Two-sorted metric temporal logics [1]

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Temporal logic has been successfully used for modeling and analyzing the behavior of reactive and concurrent systems. Standard temporal logic is inadequate for real-time applications because it only deals with qualitative timing properties. This is overcome by metric temporal logics which offer a uniform logical framework in which both qualitative and quantitative timing properties can be expressed by making use of a parameterized operator of relative temporal realization.

In this paper we deal with completeness issues for basic systems of metric temporal logic — despite their relevance, such issues have been ignored or only partially addressed in the literature. We view metric temporal logics as two-sorted formalisms having formulae ranging over time instants and parameters ranging over an (ordered) abelian group of temporal displacements. We first provide an axiomatization of the pure metric fragment of the logic, and prove its soundness and completeness. Then, we show how to obtain the metric temporal logic of linear orders by adding an ordering over displacements. Finally, we consider general metric temporal logics allowing quantification over algebraic variables and free mixing of algebraic formulae and temporal propositional symbols.

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1 Introduction

Logic-based methods for representing and reasoning about temporal information have proved to be highly beneficial in the area of formal specifications. In this paper we consider their application to the specification of real-time systems. Timing properties play a major role in the specification of reactive and concurrent software systems that operate in real-time. They constrain the interactions between different components of the system as well as between the system and its environment, and minor changes in the precise timing of interactions may lead to radically different behaviors. Temporal logic has been successfully used for modeling and analyzing the behavior of reactive and concurrent systems (cf. Manna and Pnueli [11] and Ostroff [15]). It supports semantic model checking, which can be used to verify consistency of specifications, and to check positive and negative examples of system behavior against specifications; it also supports pure syntactic deduction, which may be used to prove properties of systems. Unfortunately, most common representation languages in the area of formal specifications are inadequate for real-time applications, because they lack an explicit and quantitative representation of time. In recent years, some of them have been extended to cope with real-time aspects. In this paper, we focus on *metric temporal logics* which provide a uniform framework in which both qualitative and quantitative timing properties of real-time systems can be expressed.

The idea of a logic of positions (topological, or metric, logic) has originally been formulated by Rescher and Garson [16]. They defined the basic features of the logic, and showed how to give it a temporal interpretation. The logic of positions extends propositional logic with a parameterized operator P_α of positional realization. Such an operator allows one to constrain the truth value of a proposition at position α. The parameter α denotes either (i) an absolute position or (ii) a displacement with respect to the current position which is left implicit. According to interpretation (ii), $P_\alpha q$ is true at the position i if and only if q is true at a position j at distance α from i. In [16], Rescher and Garson introduced two axiomatizations of the logic of positions that differ from each other in the interpretation of parameters. Later, Rescher and Urquhart [17] proved the soundness and completeness of the axiomatization based on an absolute interpretation of parameters through a reduction to monadic quantification theory. Independently, a metric temporal logic has been developed by Koymans [10] to support the specification and verification of real-time systems. He extended the standard model for temporal logic based on point structures with a distance function that measures, for any pair of time points, how far they are apart in time. He provided the logic with a sound axiomatization, but no proof of completeness was given.

The main issues to confront in developing a metric temporal logic for executable specifications are:

Expressiveness (definability). Is the metric temporal logic expressive enough to express both the properties of the underlying temporal structure and the timing requirements of the specified real-time systems?

Soundness and completeness. Is the metric temporal logic equipped with a sound and complete axiomatization?

Decidability. Which properties of the specified real-time system can be automatically verified? Most temporal logics for real-time systems proposed in the literature cannot be decided (cf. Henzinger [7]). Some of them recover decidability sacrificing completeness.

Executability. How can we prove the consistency and adequacy of specifications? In principle, decidability proof methods (e.g. via Büchi automata) outline an effective procedure to prove the satisfiability and/or validity of a formula. But as soon as certain assumptions about the nature of the temporal domain and the available set of primitive operations are relaxed, the satisfiability/validity problem becomes undecidable (Alur and Henzinger

An alternative approach consists in looking at metric temporal logics as particular polymodal logics and supporting derivability by means of proof procedures for nonclassical logics or via translations in first-order theories (cf. D'Agostino et al. [4], and Ohlbach [14]). In this case, providing the logic with a sound and complete axiomatization becomes a central issue.

The aim of this paper is to explore completeness issues of metric temporal logic; we do this by starting with a very basic system, and build on it either by adding axioms or by enriching the underlying structures. We view metric temporal logics as two-sorted logics having both formulae and parameters; formulae are evaluated at time instants while parameters take values in an (ordered) abelian group of temporal displacements. In Section 2, we define a minimal metric logic that can be seen as the metric counterpart of minimal tense logic, and we provide it with a sound and complete axiomatization. In Section 3, we characterize the class of two-sorted frames with a linearly ordered temporal domain. In Section 4, we extend our systems with the ability to mix temporal and displacement formulae to make their logical machinery sufficiently powerful. The conclusions provide an assessment of the work and they outline further directions of research, including the possibility of using the proposed two-sorted framework for characterizing a variety of metric temporal logics simply by changing the requirements on its algebraic and/or temporal components, and our ongoing work on decidability aspects of metric temporal logics.

2 The basic metric logic

In this section we define the minimal metric temporal logic MTL_0, and consider some of its natural extensions.

Language

We define a two-sorted temporal language for our basic calculus MTL_0. First, its algebraic part is built up from a non-empty set A of *constants* denoting the group elements. The set of terms over A, $T(A)$, is the smallest set such that (1) $A \subseteq T(A)$, and (2) if α, $\beta \in T(A)$ then $(\alpha + \beta)$, $(-\alpha), 0 \in T(A)$. Next, the temporal part of the language is built up from a non-empty set Φ of *proposition letters*. The set of MTL_0-formulae over Φ and A, $F(\Phi, A)$, is the smallest set such that (1) $\Phi \subseteq F(\Phi, A)$, and (2) if ϕ, $\psi \in F(\Phi, A)$ and $\alpha \in T(A)$, then $\neg\phi$, $\phi \wedge \psi$, $\Delta_\alpha \phi$ (and its dual $\nabla_\alpha \phi := \neg\Delta_\alpha\neg\phi$), $\perp \in F(\Phi, A)$. We will adopt the following notational conventions: p, q, ... denote proposition letters; ϕ, ψ, ... denote MTL_0-formulae; Σ, Γ, ... denote sets of MTL_0-formulae; α, β, ... denote algebraic terms.

Structures

We define a *two-sorted frame* to be a triple $\mathfrak{F} = (T, \mathfrak{D}; \mathrm{DIS})$, where T is the set of (time) points over which temporal formulae are evaluated, $\mathfrak{D}$ is the algebra of metric displacements in whose domain D terms take their values, and $\mathrm{DIS} \subseteq T \times D \times T$ is an accessibility relation relating pairs of points and displacements.

We require the following properties to hold for the components of two-sorted frames. First, $\mathfrak{D}$ should be an abelian group, that is, a 4-tuple $(D, +, -, 0)$ where $+$ is a binary function of *displacement composition*, $-$ is a unary function of *inverse displacement*, and 0 is the *zero displacement* constant, such that:

(i)	$\alpha + \beta = \beta + \alpha$	(commutativity of $+$)
(ii)	$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$	(associativity of $+$)

(iii)	$\alpha + 0 = \alpha$	(zero element of $+$)
(iv)	$\alpha + (-\alpha) = 0$	(inverse)

Second, we require the displacement relation DIS to respect the converse operation of the abelian group in the following sense: if $\mathrm{DIS}(i, \alpha, j)$ then $\mathrm{DIS}(j, -\alpha, i)$.

We turn a two-sorted frame $\mathfrak{F}$ into a *two-sorted model* by adding an interpretation for our algebraic terms, and a valuation for atomic temporal formulae. An *interpretation* for algebraic terms is given by a function $g : A \to D$ that is automatically extended to all terms from $T(A)$ in the following way: $g(0) = 0$, $g(\alpha + \beta) = g(\alpha) + g(\beta)$, and $g(-\alpha) = -g(\alpha)$. A *valuation* is simply a function $V : \Phi \to 2^T$. Then, we say that an equation $\alpha = \beta$ is *true* in a model $\mathfrak{M} = (T, \mathfrak{D}; \mathrm{DIS}; V, g)$ whenever $g(\alpha) = g(\beta)$. Next, *truth* of temporal formulae is defined by

$$\begin{array}{rll}
\mathfrak{M}, i \Vdash p & \text{iff} & i \in V(p) \\
\mathfrak{M}, i \Vdash \bot & & \text{never} \\
\mathfrak{M}, i \Vdash \neg\phi & \text{iff} & \mathfrak{M}, i \nVdash \phi \\
\mathfrak{M}, i \Vdash \phi \wedge \psi & \text{iff} & \mathfrak{M}, i \Vdash \phi \;\text{ and }\; \mathfrak{M}, i \Vdash \psi \\
\mathfrak{M}, i \Vdash \Delta_\alpha \phi & \text{iff} & \text{there exists } j \text{ such that } \mathrm{DIS}(i, g(\alpha), j) \text{ and } \mathfrak{M}, j \Vdash \phi.
\end{array}$$

To avoid messy complications we only consider one-sorted consequences $\Gamma \models \phi$; for algebraic formulae '$\Gamma \models \phi$' means 'for all two-sorted models $\mathfrak{M}$, if $\mathfrak{M} \models \Gamma$, then $\mathfrak{M} \models \phi$'; for temporal formulae it means 'for all models $\mathfrak{M}$, and times instants i, if $\mathfrak{M}, i \Vdash \Gamma$, then $\mathfrak{M}, i \Vdash \phi$'.

A simple example

Even though the language of MTL_0 is very poor, it already allows us to express conditions on real-time systems. As a first example, consider a communication channel C that outputs each message with a delay δ with respect to its input time, and that neither generates nor loses messages (cf. Montanari et al. [12]). C can be specified as follows:

$$out \leftrightarrow \Delta_{-\delta} in.$$

This example can easily be generalized to the case of a channel C that collects messages from n different sources $S_1, \ldots, S_n$ and outputs them with delay δ. To exclude that two input events can occur simultaneously, we add the constraint:

$$\forall i, j\, \neg(in(i) \wedge in(j) \wedge i \neq j),$$

which is shorthand for

$$\neg(in(1) \wedge in(2)) \wedge \ldots \wedge \neg(in(n-1) \wedge in(n)).$$

Then the behavior of C is specified by the formula

$$\forall i\,(out(i) \leftrightarrow \Delta_{-\delta} in(i)),$$

which is shorthand for a finite conjunction.

Notice that preventing input events from occurring simultaneously also guarantees that output events do not occur simultaneously.

Suppose now that C outputs the messages it receives from $S_1, \ldots, S_n$ with delays $\delta_1, \ldots, \delta_n$, respectively. Constraining input events not to occur simultaneously no longer guarantees that there are no conflicts at output time. A simple strategy of conflict resolution consists in assigning a different priority to messages coming from different knowledge sources, so that, when a conflict occurs, C only outputs the message with highest priority. Accordingly, the specification of C is modified, preserving the requirement that it does not generate messages, but relaxing the requirement that it does not lose messages.

Assume that $S_1, \ldots, S_n$ are listed in decreasing order of priority. The behavior of C can be specified as follows:

$$\forall i\,(out(i) \leftrightarrow (\Delta_{-\delta_i} in(i) \wedge \neg\exists j\,(\Delta_{-\delta_j} in(j) \wedge j < i)))$$

which is a shorthand for

$$(out(1) \leftrightarrow \Delta_{-\delta_1} in(1)) \wedge (out(2) \leftrightarrow (\Delta_{-\delta_2} in(2) \wedge \neg\Delta_{-\delta_1} in(1))) \wedge \ldots \wedge$$

$$(out(n) \leftrightarrow (\Delta_{-\delta_n} in(n) \wedge (\neg\Delta_{-\delta_1} in(1) \wedge \ldots \wedge \neg\Delta_{-\delta_{n-1}} in(n-1))).$$

More complex examples are given in later sections.

Axioms

Our basic calculus MTL_0 has two components. On the one hand it has the usual laws of algebraic logic to deal with the displacements:

(Ref)	$\vdash \alpha = \alpha$, for all terms α	(reflexivity)
(Sym)	$\vdash \alpha = \beta \implies \vdash \beta = \alpha$	(symmetry)
(Tra)	$\vdash \delta = \alpha,\ \alpha = \beta \implies \vdash \delta = \beta$	(transitivity)
(Rep)	$\vdash \alpha = \beta \implies \vdash \delta(\alpha/x) = \delta(\beta/x)$	(replacement)
(Sub)	$\vdash \alpha = \beta \implies \vdash \alpha(\delta/x) = \beta(\delta/x)$	(substitution),

as well as the above axioms (i)–(iv) for abelian groups. Here, $\beta(\alpha/x)$ denotes the result of substituting α for all occurrences of x in β.

The second component of MTL_0 governs the temporal aspect of our structures; its axioms are the usual axioms of propositional logic plus

(Ax1)	$\nabla_\alpha(p \to q) \to (\nabla_\alpha p \to \nabla_\alpha q)$	(normality)
(Ax2)	$p \to \nabla_\alpha \Delta_{-\alpha} p$,	(symmetry)

and its rules are modus ponens and

(NEC)	$\vdash \phi \implies \vdash \nabla_\alpha \phi$	(necessitation rule for ∇_α)
(REP)	$\vdash \phi \leftrightarrow \psi \implies \vdash \chi(\phi/p) \leftrightarrow \chi(\psi/p)$ where (ϕ/p) denotes substitution of ϕ for the variable p	(replacement)
(LIFT)	$\vdash \alpha = \beta \implies \vdash \nabla_\alpha \phi \leftrightarrow \nabla_\beta \phi$	(transfer of identities).

Axiom (Ax1) is the usual distribution axiom; axiom (Ax2) expresses that a displacement α is the converse of a displacement $-\alpha$. The rules (NEC) and (REP) are familiar from modal logic, and the rule (LIFT) allows us to transfer provable algebraic identities from the displacement domain to the temporal domain.

A *derivation in* MTL_0 is a sequence of terms and/or formulae $\sigma_1, \ldots, \sigma_n$ such that each σ_i $(1 \leq i \leq n)$ is either an axiom, or obtained from $\sigma_1, \ldots, \sigma_{n-1}$ by applying one of the derivation rules of MTL_0. We write $\vdash_{MTL_0} \sigma$ to denote that there is a derivation in MTL_0 that ends in σ. It is an immediate consequence of this definition that $\vdash_{MTL_0} \alpha = \beta$ iff $\alpha = \beta$ is provable (in algebraic logic) from the axioms of abelian groups only: whereas we can lift algebraic information from the displacement domain to the temporal domain using the (LIFT) rule, there is no way in which we can import temporal information into the displacement domain. As with consequences, we only consider one-sorted inferences '$\Gamma \vdash \phi$'.

Completeness

In this subsection we prove completeness for the basic calculus MTL_0. Our strategy will be to construct a canonical-like model by taking the free abelian group over our algebraic elements as the displacement component, by taking the familiar canonical model as the temporal component, and by linking the two in a suitable way.

The displacement domain

Recall that $T(A)$ is the collection of all algebraic terms built up from the elements of A. Define a congruence relation θ on $T(A)$ by taking

$$(\alpha, \beta) \in \theta \ \text{ iff } \ \vdash_{MTL_0} \alpha = \beta.$$

Then the *canonical displacement domain* $\mathfrak{D}^0$ is constructed by taking

$$\begin{aligned} D^0 &= T(A)/\theta \\ \alpha/\theta + \beta/\theta &= (\alpha + \beta)/\theta \\ -\alpha/\theta &= (-\alpha)/\theta \\ 0 &= 0/\theta. \end{aligned}$$

That $\mathfrak{D}^0$ is indeed an abelian group is easily shown using the defining axioms and rules of MTL_0. The group $\mathfrak{D}^0$ is known as the *free abelian group over* A (cf. Burris and Sankappanavar [3]).

We interpret our terms using the *canonical mapping* $g : T(A) \to \mathfrak{D}^0$ defined by $\alpha \mapsto \alpha/\theta$.

The temporal domain

A set of MTL_0-formulae is *maximal* MTL_0*-consistent* (or: an MCS) if it is MTL_0-consistent and it does not have proper MTL_0-consistent extensions. The *canonical temporal domain* T^0 is constructed by taking

$$T^0 = \{\Sigma \mid \Sigma \text{ is maximal } MTL_0\text{-consistent } \}.$$

Define a *canonical valuation* V^0 by putting $V^0(p) = \{\Sigma \mid p \in \Sigma\}$.

The canonical model for MTL_0

We almost have all the ingredients to define a canonical model for MTL_0; we only need to define a displacement relation $\mathrm{DIS}^0 \subseteq T^0 \times D^0 \times T^0$. This is done as follows:

$$\begin{aligned} \mathrm{DIS}^0(\Sigma, \alpha/\theta, \Gamma) \ \text{ iff } & \text{ for every formula } \gamma,\ \gamma \in \Gamma \text{ implies } \Delta_\alpha \gamma \in \Sigma \\ & \text{(equivalently: for all } \sigma \text{, if } \nabla_\alpha \sigma \in \Sigma \text{ then } \sigma \in \Gamma). \end{aligned}$$

Note that if $(\alpha, \beta) \in \theta$, then $\vdash \alpha = \beta$, hence $\vdash \nabla_\alpha \phi \leftrightarrow \nabla_\beta \phi$ by the (LIFT) rule, for all formulae ϕ. From this one easily derives that the definition of DIS^0

does not depend on the representative we take for α/θ.

Also, $\mathrm{DIS}^0(\Sigma, \alpha/\theta, \Gamma)$ implies $\mathrm{DIS}^0(\Gamma, -\alpha/\theta, \Sigma)$: if $\mathrm{DIS}^0(\Sigma, \alpha/\theta, \Gamma)$ and $\sigma \in \Sigma$, then $\nabla_\alpha \Delta_{-\alpha}\sigma \in \Sigma$ by axiom (Ax2), hence $\Delta_{-\alpha}\sigma \in \Gamma$.

Then, the *canonical model* for MTL_0 is the model $\mathfrak{M}^0 = (T^0, \mathfrak{D}^0; \mathrm{DIS}^0; V^0, g)$.

Theorem 2.1 (Completeness) *MTL_0 is sound and complete for the class of all MTL_0-frames.*

Proof. Proving soundness is left to the reader. To prove completeness we show that every consistent set of MTL_0-formulae is satisfiable in a model based on a two-sorted frame.

Let Σ be a MTL_0-consistent set of formulae; by standard techniques we can extend it to a maximal MTL_0-consistent set Σ^+ that lives somewhere in the canonical model $\mathfrak{M}^0$ for MTL_0. To complete the proof of the theorem it suffices to establish the following Truth Lemma. For all MTL_0-formulae ϕ and all $\Sigma \in T^0$:

$$\phi \in \Sigma \ \text{ iff } \ \mathfrak{M}^0, \Sigma \Vdash \phi.$$

The proof of the lemma is by induction on ϕ. The atomic case is immediate from the definition of V^0, and the boolean cases are immediate from the induction hypothesis. So consider a modal formula $\Delta_\alpha \phi$.

($\Leftarrow$) Assume $\Sigma \Vdash \Delta_\alpha \phi$. Then there exists Γ such that $\mathrm{DIS}^0(\Sigma, \alpha/\theta, \Gamma)$ and $\Gamma \Vdash \phi$. By induction hypothesis $\phi \in \Gamma$, so $\Delta_\alpha \phi \in \Sigma$.

($\Rightarrow$) If $\Delta_\alpha \phi \in \Sigma$, then, to prove $\Sigma \Vdash \Delta_\alpha \phi$, we need to find a Γ with $\phi \in \Gamma$ and $\mathrm{DIS}^0(\Sigma, \alpha/\theta, \Gamma)$. Such a Γ exists if we can show that $\{\phi\} \cup \{\psi \mid \nabla_\alpha \psi \in \Sigma\}$ is consistent — but this can be done by standard modal means.

This completes the proof of the Truth Lemma, and hence the proof of the completeness theorem. $\dashv$

Imposing additional constraints

For many purposes two-sorted frames as we have studied them so far are too simple. In particular, they don't satisfy all the natural conditions one may want to impose on the displacement relation. Examples of such properties that arise in application areas such as real-time system specification include

Transitivity: $\forall i, j, k, \alpha, \beta\, (\mathrm{DIS}(i, \alpha, j) \wedge \mathrm{DIS}(j, \beta, k) \rightarrow \mathrm{DIS}(i, \alpha + \beta, k))$

Quasi-functionality: $\forall i, j, j', \alpha\, (\mathrm{DIS}(i, \alpha, j) \wedge \mathrm{DIS}(i, \alpha, j') \rightarrow j = j')$
Reflexivity: $\forall i\, \mathrm{DIS}(i, 0, i)$
Antisymmetry: $\forall i, j, \alpha\, (\mathrm{DIS}(i, \alpha, j) \wedge \mathrm{DIS}(j, \alpha, i) \rightarrow i = j \wedge \alpha = 0)$.

As in standard modal and temporal logic only some of the natural properties we want to impose on structures are expressible. In particular, the first three of the above properties are expressible in metric temporal logic, as follows (cf. Montanari et al. [12]):

(Ax3) $\nabla_{\alpha+\beta} p \rightarrow \nabla_\alpha \nabla_\beta p$ (transitivity)
(Ax4) $\Delta_\alpha p \rightarrow \nabla_\alpha p$ (quasi-functionality w.r.t. the 3rd argument)
(Ax5) $\nabla_0 p \rightarrow p$ (reflexivity)

In the case of Transitivity, Quasi-functionality, and Reflexivity we are able to extend the basic completeness result fairly effortlessly because the corresponding temporal formulae are so-called Sahlqvist formulae. And the important feature of Sahlqvist formulae is that they are *canonical* in the sense that they are validated by the frame underlying the canonical model defined in the proof of Theorem 2.1 (cf. Goldblatt [6] for analogous arguments in standard modal and temporal logic, or De Rijke and Venema [19] for the general picture). As a consequence we have the following:

Theorem 2.2 (Completeness) *Let $X \subseteq \{\mathrm{Ax3}, \mathrm{Ax4}, \mathrm{Ax5}\}$. Then MTL_0X is complete with respect to the class of frames satisfying the properties expressed by the axioms in X.*

Further natural properties like

Euclidicity: $\forall i, j, k, \alpha, \beta\, ((\mathrm{DIS}(i, \alpha, j) \wedge \mathrm{DIS}(i, \alpha + \beta, k)) \rightarrow \mathrm{DIS}(j, \beta, k))$,

which is represented in metric temporal logic by the formula (Montanari et al. [12]):

$$\Delta_\alpha \nabla_\beta p \rightarrow \nabla_{\alpha+\beta} p$$

can already be derived from $MTL_0\mathrm{Ax3}$.

In the case of Antisymmetry, we have to do more work. First of all, Antisymmetry is not expressible in the basic metric language. One can use a standard unfolding argument to prove this claim (as in ordinary modal logic). Despite the undefinability of Antisymmetry, we can prove a completeness result for the class of antisymmetric two-sorted frames: we will now show that MTL_0 is complete with respect to such frames; we use a technique based on Burgess' chronicle construction (cf. Burgess [2]).

Definition 2.3 Below we write $\rightsquigarrow_\alpha$ for the canonical displacement relation defined in the proof of Theorem 2.1: $\Sigma \rightsquigarrow_\alpha \Gamma$ if for all $\gamma \in \Gamma$, $\Delta_\alpha \gamma \in \Sigma$.

Let $\mathfrak{F} = (T, \mathfrak{D}; \mathrm{DIS})$ be a two-sorted frame, and g an interpretation of the algebraic terms on $\mathfrak{F}$. A *chronicle* τ on $\mathfrak{F}$ and g is a function τ such that τ assigns to each $i \in T$ an MCS $\tau(i)$.

A chronicle τ is *coherent* if for all α, $\mathrm{DIS}(i, \alpha, j)$ implies $\tau(i) \rightsquigarrow_\alpha \tau(j)$. Moreover, τ is *prophetic* (resp. *historic*) if it is coherent and satisfies condition 1 (resp. 2):

(i) if $\Delta_\alpha \phi \in \tau(i)$, then there exists j such that $\mathrm{DIS}(i, g(\alpha), j)$, and $\phi \in \tau(j)$;
(ii) if $\Delta_{-\alpha} \phi \in \tau(i)$, then there exists j such that $\mathrm{DIS}(j, g(\alpha), i)$, and $\phi \in \tau(j)$.

Finally, τ is *perfect* if it is both prophetic and historic.

Let V be a valuation in $(T, \mathfrak{D}; \mathrm{DIS}; g)$. The *induced chronicle* is a function τ_V such that $\tau_V(i) = \{\phi \mid i \in V(\phi)\}$, for each $i \in T$. It is easy to see that τ_V is always perfect. Conversely, if τ is a perfect chronicle, then it naturally induces a valuation V_τ defined by $V_\tau(p) = \{i \mid p \in \tau(i)\}$.

Lemma 2.4 *Let τ be a perfect chronicle on a two-sorted frame $(T, \mathfrak{D}; \mathrm{DIS})$. If $V = V_\tau$ is the valuation induced by τ, then $\tau = \tau_V$ is the chronicle induced by V, that is, $V(\phi) = \{i \mid \phi \in \tau(i)\}$. Any member of any $\tau(i)$ is thus satisfiable in $(T, \mathfrak{D}; \mathrm{DIS}; g)$.*

By definition, MTL_0 is complete for the class of all antisymmetric two-sorted frames iff every consistent formula ϕ is satisfiable on a model based on an antisymmetric two-sorted frame. By Lemma 2.4, this is equivalent to the existence of a perfect chronicle τ on some anti-symmetric two-sorted frame $(T, \mathfrak{D}; \mathrm{DIS})$ and an interpretation g such that $\phi \in \tau(i)$ for some $i \in T$. We now construct such T, $\mathfrak{D}$, DIS, g and τ.

Let T_∞ be a countably infinite set of time instants, and M the set of tuples $(T_n, \mathfrak{D}, \mathrm{DIS}_n, g, \tau_n)$ such that:

(a) T_n is a non-empty finite subset of T_∞;
(b) $\mathfrak{D}$ is the free abelian group over the set A, and g is the canonical interpretation, as in the proof of Theorem 2.1;
(c) $\mathrm{DIS}_n \subseteq T_n \times D^0 \times T_n$ is antisymmetric;
(d) τ_n is a coherent chronicle on $(T_n, \mathfrak{D}; \mathrm{DIS}_n)$ and g.

Definition 2.5 We say that a 5-tuple $\mu_n = (T_n, \mathfrak{D}, \mathrm{DIS}_n, \tau_n, g)$ in M *is extended by* a 5-tuple $\mu_m = (T_m, \mathfrak{D}, \mathrm{DIS}_m, \tau_m, g)$ in M if: (1) $T_n \subseteq T_m$; (2) $\mathrm{DIS}_n = \mathrm{DIS}_m \cap (T_n \times D^0 \times T_n)$; and (3) $\tau_n \subseteq \tau_m$.

A conditional requirement of the form specified in Definition 2.3 (1) (resp.

(2)) is called *unborn* for $\mu_n = (T_n, \mathfrak{D}, \mathrm{DIS}_n, \tau_n, g) \in M$, if its antecedent is not fulfilled. This is the case when $i \notin T_n$, or $i \in T_n$, but $\Delta_\alpha \phi \notin \tau_n(i)$ (resp. $\Delta_{-\alpha}\phi \notin \tau_n(i)$). It is called *alive* for μ_n if its antecedent is fulfilled, but its consequent is not. This is the case when $i \in T_n$ and $\Delta_\alpha\phi \in \tau_n(i)$ (resp. $\Delta_{-\alpha}\phi \in \tau_n(i)$), but there is no $j \in T_n$ such that $\mathrm{DIS}_n(i, g(\alpha), j)$ (resp. $\mathrm{DIS}_n(j, g(\alpha), i)$), and $\phi \in \tau_n(j)$. Finally, it is called *dead* if its consequent is fulfilled.

Lemma 2.6 *Consider $\mu_n = (T_n, \mathfrak{D}, \mathrm{DIS}_n, \tau_n, g) \in M$. For any requirement as in Definition 2.3 (1) (resp. (2)) which is alive for μ_n, there exists an extension $\mu_m \in M$ for which it is dead.*

Proof. Consider a requirement as in Definition 2.3 (1). Assume $i \in T_n$ and $\Delta_\alpha\phi \in \tau_n(i)$. By the proof of Theorem 2.1 there exists an MCS Γ such that $\tau_n(i) \rightsquigarrow_\alpha \Gamma$ and $\phi \in \Gamma$. Define μ_m as follows:

- $T_m = T_n \cup \{j\}$;
- $\mathrm{DIS}_m = \mathrm{DIS}_n \cup \{(i, \beta, j)\}$;
- $\tau_m = \tau_n \cup \{(j, \Gamma)\}$. $\dashv$

Theorem 2.7 (Completeness) *MTL_0 is complete with respect to the class of all antisymmetric two-sorted frames.*

Proof. Let ϕ_0 be a consistent formula. We construct an antisymmetric two-sorted frame $\mathfrak{F} = (T, \mathfrak{D}; \mathrm{DIS})$, an interpretation g, and a perfect chronicle τ on $\mathfrak{F}$ and g such that $\phi_0 \in \tau(i_0)$ for some $i_0 \in T$.

First, let $\mathfrak{D}$ be the free algebra over the set A, and g the canonical interpretation. Second, take a countably infinite set S, and fix an enumeration $i_0, i_1, \ldots$ of S, and an enumeration $\phi_0, \phi_1, \ldots$ of all formulae. Then, to each conditional requirement of the form specified in Definition 2.3 (1) (resp. (2)), with $i = i_n$ and $\phi = \phi_m$, we assign the number $2 \cdot 5^n \cdot 7^m$ (resp. $3 \cdot 5^n \cdot 7^m$). Moreover, we take an MCS Γ with $\phi_0 \in \Gamma$, and define $\mu_0 = (T_0, \mathfrak{D}, \mathrm{DIS}_0, \tau_0, g)$, where $T_0 = \{i_0\}$, $\mathrm{DIS}_0 = \emptyset$, and $\tau_0 = \{(i_0, \Gamma)\}$. If μ_n is defined, we consider the requirement with the least code number among all requirements which are alive for μ_n. By Lemma 2.6 we can choose an extension μ_{n+1} of μ_n for which that requirement is dead.

Let T, DIS and τ be respectively defined as follows: $T = \bigcup_n T_n$, $\mathrm{DIS} = \bigcup_n \mathrm{DIS}_n$, and $\tau = \bigcup_n \tau_n$. $(T, \mathfrak{D}; \mathrm{DIS})$ is an antisymmetric two-sorted frame and τ is a perfect chronicle on this frame and g. $\dashv$

When metric temporal logic is employed for specifying real-time systems, one further condition is usually imposed on the displacement relation. Since the behavior of real-time systems is essentially modeled in terms of infinite sequences of states/events, it is natural to require the closure of the temporal domain under displacements. Such a requirement is captured by imposing seriality of the displacement relation:

Seriality: $\forall i, \alpha \exists j \, \mathrm{DIS}(i, \alpha, j)$,

which can be axiomatized as

(Ax6) $\nabla_\alpha p \to \Delta_\alpha p$ (or, equivalently, $\Delta_\alpha \top$). (seriality)

Again, the basic completeness result can be extended without effort because the corresponding temporal formula is a Sahlqvist formula. Moreover, it is interesting to study the interplay between Seriality and the properties of Transitivity, Quasi-functionality and Reflexivity.

The addition of Seriality turns Quasi-functionality into Functionality:

$$\Delta_\alpha p \leftrightarrow \nabla_\alpha p.$$

Therefore, each occurrence of Δ_α can be replaced by ∇_α. This allows us, for instance, to merge Transitivity and Euclidicity

$$\nabla_{\alpha+\beta} p \leftrightarrow \nabla_\alpha \nabla_\beta p.$$

Moreover, it is easy to see that the addition of Seriality forces ∇_α and $\neg$ to commute

$$\nabla_\alpha \neg p \leftrightarrow \neg \nabla_\alpha p.$$

Given the Distributivity of ∇_α over $\wedge$, we conclude that ∇_α distributes over all truth functional connectives.

3 Two-sorted frames based on ordered groups

For a variety of application purposes, our basic calculus and its semantics need to be extended with orderings. In particular, a linear order on the temporal domain is needed in many application areas; for instance, in real-time specifications we want to guarantee that between any two time instants there is a unique displacement. In the following, we achieve this by adding a total ordering on the displacement domain D.

In the definition of a two-sorted frame we replace the abelian component by an *ordered* abelian group. That is, by a structure $\mathfrak{D} = (D, +, -, 0, <)$, where

$(D, +, -, 0)$ is an abelian group, and $<$ is an irreflexive, asymmetric, transitive and linear relation that satisfies the comparability property $(viii)$ below:

(v)	$\neg(\alpha < \alpha)$
(vi)	$\neg(\alpha < \beta \wedge \beta < \alpha)$
(vii)	$\alpha < \beta \wedge \beta < \gamma \rightarrow \alpha < \gamma$
$(viii)$	$\alpha < \beta \vee \alpha = \beta \vee \beta < \alpha.$

Next, there are two axioms expressing the relation between $+$ and $-$, and $<$:

(ix)	$\alpha < \beta \rightarrow \alpha + \gamma < \beta + \gamma$
(x)	$\alpha < \beta \rightarrow -\beta < -\alpha.$

One can use various languages to talk about ordered abelian groups. We do not have any clear preference, as long as the language used can be equipped with a complete axiomatization. We will simply use full first-order logic over $=, <$ to reason about the ordered abelian component of our two-sorted frames.

To be precise, our metric temporal language for talking about two-sorted frames based on an ordered abelian group, has a first-order component built up from terms in $T(A)$ and predicate symbols $=$ and $<$; its temporal component is as before.

The interpretation of this language on two-sorted frames based on an ordered abelian group is fairly straightforward: the first-order component is interpreted on the group, and the temporal component on the temporal domain. Validity in this language is easily axiomatized; for the displacement component we take the axioms and rules of identity, ordered abelian groups, strict linear order together with any complete calculus for first-order logic; and for the temporal component we take the same axioms as in the case of MTL_0: axioms (Ax1), (Ax2) and the rules modus ponens, (NEC), (REP) and (LIFT). Let MTL_1 denote the resulting two-sorted calculus.

Theorem 3.1 (Completeness) *MTL_1 is complete with respect to the class of two-sorted frames based on ordered abelian groups.*

Proof. We can simply repeat the proof of Theorem 2.1 here, and replace the free algebra construction of the displacement domain by a Henkin construction for first-order logic. $\dashv$

3.1 Deriving a temporal ordering

Given that we have an ordering $<$ on the algebraic component of our frames, a natural definition for an ordering $\ll$ on the temporal frame suggests itself:

$$i \ll j \quad \text{iff} \quad \text{for some } \alpha > 0,\ \mathrm{DIS}(i, \alpha, j). \tag{1}$$

So i and j are $\ll$-related if there exists a positive displacement between them. Using the relation $\ll$, we can define the qualitative operators F, P of non-metric temporal logic as follows:

$$\mathfrak{M}, i \Vdash F\phi := \exists j\, (i \ll j \wedge j \Vdash \phi) \text{ and } \mathfrak{M}, i \Vdash P\phi := \exists j\, (j \ll i \wedge j \Vdash \phi).$$

We will not consider this extension in the present paper.

Additional properties.

The definition of $\ll$ given in (1) does not produce a temporal ordering with all the natural properties that we usually expect it to have. In particular, unless we put further restrictions on the relation of temporal displacement, $\ll$ will not be a strict linear order, and there may be time instants without a unique temporal distance between them.

To repair this situation, we assume that the displacement relation DIS satisfies the following properties: transitivity, quasi-functionality, reflexivity (as defined in Section 2), and total connectedness and quasi-functionality w.r.t. the second argument:

(xi) $\quad \forall i, j \exists \alpha\, \mathrm{DIS}(i, \alpha, j)$ (total connectedness)

(xii) $\quad \forall i, j, \alpha, \beta\, (\mathrm{DIS}(i, \alpha, j) \wedge \mathrm{DIS}(i, \beta, j) \rightarrow \alpha = \beta)$
(quasi-functionality w.r.t. the 2nd argument).

Given these assumptions on the displacement relation, we can show that the temporal relation $\ll$ as defined in (1) is a strict linear order. To see that $\ll$ is transitive, assume that $i \ll j \ll k$. Then there exist α, β with $\mathrm{DIS}(i, \alpha, j)$ and $\mathrm{DIS}(j, \beta, k)$. Hence $\mathrm{DIS}(i, \alpha + \beta, k)$ and $i \ll k$.

For irreflexivity, assume $i \ll i$. Then $\mathrm{DIS}(i, \alpha, i)$ for some $\alpha > 0$. By reflexivity of DIS, $\mathrm{DIS}(i, 0, i)$, hence, by quasi-functionality of the second argument, $\alpha = 0$ — a contradiction.

For asymmetry, assume $i \ll j \ll i$. Then $\mathrm{DIS}(i, \alpha, j)$ and $\mathrm{DIS}(j, \beta, i)$ for some α, $\beta > 0$. Then $\mathrm{DIS}(j, -\alpha, i)$ and so $\beta = -\alpha$, by quasi-functionality of the second argument again, which yields a contradiction.

Finally, to prove totality, take any two i, j. By total connectedness there exists α such that $\mathrm{DIS}(i, \alpha, j)$. By axiom $(viii)$, $\alpha > 0 \vee \alpha = 0 \vee 0 > \alpha$. If $\alpha > 0$, then $i \ll j$. If $\alpha = 0$, then by quasi-functionality and reflexivity of DIS, $i = j$. And if $\alpha < 0$, then $-\alpha > 0$ and $\mathrm{DIS}(j, -\alpha, i)$, so $j \ll i$.

Call a two-sorted frame *nice* if it is transitive, reflexive, totally-connected, and quasi-functional in both the 2nd and 3rd argument of its displacement relation; a model is *nice* if it is based on an nice frame.

The next obvious question is: can we characterize the nice frames in the language of MTL_1? The answer is 'no'. To see this, we adapt two truth preserving constructions from standard modal logic to the present setting. For the sake of simplicity, we confine ourselves to frames that share the same displacement domain; however, the definitions are easily generalized to the general case.

Definition 3.2 Let $\mathfrak{F} = (T, \mathfrak{D}; \mathrm{DIS})$ and $\mathfrak{F}' = (T', \mathfrak{D}; \mathrm{DIS}')$ be two-sorted frames. The *disjoint union* of $\mathfrak{F}$ and $\mathfrak{F}'$ is the two-sorted frame $\mathfrak{F} \uplus \mathfrak{F}' = (T'', \mathfrak{D}, \mathrm{DIS}'')$. Here, T'' is the disjoint union of T and T', while the displacement relation DIS'' is just the disjoint union of DIS and DIS'.

Theorem 3.3 *Let $\mathfrak{F}$ and $\mathfrak{F}'$ be two-sorted frames, and $\mathfrak{F} \uplus \mathfrak{F}'$ their disjoint union. For all algebraic terms α, β, if $\mathfrak{F} \models \alpha = \beta$ and $\mathfrak{F}' \models \alpha = \beta$, then $\mathfrak{F} \uplus \mathfrak{F}' \models \alpha = \beta$, and, for all formulae ϕ, if $\mathfrak{F} \models \phi$ and $\mathfrak{F}' \models \phi$, then $\mathfrak{F} \uplus \mathfrak{F}' \models \phi$.*

Theorem 3.4 *There is no modal formula ϕ that expresses total connectedness of two-sorted frames.*

Proof. We prove the claim by showing that the existence of such a formula would violate preservation of truth under disjoint union. An intuitive account of this negative conclusion can be given noticing that disjoint unions are not totally connected frames "by definition".

Suppose that there exists a formula ϕ expressing total connectedness. By Theorem 3.3, it follows that ϕ is valid in the disjoint union $\mathfrak{F} \uplus \mathfrak{F}' = (T'', \mathfrak{D}; \mathrm{DIS}'')$ of any two frames $\mathfrak{F}$ and $\mathfrak{F}'$ validating ϕ. Take $i \in \mathfrak{F}$ and $j \in \mathfrak{F}'$; by definition of $\mathfrak{F} \uplus \mathfrak{F}'$, it follows that there exists no $\alpha \in \mathfrak{D}$ such that $\mathrm{DIS}''(i, \alpha, j)$. $\dashv$

Definition 3.5 Let $\mathfrak{F} = (T, \mathfrak{D}; \mathrm{DIS})$ and $\mathfrak{F}' = (T', \mathfrak{D}; \mathrm{DIS}')$ be two-sorted frames. A *bounded morphism* from $\mathfrak{F}$ to $\mathfrak{F}'$ is a mapping $f : T \to T'$ such that:

(i) if $\mathrm{DIS}(i, \alpha, j)$, then $\mathrm{DIS}'(f(i), \alpha, f(j))$;
(ii) if $\mathrm{DIS}'(f(i), \alpha, j')$, then for some $j \in T$ both $f(j) = j'$ and $\mathrm{DIS}(i, \alpha, j)$ hold.

Theorem 3.6 *Let $\mathfrak{F}$ and $\mathfrak{F}'$ be two-sorted frames, and f a surjective bounded* *morphism from $\mathfrak{F}$ to $\mathfrak{F}'$. For all algebraic terms α, β, if $\mathfrak{F} \models \alpha = \beta$, then $\mathfrak{F}' \models \alpha = \beta$. And, for all formulae ϕ, if $\mathfrak{F} \models \phi$, then $\mathfrak{F}' \models \phi$.*

Theorem 3.7 *There is no modal formula ϕ that expresses quasi-functionality w.r.t. the second argument of the displacement relation.*

Proof. We prove the claim by showing that the existence of such a formula would violate preservation of truth under bounded morphisms. Suppose that there exists a formula ϕ expressing quasi-functionality with respect to the second argument of the accessibility relation.

Consider the two-sorted frames $\mathfrak{F} = (T, \mathfrak{D}; \mathrm{DIS})$ and $\mathfrak{F}' = (T', \mathfrak{D}; \mathrm{DIS}')$ such that $T = \{i_1, i_2, i_3, i_4, j_1, j_2, j_3, j_4\}$, $T' = \{i', j'\}$, DIS contains (i_1, 1, j_1), (i_1, 2, j_3), (i_2, 2, j_1), (i_2, 1, j_3), (i_3, 1, j_2), (i_3, 2, j_4), (i_4, 1, j_4), and (i_4, 2, j_2), together with the converse triplets (j_1, -1, i_1), (j_3, -2, i_1), and so on, while $\mathrm{DIS}' = \{(i', 1, j'), (i', 2, j'), (j', -2, i'), (j', -1, i')\}$. Clearly, $\mathfrak{F}$ satisfies the requirement of quasi-functionality, while $\mathfrak{F}'$ does not.

Now, consider the mapping $f : T \to T'$ defined by $f(i_1) = f(i_2) = f(i_3) = f(i_4) = i'$, $f(j_1) = f(j_2) = f(j_3) = f(j_4) = j'$. It is easy to verify that f is a surjective bounded morphism. Then, from $\mathfrak{F} \models \phi$ Theorem 3.6 allows us to infer that $\mathfrak{F}' \models \phi$, and we have a contradiction. $\dashv$

Enriching the language.

Given that nice frames cannot be characterized in the language of MTL_1, a possible way out consists in enriching the language to enable us to express the properties of total connectedness and quasi-functionality of the displacement relation in its 2nd argument. We briefly show that those properties can actually be expressed by adding to the language the future and past operators F, P, the difference operator $\mathcal{D}$, and by allowing that information from the temporal domain is lifted to the displacement domain by permitting the two languages to be mixed.

First, the *difference operator* (de Rijke [18]) is a unary modal operator $\mathcal{D}$ that allows us to model unbounded jumps. Its semantic interpretation is defined as follows:

$$(\mathfrak{F}, V), i \Vdash \mathcal{D}\phi \text{ iff } \exists j \, (j \neq i \wedge (\mathfrak{F}, V), j \Vdash \phi),$$

with dual $\overline{\mathcal{D}}$:

$$(\mathfrak{F}, V), i \Vdash \overline{\mathcal{D}}\phi \text{ iff } \forall j \, (j \neq i \to (\mathfrak{F}, V), j \Vdash \phi).$$

The difference operator and its dual allow us to define three derived unary operators $\mathcal{E}$, its dual $\mathcal{A}$, and $\mathcal{U}$ that respectively model truth in at least one world, truth in all worlds, and truth in a unique world:

$$\mathcal{E}\phi \equiv \mathcal{D}\phi \vee \phi,\ \mathcal{A}\phi \equiv \overline{\mathcal{D}}\phi \wedge \phi,\ \text{and } \mathcal{U}\phi \equiv \mathcal{E}(\phi \wedge \neg\mathcal{D}\phi).$$

In a language in which the algebraic and temporal formulae may be mixed, properties (xi) and (xii) can be expressed by means of the qualitative operators F, P and $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{U}$ as follows:

(Ax7) $\mathcal{D}p \rightarrow Fp \vee Pp$ (total connectedness of DIS)
(Ax8) $\mathcal{U}p \wedge \mathcal{U}q \rightarrow (\mathcal{E}(p \wedge \Delta_\alpha q) \wedge \mathcal{E}(p \wedge \Delta_\beta q) \rightarrow \alpha = \beta)$
(quasi-functionality of DIS w.r.t. the 2nd argument).

However, we prefer to remain within the original language of MTL_1 and reason about nice frames there, mainly because adding the axioms Ax7 and Ax8 forces us to give up the simplicity of the basic calculus and to include non-standard derivation rules to govern the difference operator. As we will show below, the logic of nice frames can be captured in the original language.

Completeness for nice frames.

Instead of increasing the expressive power of metric temporal logic, we leave it as it stands, and prove a completeness result for nice frames in the old language. We will do this in two steps. We first prove completeness with respect to totally connected frames via some sort of generated submodel construction, and then we prove the full result.

Here's the idea for the case of total connectedness. Let $\mathfrak{F} = (T, \mathfrak{D}; \mathrm{DIS})$ be a two-sorted frame. The *master relation* on $\mathfrak{F}$ is defined by

$$(i, j) \in \mathit{Master} \quad \text{iff} \quad (i, j) \in (\ll \cup \gg)^*.$$

Thus i, j are in the master relation iff there exists a zigzag path along the displacement relation from i to j in the following sense:

$$\mathrm{DIS}(i, \alpha_1, j_1), \mathrm{DIS}(j_1, \alpha_2, j_2), \ldots, \mathrm{DIS}(j_n, \alpha_{n+1}, j),$$

where $\alpha_1, \ldots, \alpha_n \in D$, and $j_1, \ldots, j_n \in T$.

A *point-generated component* of a model $\mathfrak{M} = (T, \mathfrak{D}; \mathrm{DIS}; V, g)$ is a model $(T'$,

$\mathfrak{D}$; DIS$'$; g, V') such that for some $i \in T$,

- $T' = \{j \in T \mid (i,j) \in \mathit{Master}\}$
- $\mathrm{DIS}' = \mathrm{DIS} \cap (T' \times D \times T')$
- $V'(p) = V(p) \cap T'$, for all p.

Proposition 3.8 *Let $\mathfrak{M}'$ be a point-generated component of a model $\mathfrak{M}$ based on a two-sorted frame with ordered abelian group. If $\mathfrak{M}$ has a transitive displacement relation, then $\mathfrak{M}'$ has a transitive and totally connected displacement relation.*

Lemma 3.9 *Let $\mathfrak{M}'$ be a point-generated component of a two-sorted model $\mathfrak{M}$. Then $\mathfrak{M}'$ satisfies exactly the same algebraic formulae as $\mathfrak{M}$. Moreover, for all $i \in T'$ and for all temporal formulae ϕ we have $\mathfrak{M}, i \Vdash \phi$ iff $\mathfrak{M}', i \Vdash \phi$.*

MTL_1Ax3 extends MTL_1 with the transitivity axiom $\nabla_{\alpha+\beta}p \to \nabla_\alpha \nabla_\beta p$.

Theorem 3.10 (Completeness) *MTL_1Ax3 is sound and complete with respect to the class of two-sorted frames based on ordered abelian groups whose displacement relation is transitive and totally connected.*

Proof. We only prove completeness, and to establish this it suffices to show that every MTL_1Ax3-consistent set of formulae is satisfiable in a model based on a frame of the right kind.

Let Γ be a MTL_1Ax3-consistent set of formulae. By a Sahlqvist style argument (cf. Theorem 2.2) it is easily seen that Γ is satisfiable in a model $\mathfrak{M}$ based on a two-sorted frame with a transitive displacement relation, say at a time instant i. Let $\mathfrak{M}'$ be a point-generated component of $\mathfrak{M}$ that contains i. By Proposition 3.8 $\mathfrak{M}'$ has a transitive and totally connected displacement relation, and by Lemma 3.9 we have $\mathfrak{M}', i \Vdash \Gamma$, as required. $\dashv$

To prove completeness w.r.t. the class of nice frames, we need to carry out a second construction. First, call a two-sorted frame *almost nice* if it is transitive, reflexive, totally-connected, and quasi-functional in the 3rd argument of its displacement relation; a model is *almost nice* if it is based on an almost nice frame. So a frame is nice if it is almost nice and quasi-functional in the 2nd argument of its displacement relation.

Now, to build a nice model we will take an almost nice model and carefully unfold it. To be precise, let $\mathfrak{M} = (T, \mathfrak{D}; \mathrm{DIS}; V, g)$ be an almost nice model, and let $i \in T$. The *i-stratification of* $\mathfrak{M}$ is the model $\mathfrak{M}' = (T', \mathfrak{D}; \mathrm{DIS}'; V', g)$ which is defined as follows:

$$T' = \{(0,i)\} \cup \{(\alpha, j) \mid \mathrm{DIS}(i, \alpha, j) \ \text{ in } \mathfrak{M}\}$$

$$
\begin{aligned}
\mathrm{DIS}_0 &= \{((0,i),\alpha,(\alpha,j)) \mid (\alpha,j)\in T'\}\cup\{((\alpha,j),-\alpha,(0,i)) \mid (\alpha,j)\in T'\}\\
\mathrm{DIS}_1 &= \{((\alpha,j),\beta-\alpha,(\beta,k)) \mid (\alpha,j),(\beta,k)\in T'\}\\
\mathrm{DIS}' &= \mathrm{DIS}_0\cup\mathrm{DIS}_1\\
V'(p) &= \{(\alpha,j)\in T' \mid j\in V(p)\}.
\end{aligned}
$$

Observe that $\mathrm{DIS}_0 \subseteq \mathrm{DIS}_1$.

Proposition 3.11 *Let $\mathfrak{M}$ be an almost nice model, and let $i \in \mathfrak{M}$. The i-stratification of $\mathfrak{M}$ is nice.*

Proof. We first observe that for any pairs (α, j), $(\gamma, k) \in T'$, and $\beta \in \mathfrak{D}$, if it holds that $\mathrm{DIS}'((\alpha,j),\beta,(\gamma,k))$ then $\beta = \gamma - \alpha$.

Now, to prove the proposition, we have to check the nice-ness properties. First of all, we show that $\mathrm{DIS}'((\alpha,j),\beta,(\gamma,k))$ implies $\mathrm{DIS}'((\gamma,k),-\beta,(\alpha,j))$. By the observation $\beta = \gamma - \alpha$. Also, (α, j), $(\gamma, k) \in T'$ implies $\mathrm{DIS}'((\gamma,k),\alpha-\gamma,(\alpha,j))$, that is, $\mathrm{DIS}'((\gamma,k),-\beta,(\alpha,j))$.

Next, we show that DIS' is reflexive. As $\mathfrak{M}$ is assumed to be reflexive, we have $\mathrm{DIS}(i,0,i)$, hence $\mathrm{DIS}((0,i),0,(0,i))$. As to other points $(\alpha,j) \in T'$, $\mathrm{DIS}_1((\alpha,j),\alpha-\alpha,(\alpha,j))$, by definition of DIS_1, and thus $\mathrm{DIS}'((\alpha,j),0,(\alpha,j))$.

To see that DIS' is quasi-functional with respect to its 3rd argument, assume that both $\mathrm{DIS}'((\alpha,j),\beta,(\gamma,k))$ and $\mathrm{DIS}'((\alpha,j),\beta,(\gamma',k'))$ hold. We need to show that $\gamma = \gamma'$ and $k = k'$. First of all, $\beta = \gamma - \alpha = \gamma' - \alpha$, hence $\gamma = \gamma'$. Therefore, $\mathrm{DIS}(i,\gamma,k)$ and $\mathrm{DIS}(i,\gamma,k')$. So by the assumption that DIS is quasi-functional in its 3rd argument, $k = k'$.

Given that $\mathfrak{M}$ is total, the totality of its i-stratifications is immediate.

Transitivity of $\mathfrak{M}'$ may be established as follows: assume that

$$\mathrm{DIS}'((\alpha,j),\beta,(\gamma,k)) \text{ and } \mathrm{DIS}'((\gamma,k),\beta',(\delta,l))$$

both hold. Then $\mathrm{DIS}'((\alpha,j),\delta-\alpha,(\delta,l))$. As $\beta+\beta' = (\gamma-\alpha)+(\delta-\gamma)$, we are done.

Finally, to prove quasi-functionality of DIS' in its 2nd argument, assume that we have both $\mathrm{DIS}'((\alpha,j),\beta,(\gamma,k))$ and $\mathrm{DIS}'((\alpha,j),\beta',(\gamma,k))$. It follows that $\beta = \gamma - \alpha = \beta'$. $\dashv$

Proposition 3.12 *Let $\mathfrak{M}$ be an almost nice model, and let $\mathfrak{M}'$ be an i-stratification of $\mathfrak{M}$. For all formulae ϕ, j in $\mathfrak{M}$, and (α, j) in $\mathfrak{M}'$, we have $\mathfrak{M}, j \Vdash \phi$ iff $\mathfrak{M}', (\alpha, j) \Vdash \phi$.*

Proof. This is by induction on ϕ. The base case and the boolean cases are trivial. So consider a temporal formula $\Delta_\gamma \psi$. Assume first that $j \Vdash \Delta_\gamma \psi$. Then there exists k with $\mathrm{DIS}(j, \gamma, k)$. Now, let α be such that $(\alpha, j) \in T'$. Then $\mathrm{DIS}(i, \alpha, j)$, and hence $\mathrm{DIS}(i, \alpha + \gamma, k)$ and $(\alpha + \gamma, k) \in T'$. By definition, $\mathrm{DIS}_0((0,i), \alpha, (\alpha, j))$ and $\mathrm{DIS}_0((0,i), \alpha+\gamma, (\alpha+\gamma, k))$. But then

$$\mathrm{DIS}'((\alpha, j), \gamma, (\alpha+\gamma, k)).$$

By induction hypothesis, $(\alpha+\gamma, k) \Vdash \psi$, hence $(\alpha, j) \Vdash \Delta_\gamma \psi$.

Conversely, assume that $(\alpha, j) \Vdash \Delta_\gamma \psi$. Then there exists $(\beta, k) \in T'$ such that both $\mathrm{DIS}'((\alpha, j), \gamma, (\beta, k))$ and $(\beta, k) \Vdash \psi$ hold. Hence $\gamma = \beta - \alpha$. By construction we must have $\mathrm{DIS}(i, \alpha, j)$ and $\mathrm{DIS}(i, \beta, k)$ and hence $\mathrm{DIS}(j, \beta - \alpha, k)$. As $k \Vdash \psi$ (by induction hypothesis) and $\gamma = \beta - \alpha$, this implies $j \Vdash \Delta_\gamma \psi$, as required. $\dashv$

We are ready now for a completeness result for the class of nice frames. Let MTL_2 denote the extension of MTL_1 with axioms Ax3, Ax4 and Ax5 (expressing transitivity, quasi-functionality of DIS in its 3rd argument, and reflexivity, respectively). By an easy adaptation of the proof of Theorem 3.10, MTL_2 is sound and complete w.r.t. the class of almost nice frames.

Theorem 3.13 (Completeness) *MTL_2 is sound and complete with respect to the class of nice frames.*

Proof. We only show that every MTL_2-consistent set of temporal formulae is satisfiable on a nice model. Let Γ be such a set. By earlier remarks Γ is satisfiable on an almost nice model at some time instant i. Let $\mathfrak{M}'$ be the i-stratification of $\mathfrak{M}$. By Propositions 3.11 and 3.12 $\mathfrak{M}'$ is a nice model that satisfies Γ at i. $\dashv$

3.2 Adding discreteness

One natural specialization of the metric temporal logic of linear orders consists in the addition of discreteness. As with the earlier addition of an ordering, we will constrain the domain of temporal displacements to be discrete and show that the discreteness of the temporal domain necessarily follows.

The discreteness of the domain of displacements is expressed by the following axiom:

(xiii) $\forall \alpha \exists \beta\, (\alpha < \beta \wedge \forall \gamma\, (\alpha < \gamma \rightarrow (\beta = \gamma \vee \beta < \gamma)) \wedge \forall \delta\, (\delta < \alpha \rightarrow (\beta' = \delta \vee \delta < \beta')))$.

The discreteness of the temporal domain follows as shown by the following proposition.

Proposition 3.14 *Let $\mathfrak{F} = (T, \mathfrak{D}; \mathrm{DIS})$ be a two-sorted frame based on a discrete ordered abelian group $\mathfrak{D}$. For all $i, j \in T$, there exist only finitely many k such that $i \ll k \ll j$.*

Proof. Left to the reader. $\dashv$

An interesting consequence of restricting ourselves to discrete temporal domains is that bounded response and invariance properties like

$$p \to \exists x\,(0 \leq x < \delta \wedge \Delta_x q) \text{ and } p \to \forall x\,(0 \leq x < \delta \to \nabla_x q)$$

become expressible in the basic systems of metric temporal logics (devoid of quantification and mixed formulae).

The restricted quantification involved in bounded response properties can indeed be replaced by a finite disjunction of formulae of the form $\Delta_\alpha q$ (one disjunct for each displacement α — there exists a finite number of such displacements — such that $0 \leq \alpha < \delta$). Analogously, the restricted quantification involved in bounded invariance properties can be replaced by a finite conjunction of formulae of the form $\Delta_\alpha q$.

On the other hand, unrestricted quantification involved in unbounded versions of response and invariance properties like

$$p \to \exists x\,(0 < x \wedge \Delta_x q) \text{ and } p \to \forall x\,(0 < x \to \nabla_x q),$$

as well as nested quantification in the formula

$$\exists x\,(0 < x \wedge \Delta_x p \wedge \forall y\,(0 \leq y < x \to \nabla_y q))$$

cannot be captured by basic metric temporal logics. This deficiency can be overcome by using the qualitative operators F, P and/or the operators *Since* and *Until*. The above introduced properties can indeed be represented as $p \to Fp$, $p \to Gp$, and q *Until* p, respectively. However, this solution requires the addition of the axioms for the qualitative operators and of the axioms constraining the relationships between the qualitative operators and the operator of temporal realization, as well as a completeness proof for the resulting logical system. We do not consider such extensions in this paper.

4 Increased interaction

So far we have only considered simple languages that allow us to lift information from the algebraic domain to the temporal domain but not vice versa. For application purposes they have to be extended. As an example, consider an automatic reply system that, whenever it receives a message, sends an acknowledgment with a delay less than δ. Such a bounded response property can be represented by the following formula:

$$p \to \exists x\,(0 \leq x < \delta \wedge \Delta_x q),$$

where p and q denote the receipt of the message and its acknowledgment, respectively. However, the languages considered so far cannot express such conditions as they lack quantification and constrain displacements to occur as parameters of the operator of temporal realization only.

In this section, we will show how the ability of freely mixing temporal and displacement formulae enables us to exploit more complex ways of interaction between the two domains. Our first goal is to define the logic $Q\text{-}MTL_0$ and its language.

Language

Let A denote a set of algebraic constants, and X a collection of algebraic variables; a denotes a typical element of A, x a typical element of X. The set of algebraic terms $T(X \cup A)$ is built up as follows:

$$\alpha ::= 0 \mid a \mid x \mid \alpha + \alpha \mid -\alpha.$$

Using this, we define the formulae of $Q\text{-}MTL_0$:

$$\phi ::= p \mid \neg\phi \mid \alpha = \beta \mid \alpha < \beta \mid \phi \wedge \phi \mid \Delta_\alpha \phi \mid \forall x\, \phi.$$

Thus, we allow quantification over algebraic variables and free mixing of algebraic formulae and temporal propositional symbols.

Structures

Starting from an ordered two-sorted frame $\mathfrak{F} = (T, \mathfrak{D}; \mathrm{DIS})$ we arrive at a $Q\text{-}MTL_0$-model by adding a valuation V and an interpretation function g for the algebraic terms, as in Section 2. What remains to be defined is the way we evaluate our new mixed formulae at time instances. For the atomic case we stipulate the obvious definition:

$$\mathfrak{M}, i \Vdash \alpha = \beta \text{ iff } g(\alpha) = g(\beta)$$
$$\mathfrak{M}, i \Vdash \alpha < \beta \text{ iff } g(\alpha) < g(\beta).$$

Thus, the truth value of formulae of the form $\alpha = \beta$ and $\alpha < \beta$ is determined by referring only to the algebraic component.

Next, to evaluate quantified formulae $\forall x\, \phi$ at a point in time, we write $g =_x g'$ to denote that the assignments g and g' agree on all algebraic variables except maybe x. Then

$$(\mathfrak{F}; V, g), i \Vdash \forall x\, \phi \text{ iff } (\mathfrak{F}; V, g'), i \Vdash \phi,$$

for all assignments g' such that $g =_x g'$.

Remark 4.1 Note that in the traditional terminology from quantified modal logic, our semantic structures implement a *fixed-domain* approach with a *rigid* (objectual) interpretation of terms (cf. Garson [5]. Indeed, we assume that there exists a single domain of quantification for all time points which contains all the possible values for displacements.

An example

Consider a traffic light controller C. When the request button is pushed, the controller makes a pedestrian light turn green within a given time bound after which the light remains green for a certain amount of time (cf. Henzinger et al. [8]). Moreover, assume that C takes a unit of time to switch the light and that the time needed for its internal operations is negligible.

We require that C satisfies the following conditions:

(i) whenever a pedestrian pushes the request button ('request is true'), then the light is green within 5 time units and remains green for at least 10 time units (this condition guarantees that no pedestrian waits for more than 5 time units, and that he or she is given at least 10 time units to cross the road);
(ii) whenever request is true, then it is false within 20 time units (this condition ensures that the request button is reset);
(iii) whenever request has been false for 20 time units, the light is red (this condition should prevent the light from always being green).

The behavior of C can be formally specified in $Q\text{-}MTL_0$ as the conjunction of the following formulae:

$$\begin{aligned}
&\mathit{request} \;\rightarrow\; \exists x(0 < x \leq 5 \wedge \forall y(x \leq y < x+10 \rightarrow \nabla_y \mathit{lightIsGreen})),\\
&\mathit{request} \;\rightarrow\; \exists z(0 \leq z \leq 20 \wedge \Delta_z \neg \mathit{request}),\\
&\forall x(0 \leq x < 20 \rightarrow \nabla_x \neg \mathit{request}) \rightarrow \nabla_{20} \mathit{lightIsRed},
\end{aligned}$$

together with a formula stating that at each time instant the traffic light is either red or green:

$$\mathit{lightIsGreen} \leftrightarrow \neg \mathit{lightIsRed}.$$

Different implementations of C, all satisfying the given specification, can be obtained by making different assumptions about the value of temporal parameters, e.g., by varying the delay between requests and resets.

It is worth noting that, even if there are no restrictions on the frequency of requests, the above specification is appropriate only if that frequency is low; otherwise, it may happen that switching the light to red is delayed indefinitely.

To overcome this problem, we can constrain the duration of the periods during which the traffic light is green and those during which it is red. As an example, we can replace conditions (i)–(iii) by the following ones:

(iv) whenever a pedestrian pushes the request button and the light has been red for at least 20 time units, then the light is green within 5 time units and for at least 20 time units;
(v) whenever a pedestrian pushes the request button and the light has been red for x time units, with x less than 20 time units, then the light is green within $(20 - x) + 5$ time units and for at least 20 time units;
(vi) the light cannot be green for more than 20 time units;
(vii) the light must be red for at least 20 time units.

Conditions (iv)–(vii) can be specified in $Q\text{-}MTL_0$ as follows, using the event *pushButton* instead of the property *request*:

- $\mathit{pushButton} \wedge \forall x\,(-20 < x < 0 \rightarrow \nabla_x \mathit{lightIsRed}) \rightarrow$
  $\exists y\,(0 < y \leq 5 \wedge \forall z\,(y \leq z < y+20 \rightarrow \nabla_z \mathit{lightIsGreen})),$
- $\forall x\,(\mathit{pushButton} \wedge -20 < x < 0 \wedge \Delta_x \mathit{lightIsGreen} \wedge \forall y(x < y \leq 0 \rightarrow \nabla_y \mathit{lightIsRed}) \rightarrow$
  $\exists z\,(0 < z \leq (20-x)+5 \wedge \forall w(z \leq w < z+20 \rightarrow \nabla_w \mathit{lightIsGreen}))),$
- $\forall x\,(\forall y(0 \leq y < x \rightarrow \nabla_y \mathit{lightIsGreen}) \rightarrow x \leq 20),$
- $\mathit{lightIsGreen} \wedge \Delta_1 \mathit{lightIsRed} \rightarrow \forall x\,(0 < x \leq 20 \rightarrow \nabla_x \mathit{lightIsRed}).$

Axioms

Our next goal is to arrive at a complete axiomatization of validity in the language of $Q\text{-}MTL_0$. To the axioms of MTL_2 (see the discussion preceding Theorem 3.13) we will add a number of axiom schemata governing the behavior of quantifiers and substitutions. First of all, we have

(Ax9) $\forall x\,(\phi \to \psi) \leftrightarrow (\forall x\,\phi \to \forall x\,\psi)$ (functionality)
(Ax10) $\phi \to \forall x\,\phi$, for x not in ϕ (elimination of vacuous quantifier)
(Ax11) $\forall x\,\phi \to \phi(\alpha/x)$, with α free for x in ϕ (universal instantiation)

and the rule:

(UG) $\vdash \phi \implies \vdash \forall x\,\phi$ (universal generalization).

We also add the Barcan formula:

(Ax12) $\forall x\,\nabla_\alpha \phi \to \nabla_\alpha \forall x\,\phi$, with $x \notin \alpha$,

where $x \notin \alpha$ denotes that $x \neq \alpha$ and x does not occur in α. Furthermore, we have the following axioms relating algebraic terms and temporal operators:

(Ax13) $\alpha = \beta \to \forall x \nabla_x \alpha = \beta$
(Ax14) $\alpha \neq \beta \to \forall x \nabla_x \alpha \neq \beta$
(Ax15) $\alpha < \beta \to \forall x \nabla_x \alpha < \beta$
(Ax16) $\alpha \not< \beta \to \forall x \nabla_x \alpha \not< \beta$.

Remark 4.2 It is worth noting that the requirement that neither $x = \alpha$ nor $x \in \alpha$ in (Ax11) is essential to guarantee the soundness of the Barcan formula, as is shown by the following example. Suppose that the Barcan formula holds without restrictions. Let x be a variable (over displacements). From axiom (Ax2), by (UG), (Ax11) and Modus Ponens, we obtain $p \to \nabla_x \Delta_{-x} p$. Then, by (UG), (Ax9), (Ax10) and Modus Ponens, it follows that $p \to \forall x\,\nabla_x \Delta_{-x} p$. Now, since the Barcan formula holds without restrictions, we obtain by Modus Ponens that $p \to \nabla_x \forall x\,\Delta_{-x} p$, which clearly is not a valid formula.

Also, axiom (Ax16) can actually be derived from the other axioms.

Remark 4.3 Note that we also have converses to (Ax13)–(Ax16):

$$\begin{aligned}\forall x\,\nabla_x(\alpha = \beta) &\to \nabla_0(\alpha = \beta), \text{ by (Ax11)}\\ &\to \alpha = \beta, \text{ by (Ax5)},\end{aligned}$$

and similarly for the other cases. As a consequence we have that for purely algebraic formulae ϕ the following equivalence is provable: $\phi \leftrightarrow \forall x\,\nabla_x \phi$.

Lemma 4.4 *Q-MTL_0 derives the following formula:*

(*T*1) $\nabla_\alpha \forall x\, \phi \to \forall x\, \nabla_\alpha \phi$, *with* $x \notin \alpha$ *(converse of the Barcan formula)*

Proof. We have

$$\begin{aligned}
&\vdash \nabla_\alpha \forall x\, \phi \to \nabla_\alpha \phi, \text{ by (Ax10), (NEC) and (Ax1)}\\
&\Rightarrow \vdash \forall x\, \nabla_\alpha \forall x\, \phi \to \forall x\, \nabla_\alpha \phi, \text{ by (UG) and (Ax8)}\\
&\Rightarrow \vdash \nabla_\alpha \forall x \phi \to \forall x \nabla_\alpha \phi, \text{again by (Ax10).}
\end{aligned}$$

Observe that (T1) together with the Barcan formula allows us to conclude that the domain of temporal displacements does not change when we move from one time point to another. $\dashv$

Completeness

To prove a completeness result for *Q-MTL_0* we can follow the general pattern of the completeness proofs given in Sections 2 and 3, but the presence of mixed formulae complicates some of the details. We use a variant of Hughes and Cresswell's [9] method for proving axiomatic completeness in the presence of the Barcan formula.

First, a *Henkin-formula* with respect to a variable y is defined as follows

(i) Any formula of the form $\exists x\, \phi \to \phi(y/x)$ is a Henkin formula with respect to y.
(ii) If ψ is a Henkin-formula with respect to y, χ is any formula not containing y free, and α is an algebraic term not containing y, then $\Delta_\alpha \chi \to \Delta_\alpha(\chi \wedge \psi)$ is a Henkin-formula with respect to y.

Henkin-formulae that differ only in that each is a Henkin-formula with respect to a different variable will be said to have the same *Henkin-form*. A set of formulae has the *Henkin-property* if it contains at least one Henkin formula of every Henkin-form.

Lemma 4.5 *If ψ is a Henkin-formula with respect to y, then $\vdash \exists y\, \psi$.*

Proof. We argue by induction on Henkin-formulae. If ψ is of the form $\exists x\, \phi \to \phi(y/x)$, then, using the validity of $\exists y\, (\exists x\, \phi \to \phi(y/x))$ for y not free in ϕ, we get $\vdash \exists y\, \psi$.

Suppose that ψ is a Henkin-formula with respect to y, and that $\vdash \exists y\, \psi$. Assume that y is not free in the formula χ and doesn't occur in the term α; we have to show that $\vdash \exists y\, (\Delta_\alpha \chi \to \Delta_\alpha(\chi \wedge \psi))$. Observe

$$
\begin{aligned}
&\vdash \exists y\, \psi \\
&\Rightarrow \vdash \Delta_\alpha \chi \to \Delta_\alpha(\chi \wedge \exists y\, \psi), \text{ by standard modal reasoning,} \\
&\Rightarrow \vdash \Delta_\alpha \chi \to \Delta_\alpha \exists y\, (\chi \wedge \psi), \text{ as } y \text{ is not free in } \chi, \\
&\Rightarrow \vdash \Delta_\alpha \chi \to \exists y\, \Delta_\alpha(\chi \wedge \psi), \text{ by the Barcan formula,} \\
&\Rightarrow \vdash \exists y\, (\Delta_\alpha \chi \to \Delta_\alpha(\chi \wedge \psi)), \text{ as } y \text{ does not occur in } \alpha \text{ and} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{is not free in } \chi. \quad \dashv
\end{aligned}
$$

Lemma 4.6 *Assume Σ is a consistent set of formulae, none of which contains any occurrence of y, and let ψ be a Henkin formula with respect to y. Then $\Sigma \cup \{\psi\}$ is consistent.*

Proof. Let $\Sigma' \subseteq \Sigma$ be finite. It suffices to show that $\Sigma' \cup \{\psi\}$ is consistent. Suppose otherwise. Then

$$
\begin{aligned}
\vdash \bigwedge \Sigma' \to \neg\psi &\Rightarrow \vdash \bigwedge \Sigma' \to \forall y\, \neg\psi \\
&\Rightarrow \vdash \exists y\, \psi \to \neg \bigwedge \Sigma' \\
&\Rightarrow \vdash \neg \bigwedge \Sigma', \text{ by Lemma 4.5,}
\end{aligned}
$$

which contradicts the consistency of Σ. $\dashv$

Lemma 4.7 *Every consistent formula is contained in a maximal consistent set with the Henkin-property.*

Proof. This is standard; use Lemma 4.6. $\dashv$

Lemma 4.8 *Let Σ be a maximal consistent set of formulae with the Henkin-property. Let $\Delta_\alpha\psi \in \Sigma$. Then there exists a maximal consistent set of formulae Γ with the Henkin-property such that*

$$\{\psi\} \cup \{\chi \mid \nabla_\alpha \chi \in \Sigma\} \subseteq \Gamma.$$

Proof. Define $\Gamma_0 := \{\psi\}$. Take the first Henkin-form in some fixed enumeration of all Henkin-forms, and enumerate the Henkin-formulae of the first form as $\delta_{11}, \ldots, \delta_{1n}, \ldots$. By assumption Σ has the Henkin-property, hence it contains a formula of the form $\Delta_\alpha\psi \to \Delta_\alpha(\psi \wedge \delta_{1i_1})$. Put $\Gamma_1 := \Gamma_0 \cup \{\delta_{1i_1}\}$.

In general, given that for the first m Henkin-forms we have added Henkin-formulae $\delta_{1i_1}, \ldots, \delta_{mi_m}$, we consider a formula $\delta_{(m+1)i_{(m+1)}}$ of the $(m+1)$-th form, which is such that

$$\Delta_\alpha(\psi \wedge \delta_{1i_1} \wedge \ldots \wedge \delta_{mi_m}) \to \Delta_\alpha(\psi \wedge \delta_{1i_1} \wedge \ldots \wedge \delta_{mi_m} \wedge \delta_{(m+1)i_{(m+1)}})$$

is in Σ, and obtain Γ_{m+1} as $\Gamma_m \cup \{\delta_{(m+1)i_{(m+1)}}\}$. Let $\Gamma' = \bigcup_m \Gamma_m$. Then Γ' has the Henkin-property.

Next, add $\{\chi \mid \nabla_\alpha \chi \in \Sigma\}$ to Γ' to obtain Γ''. This can be done without destroying consistency, for consider a finite subset of Γ'':

$$\{\psi, \delta_1, \ldots, \delta_n, \chi_1, \ldots, \chi_m\}, \tag{2}$$

where ψ is the initial formula put into Γ_0, each of the δ's is a Henkin formula put into Γ' to give it the Henkin property, and each χ is such that $\nabla_\alpha \chi \in \Sigma$. Now, it is easy to see that $\Delta_\alpha(\psi \wedge \delta_1 \wedge \ldots \wedge \delta_n) \in \Sigma$, and hence

$$\{\Delta_\alpha(\psi \wedge \delta_1 \wedge \ldots \wedge \delta_n), \nabla_\alpha \chi_1, \ldots, \nabla_\alpha \chi_m\} \subseteq \Sigma. \tag{3}$$

By hypothesis Σ is consistent, hence (3) is consistent, and therefore (by modal reasoning), (2) is consistent.

Finally, increase Γ'' to a maximal consistent set Γ in the usual way. $\dashv$

We can now embark on the completeness proof for *Q-MTL*$_0$. Let Σ be a maximal *Q-MTL*$_0$-consistent set of formulae that has the Henkin property. Using Σ we will define a canonical model $\mathfrak{M}^0 = (T^0, \mathfrak{D}^0; \mathrm{DIS}^0; V^0, g)$ as follows.

The displacement domain

Using a Henkin construction, we build a displacement domain $\mathfrak{D}^0$ from Σ. In this domain the (displacement) objects are equivalences classes of terms modulo the congruence relation θ, where θ is 'provable equality according to Σ': $(\alpha, \beta) \in \theta$ iff $\Sigma \vdash \alpha = \beta$. The interpretation function $g : T(X \cup A) \to \mathfrak{D}^0$ is defined in the obvious way by putting $g(\alpha) = \alpha/\theta$.

The displacement relation

Define the relation DIS^0 as in the unquantified case: for maximal consistent sets Γ_1, Γ_2, and for every term $\gamma \in T(X \cup A)$, define

$$\begin{aligned} \mathrm{DIS}^0(\Gamma_1, g(\gamma), \Gamma_2) \text{ iff } & \text{for every formula } \sigma,\ \sigma \in \Gamma_2 \text{ implies } \Delta_\gamma \sigma \in \Gamma_1 \\ & (\text{equivalently: for all } \sigma, \text{ if } \nabla_\gamma \sigma \in \Gamma_1 \text{ then } \sigma \in \Gamma_2). \end{aligned}$$

The temporal domain

The *canonical temporal domain* T^0 consists of all maximal consistent sets Γ with the Henkin-property such that for some α, $\mathrm{DIS}^0(\Sigma, g(\alpha), \Gamma)$. Define the *canonical valuation* V^0 by putting $V^0(p) = \{\Gamma \mid p \in \Gamma\}$, for all proposition letters p.

Lemma 4.9 *For all $\Gamma \in T^0$, and all algebraic terms α, β we have that $(\alpha = \beta) \in \Gamma$ iff $(\alpha = \beta) \in \Sigma$, and similarly for formulae of the form $\alpha < \beta$.*

Proof. As $\Gamma \in T^0$, we have $\mathrm{DIS}^0(\Sigma, g(\gamma), \Gamma)$ for some γ. Then $(\alpha = \beta) \in \Gamma$ implies $\Delta_\gamma(\alpha = \beta) \in \Sigma$, and so $(\alpha = \beta) \in \Sigma$ by axiom (Ax14) and universal instantiation. Conversely, $(\alpha = \beta) \in \Sigma$ implies $\nabla_\gamma(\alpha = \beta) \in \Sigma$, by axiom (Ax13), implies $(\alpha = \beta) \in \Gamma$. $\dashv$

Theorem 4.10 (Completeness) *Q-MTL_0 is sound and complete for the class of all Q-MTL_0-frames.*

Proof. Take a consistent formula ϕ, and let Σ be a maximal consistent extension of $\{\phi\}$ with the Henkin-property. Construct the canonical model $\mathfrak{M}^0$ for Σ as defined above. To establish the completeness of Q-MTL_0 we need to check that $\mathfrak{M}^0$ validates the axioms of Q-MTL_0, but this is clear. On top of that we need a truth lemma for Q-MTL_0.

We first treat the case of atomic algebraic formulae. Let $\Gamma \in T^0$; then, for some γ, $\mathrm{DIS}^0(\Sigma, g(\gamma), \Gamma)$. Then $(\alpha = \beta) \in \Gamma$ iff $(\alpha = \beta) \in \Sigma$ (by Lemma 4.9) iff $g(\alpha) = g(\beta)$ iff $\Gamma \Vdash \alpha = \beta$, as required.

The remaining atomic cases and the boolean cases are straightforward. The case of the universal quantifier is the same as in standard completeness proofs for first-order logic. So let us consider the case of ∇_α. We have to show that

$$\nabla_\alpha \phi \in \Gamma_1, \text{ iff } \forall \Gamma_2 \left(\mathrm{DIS}^0(\Gamma_1, g(\alpha), \Gamma_2) \text{ implies } \phi \in \Gamma_2\right),$$

where Γ_1, $\Gamma_2 \in T^0$. The 'if' part follows immediately from the definition of DIS^0. To prove the 'only if' part, assume $\nabla_\alpha \phi \notin \Gamma_1$; then $\neg \nabla_\alpha \phi \in \Gamma_1$. By Lemma 4.8 the set $\{\psi \mid \nabla_\alpha \psi \in \Gamma_1\} \cup \{\neg\phi\}$ can be extended to a maximal consistent set Γ_2 with the Henkin-property. Clearly, $\Gamma_1 \in T^0$; moreover, $\mathrm{DIS}^0(\Gamma_1, g(\alpha), \Gamma_2)$ implies $\Gamma_2 \in T^0$, by axiom (Ax3). Finally, $\{\psi \mid \nabla_\gamma \psi \in \Gamma_1\}$ is a subset of Γ_2, so $\mathrm{DIS}^0(\Gamma_1, g(\gamma), \Gamma_2)$ holds, as required. $\dashv$

Enriching the temporal component.

For most application purposes the language of Q-MTL_0 (or a minor extension thereof) suffices. However, if full quantificational force of the temporal domain is required, the above techniques can easily be extended, as we will demonstrate now.

We consider a rich language in which the temporal component is based on a full first-order language instead of a propositional one. We consider the system $Q\text{-}MTL_1$.

The *language* $Q\text{-}MTL_1$ is built up using algebraic terms specified by

$$\alpha ::= 0 \mid a \mid x \mid \alpha + \alpha \mid -\alpha,$$

as before, and using a disjoint collection of 'temporal' variables S, typically denoted with s, t, $\ldots$; these are the variables that we will quantify over in the quantified temporal part of our language. Next, we define the formulae of $Q\text{-}MTL_1$:

$$\phi ::= Rt_1 \ldots t_n \mid \neg\phi \mid \alpha = \beta \mid \alpha < \beta \mid \phi \wedge \phi \mid \Delta_\alpha \phi \mid \forall x\, \phi \mid \forall s\, \phi.$$

Thus, we can quantify using displacement variables x, or using 'temporal' variables s.

The *models* of $Q\text{-}MTL_1$ are structures of the form

$$\mathfrak{M} = (T, \mathfrak{D}; \mathrm{DIS}; O, V, g).$$

O is the domain of individual objects; the function V assigns a member of O to each individual temporal variable. For every n-ary (temporal) predicate letter R, $V(R)$ is a collection of $(n+1)$-tuples $(u_1, \ldots, u_n, w)$, where u_1, $\ldots$, $u_n \in O$ and $w \in T$.

Given this set-up, we can calculate the truth value for all formulae ϕ in the following manner (we only list the novel cases):

$$\begin{array}{rcl}
(\mathfrak{F}; O, V, g), i \Vdash R(s_1, \ldots, s_n) & \text{iff} & (V(s_1), \ldots, V(s_n), i) \in V(R) \\
(\mathfrak{F}; O, V, g), i \Vdash \forall x\, \phi & \text{iff} & (\mathfrak{F}; O, V, g'), i \Vdash \phi \text{ for all assignments } g' \\
 & & \text{such that } g =_x g' \\
(\mathfrak{F}; O, V, g), i \Vdash \forall s\, \phi & \text{iff} & (\mathfrak{F}; O, V', g), i \Vdash \phi \text{ for all valuations } V' \\
 & & \text{such that } V =_s V' \\
(\mathfrak{F}; O, V, g), i \Vdash \Delta_\alpha \phi & \text{iff} & (\mathfrak{F}; O, V, g), j \Vdash \phi \text{ for some time} \\
 & & \text{instant } j \text{ with } \mathrm{DIS}(i, g(\alpha), j).
\end{array}$$

Remark 4.11 Observe that, just as with $Q\text{-}MTL_0$ models, in $Q\text{-}MTL_1$ models, the displacement domain is constant over all time instants, as are the truth values of the purely algebraic formulae. And the newly added individual objects domain is constant across all time instants, but, of course, (purely) temporal formulae may differ in truth value from one time instance to another.

Next, we specify the *axioms* of $Q\text{-}MTL_1$. To the axioms of $Q\text{-}MTL_0$ we add quantificational axioms for the temporal quantifiers, as well as the rule of universal generalization and the Barcan formula for the temporal quantifiers:

(Ax9′)	$\forall s\,(\phi \to \psi) \leftrightarrow (\forall s\,\phi \to \forall s\,\psi)$	(functionality)
(Ax10′)	$\phi \to \forall s\,\phi$, for s not in ϕ	(elimination of vacuous quantifier)
(Ax11′)	$\forall s\,\phi \to \phi(t/s)$, with t free for s in ϕ	(universal instantiation)

and the rule:

(UG′) $\quad \vdash \phi \implies \vdash \forall s\,\phi$ (universal generalization).

We also add the Barcan formula:

(Ax12′) $\quad \forall s\,\nabla_\alpha \phi \to \nabla_\alpha \forall s\,\phi$.

Theorem 4.12 (Completeness) *$Q\text{-}MTL_1$ is sound and complete.*

Sketch of the proof. To establish the completeness of $Q\text{-}MTL_1$ using the proof technique of Theorem 4.10 we need to adopt the notions of a Henkin-formula and a Henkin-form (page 27) as follows. Let r be either a displacement variable or a temporal variable.

(i) Any formula of the form $\exists x\,\phi \to \phi(y/x)$ is a Henkin formula with respect to y.
(ii) Any formula of the form $\exists s\,\phi \to \phi(t/s)$ is a Henkin formula with respect to t.
(iii) If ψ is a Henkin-formula with respect to y, χ is any formula not containing y free, and α is an algebraic term not containing y, then $\Delta_\alpha \chi \to \Delta_\alpha(\chi \wedge \psi)$ is a Henkin-formula with respect to y.
(iv) If ψ is a Henkin-formula with respect to t, χ is any formula not containing t free, then $\Delta_\alpha \chi \to \Delta_\alpha(\chi \wedge \psi)$ is a Henkin-formula with respect to t.

As before, Henkin-formulae that differ only in that each is a Henkin-formula with respect to a different variable of the same sort (i.e., either they are all displacement variables, or all temporal variables) will be said to have the same Henkin-form. A set of formula has the Henkin-property if it contains at least one Henkin formula of every Henkin-form.

We leave it to the reader to verify that given the above notions of Henkin-formula, Henkin-form, and Henkin-property, Lemma's 4.5–4.8 remain valid.

The canonical model for $Q\text{-}MTL_1$ is built up in the same way as for $Q\text{-}MTL_0$, except for the fact that we need to specify a domain of individual objects O and a valuation V; the former will simply be the collection of all temporal variables, and the latter is defined by $V(R) = \{(u_1, \ldots, u_n, \Gamma) \mid R(u_1, \ldots, u_n) \in \Gamma\}$, where R is an n-ary predicate symbol. With this modification a truth lemma can be established as in the proof of Theorem 4.10. $\dashv$

5 Conclusion and further developments

In this paper we have proved completeness results for basic systems of metric temporal logic. We started with the minimal calculus and showed how to extend it to obtain the logic of two-sorted frames with a linear temporal order in which there exists a unique temporal distance between any two time instants. After that we considered general metric temporal logics allowing quantification over algebraic and temporal variables and free mixing of algebraic and temporal formulae.

We traced a sort of preferred path from the minimal metric temporal logic MTL_0 to the quantified metric temporal logic $Q\text{-}MTL_0$, passing through the (unquantified) metric temporal logic of linear orders MTL_2. In fact, the proposed two-sorted framework allows one to characterize a variety of metric temporal logics simply by weakening or strengthening the requirements on the algebraic and temporal components and their interaction. For example, in certain application areas it seems natural to abandon the requirement that the displacement relation is quasi-functional with respect to its third argument; one situation where this comes up is in the use of our metric temporal logics for specifying the spatial behavior of read and write heads of a hard disk. Developing this more liberal approach to interpreting metric temporal languages is part of our ongoing research.

In this paper we have not discussed decidability issues. It is known that a negative result holds for $Q\text{-}MTL_0$. Burgess [2] shows that the decision problem for quantified metric temporal logic is equivalent to that for the set of all universal monadic second-order formulae true in all ordered abelian groups, and he proves that the decision problem for the validity of first-order formulae involving a single binary predicate, which is known to be undecidable, can be reduced to this equivalent problem.

As to the decidability question for propositional metric temporal logics, we are currently studying links between (propositional) metric temporal logics and versions of *propositional dynamic logic* with a view to importing results and techniques on decidability from the latter. Roughly, our strategy is the following. We re-interpret the propositional metric language on multi-modal models of the form $(W, \{R_\alpha \mid \alpha \text{ is an algebraic term }\}, V)$; and in such models the semantics of a modal operator Δ_α is based the relation R_α as follows. For atomic displacements a, R_a is arbitrary, and for more complex terms we have

$$\begin{array}{rclcl} R_{\alpha+\beta} & = & R_\alpha \circ R_\beta & = & \{i,k) \mid \exists j\,(R_\alpha(i,j) \wedge R_\beta(j,k))\} \\ R_{-\alpha} & = & (R_\alpha)^{-1} & = & \{(i,j) \mid R_\alpha(j,i)\} \\ R_0 & = & I & = & \{(i,i) \mid i \in W\}. \end{array}$$

(We need to impose certain further restrictions such as $R_\alpha \circ R_\beta = R_\beta \circ R_\alpha$, but these need not concern us here.) To prove the decidability of a metric temporal logic, one should then show that it has the finite model property with respect to the above multi-modal models, and the key tool in doing so will be (an adaptation of) the filtration method familiar from modal and dynamic logic (cf. Goldblatt [6]). We plan to report on this work in a later publication.

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