On the Complexity of Fragments of the Modal Logic of Allen's Relations over Dense Structures

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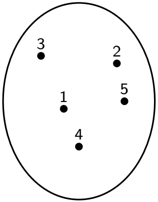
Temporal logics in computer science

- Temporal logics play a major role in computer science
  - automated system verification
- Temporal logics can be viewed as (multi-)modal logics:



set of worlds
primitive temporal entities:
time points/instants

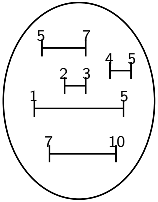
accessibility relations

$\longrightarrow$: next (the successor relation $+1$)

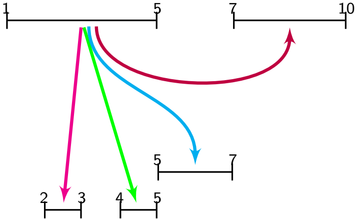
$\longrightarrow^{\star}$: eventually (the ordering relation $<$)

A different approach: from points to intervals

- worlds are intervals (time periods — pairs of points)



set of worlds
primitive temporal entities: time intervals/periods



accessibility relations
all binary relations between pairs of intervals

Motivations

- there are properties which are intrinsically related to intervals (instead of points)

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- true over a given interval of time
- not true over all other intervals
  (beginning/ending intervals, sub-intervals, super-intervals, overlapping intervals, etc.)

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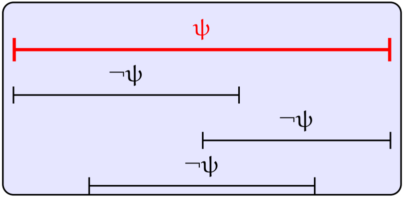
Some philosophical and logical paradoxes disappear:

- Zeno's flying arrow paradox ("if at each instant the flying arrow stands still, how is movement possible?")
- The dividing instant dilemma ("if the light is on and it is turned off, what is its state at the instant between the two events?")

The distinctive features of interval temporal logics

Truth of formulae is defined over intervals (not points).



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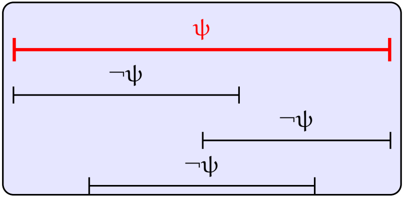

Interval temporal logics are very expressive (compared to point-based temporal logics)

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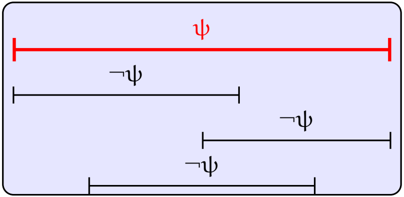

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In general, there is no reduction of the satisfiability/validity in interval temporal logics to monadic second-order logic, and therefore Rabin's theorem is not applicable here

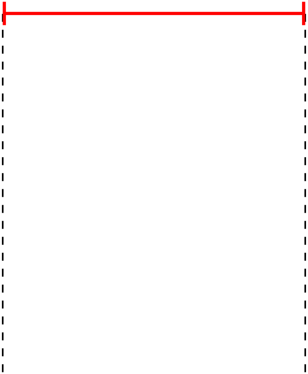
Outline

An introduction to Interval Temporal Logics

Halpern-Shoham's modal logic HS

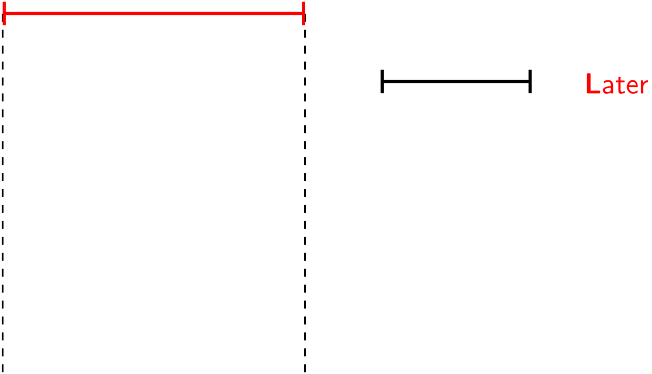
HS over dense linear orders

Binary ordering relations between intervals on linear orders



J. F. Allen
Maintaining knowledge about temporal intervals
Communications of the ACM, 1983

Binary ordering relations between intervals on linear orders



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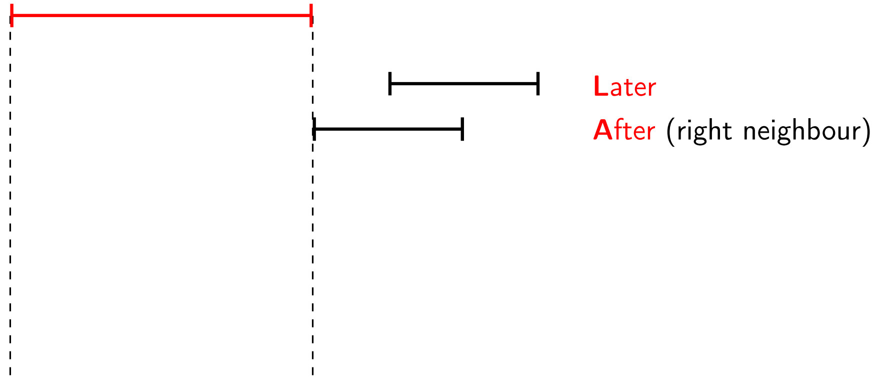

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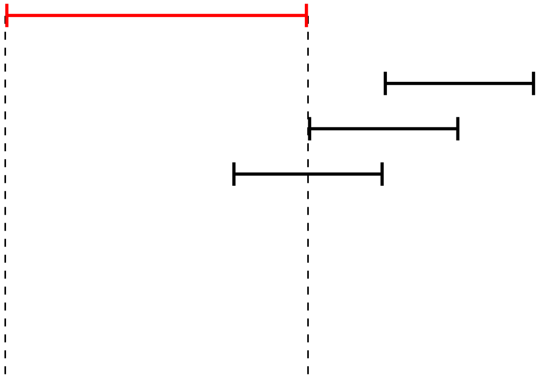

Later

After (right neighbour)

Overlaps (to right)

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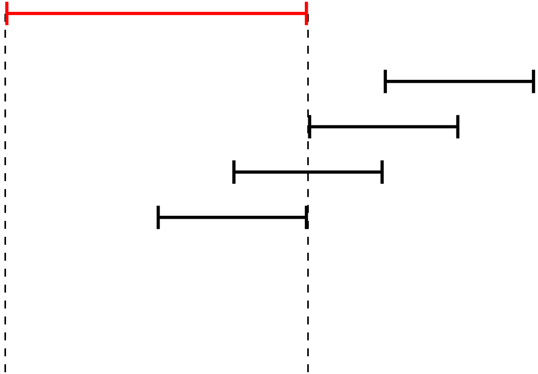

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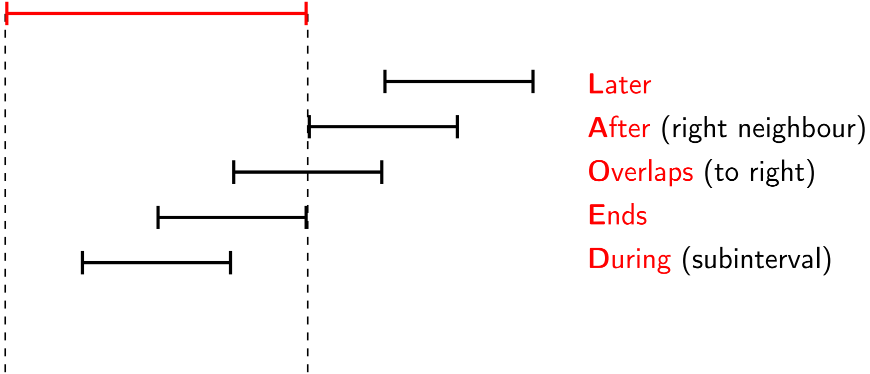

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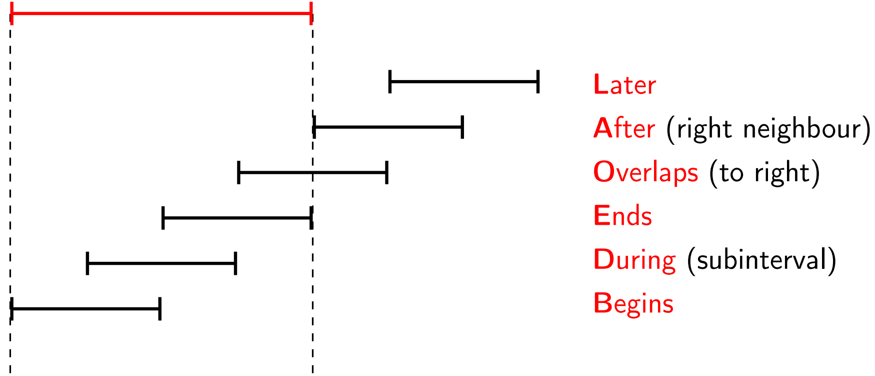

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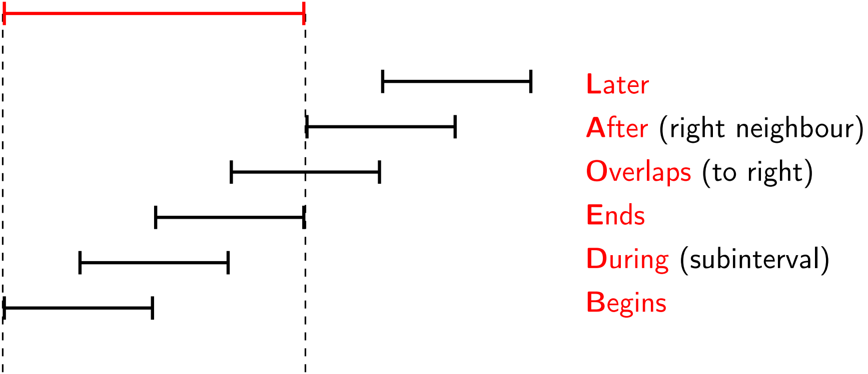

6 relations + their inverses + equality = 13 Allen's relations

J. F. Allen
Maintaining knowledge about temporal intervals
Communications of the ACM, 1983

Halpern-Shoham's modal logic of Allen's relations

interval relations give rise to modal operators → HS logic

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Syntax:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle X \rangle \varphi$$

$$\langle X \rangle \in \{\langle A \rangle, \langle L \rangle, \langle B \rangle, \langle E \rangle, \langle D \rangle, \langle O \rangle, \langle \overline{A} \rangle, \langle \overline{L} \rangle, \langle \overline{B} \rangle, \langle \overline{E} \rangle, \langle \overline{D} \rangle, \langle \overline{O} \rangle\}$$

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interval relations give rise to modal operators → **HS logic**

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Models:

$$\mathbf{M} = \langle \mathbb{I}(\mathbb{D}), V \rangle$$

$$V : \mathbb{I}(\mathbb{D}) \mapsto 2^{\mathcal{AP}}$$

$\mathcal{AP}$ atomic propositions (over intervals)

Formal semantics of HS

$\langle B \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle B \rangle \phi$ iff there exists d_2 such that $d_0 \leqslant d_2 < d_1$ and $\mathbf{M}, [d_0, d_2] \Vdash \phi$

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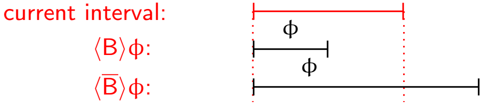

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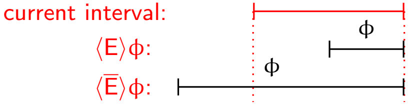

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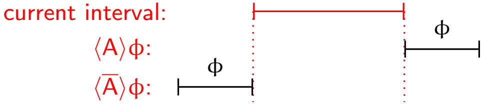

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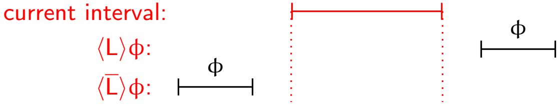

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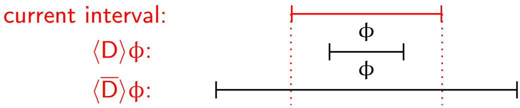

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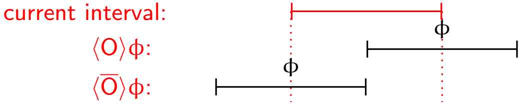

(Un)decidability of HS fragments: main parameters

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(Un)decidability of HS fragments depends on two factors:

- the set of interval modalities;
- the class of interval structures (linear orders) over which the fragment is interpreted

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I. Shapirovsky
On PSPACE-decidability in Transitive Modal Logic
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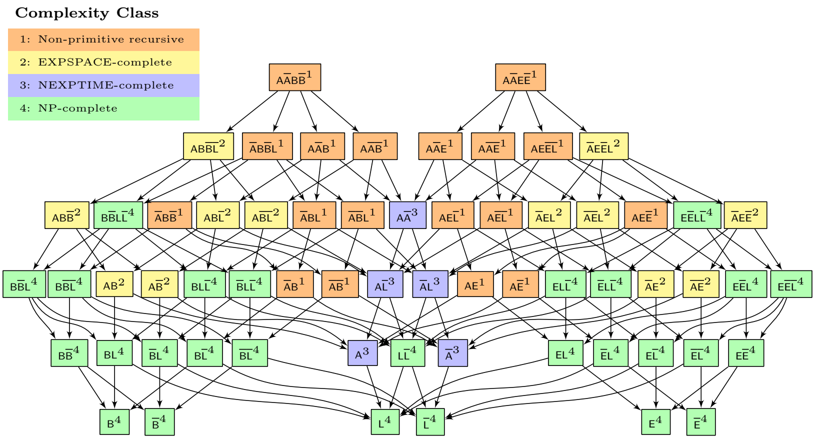
It is unknown, when D is interpreted over the class of all linear orders

HS fragments over strongly discrete linear orders (done)

- We already identified all HS fragments with a decidable satisfiability problem over the class of strongly discrete linear orders and over its relevant subclasses (the class of finite linear orders, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{Z}^-$)
- We classify them in terms of both their relative expressive power and their complexity, which ranges from NP-completeness to non-primitive recursiveness

D, Bresolin, D. Della Monica, A. Montanari, P. Sala, and G. Sciavicco
Interval temporal logics over strongly discrete linear orders: Expressiveness and complexity
Theorerical Computer Science, 2014

The complete picture for finite linear orders



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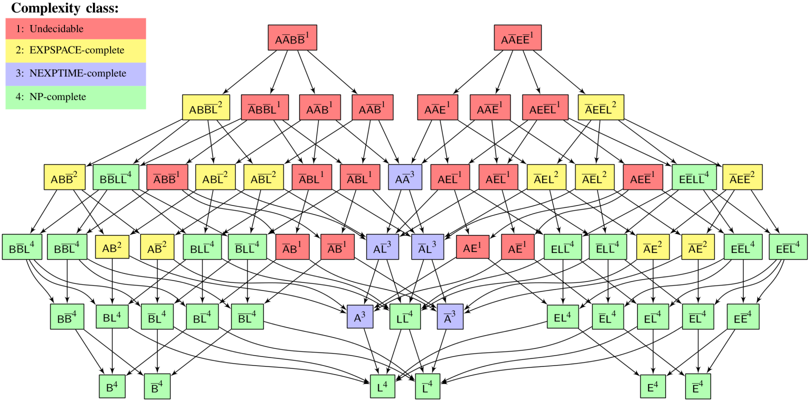

Dense case: expressively different HS fragments

There are precisely 9 different optimal definabilities that hold among HS modalities in the dense case:

$\langle \mathsf{L} \rangle p \equiv \langle \mathsf{A} \rangle \langle \mathsf{A} \rangle p;$
$\langle \mathsf{L} \rangle p \equiv \langle \overline{\mathsf{B}} \rangle [\mathsf{E}] \langle \overline{\mathsf{B}} \rangle \langle \mathsf{E} \rangle p;$
$\langle \mathsf{L} \rangle p \equiv \langle \mathsf{O} \rangle (\langle \mathsf{O} \rangle \top \wedge [\mathsf{O}] \langle \mathsf{D} \rangle \langle \mathsf{O} \rangle p);$
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$\langle \mathsf{O} \rangle p \equiv \langle \mathsf{E} \rangle \langle \overline{\mathsf{B}} \rangle p;$
$\langle \mathsf{D} \rangle p \equiv \langle \mathsf{E} \rangle \langle \mathsf{B} \rangle p$

As a consequence, only 966 HS fragments are expressively different, out of 4096 different subsets of Allen's modalities.

Decidable fragments

Of the 966 expressively different HS fragments, we already know that 146 are decidable, thanks to the following results:

Undecidability: each fragment containing (as definable) O, AD, or $A\bar{D}$ is undecidable

D. Bresolin, D. Della Monica, V. Goranko, A. Montanari, G. Sciavicco
The dark side of ITL: marking the undecidability border
Annals of Mathematics and Artificial Intelligence, 2014

Non-primitive recursive: $A\bar{A}B\bar{B}$, $A\bar{A}B$, and $A\bar{A}\bar{B}$ are decidable, but non-primitive recursive

A. Montanari, G. Puppis, and P. Sala:
Decidability of the Interval Temporal Logic $A\bar{A}B\bar{B}$ over the Rationals
MFCS 2014

Decidable fragments - cont'd

EXPSPACE-completeness: $AB\overline{B}\overline{L}$ is in EXPSPACE and each fragment containing AB or $A\overline{B}$ is EXPSPACE-hard

D. Bresolin, A. Montanari, P. Sala, and G. Sciavicco:
What's Decidable about Halpern and Shoham's Interval Logic? The Maximal Fragment $A\overline{L}B\overline{B}$
LICS 2011

NEXPTIME-completeness: $A\overline{A}$ is in NEXPTIME, and both A and $\overline{A}$ are NEXPTIME-hard

D. Bresolin, A. Montanari, P. Sala, and G. Sciavicco
Optimal Tableau Systems for Propositional Neighborhood Logic over All, Dense, and Discrete Linear Orders
TABLEAUX 2011

Decidable fragments - cont'd

PSPACE-completeness: each sub-fragment of $\mathrm{B\overline{B}D\overline{D}L\overline{L}}$ that contains (as definable) D or $\mathrm{\overline{D}}$ is PSPACE-complete

A. Montanari, G. Puppis, and P. Sala
A Decidable Spatial Logic with Cone-Shaped Cardinal Directions
CSL 2009

Completing the picture: $\mathsf{B\overline{B}L\overline{L}}$

$\mathsf{B\overline{B}L\overline{L}}$ and all its fragments are NP-complete (they are at least as expressive as propositional logic, and thus NP-hardness easily follows)

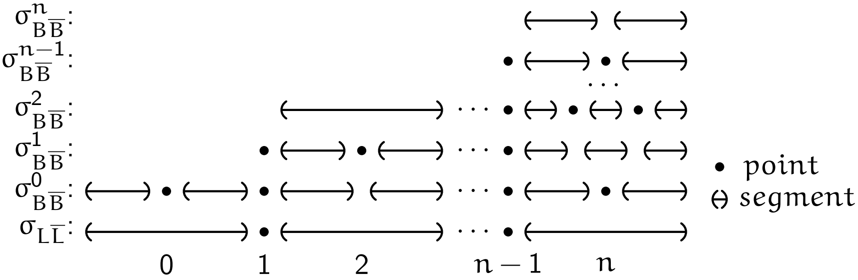
Proof: we introduce a suitable notion of pseudo-model for formulae of $\mathsf{B\overline{B}L\overline{L}}$ and we show that each satisfiable formula φ admits a pseudo-model of size at most $P(|\varphi|)$, for some polynomial P

We start with the fragment $\mathsf{L\overline{L}}$ and then we move to full $\mathsf{B\overline{B}L\overline{L}}$

- We associate with every point x the set of its $L\overline{L}$-requests, and we partition the domain of the model into a finite number of clusters of points with the same set of $L\overline{L}$-requests
- Since both $\langle \mathsf{L} \rangle$ and $\langle \overline{\mathsf{L}} \rangle$ are transitive, the set of $L\overline{L}$-requests is monotone with respect to the ordering of points: every cluster consists of either a single *point* or a *segment* of the domain and the number n of clusters is linear in $|\varphi|$ (see the figure)
- The sequence of clusters must satisfy a number of consistency and fulfilling conditions

Completing the picture: $B\overline{B}L\overline{L}$ - cont'd

- By guessing an $L\overline{L}$-sequence and then checking it for consistency and fulfillment, we can easily obtain an NP procedure to decide the satisfiability of a formula in $L\overline{L}$
- For any given point, the set of $B\overline{B}$-requests is monotone, and thus we can partition the intervals starting at any point of an $L\overline{L}$-cluster into a linear number of $B\overline{B}$-clusters (refining the original partition)



Completing the picture: $\overline{AB}$ and $\overline{A}B$

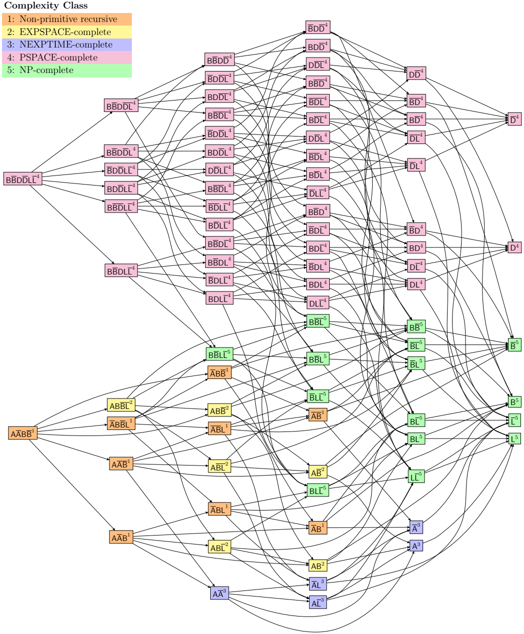
All HS fragments that contain $\overline{AB}$ or $\overline{A}B$ are non-primitive recursive

Proof: a reduction from the non-termination problem for lossy counter machines to the satisfiability problem for $\overline{A}B$ over the class of all dense linear orders

The non-termination problem for lossy counter machines is the problem of deciding whether a lossy counter machine $\mathcal{A}$ has at least one infinite run starting with the initial configuration $(q_0, \bar{0})$. This problem is known to be non-primitive recursive

P. Schnoebelen
Lossy Counter Machines Decidability Cheat Sheet
RP 2010

The complete picture for dense linear orders



Conclusions and future work

- We identified all HS fragments that turn out to be decidable over the class of dense linear orders and we classified them in terms of both their relative expressive power and complexity
  - 146 expressively-different decidable fragments
  - complexity ranges from NP to non-primitive recursive

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Thank you!