) \rightarrow \mathtt{p}(\mathtt{x})),$$

where $\mathtt{even}(\mathtt{x})$ stands for $\mathtt{T}^0(\mathtt{x}) \wedge \exists \mathtt{y}(\downarrow_0 (\mathtt{y}) = \mathtt{x})$.

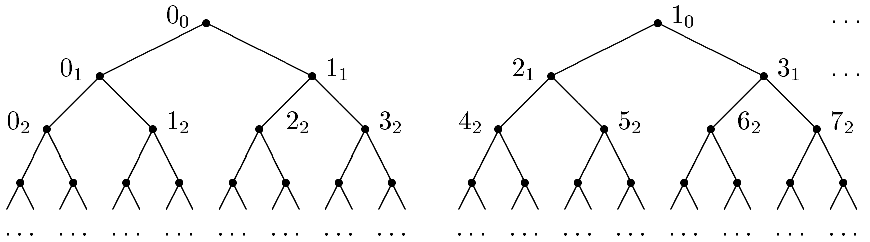

FIG. 3. A downward unbounded (2-refinable) structure (i_j stands for $(+_j 1)^i(0_j)$).

3.2 The theory of downward unbounded layered structures

As for the upward case, we first formally define downward unbounded layered structures, and then show how to interpret $MSO[<, \downarrow_0, \dots, \downarrow_{k-1}]$ over them.

DEFINITION 3.3
A downward unbounded ω-layered k-refinable metric temporal structure is a triplet $\langle \bigcup_{i \geq 0} T^i, \downarrow, < \rangle$, where

- $\{T^i\}_{i \geq 0}$ are pairwise disjoint enumerable sets;
- $\downarrow$: $\{0, \dots, k-1\} \times \bigcup_{i \geq 0} T^i \to \bigcup_{i \geq 1} T^i$ is a bijection such that for $i \geq 0$, $\downarrow|_{T^i}$: $\{0, \dots, k-1\} \times T^i \to T^{i+1}$ is a bijection;
- $<$ is the transitive closure of the relation $\tilde{<}$ on $\bigcup_{i \geq 0} T^i$ such that
  - $\langle T^0, \tilde{<}|_{T^0 \times T^0} \rangle$ is isomorphic to the set of natural numbers $\mathbb{N}$ with the standard ordering;
  - $t \tilde{<} \downarrow(j, t)$, for $0 \leq j \leq k-1$;
  - $\downarrow(j, t) \tilde{<} \downarrow(j+1, t)$, for $0 \leq j \leq k-2$;
  - if $t \tilde{<} t'$ and $t' \not\in \downarrow^n(t)$ (for $n \geq 0$), then $\downarrow(j, t) \tilde{<} t'$, for $0 \leq j \leq k-1$.

As in the case of Definition 3.1, it can be shown that each layer is isomorphic to the natural numbers. An example of downward unbounded layered structure is shown in Figure 3. It is easy to see that the domain $\bigcup_{i \geq 0} T^i$ is structured by $\downarrow$ in such a way that each time point of T_0 is the root of a leafless, perfectly balanced k-ary tree. The total ordering $<$ of the domain is induced by the preorder visit of the structure.

The domain of the interpretation of $MSO[<, \downarrow_0, \dots, \downarrow_{k-1}]$ over a downward unbounded layered structure $\langle \bigcup_{i \geq 0} T^i, \downarrow, < \rangle$ is $\bigcup_{i \geq 0} T^i$; the relational symbol $<$ of $MSO[<, \downarrow_0, \dots, \downarrow_{k-1}]$ is interpreted as the total ordering of the domain; the interpretation of $\downarrow_j(x)$ is $\downarrow(j, x)$. Notice that, unlike the case of upward unbounded layered structures, the interpretation of $\downarrow_{\mathrm{j}}$ over downward unbounded layered structures is always defined.

Remark. We would like to point out a difference between the definition of the ordering relation $<$ in the two classes of structures. In both the downward and upward unbounded structures, the relation $<$ is defined in such a way that for each $i \geq 0$, the pair $\langle T^i, <|_{T^i \times T^i} \rangle$ is isomorphic to the pair $\langle \mathbb{N}, < \rangle$ (in the upward case, this is actually true also for the whole structure $\langle \bigcup_{i \geq 0} T^i, < \rangle$). Moreover, in both structures the zero element for $\langle \bigcup_{i\geq 0} T^i, < \rangle$ coincides with the zero element for $\langle T^0, <|_{T^0 \times T^0} \rangle$. Nevertheless, we cannot use the same definition of $<$ for both classes of structures: if we ordered downward unbounded structures in the same way of upward unbounded ones, we would loose the zero element, and vice versa. As a matter of fact, the topologies of upward and downward unbounded layered structures are intrinsically different and cannot be modeled in the same way. The basic ordering relation $<$ of both upward and downward layered structures can be exploited to encode a variety of well-known ordering as shown in [22].

4 A uniform framework for time and states

Logic and computer science communities have traditionally followed a different approach to the problem of representing and reasoning about time and states (both approaches actually date back to Prior's different grades of tense-logical involvement [27]). Research in philosophy, linguistics, and mathematical logic resulted in a family of (metric) tense logics that take *time* as a primitive notion and define (*timed*) *states* as sets of atomic propositions which are true at given times, e.g. [2, 6, 12, 14, 28]. In the last year, a few papers demonstrated the possibility of successfully exploiting metric (possibly layered) tense logics in computer science, e.g. [7, 15, 20, 21, 24]. The present work further develops this research direction. Most research in computer science, however, concentrated on the so–called temporal logics of programs, which have been largely used in computer science to specify and verify reactive and concurrent systems, e.g. [10, 18, 19, 29]). In order to deal with real-time systems, such logics have been provided with a metric of time, e.g. [1]. The resulting temporal logics, called *real-time logics*, take *state* as a primitive notion, and define *time* as an attribute of states. More precisely, given an ordered set of states $\mathcal{S}$ and an ordered set of time points $\mathcal{T}$, real-time logics define a weakly monotonic function $f : \mathcal{S} \rightarrow \mathcal{T}$ that associates a time with each state (notice that it may happen that there exist pairs of states $s_i, s_j \in \mathcal{S}$ such that $s_i < s_j$ and $f(s_i) = f(s_j)$).

In this section, we show how the theory of timed state sequences, underlying real-time logics, can be embedded into both the theory of upward unbounded layered structures and the theory of downward unbounded ones. While the embedding into downward unbounded structures is rather straightforward, the embedding into upward unbounded ones is less immediate and it requires some ingenuity. Our fundamental claim is that any pair of distinct states, belonging to the same course of events, can always be temporally ordered, provided that we can refer to a sufficiently fine temporal domain. Furthermore, we claim that the ordering between states, which are temporally indistinguishable with respect to the considered domain, is actually induced by their temporal ordering with respect to a finer domain, with respect to which their are temporally distinguishable. Notice that, since it is not possible to fix a priori any bound on how fine a domain must be to allow one to temporally order a given pair of states, we need to have an infinite number of temporal domains at our disposal. Indeed, given a homogeneous metric and layered temporal structure, a finite number of layers does not allow one to temporally order a finite, but arbitrarily large, set of simultaneous states. More generally, given an infinite set of states $\mathcal{S}$ and a temporal domain T^i, homogeneous, finitely-layered metric temporal structures do not allow one to express timing functions $f : T^i \rightarrow 2^{\mathcal{S}}$ having a finite, but not limited, image.

We start by introducing the theory of *timed* state sequences [1]. It is obtained from the classical theory of state sequences [18] by the addition of a suitable temporal domain, with appropriate primitive operations, and a mapping that associates a time with every state. Let $\langle \mathcal{S}, \leq_s \rangle$, with $\mathcal{S} = \{s_i : i = 0, 1, \ldots\}$, be a discrete, linearly ordered set of states (isomorphic to the set of natural numbers $\langle \mathbb{N}, \leq \rangle$) and let $\langle \mathcal{T}, \leq_t \rangle$, with $\mathcal{T} = \{t_i : i = 0, 1, \ldots\}$, be a discrete, linearly ordered temporal domain (isomorphic to $\langle \mathbb{N}, \leq \rangle$). Given a finite set $P = \{p_1, \ldots, p_n\}$ of set variable (predicate) symbols, a *timed state sequence* $\langle \sigma, \rho \rangle$ is a pair of functions $\sigma : \mathcal{S} \to 2^P$ and $\rho : \mathcal{S} \to \mathcal{T}$, where ρ is weakly monotonic, that is, for all pairs s_1, s_2, $s_1 \leq_s s_2$ implies $\rho(s_1) \leq_t \rho(s_2)$.

Definition 4.1
The second-order language $\mathcal{L}^2_T$, with the two sorts *State* and *Time*, is built up as follows.
The *vocabulary* of $\mathcal{L}^2_T$ consists of set variable (predicate) symbols $p_1, \ldots, p_n$, individual variable symbols $x_s, y_s, \ldots$ and the binary relational symbol $\leq_s$ over the state sort; individual variable symbols $x_t, y_t, \ldots$, the unary function symbol $+1$, and the binary relational symbol $\leq_t$ over the time sort; the unary function symbol f from the state sort to the time sort.
Atomic formulas have the form $p_i(x_s)$, $x_s \leq_s y_s$, $w_t \leq_t z_t$, $w_t =_t z_t + 1$, $z_t =_t f(x_s)$, with x_s, y_s individual state variables and z_t, w_t individual time variables.
formulas are obtained from atomic formulas by using the boolean connectives $\wedge$, $\vee$, $\neg$, and $\to$, and the quantifiers $\exists$ and $\forall$ ranging over state and time individual variables and over state set variables.

The intended interpretations for the state and time sorts are $\mathcal{S}$ and $\mathcal{T}$, respectively, and the primitives are interpreted in the standard way. Given a formula $\phi \in \mathcal{L}^2_T$, with free set variable symbols $p_1, \ldots, p_n$, an interpretation $\mathcal{I}$ for ϕ specifies n sets $p_1^{\mathcal{I}}, \ldots, p_n^{\mathcal{I}} \subseteq \mathcal{S}$ and the monotonic function $f : \mathcal{S} \to \mathcal{T}$. Every interpretation $\mathcal{I}$ can be viewed as a timed state sequence $\langle \sigma, \rho \rangle$, with $\rho = f^{\mathcal{I}}$.

4.1 The embedding into upward unbounded layered structures

In this section, we show how the theory of timed state sequences can be proved to be a fragment of the theory of upward unbounded layered structures. To this end, we need to properly characterize the set of states, the set of times, and the mapping from states to times.

Let *state* and *time* be two set variable symbols, $P = \{p_1, \ldots, p_n\}$ be a finite set of set variable symbols, and $\mathcal{U} = \langle \bigcup_{i \geq 0} T_i, \downarrow, < \rangle$ be an upward unbounded (2-refinable) layered structure. Moreover, let $\mathcal{I}$ be an interpretation (over $\mathcal{U}$) for *state*, *time* and the set variable symbols in P such that:

(1) $state^{\mathcal{I}}$ is an infinite set;
(2) for any $t, t' \in time^{\mathcal{I}}$, t is not an ancestor of t';
(3) for any $s, s' \in state^{\mathcal{I}}$, s is not an ancestor of s';
(4) for any $s \in state^{\mathcal{I}}$, if there exists $t \in time^{\mathcal{I}}$ such that t is an ancestor of s, then, for each $s' \in state^{\mathcal{I}}$ such that $s' \leq s$, there exists an ancestor $t' \in time^{\mathcal{I}}$ of s'.

Such an interpretation is called a *linearizable* interpretation[1]. We can equivalently characterize linearizable interpretations as (all and only) those interpretations that satisfy the following formula (hereafter, we will refer to such a formula as θ):

$$\forall \mathtt{x}(\mathtt{state}(\mathtt{x}) \rightarrow \exists \mathtt{y}(\mathtt{state}(\mathtt{y}) \wedge \mathtt{x} < \mathtt{y}))\wedge \tag{4.1}$$

$$\forall \mathtt{x}, \mathtt{y}((\mathtt{time}(\mathtt{x}) \wedge \mathtt{time}(\mathtt{y}) \wedge \mathtt{ancestor}(\mathtt{x}, \mathtt{y})) \rightarrow \mathtt{x} = \mathtt{y})\wedge \tag{4.2}$$

$$\forall \mathtt{x}, \mathtt{y}((\mathtt{state}(\mathtt{x}) \wedge \mathtt{state}(\mathtt{y}) \wedge \mathtt{ancestor}(\mathtt{x}, \mathtt{y})) \rightarrow \mathtt{x} = \mathtt{y})\wedge \tag{4.3}$$

$$\begin{gathered} \forall \mathtt{x}((\mathtt{state}(\mathtt{x}) \wedge \exists \mathtt{y}(\mathtt{time}(\mathtt{y}) \wedge \mathtt{ancestor}(\mathtt{y}, \mathtt{x}))) \rightarrow \\ \forall \mathtt{y}((\mathtt{state}(\mathtt{y}) \wedge \mathtt{y} < \mathtt{x}) \rightarrow \exists \mathtt{z}(\mathtt{time}(\mathtt{z}) \wedge \mathtt{ancestor}(\mathtt{z}, \mathtt{y})))), \end{gathered} \tag{4.4}$$

where $\mathtt{ancestor}(\mathtt{y}, \mathtt{x})$ is a shorthand for the formula:

$$\exists \mathtt{p}(\mathtt{p}(\mathtt{x}) \wedge \phi(\mathtt{y}, \mathtt{p}) \wedge \forall \mathtt{q}(\phi(\mathtt{y}, \mathtt{q}) \rightarrow \forall \mathtt{y}(\mathtt{p}(\mathtt{y}) \rightarrow \mathtt{q}(\mathtt{y})))),$$

including two occurrences of the formula $\phi(\mathtt{y}, \mathtt{p})$, with free individual variable $\mathtt{y}$ and free set variable $\mathtt{p}$, defined as follows:

$$\mathtt{p}(\mathtt{y}) \wedge \forall \mathtt{z}, \mathtt{w}((\mathtt{p}(\mathtt{z}) \wedge \bigvee_{\mathtt{j}=0}^{\mathtt{k}-1} \downarrow_{\mathtt{j}} (\mathtt{z}) = \mathtt{w}) \rightarrow \mathtt{p}(\mathtt{w})).$$

The formula $\phi(\mathtt{y}, \mathtt{p})$, together with the conjunct $\forall \mathtt{q}(\phi(\mathtt{y}, \mathtt{q}) \rightarrow \forall \mathtt{y}(\mathtt{p}(\mathtt{y}) \rightarrow \mathtt{q}(\mathtt{y})))$, guarantees minimality: $\mathtt{p}$ holds *exactly* at $\mathtt{y}$ and all of its descendants; from this, the third conjunct $\mathtt{p}(\mathtt{x})$ forces $\mathtt{x}$ to be a descendant of $\mathtt{y}$.

The idea underlying the notion of linearizable interpretation is that the elements of $state^{\mathcal{I}}$ and $time^{\mathcal{I}}$ represent states and times, respectively. Since the pair $\langle \bigcup_{i \geq 0} T_i, < \rangle$ is isomorphic to $\langle \mathbb{N}, < \rangle$, $state^{\mathcal{I}}$ and $time^{\mathcal{I}}$ are linearly ordered by the restriction of the ordering $\leq$ of $\mathcal{U}$ to $state^{\mathcal{I}}$ and $time^{\mathcal{I}}$, respectively. If time does not progress after a given state, $time^{\mathcal{I}}$ is finite; otherwise, $time^{\mathcal{I}}$ is infinite.

For each state $s \in state^{\mathcal{I}}$, if there exists an element t belonging to the set $time^{\mathcal{I}}$ which is an ancestor of s, we say that s holds at time t. Conditions (2) and (4) imply that if such an element exists, it is unique. If $time^{\mathcal{I}}$ is finite, with $|time^{\mathcal{I}}| = n$, then the successor of the n-th instant of $time^{\mathcal{I}}$ is associated with infinitely many states and a time ancestor is missing for those states. A linearizable interpretation $\mathcal{I}$ *induces* a pair of functions $\sigma_{\mathcal{I}} : \mathcal{S}_{\mathcal{I}} \rightarrow 2^P$ and $\rho_{\mathcal{I}} : \mathcal{S}_{\mathcal{I}} \rightarrow \mathcal{T}_{\mathcal{I}}$, with

$$\mathcal{S}_{\mathcal{I}} = state^{\mathcal{I}} \quad \text{and} \quad \mathcal{T}_{\mathcal{I}} = time^{\mathcal{I}} \cup \{t : t \in \textstyle\bigcup_{i \geq 0} T_i \wedge max(time^{\mathcal{I}}) < t\},$$

which is in fact a timed state sequence, where

- $\sigma_{\mathcal{I}}(s) = \{p : p \in P \text{ and } s \in p^{\mathcal{I}}\}$;
- $\rho_{\mathcal{I}}(s) = \begin{cases} t & \text{if } t \text{ is an ancestor of } s; \\ max(time^{\mathcal{I}}) + 1 & \text{if } s \text{ has no ancestors in } time^{\mathcal{I}}. \end{cases}$

The relationships between linearizable interpretations and timed state sequences are precisely stated by the following proposition.

[1] Condition (3) is not really necessary; we add it to make the proposed characterization more intuitive.

PROPOSITION 4.2
The following properties hold:

(a) if $\mathcal{I}$ is a linearizable interpretation, then $\rho_{\mathcal{I}}$ is weakly monotonic;

(b) for each timed state sequence $\langle \sigma, \rho \rangle$, there exists a linearizable interpretation $\mathcal{I}$ such that $\langle \sigma_{\mathcal{I}}, \rho_{\mathcal{I}} \rangle$ and $\langle \sigma, \rho \rangle$ are equal, up to isomorphisms of their respective domains.

The proof is rather straightforward, and thus omitted.

Let us show now how to translate an $\mathcal{L}^2_T$-formula ϕ into an equisatisfiable $MSO[<, \downarrow_0, \downarrow_1]$-formula $\tau(\phi)$ interpreted over an upward unbounded (2-refinable) layered structure. To this purpose, it is useful to introduce a successor relation $\mathtt{T\text{-}Succ(x,y)}$ over $\mathcal{T}_{\mathcal{I}}$, such that $\mathtt{T\text{-}Succ(x,y)}$ holds if and only if $\mathtt{x}$ and $\mathtt{y}$ (in fact their interpretation) belong to $\mathcal{T}_{\mathcal{I}}$ and the successor of $\mathtt{x}$, with respect to $\mathcal{T}_{\mathcal{I}}$, is $\mathtt{y}$, as a shorthand for the following $MSO[<, \downarrow_0, \downarrow_1]$-formula:

$$(\mathtt{time(x)} \wedge \exists \mathtt{z}(\mathtt{time(z)} \wedge \mathtt{x} < \mathtt{z}) \wedge \mathtt{time(y)} \wedge \forall \mathtt{z}((\mathtt{time(z)} \wedge \mathtt{x} < \mathtt{z}) \rightarrow \mathtt{y} \leq \mathtt{z})))$$
$$\vee(\forall \mathtt{z}(\mathtt{time(z)} \rightarrow \mathtt{z} \leq \mathtt{x}) \wedge \mathtt{x} < \mathtt{y} \wedge \forall \mathtt{z}(\mathtt{x} < \mathtt{z} \rightarrow \mathtt{y} \leq \mathtt{z}))$$

Notice that if $time^{\mathcal{I}}$ is infinite, then the T-successor of the i-th element of $time^{\mathcal{I}}$ is the $(i+1)$-th element of $time^{\mathcal{I}}$. Otherwise, if $|time^{\mathcal{I}}| = n$ and t is greater than or equal to the n-th element of $time^{\mathcal{I}}$, then the T-successor of t is $t+1$, that is, the successor of t in $\mathcal{U}$. It is also useful to define a binary relation the $\mathtt{At\text{-}Time(x,y)}$, such that $\mathtt{At\text{-}Time(x,y)}$ holds if and only if $\mathtt{x}$ is a state, $\mathtt{y}$ is a time, and $\mathtt{x}$ holds at $\mathtt{y}$, as a shorthand for the following $MSO[<, \downarrow_0, \downarrow_1]$-formula:

$$\mathtt{state(x)} \wedge ((\mathtt{time(y)} \wedge (\mathtt{ancestor(y,x)}) \vee (\neg \exists \mathtt{z}(\mathtt{time(z)}) \wedge \mathtt{y} = \mathtt{0}) \vee$$
$$(\neg \exists \mathtt{z}(\mathtt{ancestor(z,x)}) \wedge \exists \mathtt{z}(\mathtt{time(z)} \wedge \forall \mathtt{w}(\mathtt{time(w)} \rightarrow \mathtt{w} \leq \mathtt{z}) \wedge \mathtt{T} - \mathtt{Succ(z,y)))),$$

where $\mathtt{y} = \mathtt{0}$ is a shorthand for $\neg \exists \mathtt{z}(\mathtt{z} < \mathtt{y}))$.

The mapping $\tau : \mathcal{L}^2_T \rightarrow MSO[<, \downarrow_0, \downarrow_1]$ is inductively defined as follows:

- $\tau(\mathtt{p}(\mathtt{x_s}))$ is $\mathtt{state}(\mathtt{x_s}) \wedge \mathtt{p}(\mathtt{x_s})$;
- $\tau(\mathtt{x_s} \leq_{\mathtt{s}} \mathtt{y_s})$ is $\mathtt{state}(\mathtt{x_s}) \wedge \mathtt{state}(\mathtt{y_s}) \wedge (\mathtt{x_s} < \mathtt{y_s} \vee \mathtt{x_s} = \mathtt{y_s})$;
- $\tau(\mathtt{w_t} \leq_{\mathtt{t}} \mathtt{z_t})$ is $\mathtt{time}(\mathtt{w_t}) \wedge \mathtt{time}(\mathtt{z_t}) \wedge (\mathtt{w_t} < \mathtt{z_t} \vee \mathtt{w_t} = \mathtt{z_t})$;
- $\tau(\mathtt{w_t} =_{\mathtt{t}} \mathtt{z_t} + 1)$ is $\mathtt{time}(\mathtt{w_t}) \wedge \mathtt{time}(\mathtt{z_t}) \wedge \mathtt{T\text{-}Succ}(\mathtt{z_t}, \mathtt{w_t})$;
- $\tau(\mathtt{z_t} =_{\mathtt{t}} \mathtt{f}(\mathtt{x_s}))$ is $\mathtt{state}(\mathtt{x_s}) \wedge \mathtt{time}(\mathtt{z_t}) \wedge \mathtt{At\text{-}Time}(\mathtt{x_s}, \mathtt{z_t})$;
- τ commutes with the boolean connectives $\wedge, \vee$ and $\neg$;
- if ϕ is $\exists \mathtt{x_s} \psi$ (resp. $\forall \mathtt{x_s} \psi$), then $\tau(\phi)$ is $\exists \mathtt{x_s}(\mathtt{state}(\mathtt{x_s}) \wedge \tau(\psi))$ (resp. $\tau(\phi)$ is $\forall \mathtt{x_s}(\mathtt{state}(\mathtt{x_s}) \rightarrow \tau(\psi))$;
- if ϕ is $\exists \mathtt{x_t} \psi$ (resp. $\forall \mathtt{x_t} \psi$), then $\tau(\phi)$ is $\exists \mathtt{x_t}(\mathtt{time}(\mathtt{x_t}) \wedge \tau(\psi))$ (resp. $\tau(\phi)$ is $\forall \mathtt{x_t}(\mathtt{time}(\mathtt{x_t}) \rightarrow \tau(\psi))$;
- if ϕ is $\exists \mathtt{p} \psi$ (resp. $\forall \mathtt{p} \psi$), then $\tau(\phi)$ is $\exists \mathtt{p}(\forall \mathtt{y}(\mathtt{p(y)} \rightarrow \mathtt{state(y)}) \wedge \tau(\psi))$ (resp. $\tau(\phi)$ is $\forall \mathtt{p}(\forall \mathtt{y}(\mathtt{p(y)} \rightarrow \mathtt{state(y)}) \rightarrow \tau(\psi)))$.

The relationship between an $\mathcal{L}^2_T$-formula and its translation $\tau(\phi)$ is expressed by the following theorem, whose proof directly follows from the given definitions.

THEOREM 4.3
Given an $\mathcal{L}^2_T$-formula ϕ, an interpretation $\mathcal{I}$ satisfies the formula $\tau(\phi) \wedge \theta \in MSO[<, \downarrow_0, \downarrow_1]$ iff it is linearizable and the timed state sequence $\langle \sigma_{\mathcal{I}}, \rho_{\mathcal{I}} \rangle$ induced by $\mathcal{I}$ satisfies ϕ.

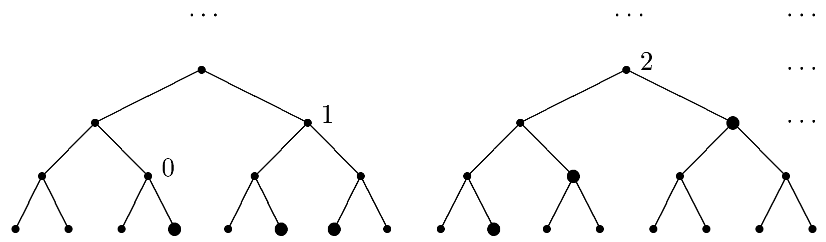

FIG. 4. A linearizable interpretation: big circles are states, labelled nodes are times.

In the theory of timed state sequences, where many states may hold at the same time, saying that a state holds at time i can be meant to be an abstraction for the fact that such a state can be arbitrarily placed in the (right-open) interval $[i, i+1)$. The stratification of domains gives means for reducing the uncertainty involved in this abstraction. The encoding of times and states we have provided allows us to express facts like "a state s holds at a time instant belonging to the first half of interval $[i, i+1)$". More formally, we could extend $\mathcal{L}_T^2$ with a family of predicate symbols $(\downarrow_{\frac{p}{2^q}})_{q\geq 0, 0\leq p\leq 2^q-1}$ such that $\downarrow_{\frac{p}{2^q}}(\mathtt{x})$ is true for a state $\mathtt{x}$ if $\mathtt{x}$ is placed in the time interval $[i+\frac{p}{2^q}, i+\frac{p+1}{2^q})$. If $a_1, \ldots, a_q$ are the coefficients of the binary representation of $0 \leq p \leq 2^q$, the formula $\downarrow_{\frac{p}{2^q}}(\mathtt{x})$ can be seen as a shorthand for the formula:

$$\exists \mathtt{x}_1, \ldots, \mathtt{x}_q(\mathtt{At-Time}(\mathtt{x}, \mathtt{x}_1) \wedge \bigwedge_{\mathtt{i}=1}^{\mathtt{q}} \downarrow_{\mathtt{a}_\mathtt{i}}(\mathtt{x}_\mathtt{i}) = \mathtt{x}_{\mathtt{i}+1} \wedge \mathtt{ancestor}(\mathtt{x}, \mathtt{x}_\mathtt{q}))$$

In summary, we have shown that the theory $MSO[<, \downarrow_0, \downarrow_1]$ over upward unbounded layered structures is expressive enough to capture the theory of timed state sequences. In fact, our embedding shows that the theory of timed state sequences can be (properly) extended with predicates $\downarrow_{\frac{p}{2^q}}(\mathtt{x})$, preserving decidability.

4.2 The embedding into downward unbounded layered structures

In this section, we show how the theory of timed state sequences can be embedded into the theory of downward unbounded layered structures. The idea is similar to that exploited in the upward case; the only difference is that the temporal domain is the coarsest layer T^0 instead of being described by a predicate like *time*.

Let *state* be a set variable symbol, P a finite set of predicate symbols and $\mathcal{DU} = \langle \bigcup_{i\geq 0} T_i, \downarrow, < \rangle$ a downward unbounded 2-refinable structure. Moreover, let $\mathcal{I}$ be an interpretation (over $\mathcal{DU}$) for *state* and predicate symbols in P such that

1. $state^{\mathcal{I}}$ is an infinite set;
2. there exists a least element in $state^{\mathcal{I}}$ (with respect to the $<$ relation of $\mathcal{DU}$);
3. for any $s, s' \in state^{\mathcal{I}}$, s is not an ancestor of s';
4. for any $t \in T^0$, if there exist $t' \in T^0$ and $s' \in state^{\mathcal{I}}$ such that $t < t'$ and t' is an ancestor of s', then there are only finitely many $s \in state^{\mathcal{I}}$ such that t is an ancestor of s.

Such an interpretation is called a *linearizable* interpretation. The idea is that the elements of $state^{\mathcal{I}}$ are states and that the time at which a state $s \in state^{\mathcal{I}}$ holds is i, whenever the upmost ancestor of s (i.e. the ancestor of s belonging to T^0) is the i-th element of T^0 (we recall that T^0 is isomorphic to $\mathbb{N}$). A linearizable interpretation $\mathcal{I}$ *induces* a pair of functions $\sigma_{\mathcal{I}} : \mathcal{S}_{\mathcal{I}} \to 2^P$ and $\rho_{\mathcal{I}} : \mathcal{S}_{\mathcal{I}} \to \mathcal{T}_{\mathcal{I}}$, with

$$\mathcal{S}_{\mathcal{I}} = state^{\mathcal{I}} \quad \text{and} \quad \mathcal{T}_{\mathcal{I}} = T^0,$$

which is in fact a timed state sequence, where

- $\sigma_{\mathcal{I}}(s) = \{p : p \in P, \text{ and } s \in p^{\mathcal{I}}\}$;
- $\rho_{\mathcal{I}}(s) = t$ if t is an ancestor of s.

Also in the case of downward unbounded structures, the analogue of Proposition 4.2 holds. A formula ϕ of $\mathcal{L}^2_T$ can be translated into an equisatisfiable formula of $MSO[<, \downarrow_0, \downarrow_1]$ by exploiting the map $\tau : \mathcal{L}^2_T \to MSO[<, \downarrow_0, \downarrow_1]$ which has been defined for the upward unbound case, provided that $\mathtt{time}(\mathtt{x})$, $\mathtt{At\text{-}Time}(\mathtt{x},\mathtt{y})$ and $\mathtt{T\text{-}Succ}(\mathtt{x},\mathtt{y})$ are modified as follows:

$\mathtt{time}(\mathtt{x})$ (i.e. $\mathtt{x} \in \mathtt{T}^0$) is a shorthand for

$$\neg\exists \mathtt{y}(\bigvee_{\mathtt{i}=0}^{1} \downarrow_{\mathtt{i}} (\mathtt{y}) = \mathtt{x});$$

$\mathtt{At\text{-}Time}(\mathtt{x},\mathtt{y})$ (i.e. the uppermost ancestor of $\mathtt{x}$ is $\mathtt{y}$) is a shorthand for

$$\mathtt{state}(\mathtt{x}) \wedge \mathtt{time}(\mathtt{y}) \wedge \mathtt{ancestor}(\mathtt{y},\mathtt{x});$$

$\mathtt{T\text{-}Succ}(\mathtt{x},\mathtt{y})$ (i.e. the successor of $\mathtt{x}$ is $\mathtt{y}$) is a shorthand for

$$\mathtt{time}(\mathtt{x}) \wedge \mathtt{time}(\mathtt{y}) \wedge \mathtt{x} < \mathtt{y} \wedge \forall \mathtt{z}((\mathtt{time}(\mathtt{z}) \wedge \mathtt{z} > \mathtt{x}) \to \mathtt{z} \geq \mathtt{y}).$$

THEOREM 4.4
Given a $\mathcal{L}^2_T$-formula ϕ, a linearizable interpretation $\mathcal{I}$ satisfies $\tau(\phi) \in MSO[<, \downarrow_0, \downarrow_1]$ iff the timed state sequence $\langle \sigma_{\mathcal{I}}, \rho_{\mathcal{I}} \rangle$ induced by $\mathcal{I}$ satisfies ϕ.

As in the upward unbounded case, this allows us to conclude that the theory $MSO[<, \downarrow_0, \downarrow_1]$ over downward unbounded layered structures is expressive enough to capture the theory of timed state sequences.

5 Decidability of ω-layered structures

In order to guarantee the usefulness of the proposed theories of ω-layered metric temporal structures as formal tools, it is necessary to show some basic decidability properties. The decidability problem for the pure metric (non-granular) fragment has been addressed by Alur and Henzinger in [1]. They showed that, under suitable assumptions about the temporal domain and the associated operations, the satisfiability problem for the theories of flat metric temporal structures is decidable. This problem can be reduced to the decidability problem for the decidable theory $MSO[<]$. A first extension of their results, aiming at dealing with time granularity, has been presented

in [24]. Such an extension allows one to treat situations in which a finite number of coarsenings/refinements of the temporal domain is sufficient, and the relationships between time points belonging to different layers can be modeled by means of a simple binary relation of projection. The key idea to deal with the resulting finitely-layered metric temporal structures is to reformulate the decidability problem into an equivalent one relative to the finest layer. As in the original work by Alur and Henzinger, the basic tool for proving decidability is the theory $MSO[<]$, and the basic engine is Büchi theorem on the decidability of regular ω-languages.
In this paper, we deal with the more general case of ω-layered (k-refinable) metric temporal logics whose temporal structure consists of infinitely many temporal layers. We first prove that the theory of upward unbounded layered structures is decidable. To deal with such structures we use the theory $MSO[<, pow_k, hp_k]$. The decidability of $MSO[<, pow_k, hp_k]$ is shown by using a more powerful basic engine, namely the decidability of ω-languages recognized by k-ary Systolic Tree Automata. Such class of ω-languages has been recently proved to properly extend the class of regular ω-languages maintaining the same decidability properties [25, 26]. Successively, we deal with the problem of deciding downward unbounded layered structures. We prove that the decidability of the satisfiability problem for the theory of such structures can be reduced to the decidability of the satisfiability problem for $MSO[<_k, succ_0, \ldots, succ_{k-1}]$, the well-known monadic second-order decidable theory of k successors [30]. In this case the basic decidability engine is Rabin's Tree Theorem.

5.1 Decidability of upward unbounded layered structures

We prove that the theory of upward unbounded layered structures is decidable, by defining a translation function τ that maps $MSO[<, \downarrow_0, \ldots, \downarrow_{k-1}]$-formulas into equisatisfiable $MSO[<, pow_k, hp_k]$-formulas. First of all, observe that any two upward unbounded layered structures are isomorphic. In fact, given $\mathcal{T}' = \langle \bigcup_{i\geq 0} T'^i, \downarrow', \leq' \rangle$ and $\mathcal{T}'' = \langle \bigcup_{i\geq 0} T''^i, \downarrow'', \leq'' \rangle$, the mapping $f : \bigcup_{i\geq 0} T'^i \rightarrow \bigcup_{i\geq 0} T''^i$ that associates the j-th element of T'^i with the j-th element of T''^i is a bijection which preserves projection and ordering. Therefore, it follows that a formula ϕ is satisfiable under an interpretation $\mathtt{p}_1^{\mathcal{I}}, \ldots \mathtt{p}_\mathtt{m}^{\mathcal{I}} \subseteq \bigcup_{i\geq 0} T'^i$ (where $\mathtt{p}_1, \ldots, \mathtt{p}_\mathtt{m}$ are the free predicates symbols occurring in ϕ) if and only if ϕ is satisfiable under the interpretation $f(\mathtt{p}_1^{\mathcal{I}}), \ldots, f(\mathtt{p}_\mathtt{m}^{\mathcal{I}}) \subseteq \bigcup_{i\geq 0} T''^i$. This property allows us to replace (parametrically on k) the class of upward unbounded layered structures by a suitable single *concrete* structure which can be easily encoded into $\mathbb{N}$.

The *concrete* upward unbounded layered structure is

$$\mathcal{C} = \langle \bigcup_{i\geq 0} T^{i\mathcal{C}}, \downarrow^{\mathcal{C}}, <^{\mathcal{C}} \rangle, \text{ where}$$

- $T^{i\mathcal{C}} = \{k^i + nk^{i+1} : n \geq 0\}$;
- $\downarrow^{\mathcal{C}}$: $\{0, \ldots, k-1\} \times \bigcup_{i\geq 1} T^{i\mathcal{C}} \rightarrow \bigcup_{i\geq 0} T^{i\mathcal{C}}$ maps the pair $(j, k^i + nk^{i+1})$ into $k^{i-1} + (j + nk)k^i$, for $0 \leq j \leq k-1$ and $i \geq 0$;
- $<^{\mathcal{C}}$ is the restriction to $\bigcup_{i\geq 0} T^{i\mathcal{C}}$ of the usual ordering on natural numbers.

Notice that the elements of the concrete structure are all (and only) the natural

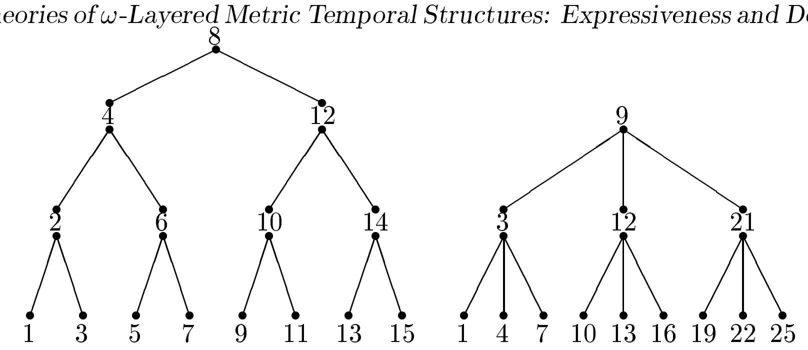

FIG. 5. The trees associated with numbers 8 (for $k = 2$) and 9 (for $k = 3$).

numbers having 1 as the least non-null coefficient in their k-ary representation. Notice also that, given a natural number

$$y = a_n k^n + a_{n-1} k^{n-1} + \ldots + a_m k^m, \text{ with } m > 0 \text{ and } a_m = 1,$$

the j-th child x of y (with respect to the projection $\downarrow^{\mathcal{C}}$), with $0 \leq j \leq k-1$, is

$$x = a_n k^n + a_{n-1} k^{n-1} + \ldots + j k^m + 1 k^{m-1}.$$

The trees associated with numbers 8 (for $k = 2$) and 9 (for $k = 3$) are shown in Figure 5. The following proposition proves that the *concrete* upward unbounded layered structure is well-defined.

PROPOSITION 5.1
$\mathcal{C}$ is an upward unbounded layered structure.

PROOF. For $i, j \geq 0$, with $i \neq j$, $T^{i\mathcal{C}}$ and $T^{j\mathcal{C}}$ are disjoint. In fact, if $t \in T^{i\mathcal{C}}$ and the k-ary representation of t is $a_n k^n + \ldots + a_0 k^0$, then $a_i = 1$ and $a_l = 0$ for $0 \leq l < i$. In order to prove that the concrete structure satisfies the required ordering properties, we list some useful properties of the elements of the concrete structure (for notational simplicity, we use the relation $<$ instead of its restriction $<^{\mathcal{C}}$):

a. if $t = k^i + nk^{i+1}$ and t is an ancestor of t', then $t + \alpha(t) \leq t' \leq t + \beta(t)$, where

$$\alpha(t) = 1 - k^i \text{ and } \beta(t) = k^{i+1} - k^i - k + 1;$$

b. if $t, t' \in T^{i\mathcal{C}}$ and $t < t'$, then $t + \beta(t) < t' + \alpha(t')$;
c. if $\downarrow^{\mathcal{C}} (0, t) = t'$, then $t' + \beta(t') < t$;
d. if $\downarrow^{\mathcal{C}} (1, t) = t'$, then $t < t' + \alpha(t')$.

With reference to the properties of $\tilde{<}$, given in Definition 3.1, we have:

1. $\downarrow^{\mathcal{C}} (0, t) < t$ by property c and, for $j > 0$, $t <\downarrow^{\mathcal{C}} (j, t)$ by properties a, b, and d.
2. for $j > 0$, $\downarrow^{\mathcal{C}} (j, t) <\downarrow^{\mathcal{C}} (j + 1, t)$ by properties a and b.
3. Assume that $t < t'$ and t' is not an ancestor (with respect to $\downarrow^{\mathcal{C}}$) of t. We first consider the case where t is an ancestor of t'. Let $t_0 \ldots t_{k-1}$ be the children of t.

Since $t > t_0 + \beta(t_0)$ (by property c) and, for $i > 0$, $t < t_i + \alpha(t_i)$ (by properties d and point 2), it follows that there exists t_j, with $1 \leq j \leq k-1$, which is an ancestor of (or is equal to) t'; this allows us to conclude that $t < t_j + \alpha(t_j) \leq\downarrow^{\mathcal{C}} (0, t')$.
Consider now the case where t' is not an ancestor of t. In this case, let s be the lowest common ancestor of t and t' (which differs from both t and t') and let $s_0 \ldots s_{k-1}$ be the children of s. Since, for $0 \leq i \leq k-2$, $s_i + \beta(s_i) < s_{i+1} + \alpha(s_{i+1})$ (by property b and point 2), from $t < t'$ it follows that there exist i and j, with $i < j$, such that s_i is an ancestor of t and s_j is an ancestor of t'. This allows us to conclude that $t \leq s_i + \beta(s_i) < s_j + \alpha(s_j) \leq\downarrow^{\mathcal{C}} (0, t')$.

4. The fourth property 4 of $\tilde{<}$ (cf. Definition 3.1) can be proved analogously.

■

In order to define a mapping from $MSO[<, \downarrow_0, \ldots, \downarrow_{k-1}]$-formulas into $MSO[<, pow_k, hp_k]$-formulas, we preliminary introduce some $MSO[<, pow_k, hp_k]$-predicates, which will play a major role in the proof of Lemma 5.2.
We first define $k-1$ predicates $\mathtt{L_j}$, with $1 \leq j \leq k-1$, such that $\mathtt{L_j(x)}$ holds if and only if j is the least non-null coefficient in the k-ary representation of $\mathtt{x}$ (notice that $\mathtt{L_{k-1}} = \mathtt{hp_k}$). For $j = 1, \ldots, k-1$, $\mathtt{L_j(x)}$ can be defined as follows:

$$\exists \mathtt{x_{k-1}}, \ldots, \mathtt{x_j}(\mathtt{hp_k}(\mathtt{x_{k-1}}) \wedge \mathtt{x} = \mathtt{x_j} \wedge \bigwedge_{\mathtt{i=j+1}}^{\mathtt{k-1}} \mathtt{x_{i-1}} = \mathtt{pow_k}(\mathtt{x_i})). \tag{5.1}$$

Furthermore, for any pair of natural numbers $\mathtt{x}$ and $\mathtt{y}$ such that both $\mathtt{L_1(x)}$ and $\mathtt{L_1(y)}$ hold, $\mathtt{x}$ is the 0-th (i.e. the leftmost) child of $\mathtt{y}$ (notationally, $\mathtt{y} \xrightarrow{0} \mathtt{x}$) if and only if $x = max\{w : w < y \wedge pow_k(w) = pow_k(y)\}$. Hence, we define $\mathtt{y} \xrightarrow{0} \mathtt{x}$ as follows:

$$\begin{gathered} \mathtt{L_1(y)} \wedge \mathtt{L_1(x)} \wedge \mathtt{x} < \mathtt{y} \wedge \mathtt{pow_k(y)} = \mathtt{pow_k(x)} \wedge \\ \forall \mathtt{w}((\mathtt{L_1(w)} \wedge \mathtt{w} < \mathtt{y} \wedge \mathtt{pow_k(y)} = \mathtt{pow_k(w)}) \rightarrow \mathtt{w} \leq \mathtt{x}). \end{gathered} \tag{5.2}$$

Analogously, for $\mathtt{x}$ and $\mathtt{y}$ such that both $\mathtt{L_1(x)}$ and $\mathtt{L_1(y)}$ hold, x is the j-th child of y (notationally, $\mathtt{y} \xrightarrow{\mathtt{j}} \mathtt{x}$), with $1 \leq j \leq k-1$, if and only if $\mathtt{L_j(pow_k(x))} \wedge \mathtt{y} = (\mathtt{pow_k})^{\mathtt{j}}(\mathtt{x})$, where $(\mathtt{pow_k})^{\mathtt{j}}(\mathtt{x})$ is a shorthand for $\mathtt{pow_k}$ repeatedly applied to $\mathtt{x}$ j times. Therefore, $\mathtt{y} \xrightarrow{\mathtt{j}} \mathtt{x}$, with $0 < j \leq k-1$) can be defined as follows

$$\begin{gathered} \mathtt{L_1(y)} \wedge \mathtt{L_1(x)} \wedge (\exists \mathtt{x_j}, \ldots, \mathtt{x_1}(\mathtt{x_j} = \mathtt{pow_k(x)} \wedge \\ \mathtt{L_j(x_j)} \wedge \mathtt{x_1} = \mathtt{y} \wedge \textstyle\bigwedge_{\mathtt{i=2}}^{\mathtt{j}} \mathtt{x_{i-1}} = \mathtt{pow_k(x_i)})). \end{gathered} \tag{5.3}$$

The (inductively) defined map $\tau : MSO[<, \downarrow_0, \ldots, \downarrow_{k-1}] \rightarrow MSO[<, pow_k, hp_k]$ is as follows:

- if ϕ is either $\mathtt{x} < \mathtt{y}$ or $\mathtt{x} = \mathtt{y}$, then $\tau(\phi)$ is $\phi \wedge \mathtt{L_1(x)} \wedge \mathtt{L_1(y)}$;
- if ϕ is $\downarrow_{\mathtt{i}} (\mathtt{x}) = \mathtt{y}$, then $\tau(\phi)$ is $\mathtt{x} \xrightarrow{\mathtt{i}} \mathtt{y}$;
- if ϕ is $\mathtt{p(x)}$, then $\tau(\phi)$ is $\mathtt{p(x)} \wedge \mathtt{L_1(x)}$;
- if ϕ is $\neg\psi$, then $\tau(\phi)$ is $\neg\tau(\psi)$;
- if ϕ is $\psi \wedge \theta$ (resp. ϕ is $\psi \vee \theta$), then
  $\tau(\phi)$ is $\tau(\psi) \wedge \tau(\theta)$ (resp. $\tau(\phi)$ is $\tau(\psi) \vee \tau(\theta)$);

- if ϕ is $\exists \mathtt{x}\psi$ (resp. $\forall \mathtt{x}\psi$), then
  $\tau(\phi)$ is $\exists \mathtt{x}(\mathtt{L_1}(\mathtt{x}) \wedge \tau(\psi))$ (resp. $\tau(\phi)$ is $\forall \mathtt{x}(\mathtt{L_1}(\mathtt{x}) \rightarrow \tau(\psi))$;
- if ϕ is $\exists \mathtt{p}\psi$ (resp. $\forall \mathtt{p}\psi$), then $\tau(\phi)$ is $\exists \mathtt{p}(\forall \mathtt{y}(\mathtt{p}(\mathtt{y}) \rightarrow \mathtt{L_1}(\mathtt{y})) \wedge \tau(\psi))$
  (resp. $\tau(\phi)$ is $\forall \mathtt{p}(\forall \mathtt{y}(\mathtt{p}(\mathtt{y}) \rightarrow \mathtt{L_1}(\mathtt{y})) \rightarrow \tau(\psi))$).

LEMMA 5.2
For any $\phi \in MSO[<, \downarrow_0, \ldots, \downarrow_{k-1}]$, with $\mathtt{x_1}, \ldots, \mathtt{x_m}$ free individual variables and $\mathtt{p_1}, \ldots, \mathtt{p_n}$ free set variables of ϕ, ϕ is satisfiable iff $\tau(\phi) \in MSO[<, pow_k, hp_k]$ is satisfiable.

PROOF. For the sake of simplicity, we assume that the sets of individual and set variables of $MSO[<, \downarrow_0, \ldots, \downarrow_{k-1}]$ and $MSO[<, pow_k, hp_k]$ coincide.
($\Rightarrow$) Let ϕ be a formula, η be the valuation of individual variables, and ν be the valuation for set variables, and suppose that $(\phi)^{\eta,\nu} = true$. We prove that $(\tau(\phi))^{\eta,\nu} = true$. The proof is by induction on the structure of ϕ.

(Base case.) Let ϕ be either $\mathtt{x} < \mathtt{y}$ or $\mathtt{x} = \mathtt{y}$ (resp. $\mathtt{p}(\mathtt{x})$). By definition of τ, $\tau(\phi)$ is $\phi \wedge \mathtt{L_1}(\mathtt{x}) \wedge \mathtt{L_1}(\mathtt{y})$ (resp. $\phi \wedge \mathtt{L_1}(\mathtt{x})$). Since $\bigcup_{i \geq 0} T^{i\mathcal{C}} \subseteq \mathbb{N}$ and the relation $<^{\mathcal{C}}$ is a subset of the ordering $<$ of $\mathbb{N}$, it immediately follows that $(\tau(\phi))^{\eta,\nu} = true$.
If ϕ is $\downarrow_{\mathtt{i}} (\mathtt{x}) = \mathtt{y}$, then, from equations 5.2 and 5.3, it follows that $(\mathtt{x} \xrightarrow{\mathtt{i}} \mathtt{y})^{\eta,\nu} = true$.

(Inductive step.) The proof for the boolean connectives $\phi = \neg\psi$, $\phi = \psi \wedge \theta$, and $\phi = \psi \vee \theta$ is straightforward, and thus omitted. As for quantifiers, we only consider the case of existential quantification of individual variables expressed by the formula $\phi = \exists \mathtt{x}\psi$. The other quantifications are indeed dealt with in a similar way. Suppose that $(\phi)^{\eta,\nu} = true$. Since $\phi = \exists \mathtt{x}\psi$, this means that there exists $\overline{\eta}$ that differs from η at most in the value it assigns to $\mathtt{x}$ such that $(\psi)^{\overline{\eta},\nu} = true$. By the induction hypothesis, $(\tau(\psi))^{\overline{\eta},\nu} = true$. Finally, since $\overline{\eta}(\mathtt{x})$ belongs to the concrete domain, it follows that $(\mathtt{L_1}(\mathtt{x}))^{\overline{\eta},\nu} = true$, and therefore $(\tau(\phi))^{\overline{\eta},\nu} = true$.

($\Leftarrow$) Let ϕ be a formula, η be the valuation of individual variables, and ν be the valuation for set variables, and suppose that $\tau(\phi)^{\eta,\nu} = true$. We prove, by induction on the structure of ϕ, that there exist two valuations $\overline{\eta}$ and $\overline{\nu}$ such that, for each individual variable $\mathtt{x}$, $\overline{\eta}(\mathtt{x}) \in \bigcup_{\mathtt{i} \geq 0} \mathtt{T}^{\mathtt{i}\mathcal{C}}$, and each set variable $\mathtt{p}$, $\overline{\nu}(\mathtt{p}) \subseteq \bigcup_{\mathtt{i} \geq 0} \mathtt{T}^{\mathtt{i}\mathcal{C}}$, and $(\tau(\phi))^{\overline{\eta},\overline{\nu}} = true$ (take $\overline{\eta}(\mathtt{x}) = \eta(\mathtt{x})$, if $\eta(\mathtt{x}) \in \bigcup_{\mathtt{i} \geq 0} \mathtt{T}^{\mathtt{i}\mathcal{C}}$, and $\overline{\eta}(\mathtt{x})$ arbitrarily defined over $\bigcup_{i \geq 0} T^{i\mathcal{C}}$, if $\eta(\mathtt{x}) \notin \bigcup_{\mathtt{i} \geq 0} \mathtt{T}^{\mathtt{i}\mathcal{C}}$; take $\overline{\nu}(\mathtt{p}) = \nu(\mathtt{p}) \cap \bigcup_{\mathtt{i} \geq 0} \mathtt{T}^{\mathtt{i}\mathcal{C}}$).

(Base case.) Let ϕ be either $\mathtt{x} < \mathtt{x}'$ or $\mathtt{x} = \mathtt{x}'$ (resp. $\mathtt{p}(\mathtt{x})$). By definition of τ, $\tau(\phi)$ is equal to $\phi \wedge \mathtt{L_1}(\mathtt{x}) \wedge \mathtt{L_1}(\mathtt{y})$ (resp. $\phi \wedge \mathtt{L_1}(\mathtt{x})$). From $(\tau(\phi))^{\eta,\nu} = true$, it follows that both $(\mathtt{L_1}(\mathtt{x}))^{\eta,\nu} = true$ and $(\mathtt{L_1}(\mathtt{y}))^{\eta,\nu} = true$ (resp. $(\mathtt{L_1}(\mathtt{x}))^{\eta,\nu} = true$). This means that η always assigns values of the concrete domain to both $\mathtt{x}$ and $\mathtt{y}$ (resp. $\mathtt{x}$); hence, $(\tau(\phi))^{\overline{\eta},\overline{\nu}} = true$ and that implies $(\phi)^{\overline{\eta},\overline{\nu}} = true$.
If ϕ has the form $\downarrow_{\mathtt{i}} (\mathtt{x}) = \mathtt{y}$ and $(\mathtt{x} \xrightarrow{\mathtt{i}} \mathtt{y})^{\eta,\nu} = true$, it follows that $\eta(\mathtt{x}), \eta(\mathtt{y}) \in \bigcup_{i \geq 0} T^{i\mathcal{C}}$ and thus $(\tau(\phi))^{\overline{\eta},\overline{\nu}} = true$. From equations 5.2 and 5.3 we have $(\phi)^{\overline{\eta},\overline{\nu}} = true$.

(Inductive step). The proof for the boolean connectives is straightforward, and thus omitted. As for quantifiers, we only consider the case of existential quantification of individual variables expressed by the formula $\phi = \exists \mathtt{x}\psi$. Let $(\tau(\phi))^{\eta,\nu} = true$. Since $\phi = \exists \mathtt{x}\psi$, this means that there exists η', which differs from η at most in the value it assigns to $\mathtt{x}$, such that both $(\mathtt{L_1(x)})^{\eta',\nu} = true$ and $(\psi)^{\eta',\nu} = true$. From $(\mathtt{L_1(x)})^{\eta',\nu} = true$, it follows that $\eta'(\mathtt{x})$ belongs to $\bigcup_{i\geq 0} T^{i\mathcal{C}}$. Moreover, by the induction hypothesis, it follows that $(\psi)^{\overline{\eta'},\overline{\nu}} = true$. Hence, we can conclude that $(\exists \mathtt{x}\psi)^{\overline{\eta},\overline{\nu}} = true$. ∎

THEOREM 5.3
The theory of upward unbounded layered structures is decidable.

PROOF. It follows from Lemma 5.2 and the decidability of $MSO[<, pow_k, hp_k]$ (cf. [22]). ∎

5.2 Decidability of downward unbounded layered structures

We prove that the theory of downward unbounded layered structures is decidable, by showing that any formula $\phi \in MSO[<, \downarrow_0, \ldots, \downarrow_{k-1}]$ can be transformed into an equisatisfiable formula of $MSO[<_k, succ_0, \ldots, succ_{k-1}]$. In particular, for each $0 \leq j \leq k-1$, the function $\downarrow_j$ acts as $succ_j$, and the ordering $<$ of $MSO[<, \downarrow_0, \ldots, \downarrow_{k-1}]$ can be expressed by using the prefix order $<_k$. The *concrete* downward unbounded layered structure is $\mathcal{R} = \langle \bigcup_{i\geq 0} T^{i\mathcal{R}}, \downarrow^{\mathcal{R}}, <^{\mathcal{R}}\rangle$, where

- $T^{i\mathcal{R}} = \{v \cdot 0 \cdot w : v \in \{k-1\}^\star, w \in \{0, \ldots, k-1\}^\star, |w| = i\}$;
- $\downarrow^{\mathcal{R}} (j, v) = v \cdot j (= succ_j(v))$, for all $v \in \bigcup_{i\geq 0} T^{i\mathcal{R}}$;
- $<^{\mathcal{R}}$ is the restriction to $\bigcup_{i\geq 0} T^{i\mathcal{R}}$ of the lexicographical ordering over $\{0, \ldots, k-1\}^\star$.

If we look at a downward unbounded layered structure as an infinite sequence of leafless perfectly-balanced trees, v identifies a tree in the sequence, while w specifies a path from the root of such a tree to a specific node.

In Figure 6, we show how a downward unbounded layered structure can be encoded into the domain of interpretation of $MSO[<_k, succ_0, \ldots, succ_{k-1}]$ (i.e., $\{0, \ldots, k-1\}^\star$).

PROPOSITION 5.4
Let $\mathcal{S} = \langle \bigcup_{i\geq 0} T^i, \downarrow, <\rangle$ be a downward unbounded layered structure. For each $i \geq 0$ and $t \in T^i$, let j be such that $t = (+_i 1)^j(0_i)$. The map $cod : \bigcup_{i\geq 0} T^i \mapsto \bigcup_{i\geq 0} T^{i\mathcal{R}}$ such that

$$cod(t) = \begin{cases} (k-1)^j \cdot 0 & \text{if } i = 0; \\ (k-1)^q \cdot 0 \cdot r_1 \cdot \ldots \cdot r_i & \text{otherwise,} \end{cases}$$

where $j = k^i q + r$, with $0 \leq r < k^i$, and $r_1, \ldots, r_i$ are the coefficients of the k-ary representation of r, is an isomorphism of downward unbounded layered structures.

The proof is straightforword, and thus omitted.

The above result allows us to replace (parametrically on k) the class of downward unbounded layered structure by the concrete structure $\mathcal{R}$.

In order to map a formula $\phi \in MSO[<, \downarrow_0, \ldots, \downarrow_{k-1}]$ into an equisatisfiable formula of $MSO[<_k, succ_0, \ldots, succ_{k-1}]$, we define two auxiliary predicates $\mathtt{R}$ and $\mathtt{Dom}$ of $MSO[<_k, succ_0, \ldots, succ_{k-1}]$. These predicates are defined in such a way that $\mathtt{R(x)}$ holds if and only if $\mathtt{x}^{\mathcal{I}} \in \{k-1\}^\star$ (big circles in Figure 6), and $\mathtt{Dom(x)}$ holds if and only if $\mathtt{x}^{\mathcal{I}} \in \bigcup_{i\geq 0} T^{i\mathcal{R}}$ (which is a proper subset of $\{0, \ldots, k-1\}^\star \setminus \{k-1\}^\star$).

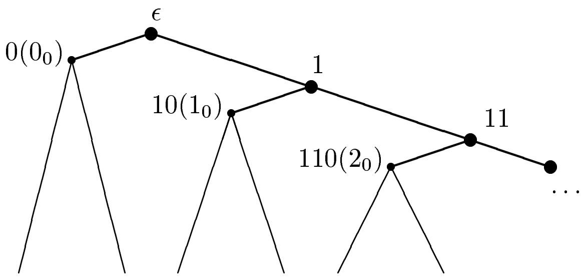

FIG. 6. The encoding of a downward unbounded (2-refinable) structure into $\{0,1\}^\star$.

$\mathtt{R(x)}$ is a shorthand for the formula:

$$\mu\,(\mathtt{p}(\epsilon) \wedge \forall \mathtt{y},\mathtt{z}((\mathtt{p(y)} \wedge \mathtt{succ_{k-1}(y) = z}) \rightarrow \mathtt{p(z)}))\,(\mathtt{x}),$$

where ϵ is the empty word, while $\mathtt{Dom(x)}$ is a shorthand for the formula:

$$\mu((\forall \mathtt{y},\mathtt{z}((\mathtt{y} = \mathtt{succ_0(z)} \wedge \mathtt{R(z)}) \rightarrow \mathtt{p(y)})\wedge$$

$$\forall \mathtt{y},\mathtt{z}((\mathtt{p(y)} \wedge \bigvee_{\mathtt{j=1}}^{\mathtt{k-1}} \mathtt{succ_j(y) = z}) \rightarrow \mathtt{p(z)})))(\mathtt{x}).$$

The mapping that transforms a formula $\phi \in MSO[<,\downarrow_0,\ldots,\downarrow_{k-1}]$ into a formula of $MSO[<_k, succ_0, \ldots, succ_{k-1}]$ is defined as follows:

if ϕ is $\mathtt{x = y}$, then $\tau(\phi)$ is $\mathtt{Dom(x)} \wedge \mathtt{Dom(y)} \wedge \mathtt{x = y}$;
if ϕ is $\mathtt{x < y}$, then $\tau(\phi)$ is
$\mathtt{Dom(x)} \wedge \mathtt{Dom(y)} \wedge (\mathtt{x} <_\mathtt{k} \mathtt{y} \vee$
$\exists \mathtt{z}(\bigvee_{\mathtt{l=0}}^{\mathtt{k-1}}(\mathtt{succ_l(z)} \leq_\mathtt{k} \mathtt{x} \wedge \bigvee_{\mathtt{s=l+1}}^{\mathtt{k-1}} \mathtt{succ_s(z)} \leq_\mathtt{k} \mathtt{y})))$;
if ϕ is $\downarrow_\mathtt{i} (\mathtt{x}) = \mathtt{y}$, then $\tau(\phi)$ is $\mathtt{Dom(x)} \wedge \mathtt{Dom(y)} \wedge \mathtt{succ_i(x) = y}$;
if ϕ is $\mathtt{p(x)}$, then $\tau(\phi)$ is $\mathtt{Dom(x)} \wedge \mathtt{p(x)}$;
if ϕ is $\neg\psi$, then $\tau(\phi)$ is $\neg\tau(\psi)$;
if ϕ is $\psi \wedge \theta$ (resp. ϕ is $\psi \vee \theta$), then
$\tau(\phi)$ is $\tau(\psi) \wedge \tau(\theta)$ (resp. $\tau(\phi)$ is $\tau(\psi) \vee \tau(\theta)$);
if ϕ is $\exists \mathtt{x}\psi$ (resp. $\forall \mathtt{x}\psi$), then
$\tau(\phi)$ is $\exists \mathtt{x}(\mathtt{Dom(x)} \wedge \tau(\psi))$ (resp. $\tau(\phi)$ is $\forall \mathtt{x}(\mathtt{Dom(x)} \rightarrow \tau(\psi))$;
if ϕ is $\exists \mathtt{p}\psi$ (resp. $\forall \mathtt{p}\psi$), then $\tau(\phi)$ is $\exists \mathtt{p}(\forall \mathtt{y}(\mathtt{p(y)} \rightarrow \mathtt{Dom(y)}) \wedge \tau(\psi))$
(resp. $\tau(\phi)$ is $\forall \mathtt{p}(\forall \mathtt{y}(\mathtt{p(y)} \rightarrow \mathtt{Dom(y)}) \rightarrow \tau(\psi)))$.

Notice that in the case where ϕ is $\mathtt{x < y}$, $\tau(\phi)$ codifies the lexicographical ordering $<$ in terms of the prefix ordering $<_k$.

LEMMA 5.5
For any $\phi \in MSO[<,\downarrow_0,\ldots,\downarrow_{k-1}]$ with $\mathtt{x_1},\ldots,\mathtt{x_m}$ free individual variables and $\mathtt{p_1},\ldots,$ p_n free set variables of ϕ, ϕ is satisfiable iff $\tau(\phi) \in MSO[<_k,\ succ_0,\ldots,succ_{k-1}]$ is satisfiable.

PROOF. The proof is by induction on the structure of ϕ. For the sake of simplicity, we assume that the sets of individual and set variables of $MSO[<, \downarrow_0, \ldots, \downarrow_{k-1}]$ and $MSO[<_k, succ_0, \ldots, succ_{k-1}]$ coincide.

($\Rightarrow$) Let η be a valuation of individual variables, and ν be a valuation of set variables, and suppose that $(\phi)^{\eta,\nu} = true$. We prove that $(\tau(\phi))^{\eta,\nu} = true$.

(Base case). The only non-trivial case is $\phi = \mathtt{x} < \mathtt{y}$. If $(\mathtt{x})^{\eta,\nu} < (\mathtt{y})^{\eta,\nu}$, then there are three possible cases:

(1) $(\mathtt{x})^{\eta,\nu}$ is an ancestor of $(\mathtt{y})^{\eta,\nu}$. This implies that there are $v_1, \ldots, v_n \in \bigcup_{i \geq 0} T^{i\mathcal{R}}$ such that $v_1 = (\mathtt{x})^{\eta,\nu}$, $v_n = (\mathtt{y})^{\eta,\nu}$, and $\downarrow(j_i, v_i) = v_{i+1}$, for each $1 \leq i \leq n-1$. Since $\bigcup_{i \geq 0} T^{i\mathcal{R}} \subset \{0, \ldots, k-1\}^\star$ and $(\mathtt{y})^{\eta,\nu} = (\mathtt{x})^{\eta,\nu} \cdot \mathtt{j_1} \cdot \ldots \cdot \mathtt{j_{n-1}}$, from the definitions of $\mathtt{Dom}$, it immediately follows that both $(\mathtt{Dom(x)})^{\eta,\nu}$ and $(\mathtt{Dom(y)})^{\eta,\nu}$ hold. Since $(\mathtt{x})^{\eta,\nu}$ is a prefix of $(\mathtt{y})^{\eta,\nu}$, $(\mathtt{x})^{\eta,\nu} <_\mathtt{k} (\mathtt{y})^{\eta,\nu}$ holds, and thus $(\tau(\phi))^{\eta,\nu} = true$.

(2) There exists $v \in \bigcup_{i \geq 0} T^{i\mathcal{R}}$ such that v is an ancestor of both $(\mathtt{x})^{\eta,\nu}$ and $(\mathtt{y})^{\eta,\nu}$. If we take the valuation η', which differs from η at most in the value v it assigns to $\mathtt{z}$, we have that $(\bigvee_{\mathtt{l=0}}^{\mathtt{k-1}} \mathtt{succ_l(z)} \leq_\mathtt{k} \mathtt{x} \wedge \bigvee_{\mathtt{s=l+1}}^{\mathtt{k-1}} \mathtt{succ_s(z)} \leq_\mathtt{k} \mathtt{y})^{\eta',\nu} = \mathtt{true}$. It follows that $(\exists z(\bigvee_{l=0}^{k-1} succ_l(z) \leq_k x \wedge \bigvee_{s=l+1}^{k-1} succ_s(z) \leq_k y))^{\eta,\nu} = true$, and thus the thesis.

(3) There exist $v, v' \in \bigcup_{i \geq 0} T^{i\mathcal{R}}$ such that v is an ancestor of $(\mathtt{x})^{\eta,\nu}$ and v' is an ancestor of $(\mathtt{y})^{\eta,\nu}$, with $v \leq v'$. If we take the valuation η', which differs from η at most in the value, which is $(k-1)^m$ (with $v = (k-1)^m \cdot 0$), it assigns to $\mathtt{z}$, then we have $(\bigvee_{\mathtt{l=0}}^{\mathtt{k-1}} \mathtt{succ_l(z)} \leq_\mathtt{k} \mathtt{x} \wedge \bigvee_{\mathtt{s=l+1}}^{\mathtt{k-1}} \mathtt{succ_s(z)} \leq_\mathtt{k} \mathtt{y})^{\eta',\nu} = \mathtt{true}$, and thus the thesis.

(Inductive step). The proofs for the boolean connectives $\phi = \neg\psi$, $\phi = \psi \wedge \theta$ and $\phi = \psi \vee \theta$ are straightforward, and thus omitted.
As for the quantifiers, let us consider the case of existential quantification of individual variables expressed by the formula $\phi = \exists \mathtt{x} \psi$. The other quantifications are dealt with in a similar way. Assume that $(\phi)^{\eta,\nu} = true$. This means that there exists $\overline{\eta}$ that differs from η at most in the value it assigns to $\mathtt{x}$ such that $(\psi)^{\overline{\eta},\nu} = true$. By the induction hypothesis, $(\tau(\psi))^{\overline{\eta},\nu} = true$. Since $\overline{\eta}(\mathtt{x})$ belongs to the concrete domain, it follows that $(\mathtt{Dom(x)})^{\overline{\eta},\nu} = true$, which allows us to conclude $(\tau(\phi))^{\eta,\nu} = true$.

($\Leftarrow$) The proof of the opposite implication is only sketched. Assume that $(\tau(\phi))^{\eta,\nu} = true$, for given η, and ν. It can be easily proved, by induction on the structure of ϕ, that there exist two valuations $\overline{\eta}$ and $\overline{\nu}$ such that, for each individual variable $\mathtt{x}$ and each set variable $\mathtt{p}$, $\overline{\eta}(\mathtt{x}) \in \bigcup_{\mathtt{i} \geq 0} \mathtt{T}^{\mathtt{i}\mathcal{R}}$, $\overline{\nu}(\mathtt{p}) \subseteq \bigcup_{\mathtt{i} \geq 0} \mathtt{T}^{\mathtt{i}\mathcal{R}}$, and $(\tau(\phi))^{\overline{\eta},\overline{\nu}} = true$ (take $\overline{\eta}(\mathtt{x}) = \eta(\mathtt{x})$ if $\eta(\mathtt{x}) \in \bigcup_{\mathtt{i} \geq 0} \mathtt{T}^{\mathtt{i}\mathcal{R}}$ and $\overline{\eta}(\mathtt{x})$ arbitrarily defined over $\bigcup_{i \geq 0} T^{i\mathcal{R}}$ if $\eta(\mathtt{x}) \notin \bigcup_{\mathtt{i} \geq 0} \mathtt{T}^{\mathtt{i}\mathcal{R}}$; take $\overline{\nu}(\mathtt{p}) = \nu(\mathtt{p}) \cap \bigcup_{\mathtt{i} \geq 0} \mathtt{T}^{\mathtt{i}\mathcal{R}}$. It is not difficult to show, by induction on the structure of ϕ, that $(\phi)^{\overline{\eta},\overline{\nu}} = true$. ■

THEOREM 5.6
The theory of downward unbounded layered structures is decidable.

PROOF. The thesis follows from Lemma 5.5 and the decidability of $MSO[<_k, succ_0, \ldots, succ_{k-1}]$ (cf. [30]). ■

6 High-level languages for ω-layered temporal structures

In [20], Montanari proposed a metric and layered temporal logic ($MLTL$ for short) for specifying granular real-time systems. Metric temporal logics, e.g., [15, 21], extend propositional logic with a parameterized operator of relative temporal realization. $MLTL$ can be viewed as the combination of a (possibly infinite) number of differently-grained metric temporal logics. It replaces the flat temporal domain of metric temporal logics with a temporal universe consisting of a set of differently-grained temporal domains. Such a temporal universe identifies the temporal domains relevant to a granular system specification and defines the relations between instants belonging to different domains. To qualify formulas with respect to the temporal universe, $MLTL$ is provided with an operator of *contextualization* that identifies the domains a given formula refers to. Within each temporal domain, it is then possible to talk about truth and falsehood of formulas at different time instants by means of a *displacement* operator. Finally, a *projection* operator is added to constrain the relationships between formulas associated with differently-grained domains. The combined use of these operators allows one to represent a granular system by properly connecting a set of differently-grained formulas [7]. In the simplest case, such a representation consists of the logical composition of a number of formulas referring to different temporal domains. In more complex cases, the projection operator is used to deal with nested quantifications of differently-grained temporal displacements (e.g., to specify the condition: "there exist some days during which the plant remains inactive for some hours"), or to specify the composition of differently-grained temporal displacements (e.g., to specify the condition: "in twenty seconds, five minutes will have passed from the occurrence of the fault").
A full account of syntax, semantics and axiomatization of $MLTL$ can be found in [7, 20, 21].

In this section, we demonstrate that the key features of $MLTL$ can actually be expressed in $MSO[<, \downarrow_0, \ldots, \downarrow_{k-1}]$. To this end, we first define a high-level second-order language for the theory of upward (resp. downward) unbounded layered structures, including suitable primitives for modeling contextualization, projection, and displacement, and then show that the resulting language is in fact as expressive as the basic one ($MSO[<, \downarrow_0, \ldots, \downarrow_{k-1}]$).

6.1 The language for upward unbounded layered structures

Let $\mathcal{UL}^2$ be the monadic second order language over the set U of unary relational symbols and the set B of binary relational symbols defined as follows. The set U contains *local zero* symbols $\mathtt{0_i}$ (for $i \geq 0$), *vertical contextualization* symbols $\mathtt{T^i}$ (for $i \geq 0$) and *horizontal contextualization* symbols $\mathtt{\Delta_i}$ (for $i \geq 0$).
The set B contains symbols $<, \downarrow_0, \ldots, \downarrow_{k-1}$, *local successor* symbols $+_\mathtt{i}\mathtt{1}$ (for $i \geq 0$), *local congruences* symbols $\equiv_{\mathtt{i,j}}$ (for $i, j \geq 0$) and an *approximate ordering* symbol $\preceq$.

The domain of interpretation for the language $\mathcal{UL}^2$ over an upward unbounded layered temporal structure $\langle \bigcup_{i \geq 0} T^i, \downarrow, < \rangle$ is $\bigcup_{i \geq 0} T^i_{i \geq 0}$; symbols $<, \downarrow_0, \ldots, \downarrow_{k-1}$ are interpreted exactly as in the case of $MSO[<, \downarrow_0, \ldots, \downarrow_{k-1}]$;
$\mathtt{0_i}$ is interpreted as the singleton containing the least element of T^i;
$+_\mathtt{i}\mathtt{1}$ is interpreted as the successor relation over T^i;

Δ_i is interpreted as the set

$$\{t : t \text{ is the } i\text{-th element of } T^j, j \geq 0\};$$

$\preceq$ is interpreted as the set

$$\{\langle t, t'\rangle : \mathit{ancestor}(t, t') \vee \mathit{ancestor}(t', t) \vee t \leq t'\},$$

where ancestor can be defined as in Section 4;
$\equiv_{\mathtt{i,j}}$ is interpreted as the congruence modulo-j (in the standard algebraic sense) over T^i, which is formally the set

$$\{\langle t, t'\rangle : t, t' \in T^i \wedge \Delta_m(t) \wedge \Delta_n(t) \wedge m \equiv_j n\}.$$

For each $i \geq 0$, $\mathtt{0_i}$ denotes the origin of the domain T^i. Vertical contextualizations $\mathtt{T^i(x)}$ and horizontal contextualizations $\mathtt{\Delta_i(x)}$ respectively restrict the range of possible values of the variable $\mathtt{x}$, constraining it to be interpreted over the domain T^i and over the set of time points at distance i from the origin of the domain they belong to. For each $i \geq 0$, the set of points satisfying $\mathtt{\Delta_i}$, together with (the obvious restriction of) $<$, is isomorphic to the structure $\langle \mathbb{N}, <\rangle$. In this respect, vertical contextualization and horizontal contextualization act in a perfectly symmetric way. For each $t, t' \in \bigcup_{i \geq 0} T^i$, $\preceq$ relates t to t' whenever $t \leq t'$ holds or either t' belongs to the projection of t over the domain t' belongs to or t belongs to the projection of t' over the domain t belongs to. In the following we shall use the notation $\mathtt{+_i(x) = y}$ instead of $\mathtt{+_i(x, y)}$.

Example 6.1
We describe a property whose specification exploits the expressive power of the operators of horizontal contextualization and projection of upward unbounded layered structures, and cannot be expressed in the theory of finitely-layered ones. For the sake of simplicity, we take $k = 2$. Consider a process of decay, where the elapsing time between two successive occurrences of a given phenomenon exponentially increases. It can be modeled in $\mathcal{UL}^2$ as follows. Let $\mathtt{p}$ and $\mathtt{q}$ be two predicates that respectively detect occurrences and non-occurrences of the considered phenomenon. With respect to each temporal domain, we assume that the phenomenon occurs at the first two time instants (that is, with delay 0); the phenomenon is not detected at the next time instant; it occurs again at the fourth instant (that is, with delay 2^0); then it is not detected for 2 time instants, but it surely happens before the end of an interval of length 2 (that is, with delay greater than or equal to 2^1 and less than 2^2); then it is not detected for 4 time instants, but it surely happens before the end of an interval of length 4 (that is, with delay greater than or equal to 2^2 and less than 2^3); and so on. If an occurrence of the phenomenon is detected at a time instant $i \in T^j$, then it occurs at least in its rightmost child, while if it is not detected at $i \in T^j$, then it is not detected at each of its children.

Formally, the process is specified by the following $\mathcal{UL}^2$-formula:

$$\begin{gathered}\forall \mathtt{x}((\Delta_0(\mathtt{x}) \vee \Delta_1(\mathtt{x})) \rightarrow \mathtt{p}(\mathtt{x}) \wedge \Delta_2(\mathtt{x}) \rightarrow \mathtt{q}(\mathtt{x}) \wedge \mathtt{p}(\mathtt{x}) \leftrightarrow \neg \mathtt{q}(\mathtt{x}) \wedge \\ \wedge \mathtt{q}(\mathtt{x}) \rightarrow (\mathtt{q}(\downarrow_0 (\mathtt{x})) \wedge \mathtt{q}(\downarrow_1 (\mathtt{x}))) \wedge \mathtt{p}(\mathtt{x}) \rightarrow \mathtt{p}(\downarrow_1 (\mathtt{x}))).\end{gathered}$$

It is easy to verify that each model of the above formula associates with each layer T^j an ω-sequence of the form $s_0 \cdot s_1 \cdot \ldots \cdot s_i \cdot \ldots$, where $s_0 = \{p\}$ and

$$s_i = \{p\} \cdot \{q\}^{2^{i-1}} \cdot w_i, \text{ with } w_i \in \{p, q\}^*, |w_i| = 2^{i-1} - 1.$$

It is possible to prove that the set of ω-sequences fulfilling the above condition is a non-regular ω-language over the alphabet $\Sigma = \{p, q\}$.

We conclude this section by proving that $\mathcal{UL}^2$ is as expressive as $MSO[<, \downarrow_0, \ldots, \downarrow_{k-1}]$. More precisely, we show that for each $i \geq 0$, the local zero $\mathtt{0_i}$, the successor function $+_\mathtt{i}\mathtt{1}$, the vertical and horizontal contextualizations $\mathtt{T^i}$ and $\mathtt{\Delta_i}$, the congruences $\equiv_{\mathtt{i,j}}$, with $j \geq 2$, and the approximate ordering $\preceq$ can be expressed in terms of $\downarrow_0, \ldots, \downarrow_{\mathtt{k-1}}$ and $<$.

Let $\phi \in \mathcal{UL}^2$. The following steps transform ϕ into an equisatisfiable formula $\phi' \in MSO[<, \downarrow_0, \ldots, \downarrow_{k-1}]$.

Step 1: removal of predicates $\mathtt{0_i}$. For each $i \geq 0$, we replace all the occurrences of $\mathtt{0_i(x)}$ (if any) by

$$\exists \mathtt{x_0}, \ldots, \mathtt{x_i}(\mathtt{x} = \mathtt{x_i} \wedge \forall \mathtt{y}\ (\mathtt{x_0} \leq \mathtt{y}) \wedge \bigwedge_{\mathtt{j=1}}^{\mathtt{i}} \downarrow_0 (\mathtt{x_j}) = \mathtt{x_{j-1}}).$$

Step 2: removal of successor functions $+_\mathtt{i}\mathtt{1}$. We replace each atomic formula $+_\mathtt{i}\mathtt{1}(\mathtt{x}) = \mathtt{y}$ by

$$\mathtt{T^i(x)} \wedge \mathtt{T^i(y)} \wedge \mathtt{x} < \mathtt{y} \wedge \forall \mathtt{z}(\mathtt{T^i(z)} \wedge \mathtt{x} < \mathtt{z} \rightarrow (\mathtt{y} \leq \mathtt{z}).$$

Step 3: removal of y-contextualizations $\mathtt{T^i}$. We replace each occurrence of atomic formulas of the form $\mathtt{T^i}(\mathrm{x})$, with $i \geq 0$, by the formula:

$$\exists \mathtt{x_0}, \ldots, \mathtt{x_i}(\mathtt{x} = \mathtt{x_i} \wedge \forall \mathtt{y} \neg \downarrow_0 (\mathtt{x_0}) = \mathtt{y} \wedge \bigwedge_{\mathtt{j=1}}^{\mathtt{i}} (\bigvee_{\mathtt{l=0}}^{\mathtt{k-1}} \downarrow_\mathtt{l} (\mathtt{x_j}) = \mathtt{x_{j-1}})).$$

Step 4: removal of x-contextualizations $\mathtt{\Delta_i}$. The removal of x-contextualizations $\mathtt{\Delta_i}$ is performed in two steps. First, for each $i > 0$, we replace each occurrence of atomic formulas of the form $\mathtt{\Delta_i(x)}$ by a formula which contains an atomic formula of the form $\mathtt{\Delta_0}(\mathrm{y})$; then, we remove from the resulting formula each occurrence of atomic formulas of the form $\mathtt{\Delta_0}(\mathrm{y})$ (possibly added at the previous step) by exploiting the formula given in equation 3.1.

For each $i > 0$, let $a_n k^n + \ldots + a_0 k^0$ be the k-ary representation of i. We replace each occurrence of atomic formulas of the form $\mathtt{\Delta_i(x)}$ by the formula:

$$\exists \mathtt{x_{n+1}}, \ldots, \mathtt{x_0}(\mathtt{\Delta_0}(\mathtt{x_{n+1}}) \wedge \mathtt{x_0} = \mathtt{x} \wedge \bigwedge_{\mathtt{j=0}}^{\mathtt{n}} \downarrow_{\mathtt{a_j}} (\mathtt{x_{j+1}}) = \mathtt{x_j}).$$

Step 5: removal of congruences $\equiv_{\mathtt{i,j}}$. We replace each occurrence of atomic formulas of the form $\equiv_{\mathtt{i,j}} (\mathrm{x}, \mathrm{y})$, with $i \geq 0$ and $j \geq 2$, by the formula:

$$\bigvee_{\mathtt{l=0}}^{\mathtt{j-1}} (\mathtt{Cong_i^{l,j}(x)} \wedge \mathtt{Cong_i^{l,j}(y)})$$

where $\mathtt{Cong_i^{m,n}}(\mathrm{x})$, with $m < n$, stands for:

$$\mu(\exists \mathtt{z},\mathtt{y}(\mathtt{0_i}(\mathtt{z}) \wedge (+_{\mathtt{i}}1)^{\mathtt{m}}(\mathtt{z}) = \mathtt{y} \wedge \mathtt{p}(\mathtt{y})) \wedge \forall \mathtt{z},\mathtt{y}(\mathtt{p}(\mathtt{z}) \wedge (+_{\mathtt{i}}1)^{\mathtt{n}}(\mathtt{z}) = \mathtt{y} \rightarrow \mathtt{p}(\mathtt{y})))(\mathtt{x})$$

where $(+_{\mathtt{i}}1)^{\mathtt{n}}(\mathtt{x}) = \mathtt{y}$ the obvious shorthand, and then we remove the occurrences of $\mathtt{0_i}$, $+_{\mathtt{i}}1$ and $\mathtt{T^i}$ by executing the operations specified in steps 1, 2 and 3, respectively.
Step 6: removal of the approximate ordering $\preceq$. We replace each occurrence of atomic formulas of the form $\mathtt{x} \preceq \mathtt{y}$ by the formula:

$$\neg\mathtt{ancestor}(\mathtt{x},\mathtt{y}) \wedge \neg\mathtt{ancestor}(\mathtt{y},\mathtt{x}) \rightarrow \mathtt{x} \leq \mathtt{y},$$

where $\mathtt{ancestor}(\mathtt{x},\mathtt{y})$ is defined in Section 4.

The relationship between the original formula $\phi \in \mathcal{UL}^2$ and the resulting formula $\phi' \in MSO[<,\downarrow_0,\ldots,\downarrow_{k-1}]$ is formally stated by the following lemma.

LEMMA 6.2
Let ϕ be an $\mathcal{UL}^2$-formula and ϕ' be the $MSO[<,\downarrow_0,\ldots,\downarrow_{k-1}]$-formula obtained from ϕ after the execution of the transformation steps 1-6. ϕ is satisfiable if and only if ϕ' is satisfiable.

The proof of the Lemma 6.2 is straightforward, and thus omitted.

6.2 The language for downward unbounded layered structures

The formulas of the high-level second-order language $\mathcal{DL}^2$ for the theory of downward unbounded layered structures differ from the ones of $\mathcal{UL}^2$ only in the fact that atomic formulas having the form $\mathtt{\Delta_i}(\mathtt{x})$ are replaced by atomic formulas having the form $\mathtt{\Delta_{i,j}}(\mathtt{x})$, for $i, j \geq 0$. The interpretation of $\mathcal{DL}^2$ over a downward unbounded layered structure differs from the interpretation of $\mathcal{UL}^2$ only in the interpretation of $\downarrow_0,\ldots,\downarrow_k$, which is given as in the case of $MSO[<,\downarrow_0,\ldots,\downarrow_k]$ (interpreted over downward unbounded layered structures), and in the interpretation for $\mathtt{\Delta_{i,j}}$ which is now the set

$$\{t : (+_m 1)^i(t') = t, (\downarrow_0)^m(t'') = t', t'' \text{ is the } j\text{-th element of } T^0, m \geq 0\}.$$

This semantic clause states that for any $m \geq 0$ and any $t \in T^m$, $\mathtt{\Delta_{i,j}}$ holds at t iff t is at distance i from the (unique) element $t' \in T^m$ belonging to the leftmost branch of the tree rooted at the (unique) element $t'' \in T^0$ which is at distance j from 0_0. It is easy to see that the predicate $\mathtt{\Delta_i}$ of $\mathcal{UL}^2$ is equivalent to $\mathtt{\Delta_{i,0}}$.

EXAMPLE 6.3
Let $\mathtt{p}$ and $\mathtt{q}$ be two predicates (denoting either a pair of events or a pair of states). The properties of downward unbounded layered structures can be exploited to constrain the temporal relationships between $\mathtt{p}$ and $\mathtt{q}$. Let us consider the case of temporal overlapping. We say that there is a temporal overlapping between $\mathtt{p}$ and $\mathtt{q}$ *with respect to a given layer T^i* if there exists a time point of T^i at which both $\mathtt{p}$ and $\mathtt{q}$ hold. Such a condition can be easily expressed as follows:

$$\exists \mathtt{x}(\mathtt{T^i}(\mathtt{x}) \wedge \mathtt{p}(\mathtt{x}) \wedge \mathtt{q}(\mathtt{x})).$$

We say that there is a temporal overlapping between $\mathtt{p}$ and $\mathtt{q}$ *with respect to the whole temporal structure* if for all time points at which both $\mathtt{p}$ and $\mathtt{q}$ hold, there exists an infinite (downward) path, starting at such a point, such that both $\mathtt{p}$ and $\mathtt{q}$ hold at each point belonging to the path. Such a condition can be expressed as follows:

$$\forall\mathtt{x}\exists\mathtt{r}(\phi(\mathtt{x},\mathtt{r}) \land \forall\mathtt{s}(\phi(\mathtt{x},\mathtt{s}) \rightarrow \forall\mathtt{z}(\mathtt{r}(\mathtt{z}) \rightarrow \mathtt{s}(\mathtt{z}))) \land \forall\mathtt{w}(\mathtt{r}(\mathtt{w}) \rightarrow \mathtt{p}(\mathtt{w}) \land \mathtt{q}(\mathtt{w}))),$$

where $\phi(\mathtt{x},\mathtt{r})$ stands for:

$$\mathtt{r}(\mathtt{x}) \land \forall\mathtt{z}(\mathtt{r}(\mathtt{z}) \rightarrow \exists\mathtt{w}\bigvee\nolimits_{\mathtt{i}=0}^{\mathtt{k}-1}(\mathtt{w} =\downarrow_{\mathtt{i}} (\mathtt{z}) \land \mathtt{r}(\mathtt{w}))).$$

From this, it immediately follows that $\mathtt{p}$ and $\mathtt{q}$ are not temporally overlapped (with respect to the whole structure) if there exists a time point such that, for each path starting from it, there exists a time point belonging to the path at which either p holds and q does not hold or p does not hold and q holds:

$$\exists\mathtt{x}\forall\mathtt{r}((\phi(\mathtt{x},\mathtt{r}) \land \forall\mathtt{s}(\phi(\mathtt{x},\mathtt{s}) \rightarrow \forall\mathtt{z}(\mathtt{r}(\mathtt{z}) \rightarrow \mathtt{s}(\mathtt{z})))) \rightarrow$$

$$\exists\mathtt{w}(\mathtt{r}(\mathtt{w}) \land ((\neg\mathtt{p}(\mathtt{w}) \land \mathtt{q}(\mathtt{w})) \lor (\mathtt{p}(\mathtt{w}) \land \neg\mathtt{q}(\mathtt{w}))))).$$

As in the upward unbounded case, it is possible to show that $\mathcal{DL}^2$ is as expressive as $MSO[<, \downarrow_0, \ldots, \downarrow_{k-1}]$. Let $\phi \in \mathcal{DL}^2$. The following steps transform ϕ into an equisatisfiable formula $\phi' \in MSO[<, \downarrow_0, \ldots, \downarrow_{k-1}]$.

Step 1: removal of local zero $\mathtt{0_i}$. For each $i \geq 0$, we replace all the occurrences of $\mathtt{0_i}(\mathtt{x})$ (if any) by

$$\exists\mathtt{x}_0, \ldots, \mathtt{x_i}(\mathtt{x} = \mathtt{x_i} \land \forall\mathtt{y}\ (\mathtt{x}_0 \leq \mathtt{y}) \land \bigwedge_{\mathtt{j}=0}^{\mathtt{i}-1} \downarrow_0 (\mathtt{x_j}) = \mathtt{x_{j+1}}).$$

Step 2: removal of successor functions $+_{\mathtt{i}}1$. It is exactly as in the case of upward unbounded layered structures.
Step 3: removal of vertical contextualizations $\mathtt{T^i}$. We replace each occurrence of atomic formulas of the form $\mathtt{T^i}(\mathtt{x})$, with $i \geq 0$, by the formula:

$$\exists\mathtt{x}_0, \ldots, \mathtt{x_i}(\mathtt{x} = \mathtt{x_i} \land \forall\mathtt{y}\ \neg(\bigvee_{\mathtt{j}=0}^{\mathtt{k}-1} \downarrow_{\mathtt{j}} (\mathtt{y}) = \mathtt{x}_0) \land \bigwedge_{\mathtt{j}=0}^{\mathtt{i}-1}(\bigvee_{\mathtt{l}=0}^{\mathtt{k}-1} \downarrow_{\mathtt{l}} (\mathtt{x_j}) = \mathtt{x_{j+1}})).$$

Step 4: removal of horizontal contextualizations $\mathtt{\Delta_{i,j}}$. The removal of horizontal contextualizations is performed recursively.
We replace each occurrence of atomic formulas of the form $\mathtt{\Delta_{0,j}}(\mathtt{x})$ by the formula:

$$\mu(\exists z, y(\forall w(z \leq w) \land (+_0 1)^j(z) = y \land p(y)) \land \forall z, y(p(z) \land \downarrow_0 (z) = y \rightarrow p(y)))(x)$$

where occurrences of $(+_0 1)^j$ are replaced as shown at step 2.
For each $i > 0$, let $a_n k^n + \ldots + a_0 k^0$ be the k-ary representation of i. The coefficients $a_n, \ldots, a_0$ specify a path starting from a node in the leftmost edge of the j-th tree and leading to a node at distance i from that edge (first disjunct of the formula below). However, (for $i \geq 1$) there exist nodes at distance i from the leftmost edge of the j-th

tree which do not belong to such a tree. The second disjunct of the formula below deals with this case.
We replace each occurrence of atomic formulas of the form $\mathtt{\Delta_{i,j}(x)}$ by the formula:

$$\exists \mathtt{x_{n+1}, \ldots, x_0 (\Delta_{0,j}(x_{n+1}) \wedge x_0 = x \wedge (\bigwedge_{j=0}^{n} \downarrow_{a_j} (x_{j+1}) = x_j))} \vee$$

$$\bigvee_{\mathtt{l=0}}^{\lfloor \log_k(i) \rfloor} \mathtt{(T^l(x) \wedge \Delta_{r_l, j+q_l}(x))}, \text{ with } \mathtt{i = q_l k^l + r_l} \text{ and } \mathtt{0 \leq r_l < k^l}$$

where $\mathtt{T^l(x)}$ is replaced as shown at step 3.
Step 5: removal of congruences $\equiv_{\mathtt{i,j}}$. It is exactly as in the case of upward unbounded layered structures.
Step 6: removal of the approximate ordering $\mathtt{x \preceq y}$. We replace each occurrence of atomic formulas of the form $\mathtt{x \preceq y}$ by the formula:

$$\neg \mathtt{ancestor(x,y)} \wedge \neg \mathtt{ancestor(y,x)} \rightarrow \mathtt{x \leq y},$$

where $\mathtt{ancestor(x,y)}$ is defined as in Section 4.

The relationship between the original formula $\phi \in \mathcal{DL}^2$ and the resulting formula $\phi' \in MSO[<, \downarrow_0, \ldots, \downarrow_{k-1}]$ is formally stated by the following lemma.

LEMMA 6.4
Let ϕ be an $\mathcal{DL}^2$-formula and ϕ' be the $MSO[<, \downarrow_0, \ldots, \downarrow_{k-1}]$-formula obtained from ϕ after the execution of the transformation steps 1-6. ϕ is satisfiable if and only if ϕ' is satisfiable.

As for Lemma 6.2, the proof is straightforward, and thus omitted.

Remark. In this section we have shown that the basic functionalities of $MLTL$ can be expressed in $MSO[<, \downarrow_0, \ldots, \downarrow_{k-1}]$. We want to point out that, however, there exist significant properties of ω-layered structures that cannot be expressed in $MSO[<, \downarrow_0, \ldots, \downarrow_{k-1}]$. As an example, it is not possible to define a binary predicate *same_layer* such that, for all $t, t' \in \bigcup_{i \geq 0} T^i$, $same\_layer(t, t')$ iff $\exists i (t \in T^i \wedge t' \in T^i)$. As a matter of fact, extending $MSO[<, \downarrow_0, \ldots, \downarrow_{k-1}]$ with the predicate *same_layer* would make the theory of downward unbounded layered structures undecidable, a result which follows from the undecidability of the extension of $MSO[<_k, succ_0, \ldots, succ_{k-1}]$ with the *equal_level* predicate E given by $E(u, v)$ iff $|u| = |v|$, with $u, v \in \{0,1\}^*$ (cf. [17]). It is possible to show that a similar undecidability result holds for the theory of upward unbounded ones.

Conclusions and further developments

In this paper, we proved the decidability of theories of metric and layered temporal structures provided with an infinite number of either arbitrarily coarse or arbitrarily fine layers by exploiting the theories $MSO[<, pow_k, hp_k]$ and $MSO[<_k, succ_0, \ldots, succ_{k-1}]$, respectively. The questions whether we can decide upward unbounded layered structures using the theory $MSO[<]$ (instead of $MSO[<, pow_k, hp_k]$)

and whether we can decide downward unbounded layered structures using $MSO[<, pow_k, hp_k]$ (instead of $MSO[<_k, succ_0, \ldots, succ_{k-1}]$) are still open (our conjecture is that both questions have a negative answer).

We are currently exploring the natural generalization to layered structures which are both upward and downward unbounded. We are also investigating the possibility of extending the achieved results to more loosely structured ω-layered metric temporal structures. In particular, we are interested in weakening the requirement that each point has k children, uniformly in each layer. As an example, keeping the number of children constant within each layer, but allowing it to change across layers, would make it possible to deal with temporal structures including the domains of days, hours, and minutes. Let us define the refinement degree of a given layer as the number of children of each point belonging to the layer. It is not difficult to show that decidability is preserved if the mapping that associates a refinement degree to each layer is constant, except for a finite number of layers. We are currently analyzing the case where such a mapping is (ultimately) periodic.

Finally, notice that the above results directly hold for metric and layered temporal logics non-axiomatically defined. Since they are decidable, one can list all their theorems and hence axiomatic completeness trivially follows, even though the axioms are not produced explicitly. In [13], Gabbay et al. showed that propositional linear temporal logic ($PLTL$) can be viewed as an expressively complete and elementarily decidable modal temporal logic counterpart of $MFO[<]$. We are looking for a suitable extension of $PLTL$ that plays the same role with respect to $MFO[<, pow_k, hp_k]$ (which properly includes $MFO[<]$).

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