Verification of infinite state systems

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Go

In this part

We reduce the model checking problem for MSO logic over

- colored semi-infinite lines
- colored infinite trees

to suitable acceptance problems respectively for

- sequential Büchi automata (Büchi Theorem exploited)
- Rabin tree automata (Rabin Theorem exploited)

In analogy to the case of the semi-infinite line, Rabin Theorem

Theorem (Rabin '69)

For any MSO-formula ψ with free variables $X_1, \ldots, X_m$, one can compute a Rabin tree automaton $\mathcal{A}_\psi$ over $\mathbb{B}^m$ such that, for every tuple of unary predicates $P_1, \ldots, P_m \subseteq \mathbb{B}^*$

$$(\mathbb{B}^*, \delta_0, \delta_1, \bar{P}) \vDash \psi[P_1/X_1, \ldots, P_m/X_m] \quad \text{iff} \quad \mathcal{T}_{2,\bar{P}} \in \mathscr{L}(\mathcal{A}_\psi)$$

can be exploited to reduce the decision problem for the MSO-theory of an *expanded* infinite complete tree $(\mathbb{B}^*, \delta_0, \delta_1, \bar{P})$ to the acceptance problem of $\mathcal{T}_{2,\bar{P}}$ (i.e., the **characteristic colored tree** encoding $(\mathbb{B}^*, \delta_0, \delta_1, \bar{P})$) for Rabin tree automata.

Definition (Acceptance problem)

The **acceptance problem** of a colored and complete infinite binary tree $\mathcal{T}$, denoted $Acc_{\mathcal{T}}$, consists in deciding, for any Rabin tree automaton $\mathcal{A}$, whether

$$\mathcal{T} \in \mathscr{L}(\mathcal{A}) \qquad (\mathcal{A} \textit{ accepts } \mathcal{T})$$

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Corollary (of RabinTheorem)

The problem of deciding the MSO-theory of a colored and complete infinite binary tree $\left(\mathbb{B}^*, \delta_0, \delta_1, \bar{P}\right)$ *is reducible to the problem* $Acc_{\mathcal{T}_{2,\bar{P}}}$.

Such a result can be easily generalized to k-ary and non-complete trees (i.e., trees with leaves).

To this end, we slightly modify the notion of tree automaton:

1. the transition relation Δ is now a subset of $Q \times C \times Q^k$
2. the automaton reads a dummy symbol $\perp$
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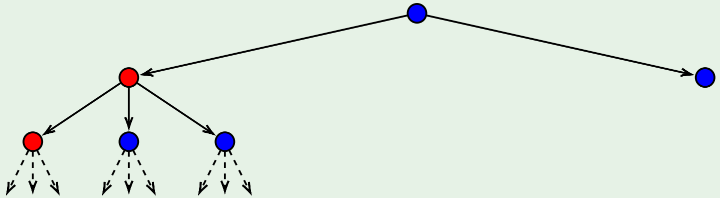
Example

Consider the ternary non-complete $\{red, blue\}$-colored tree



and the Rabin tree automaton having

- three states, r, b, and d, that signal which color was seen last
- transitions $(r/b, red, r, r, r)$, $(r/b, blue, b, b, b)$, $(r, dummy, d, d, d)$, $(b, dummy, d, d, d)$, $(d, dummy, d, d, d)$

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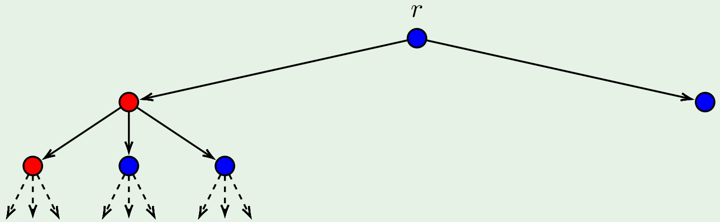

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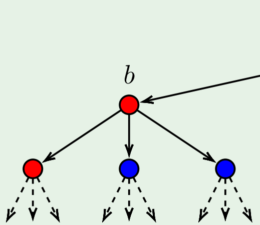

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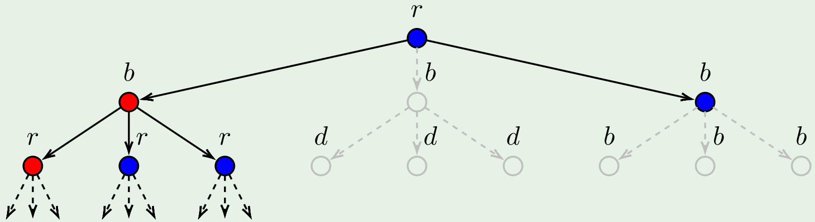

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Proposition

The problem of deciding the MSO-theory of a k-ary (possibly incomplete) colored tree $\left(D, \delta_0, ..., \delta_{k-1}, \bar{P}\right)$ is reducible to the problem $Acc_{\mathcal{T}_{k,\bar{P}}}$.

In the following, we describe a method to identify infinite colored trees $\mathcal{T}$, including incomplete ones, for which $Acc_{\mathcal{T}}$ is decidable.

Proposition

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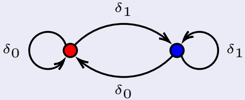

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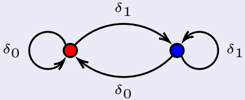
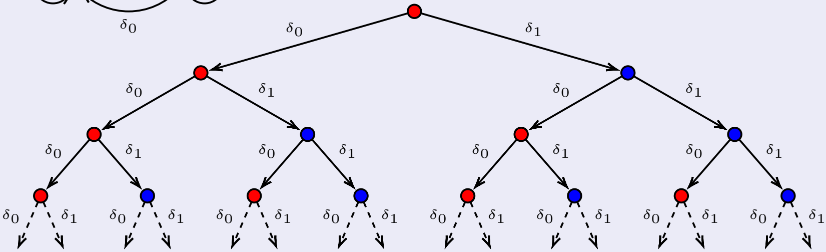
Let $\mathcal{G}$ be a finite colored graph and $\mathcal{T}$ its unfolding.





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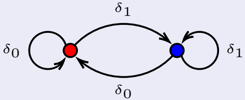
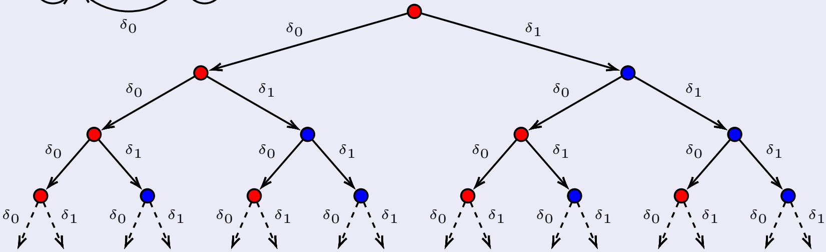
We view $\mathcal{G}$ as an automaton $\mathcal{A}_{\mathcal{T}}$ recognizing the singleton $\{\mathcal{T}\}$.





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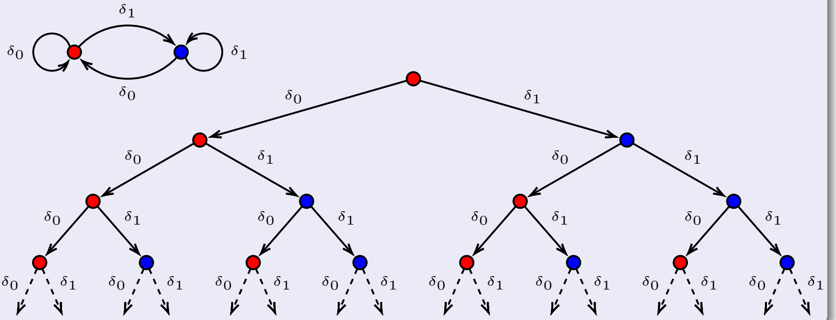
We view $\mathcal{G}$ as an automaton $\mathcal{A}_{\mathcal{T}}$ recognizing the singleton $\{\mathcal{T}\}$.

$\Rightarrow$ given any automaton $\mathcal{A}$, $\mathcal{T} \in \mathscr{L}(\mathcal{A})$ iff $\mathscr{L}(\mathcal{A}) \cap \mathscr{L}(\mathcal{A}_{\mathcal{T}}) \neq \emptyset$



Goal

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Idea

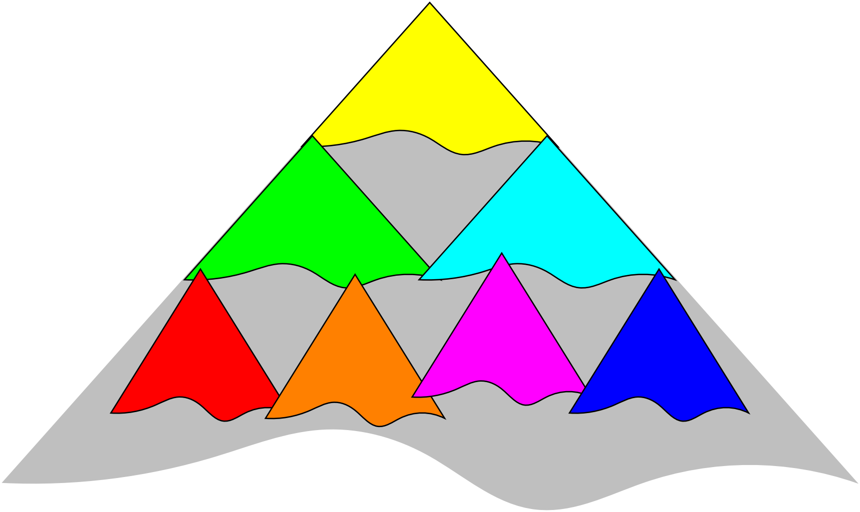
Reduce the acceptance problem of a non regular tree $\mathcal{T}$ to an equivalent acceptance problem of a regular tree $\vec{\mathcal{T}}$:

1. decompose $\mathcal{T}$ into factors
2. 'distill' the relevant features of each factor F (features describe the behavior of a given automaton $\mathcal{A}$ on F)
3. reason on the feature tree $\vec{\mathcal{T}}$ (i.e., a tree-shaped arrangement of features).

The *features* of F w.r.t. $\mathcal{A}$ are called $\mathcal{A}$-**type** of F.
The *feature tree* $\vec{\mathcal{T}}$ is called $\mathcal{A}$-**contraction** of $\mathcal{T}$.

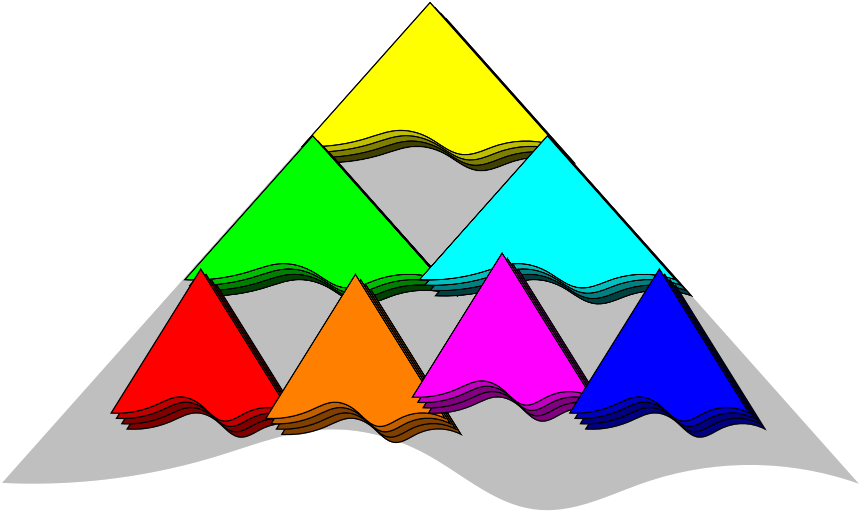
A picture of the method:

Given a tree $\mathcal{T}$, decompose it into **factors** ...



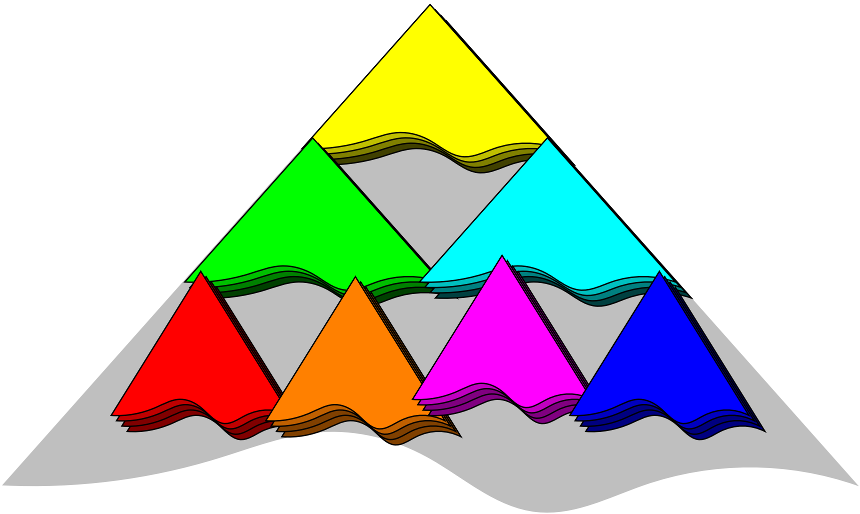
A picture of the method:

... then consider the equivalence classes
induced by the $\mathcal{A}$-**types** of the factors ...



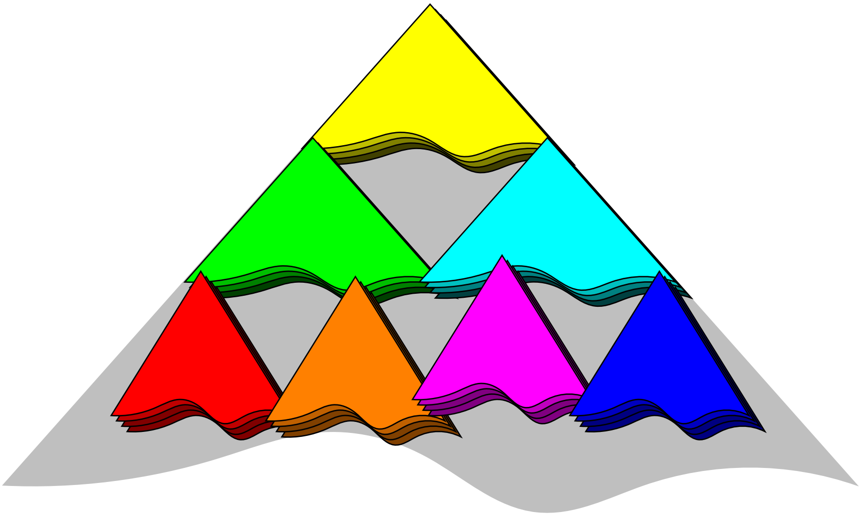
A picture of the method:

... The automaton $\mathcal{A}$ has the **same behavior**
on all trees in each equivalence class



A picture of the method:

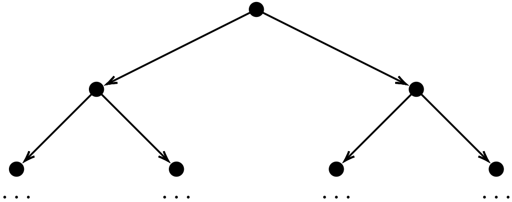
$\Rightarrow$ We replace $\mathcal{A}$ with an automaton $\vec{\mathcal{A}}$ that runs on the (possibly regular) $\mathcal{A}$-**contraction** and mimics $\mathcal{A}$



Definition (Factorization)

A **factorization** of a tree $\mathcal{T}$ is an *uncolored* tree Π such that

- $\{root(\mathcal{T})\} \subseteq \mathcal{D}om(\Pi) \subseteq \mathcal{D}om(\mathcal{T})$



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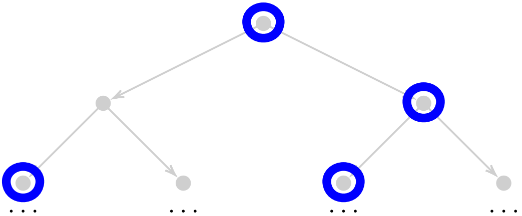

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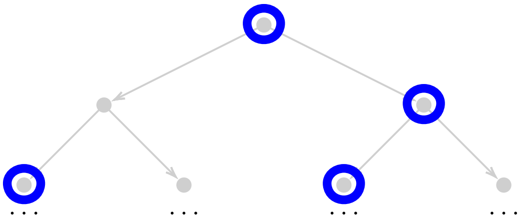

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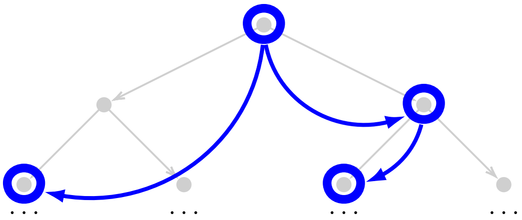

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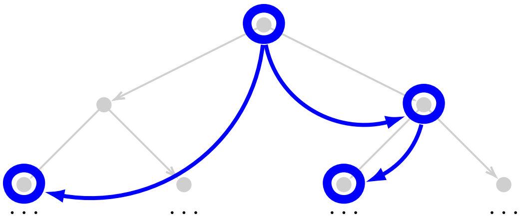

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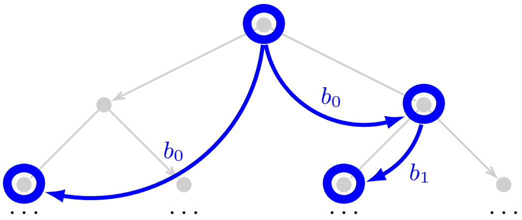

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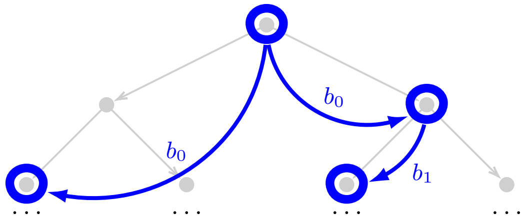

Note: Π can be a non-deterministic tree
and it can have even unbounded/infinite degree.

Definition (Factor)

For any $u \in \mathcal{D}om(\Pi)$, the **factor** $\mathcal{T}_u$ of $\mathcal{T}$ in u
is the subgraph of $\mathcal{T}$ induced by the set

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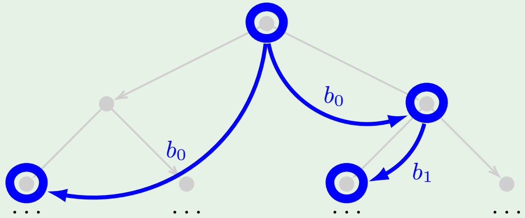

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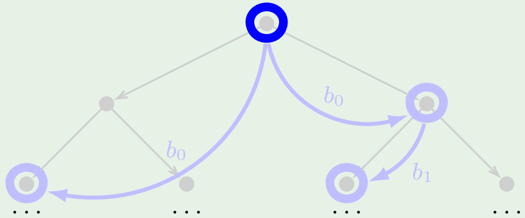

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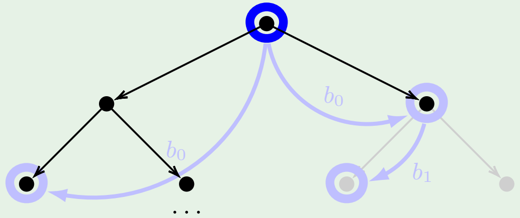

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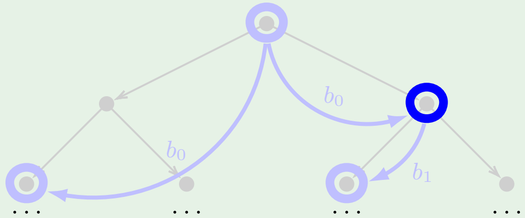

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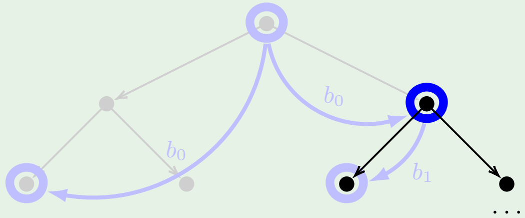

We also need to expand factors with information about the *edge labels* of the factorization Π:

Definition (Marked factor)

Given $u \in \mathcal{D}om(\Pi)$, the **marked factor** T_u^+ is obtained from $\mathcal{T}_u$ by *recoloring each leaf* u' with the *label* of the edge of Π that reaches u'.

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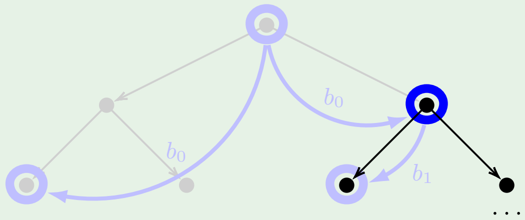

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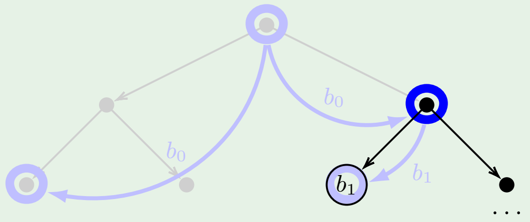

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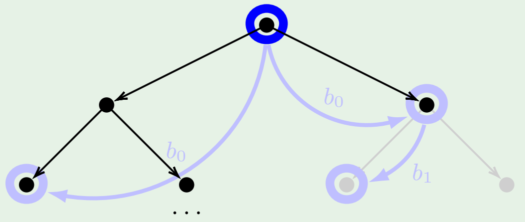

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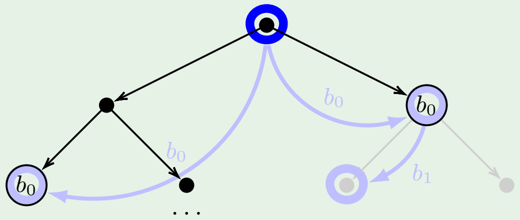
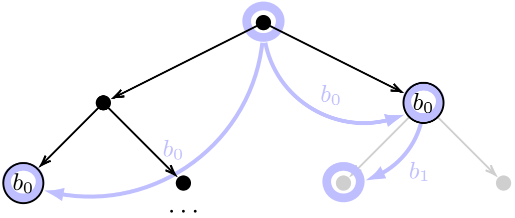


Definition ($\mathcal{A}$-Type)

Given an automaton M and a marked factor $\mathcal{T}_u^+$,
the $\mathcal{A}$-**type** $[\mathcal{T}_u^+]_{\mathcal{A}}$ is the *set of triples* of the form

$$\begin{pmatrix} \mathcal{R}(\varepsilon) \\ \{(\mathcal{T}(v), \mathcal{R}(v), \mathcal{I}mg(\mathcal{R}|\pi_v)) \,:\, v \in \mathcal{F}r(\mathcal{T})\} \\ \{\mathcal{I}nf(\mathcal{R}|\pi) \,:\, \pi \in \mathcal{B}ch(\mathcal{T})\} \end{pmatrix}$$

over all possible **partial runs** $\mathcal{R}$ of $\mathcal{A}$ on $\mathcal{T}_u^+$.



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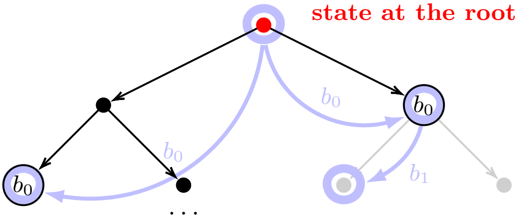

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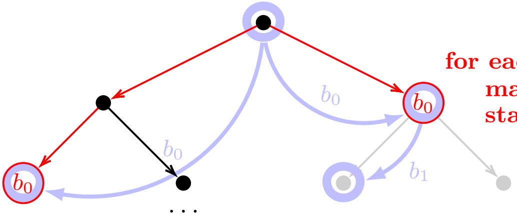

for each leaf, marker + state + states along access path

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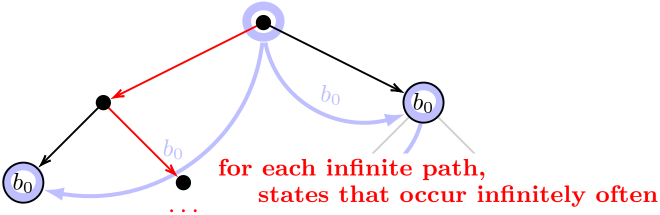
over all possible **partial runs** $\mathcal{R}$ of $\mathcal{A}$ on $\mathcal{T}_u^+$.



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$\Rightarrow$ we can arrange the $\mathcal{A}$-types of the factors of a tree
in a tree-shaped colored structure called $\mathcal{A}$-contraction.

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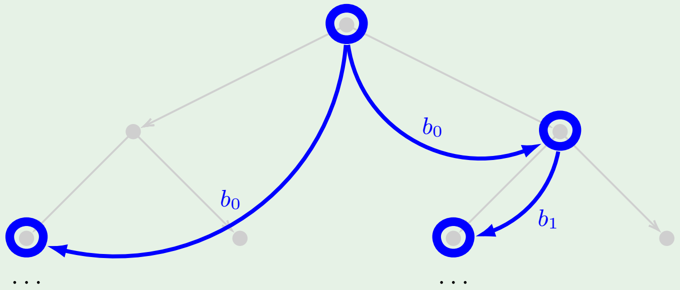

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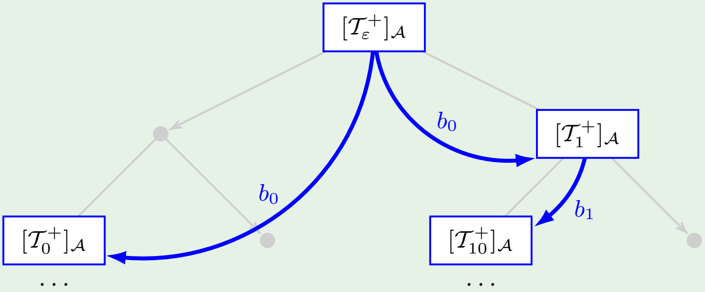

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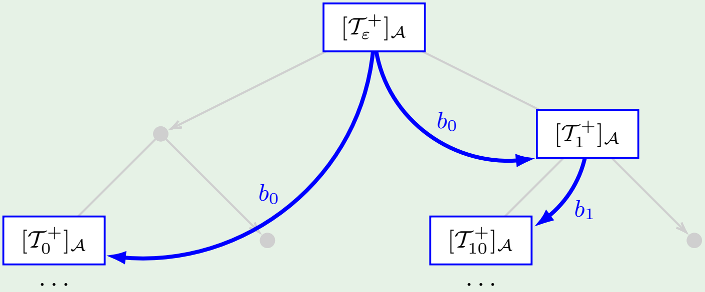

Note: $\vec{\mathcal{T}}$ may have infinite out-degree ($\Rightarrow$ non-deterministic).

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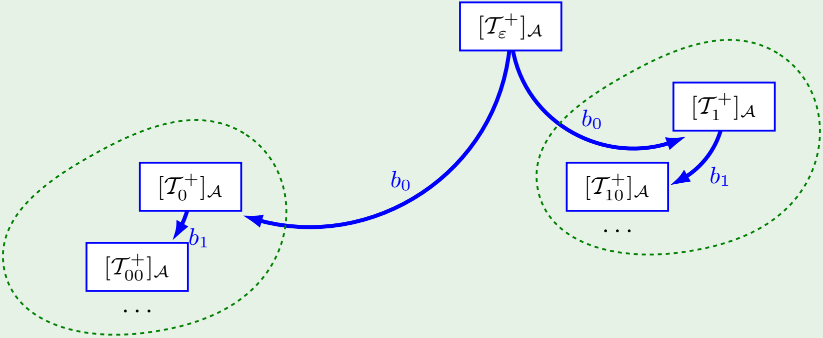

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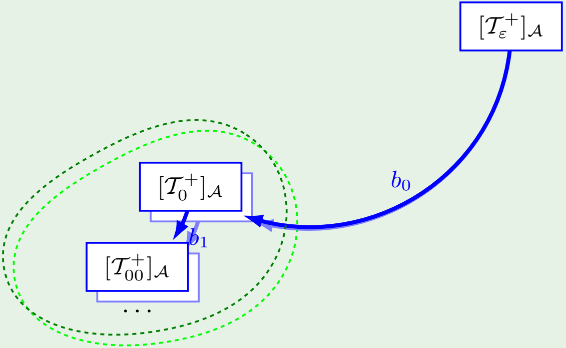

Theorem (Montanari and Puppis '04)

If a tree $\mathcal{T}$ has a deterministic $\mathcal{A}$-contraction $\vec{\mathcal{T}}$ for any automaton $\mathcal{A}$, then one can build another Rabin tree automaton $\vec{\mathcal{A}}$, called **contraction automaton**, *such that*

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Proof idea

- the input alphabet of $\vec{\mathcal{A}}$ consists of all $\mathcal{A}$-types plus a dummy symbol $\perp$ for missing b-labeled successors in $\vec{\mathcal{T}}$
- a transition of $\vec{\mathcal{A}}$ from a vertex colored by $[\mathcal{T}_u^+]_\mathcal{A}$ mimics a computation of $\mathcal{A}$ on the marked factor $\mathcal{T}_u^+$

  (note: this can be done since the $\mathcal{A}$-type $[\mathcal{T}_u^+]_\mathcal{A}$ is a finite object that completely characterizes the 'behavior' of $\mathcal{A}$ on the marked factor $\mathcal{T}_u^+$)

Corollary

Given a tree $\mathcal{T}$ and a Rabin tree automaton $\mathcal{A}$, if one can compute a (deterministic) $\mathcal{A}$-contraction $\overrightarrow{\mathcal{T}}$ with $Acc_{\overrightarrow{\mathcal{T}}}$ decidable (e.g., a regular $\mathcal{A}$-contraction), then $Acc_{\mathcal{T}}$ is decidable.

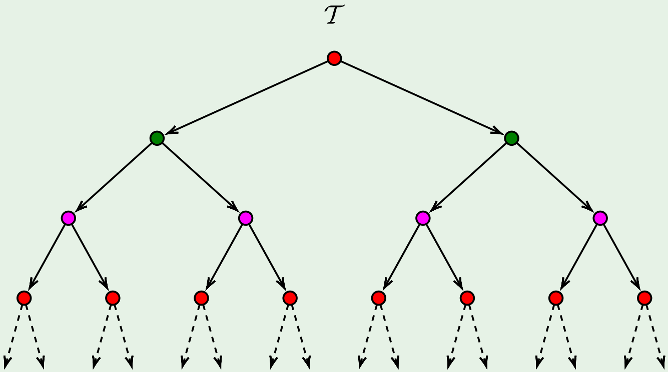
To summarize

to prove that the acceptance problem of a tree $\mathcal{T}$ is decidable:

1. provide a suitable factorization Π of $\mathcal{T}$
2. build the $\mathcal{A}$-contraction $\overrightarrow{\mathcal{T}}$ of $\mathcal{T}$ w.r.t. Π
3. show that $\overrightarrow{\mathcal{T}}$ is (bisimilar to) a deterministic tree
4. show that $Acc_{\overrightarrow{\mathcal{T}}}$ is decidable (e.g., show that $\overrightarrow{\mathcal{T}}$ is regular).

Example

Let $\mathcal{T}$ be a tree with homogeneously-colored levels



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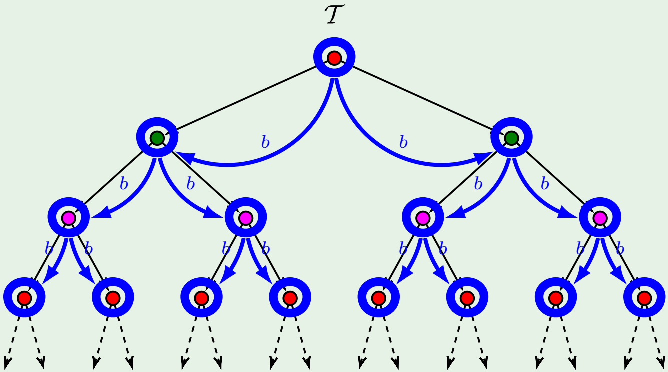

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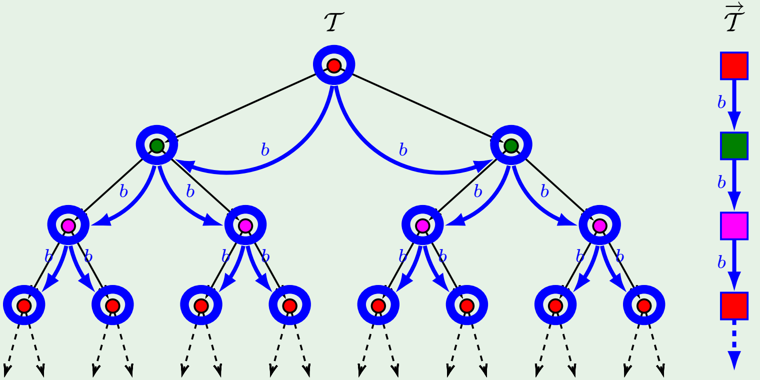
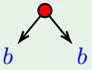
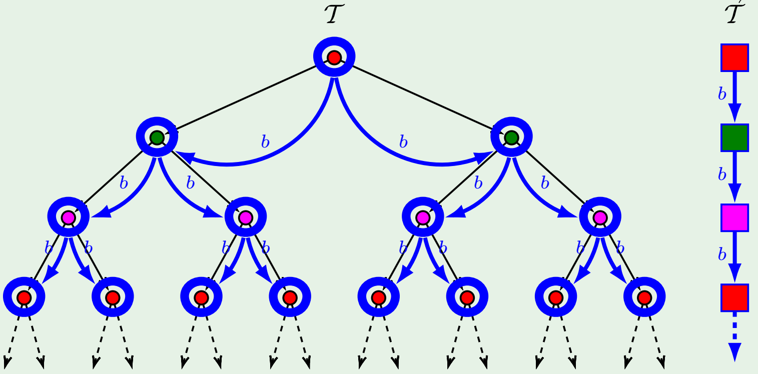


Example

Each vertex of the $\mathcal{A}$-contraction $\vec{\mathcal{T}}$

is colored with the $\mathcal{A}$-type of the corresponding factor





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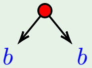
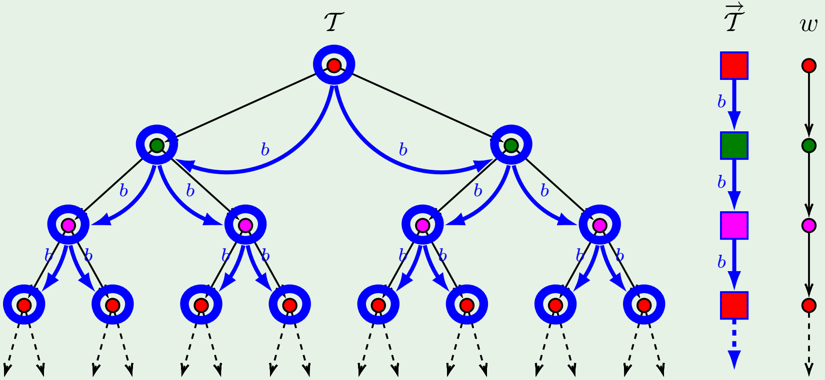


Acc_w decidable



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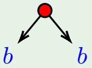

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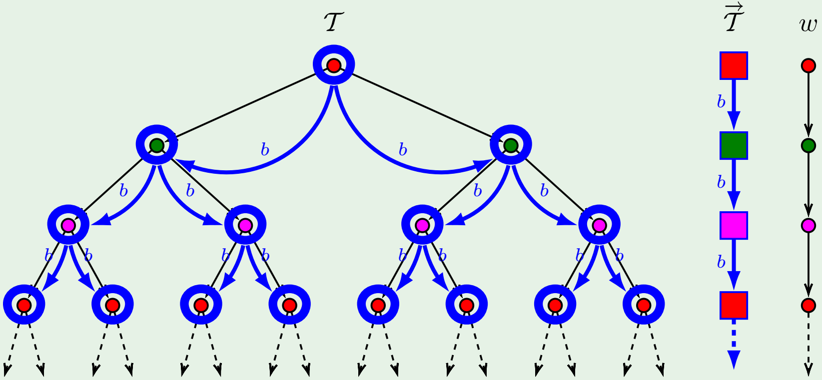

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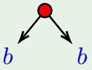

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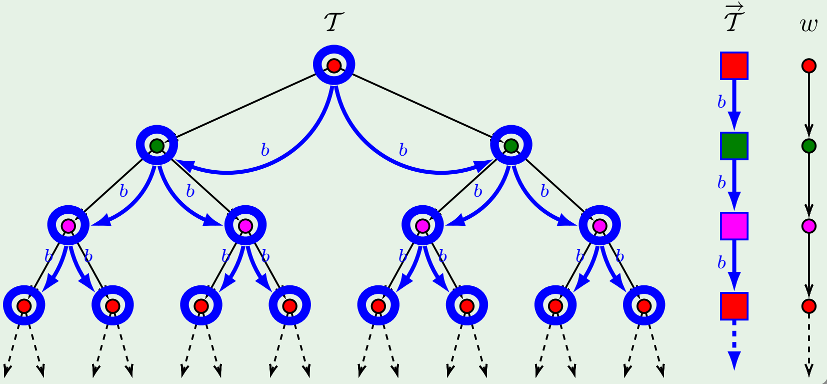

Definition (Second-order tree substitution)

The **second-order tree substitution** $\mathcal{C}[\![\mathcal{T}/x]\!]$ is the replacement of each x-colored vertex in $\mathcal{C}$ with $\mathcal{T}$.

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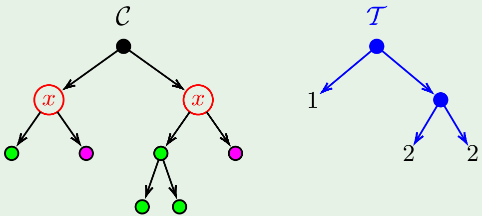

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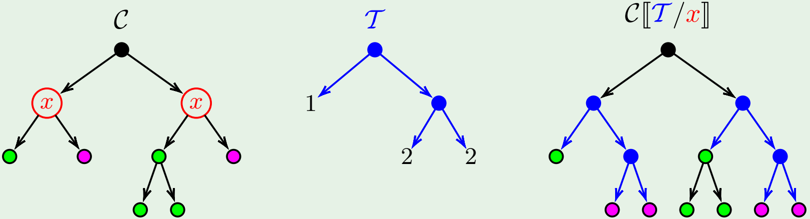

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Second-order tree substitutions respect the $\mathcal{A}$-types:

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Given an automaton $\mathcal{A}$ and a function γ such that

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there is a well-defined (computable) function $\gamma_{\mathcal{A}}$ such that

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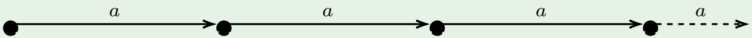
The set of all functions $\gamma_{\mathcal{A}}$ with functional composition $\circ$ is a

finite monoid

Example

Let $\mathcal{L}$ be the semi-infinite line



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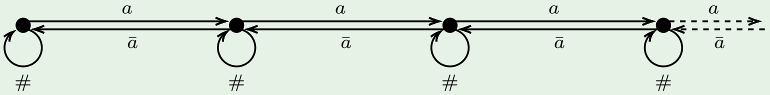

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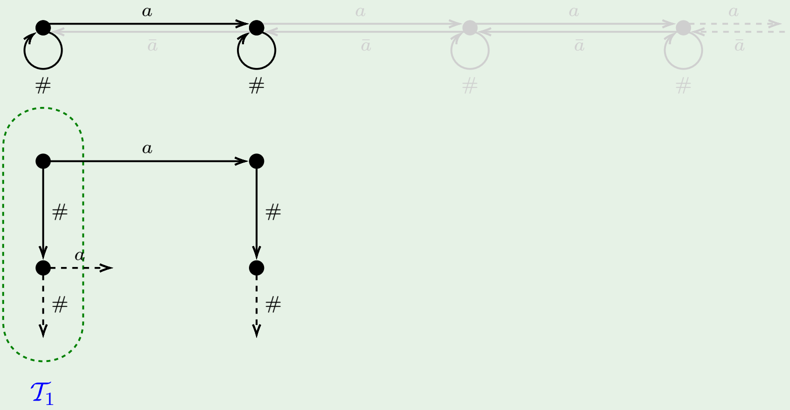

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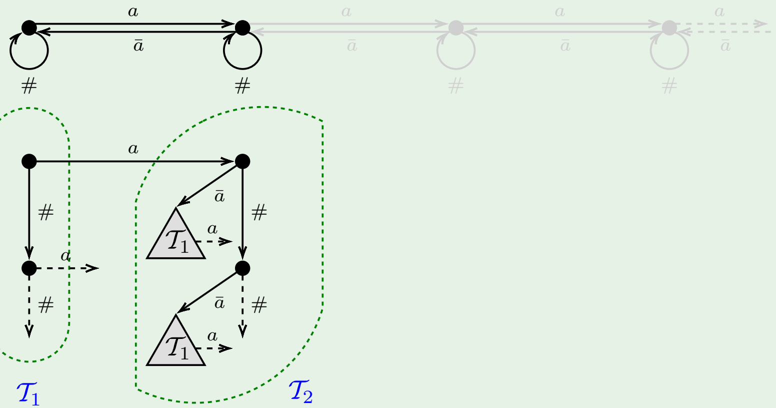

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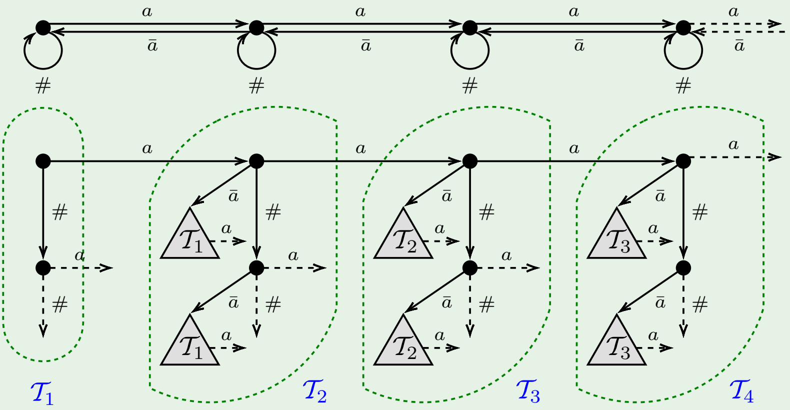
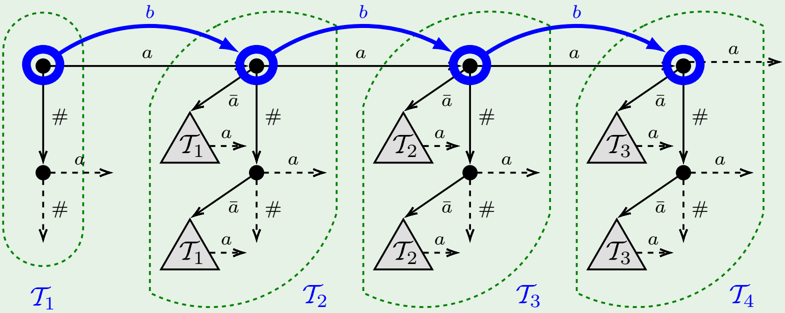


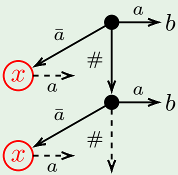
Example

We now define the following factorization:

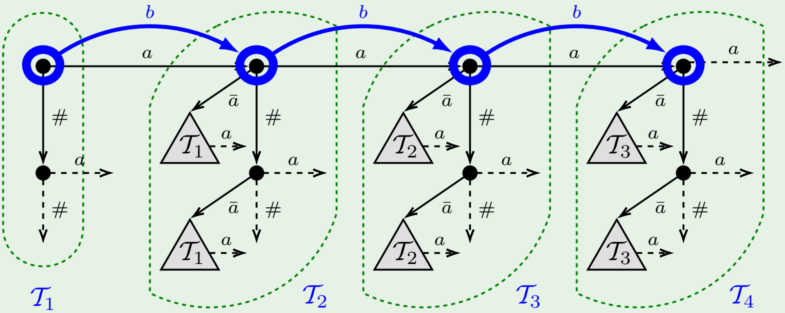


Example

Each factor $\mathcal{T}_{n+1}$ is obtained from the

second-order tree substitution in 

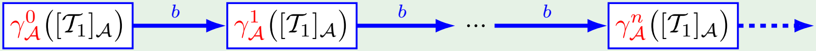
of x by $\mathcal{T}_n$.

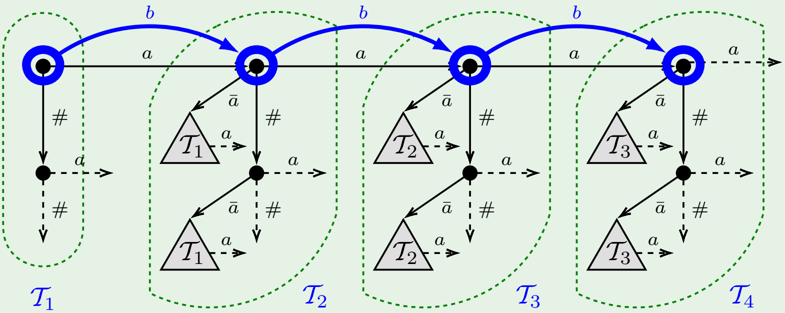


Example

We can write $\mathcal{T}_{n+1} = \gamma(\mathcal{T}_n)$, hence $[\mathcal{T}_{n+1}]_{\mathcal{A}} = \gamma^n_{\mathcal{A}}([\mathcal{T}_1]_{\mathcal{A}})$

$\Rightarrow$ the $\mathcal{A}$-contraction is a regular tree of the form





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Example

Consider the problem of deciding if $\mathcal{T} \in \mathscr{L}(\mathcal{A})$:

If $\mathcal{T}$ has an $\mathcal{A}$-contraction $\vec{\mathcal{T}}$, and
$\vec{\mathcal{T}}$ has a *regular* $\vec{\mathcal{A}}$-contraction $\vec{\vec{\mathcal{T}}}$

Then we can decide if $\vec{\vec{\mathcal{T}}} \in \mathscr{L}(\vec{\vec{\mathcal{A}}})$, $\vec{\mathcal{T}} \in \mathscr{L}(\vec{\mathcal{A}})$, and $\mathcal{T} \in \mathscr{L}(\mathcal{A})$.

Definition (Reducible trees)

It comes natural to define a

hierarchy of recursively reducible trees:

- **rank** 0 **trees** := regular trees
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Theorem

The acceptance problem of any recursively reducible tree is decidable.

Closure properties of rank n trees

By exploiting the inductive structure of rank n trees, one can show that, for any $n \in \mathbb{N}$, rank n trees are closed under:

- **rational colorings**

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- **regular tree morphisms**
  specified by a tuple of regular trees $(\mathcal{T}_x)_{x \in X}$
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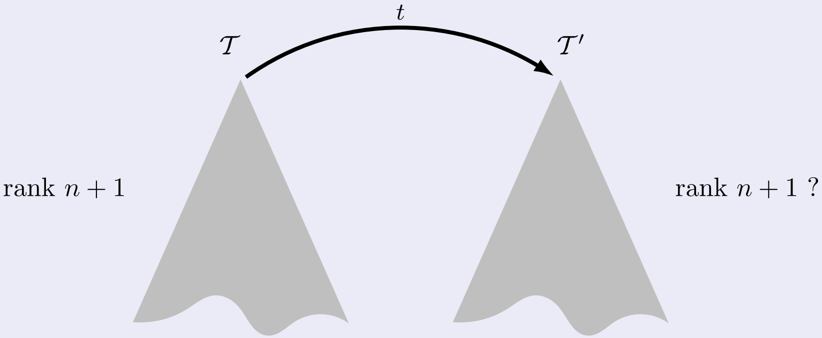
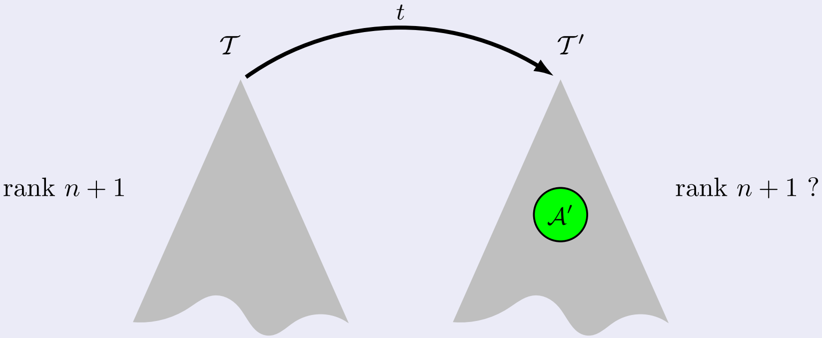
Proof idea (closure properties of rank n trees)

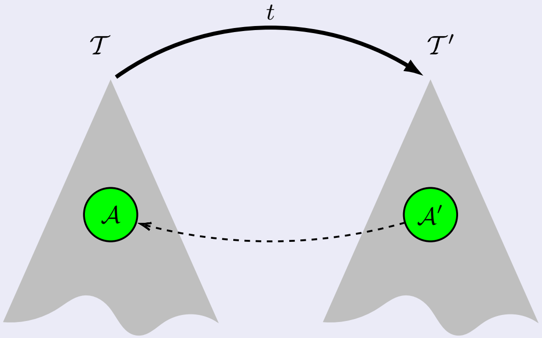


Proof idea (closure properties of rank n trees)



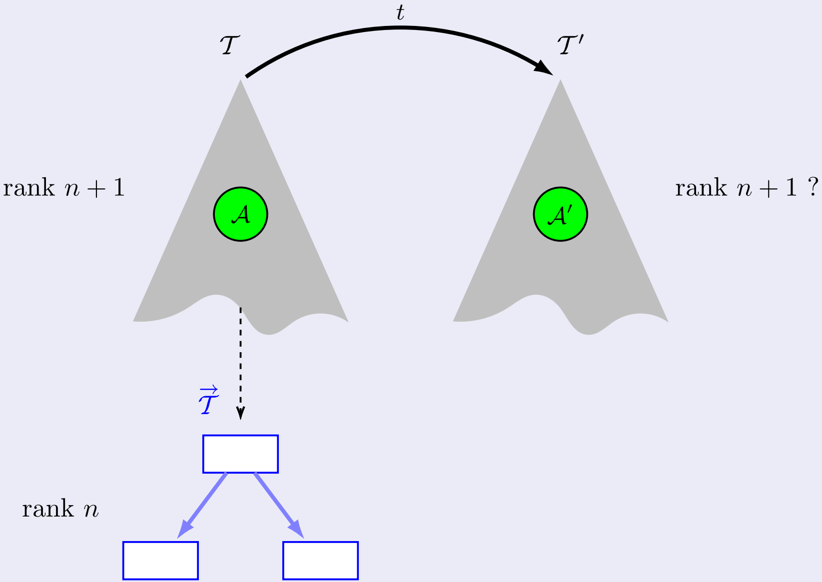
Proof idea (closure properties of rank n trees)

rank $n+1$

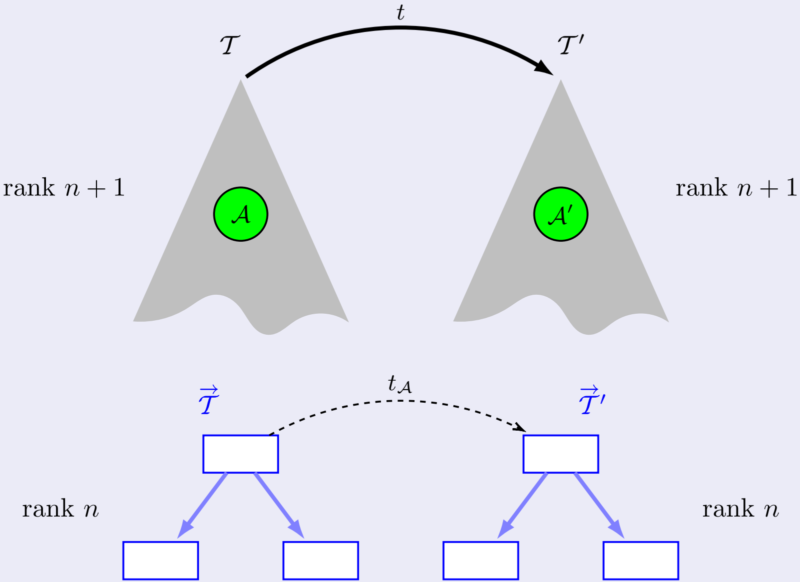


rank $n+1$?

Proof idea (closure properties of rank n trees)



Proof idea (closure properties of rank n trees)



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