Verification of infinite state systems

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Go

In this part

We present two interesting classes of transition systems:

- Context-free graphs
  these are (the connected components of) transition graphs of **pushdown systems**
- Prefix-recognizable graphs
  these are the transition graphs of **prefix rewriting systems**

We provide alternative representations of graphs in both classes and we show that their MSO-theories are decidable.

Definition (Pushdown system)

A **pushdown system** is a tuple $\mathcal{P} = (Q, \Gamma, A, \Delta)$, where:

- Q is a finite set of control states
- Γ is a finite set of stack symbols
- A is a finite set of transition labels
- $\Delta \subseteq Q \times \Gamma \times A \times Q \times \Gamma^*$ is a set of transition rules

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Configurations:
pairs in $Q \times \Gamma^*$ (state + stack content).

Transitions:
$(q, zw) \xrightarrow[\mathcal{P}]{a} (q', w'w)$ is a transition iff $(q, z, a, q', w') \in \Delta$.

Two main differences w.r.t. standard pushdown automata:

- no initial state and no initial stack symbol

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  ... we are interested in evaluating
  properties of their transition graphs

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- normalized forms of transition

  change: (q, z, a, q', z') with $q, q' \in Q$, $z, z' \in \Gamma$, $a \in A$
  push: $(q, z, a, q', z'z)$ with $q, q' \in Q$, $z, z' \in \Gamma$, $a \in A$
  pop: $(q, z, a, q', \varepsilon)$ with $q, q' \in Q$, $z \in \Gamma$, $a \in A$

  the length of the stack changes at most by one ...
  ... this is not a restriction: use *blocks of stack symbols*
  to put a generic pushdown system into normal form.

Definition (Pushdown transition graph)

The **transition graph** of a pushdown system $\mathcal{P} = (Q, \Gamma, A, \Delta)$ is the transition system $\mathcal{T} = \big(S, (\delta_a)_{a \in A}\big)$ where

- $S = Q \times \Gamma^*$
- $\big((q, w), (q', w')\big) \in \delta_a$ iff $(q, w) \xrightarrow[\mathcal{P}]{a} (q', w')$.

Note: pushdown graphs have bounded out-/in-degree.

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Definition (Connected component)

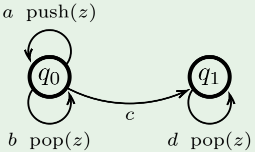
A (strongly) **connected component** of a graph $\mathcal{T}$ is a maximal subgraph of $\mathcal{T}$ such that, for every pair of vertices u, v, there exist a path π from u to v (π is allowed to traverse edges in both directions).

A **context-free graph** is a connected component of a pushdown transition graph.

Example

Consider the pushdown system $\mathcal{P} = (Q, \Gamma, A, \Delta)$, where

- $Q = \{q_0, q_1\}$
- $\Gamma = \{z\}$
- $A = \{a, b, c, d\}$
- Δ consists of the transitions (q_0, z, a, q_0, zz), $(q_0, z, b, q_0, \varepsilon)$, (q_0, z, c, q_1, z), $(q_1, z, d, q_1, \varepsilon)$



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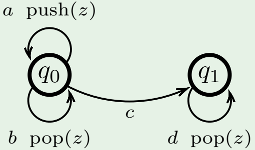
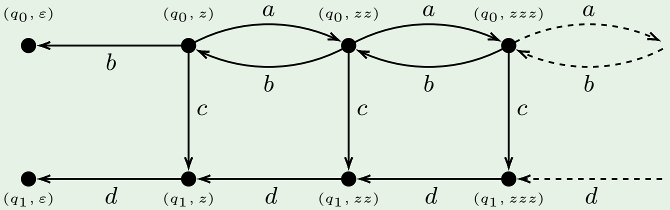


The connected component of its transition graph is depicted below:



Theorem

Transition graphs of pushdown systems and context-free graphs can be defined inside the infinite binary tree via inverse rational mappings (in fact, inverse finite mappings) and rational restrictions.

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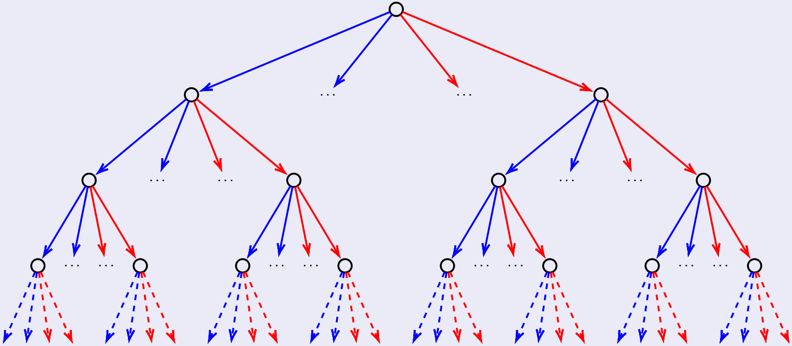
Corollary (Muller and Schupp '85)

The model checking problem for MSO logic over context-free graphs is decidable.

Proof of the theorem

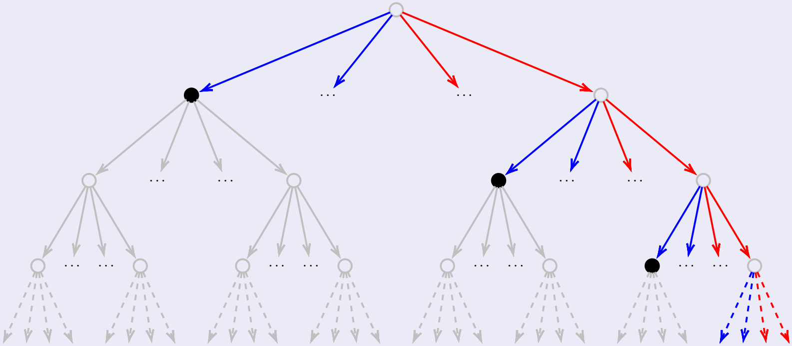
Let $\mathcal{P} = (Q, \Gamma, A, \Delta)$ be a pushdown system and let $\mathcal{T}$ be the **infinite** $Q \cup \Gamma$**-labeled tree**.



Proof of the theorem

We identify a $\mathcal{P}$-configuration (q, w) by the **reversed word** $\widetilde{w}q$ and we color the corresponding vertices of $\mathcal{T}$ by black:

$$k(black) := \Gamma^* Q$$

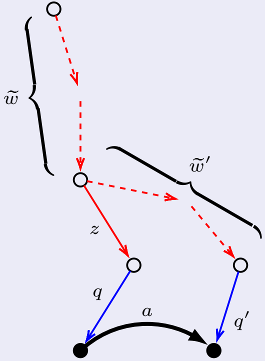


Proof of the theorem

We define a-labeled transitions via the inverse finite mapping

$$h(a) := \{\bar{q}\bar{z}\widetilde{w}'q' \ : \ (q, z, a, q', w') \in \Delta\}$$

$(q, zw) \xrightarrow[\mathcal{P}]{a} (q', w'w)$ iff $(\widetilde{w}zq, \widetilde{w}\widetilde{w}'q')$ is an a-labeled edge in $h^{-1}(\mathcal{T})$.



Proof of the theorem

Now, $h^{-1}(\mathcal{T})$, restricted to black-colored vertices, is the **transition graph** of $\mathcal{P}$.

To cope with context-free graphs, we must further restrict the domain of $h^{-1}(\mathcal{T})$.

Given a $\mathcal{P}$-configuration (q, w), we further restrict the domain of $h^{-1}(\mathcal{T})$ to the **regular** set

$$L := \widetilde{w}q \cdot \left(\bigcup_{a \in A} h(a) \cup \bigcup_{a \in A} \overline{\widetilde{h(a)}} \right)^*$$

thus obtaining the connected component of $h^{-1}(\mathcal{T})$ (i.e., the **context-free graph** of $\mathcal{P}$) that contains $\widetilde{w}q$.

Let thus now consider the class of prefix rewriting systems, which are a natural generalization of pushdown systems and, unlike them, may produce graphs with possibly infinite out-/in-degree.

Basic features:

- no more distinction between control states and stack letters (a single alphabet is used)
- less restricted forms of rewriting rules (more than one letter can be rewritten in a single transition)

Definition (Prefix rewriting system)

A **prefix rewriting system** is a tuple $\mathcal{P} = (\Gamma, L, A, \Delta)$, where:

- Γ is a finite set of symbols
- L is a regular language over Γ,
- A is a finite set of transition labels
- Δ is a finite set of rules of the form (U, a, V), where $a \in A$ and U, V *are regular languages over* Γ.

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Configurations:
all finite words in L.

Transitions:
$uw \xrightarrow[\mathcal{P}]{a} vw$ is a transition iff $\exists\, (U, a, V) \in \Delta.\ u \in U, v \in V$.

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Note: pushdown systems are special forms of prefix rewriting ones.

Definition (Prefix-recognizable graph)

A **prefix-recognizable graph** is the transition graph of a prefix rewriting system.

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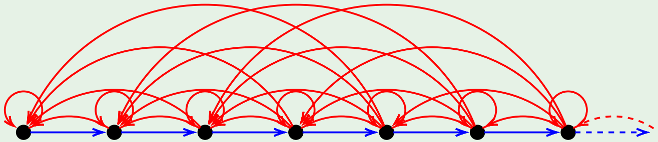
Example

Consider the prefix rewriting system $\mathcal{P} = (\Gamma, L, A, \Delta)$, where

- $\Gamma = \{z\}$
- $L = \{z\}^*$
- $A = \{succ, geq\}$
- Δ consists of the two rules $\big(\{\varepsilon\}, succ, \{z\}\big)$, $\big(\{z\}^*, geq, \{\varepsilon\}\big)$.



Theorem (Caucal '96)

Prefix-recognizable graphs are definable inside the infinite binary tree via inverse rational mappings and rational restrictions.

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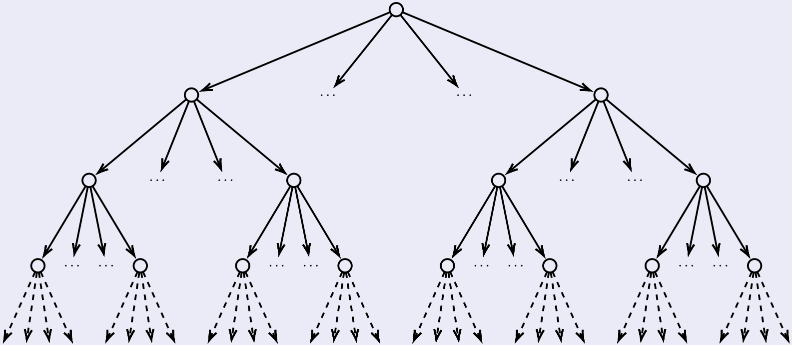
Corollary (Caucal '96)

The model checking problem for MSO logic over prefix-recognizable graphs is decidable.

Proof of the theorem

Let $\mathcal{P} = (\Gamma, L, A, \Delta)$ be a prefix rewriting system and let $\mathcal{T}$ be the **infinite Γ-labeled tree**.



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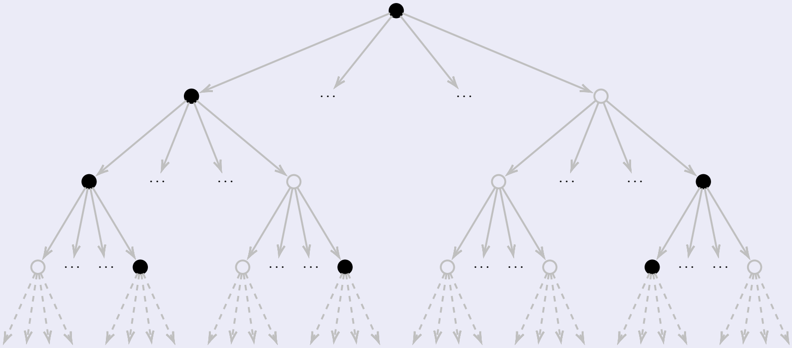
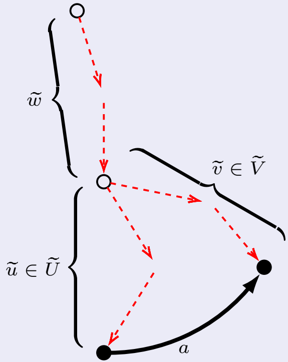


Proof of the theorem

We define a-labeled transitions via the inverse rational mapping

$$h(a) := \bigcup_{(U,a,V)\in\Delta} \overline{U} \cdot \widetilde{V}$$

$uw \xrightarrow[\mathcal{P}]{a} vw$ iff $(\widetilde{w}\widetilde{u}, \widetilde{w}\widetilde{v})$ is an a-labeled edge in $h^{-1}(\mathcal{T})$.



So far we know that

- pushdown transition graphs are obtained from the infinite binary tree via inverse finite mappings and rational restrictions.

  The converse is also true (Caucal '96):
  inverse finite mappings and rational restrictions applied to the infinite binary tree yield pushdown transition graphs.

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$\Rightarrow$ inverse finite/rational mappings and rational restrictions can be thought of as **external presentations** of pushdown/prefix-recognizable graphs.

Context-free graphs have alternative representations based on **hyperedge-replacement graph grammars**.

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Definition (Hypergraph)

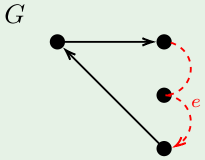
A **hyperedge** is a sequence of k vertices $(v_1, ..., v_k)$.
(an edge is a special form of hyperedge with 2 vertices only).

A **hypergraph** is a graph where edges are replaced with hyperedges (labels can be assigned to the hyperedges of a hypergraph).

A **hyperedge replacement** is the replacement of a hyperedge $e = (v_1, ..., v_k)$ in a hypergraph G with a (hyper)graph H

(a suitable marking of the vertices of H is used to identify the vertices of H that replace the vertices of G in e).

Example (Hyperedge replacement)



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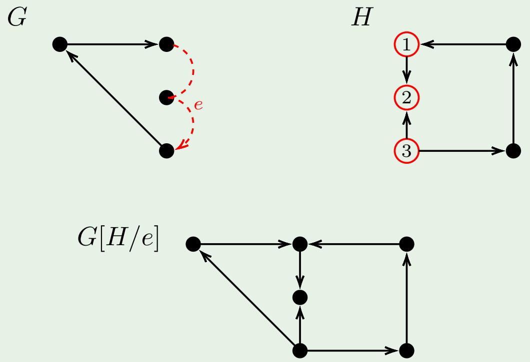

Definition (Graph grammar)

Given a finite set N of **nonterminal symbols**,
a **(hyperedge-replacement) graph grammar** is
a tuple $\mathcal{G} = (H_z)_{z \in N}$ of hypergraphs that defines
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hyperedge with the hypergraph H_z.

The **pattern graph** generated by $\mathcal{G}$ starting from
an axiom $z_0 \in N$ is the *limit* of the sequence of graphs
obtained by the repeated application of replacement rules in $\mathcal{G}$.

Note that

- *the limit operation does not take into account the nonterminal hyperedges*
- *every hyperedge is eventually replaced* (replacement order does not matter).

Example (a pattern graph)

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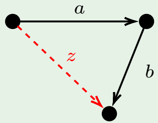

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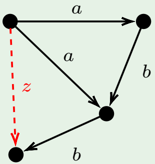

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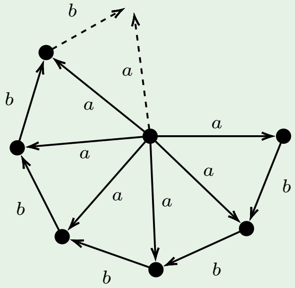

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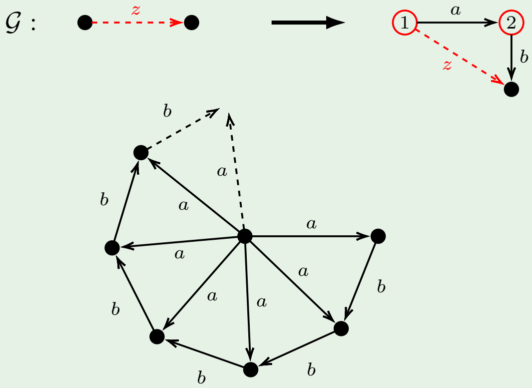

Note: pattern graphs may have infinite in-/out- degree and unconnected components ...

If we restrict ourselves to **special** graph grammars $\mathcal{G} = (H_z)_{z \in N}$, where

- nonterminal hyperedges in H_z have no repeated vertices
- vertices of nonterminal hyperedges in H_z are not marked
- every vertex of a nonterminal hyperedge in H_z is also a vertex of a terminal edge
- every terminal edge in H_z has at least one marked vertex
- for every (non-terminal symbol) $z \in N$, the pattern graph generated by $\mathcal{G}$ starting from z is a connected graph,

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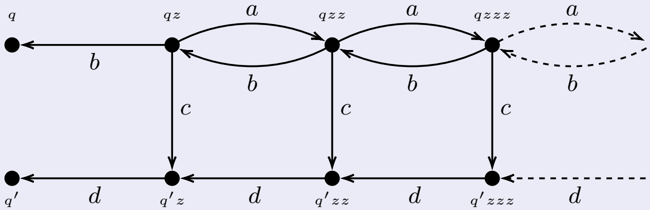
then:

Theorem (Muller and Schupp '85)

The context-free graphs are exactly the pattern graphs generated by special (hyperedge-replacement) graph grammars.

An intuitive account (one direction)

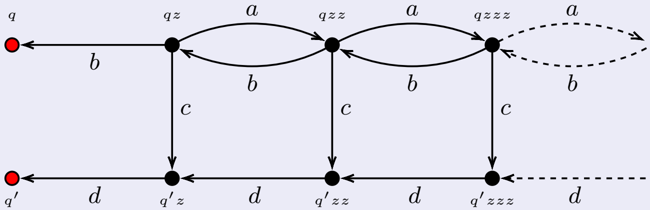
Consider the context-free graph $\mathcal{T} = (S, (\delta_a)_{a \in A})$



An intuitive account (one direction)

The **end-components** induced by $V_n = \{v \in S : |v| \geq n\}$ are

$n = 1$



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The **end-components** induced by $V_n = \{v \in S : |v| \geq n\}$ are

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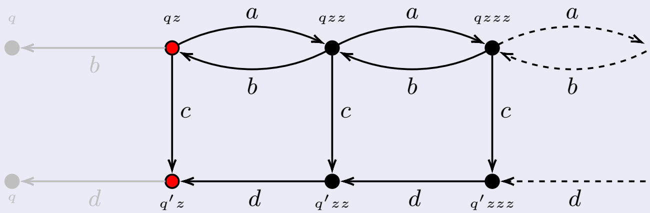

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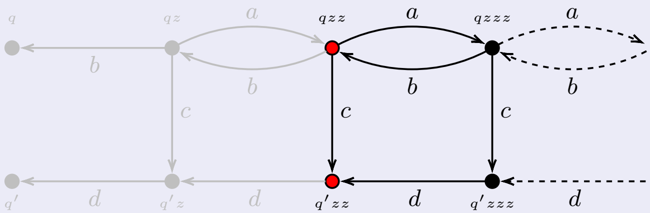

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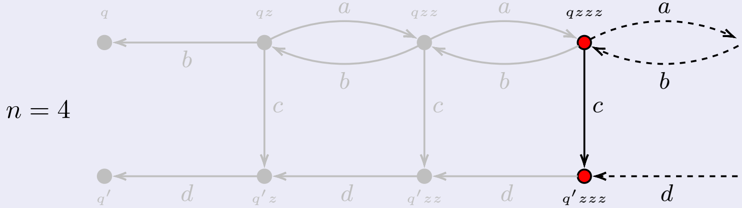

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The **end-components** induced by $V_n = \{v \in S \,:\, |v| \geq n\}$ are

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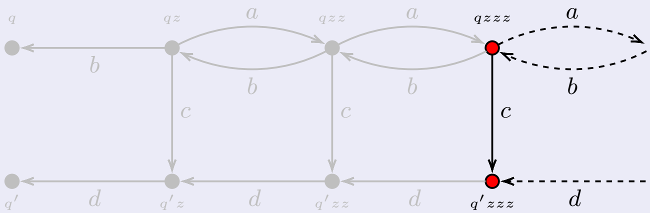

One can show that there are only finitely many non-isomorphic end-components, each one generating a distinct replacement rule

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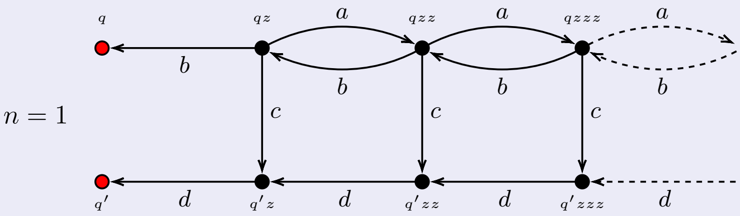

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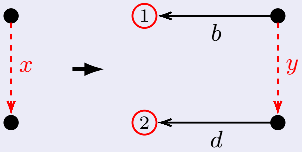

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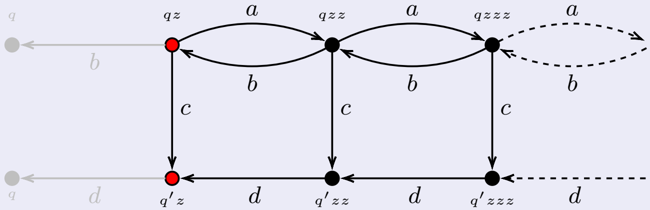

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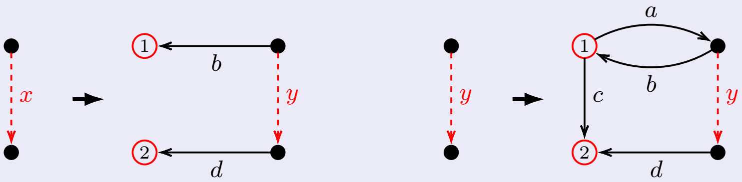

An analogous characterization holds for prefix-recognizable graphs:

Theorem (Courcelle '92)

The prefix-recognizable graphs are exactly the pattern graphs generated by **vertex-replacement** *graph grammars.*

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The prefix-recognizable graphs are exactly the pattern graphs generated by **vertex-replacement** *graph grammars.*

Moreover, we have that

Theorem

The prefix-recognizable graphs are exactly the graphs in the second level of the Caucal hierarchy, namely, the graphs generated by MSO-interpretations over infinite regular trees.

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