From I3 to I1

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Main Sources: [10] and [9]

Main Results: Introduction of ω-many new hypothesis between I3 and I1. When n is odd, than a Σ^1_n rank-into-rank elementary embedding is also Σ^1_{n+1}, but when n is even, the existence of a Σ^1_{n+1} rank-into-rank elementary embedding is stricly stronger than the existence of a Σ^1_n one.

Remember that I3 is the existence of an elementary embedding $j : V_\lambda \prec V_\lambda$ and I1 of $j : V_{\lambda+1} \prec V_{\lambda+1}$. Is there a correlation between these two (other than the trivial implication of I3 from I1)? Are they really two different axioms, or maybe also I3 implies I1? The question with an affirmative answer is the first, and in this section will be presented an infinity of axioms between I3 and I1, all strictly implying one another.

Definition 0.1. *Let $j : V_\lambda \prec V_\lambda$. Define $j^+ : V_{\lambda+1} \to V_{\lambda+1}$ as*

$$\forall A \subset V_\lambda \quad j^+(A) = \bigcup_{\beta<\lambda} j(A \cap V_\beta).$$

While it is not clear whether j^+ is an elementary embedding, every elementary embedding from $V_{\lambda+1}$ to itself is the 'plus' of its restriction to V_λ:

Lemma 0.2. *If $j : V_{\lambda+1} \prec V_{\lambda+1}$, then $(j \upharpoonright V_\lambda)^+ = j$. Thus every $j : V_{\lambda+1} \prec V_{\lambda+1}$ is defined by its behaviour on V_λ, i.e., for every $j, k : V_{\lambda+1} \prec V_{\lambda+1}$,*

$$j = k \qquad \textit{iff} \qquad j \upharpoonright V_\lambda = k \upharpoonright V_\lambda.$$

Proof. The critical sequence $\langle \kappa_n : n \in \omega \rangle$ is a subset of V_λ, so it belongs to

$V_{\lambda+1}$. But then for every $A \subseteq V_\lambda$, $\{A \cap V_{\kappa_n} : n \in \omega\} \in V_{\lambda+1}$, so

$$\begin{aligned}(j \restriction V_\lambda)^+(A) &= \bigcup_{n\in\omega}(j \restriction V_\lambda)(A \cap V_{\kappa_n}) \\ &= \bigcup_{n\in\omega} j(A \cap V_{\kappa_n}) \\ &= j(\bigcup_{n\in\omega}(A \cap V_{\kappa_n})) \\ &= j(A).\end{aligned}$$

□

It is worth noting that there is a strong connection between first-order formulas in $V_{\lambda+1}$ and second-order formulas in V_λ. In fact, all the elements of $V_{\lambda+1}$ are subsets of V_λ, so they can be replaced with relation symbols. First of all, note that since V_λ is closed by finite sequences, all the λ-sequences in $V_{\lambda+1}$ can be codified as members of $V_{\lambda+1}$.

Lemma 0.3. *Let $A \in V_{\lambda+1} \setminus V_\lambda$ and $\varphi(v_0, v_1, \ldots, v_n)$ be a formula. Fix $\hat{A}$ a relation symbol, and define $\varphi^*(v_1, \ldots, v_n)$ in the language of* LST *expanded with $\hat{A}$ as following:*

- *for every occurrence of v_0, substitute $\hat{A}$;*
- *for every non-bounded quantified variable x, substitute every occurrence of x with X, a second-order variable.*

Then for every $a_1, \ldots, a_n \in V_\lambda$

$$V_{\lambda+1} \vDash \varphi(A, a_1, \ldots, a_n) \quad \textit{iff} \quad (V_\lambda, A) \vDash \varphi^*(a_1, \ldots, a_n).$$

Proof. The proof is by induction on the complexity of φ.

If φ is atomic, or a conjunction of atomic formulas, the Lemma it's obvious.

If φ is $\exists x \in v_i\ \psi(v_0, x, v_1, \ldots, v_n)$, whether i is 0 or not, then $V_{\lambda+1} \vDash \varphi(A, a_1, \ldots, a_n)$ iff there exists $c \in V_{\lambda+1}$, $c \in a_i$ or $c \in A$, such that $V_{\lambda+1} \vDash \psi(A, c, a_1, \ldots, a_n)$ But then c must be in V_λ, and by induction we have that this happens iff $(V_\lambda, A) \vDash \psi^*(c, a_1, \ldots, a_n)$, that is $(V_\lambda, A) \vDash \varphi^*(a_1, \ldots, a_n)$.

If φ is $\forall x \in v_i\ \psi(v_0, x, v_1, \ldots, v_n)$, then suppose that $(V_\lambda, A) \vDash \varphi^*(a_1, \ldots, a_n)$ and fix $b \in a_i$ or $b \in A$ (depending on whether i is 0 or not). Therefore $(V_\lambda, A) \vDash \psi^*(b, a_1, \ldots, a_n)$ and by induction $V_{\lambda+1} \vDash \psi(A, b, a_1, \ldots, a_n)$. Since this it's true for every $b \in a_i$, $V_{\lambda+1} \vDash \varphi(A, a_1, \ldots, a_n)$.

If φ is $\exists x\ \psi(v_0, x, v_1, \dots, v_n)$, then suppose that $V_{\lambda+1} \vDash \varphi(A, a_1, \dots, a_n$, i.e. let $C \in V_{\lambda+1}$ such that $V_{\lambda+1} \vDash \psi(A, C, a_1, \dots, a_n)$. By induction $(V_\lambda, A) \vDash \psi^*(C, a_1, \dots, a_n)$ (with ψ^* a second-order formula), so $(V_\lambda, A) \vDash \exists X \varphi^*(X, a_1, \dots, a_n)$.

The $\forall x$ case is the same as the previous one. □

The previous Lemma is a key to clarify the relationship between elementary embeddings in V_λ and $V_{\lambda+1}$, and to finally prove that j^+ is a Σ_0 elementary embedding from $V_{\lambda+1}$ to itself (or, alternatively that j is a Σ^1_0 elementary embedding from V_λ to itself).

Theorem 0.4. *Let $j : V_\lambda \prec V_\lambda$. Then $j^+ : V_{\lambda+1} \to V_{\lambda+1}$ is a Δ_0-elementary embedding.*

Proof. Let $\hat{A}$ be a symbol for an 1-ary relation, $\hat{a}_1, \dots, \hat{a}_n$ symbols for constants and let φ be a formula in LST*, the language of LST expanded with $\hat{A}$ and $\hat{a}_1, \dots, \hat{a}_n$. We want to prove that

$$(V_\lambda, A, a_1, \dots, a_n) \vDash \varphi \qquad \text{iff} \qquad (V_\lambda, j^+(A), j(a_1), \dots, j(a_n)) \vDash \varphi.$$

Actually we will prove only one direction for every φ, the other one following by considering $\neg\varphi$.

First of all, we skolemize φ, so we find $f_1, \dots, f_m$ functions, $f_i : (V_\lambda)^{m_i} \to V_\lambda$, such that φ is equivalent to a formula φ^* in LST* expanded with $\hat{f}_1, \dots, \hat{f}_m$, where φ^* is $\forall x_1 \forall x_2 \dots \forall x_m\ \varphi'$, with φ' a Δ_0 formula.

Let $t_0(x_1, \dots, x_m), \dots, t_p(x_1, \dots, x_m)$ be all the terms that are in φ'. We can express as a logical formula the phrase "$t_i(x_1, \dots, x_m)$ exists": we work by induction on the tree of the term, writing a conjuction of formulas in this way

- every time that there is an occurence of a function, i.e. $\hat{f}_i(t')$, we add to the formula $\exists y\ f_i(t') = y$;
- every time that there is an occurence of a constant, i.e. $\hat{a}_i$, we add to the formula $\hat{a}_i \neq \emptyset$, if $a_i \neq \emptyset$; otherwise we don't write anything.

Then we have that

$$\begin{aligned}(V_\lambda, A, a_1, \dots, a_n, f_1, \dots, f_m) \vDash \varphi^* \quad &\text{iff} \\ &\forall \delta < \lambda\ (V_\delta, A \cap V_\delta, a'_1, \dots, a'_n, f_1 \cap V_\delta, \dots, f_m \cap V_\delta) \vDash \\ &\qquad \forall x_1, \dots, \forall x_m (\bigwedge_{i<p} t_i(x_1, \dots, x_m) \text{ exists } \to \varphi'),\end{aligned}$$

where a'_i is a_i if $a_i \in V_\delta$, otherwise is $\emptyset$. This is true because, with δ fixed, if only one term doesn't exist the formula is satisfied, and if all the terms exist, then they are a witness for the satisfaction (or not) of φ'.

So

$$
\begin{aligned}
&(V_\lambda, A, a_1, \dots, a_n) \vDash \varphi \\
&\quad \to (V_\lambda, A, a_1, \dots, a_n, f_1, \dots, f_m) \vDash \varphi^* \\
&\quad \to \forall \delta < \lambda \ (V_\delta, A \cap V_\delta, a'_1, \dots, a'_n, f_1 \cap V_\delta, \dots, f_m \cap V_\delta) \vDash \\
&\qquad\qquad\qquad\qquad \forall x_1, \dots, \forall x_m \ (\bigwedge_{i<p} t_i(x_1, \dots, x_m) \text{ exists } \to \varphi') \\
&\quad \to \forall \delta < \lambda \ (V_\delta, j(A \cap V_\delta), j(a'_1), \dots, j(a'_n), j(f_1 \cap V_\delta), \dots, j(f_m \cap V_\delta)) \vDash \\
&\qquad\qquad\qquad\qquad \forall x_1, \dots, \forall x_m \ (\bigwedge_{i<p} t_i(x_1, \dots, x_m) \text{ exists } \to \varphi') \\
&\quad \to (V_\lambda, j^+(A), j(a_1), \dots, j(a_n), j^+(f_1), \dots, j^+(f_m)) \vDash \varphi^* \\
&\quad \to (V_\lambda, j^+(A), j(a_1), \dots, j(a_n)) \vDash \varphi.
\end{aligned}
$$

$\square$

Therefore every $j : V_\lambda \prec V_\lambda$ can be extended to a unique $j^+ : V_{\lambda+1} \to V_{\lambda+1}$ that is at least a Δ_0-elementary embedding. Moreover it is possible to prove

- I3 holds iff for some λ there exists a $j : V_{\lambda+1} \to V_{\lambda+1}$ that is a Δ_0-elementary embedding;
- I1 holds iff for some λ there exists a $j : V_{\lambda+1} \prec V_{\lambda+1}$ that is a Σ_n-elementary embedding for every n

Theorem 0.4 implies that if $j, k : V_\lambda \prec V_\lambda$, then $j^+(k) : V_\lambda \prec V_\lambda$. This operation between elementary embeddings is called *application*, and we write $j^+(k) = j \cdot k$. Coupled with composition (not to be mistaken to!) they create an interesting algebra.

Theorem 0.5 (Laver, 1992 [7]). *Fix λ and let $\mathcal{E}_\lambda = \{j : V_\lambda \prec V_\lambda\}$. For all $j \in \mathcal{E}_\lambda$:*

- *The closure of $\{j\}$ in $(\mathcal{E}_\lambda, \cdot)$ is the free algebra generated by the law*

$$\textit{(Left Distributive Law)} \quad i \cdot (j \cdot k) = (i \cdot j) \cdot (i \cdot k).$$

- *The closure of $\{j\}$ in $(\mathcal{E}_\lambda, \cdot, \circ)$ is the free algebra generated by the left distributive law and the following laws:*

$$\begin{aligned}
&i \circ (j \circ k) = (i \circ j) \circ k \\
&(i \circ j) \cdot k = i \cdot (j \cdot k) \\
&i \cdot (j \circ k) = (i \cdot j) \circ (i \cdot k) \\
&i \circ j = (i \cdot k) \circ i.
\end{aligned}$$

Moreover Laver proved that the free algebra generated by the laws above satisfies the word problem. His proof used extensively I3, but later Dehornoy ([1]) managed to prove the same thing in ZFC. A proof of this can be found in the Handbook of Set Theory [3], Chapter 11.

Another interesting result in I3 regards a function on the integers. Consider $j : V_\lambda \prec V_\lambda$ with critical point κ and let $\mathcal{A}_j$ be the closure of j in $(\mathcal{E}_j, \cdot)$. Then define

$$f(n) = |\{\mathrm{crt}(k) : k \in \mathcal{A}_j,\ j^n(\kappa) < \mathrm{crt}(k) < j^{n+1}(\kappa)\}|.$$

Then $f(0) = f(1) = 0$ and $f(2) = 1$, because the simplest element of $\mathcal{A}_j$ that has a critical point not in the critical sequence of j is $((j \cdot j) \cdot j) \cdot (j \cdot j)$. However $f(3)$ is very large. Laver ([8]) proved that for any n, $f(n)$ is finite, but f dominates the Ackermann function, so f cannot be primitive recursive ([2]).

At last, the possibility of applying an elementary embedding to itself leads to interesting reflection properties of V_λ:

Lemma 0.6. *Suppose $j : V_\lambda \prec V_\lambda$ and let $\langle \kappa_n : n \in \omega\rangle$ be its critical sequence. Then for every $n \in \omega$, $V_{\kappa_n} \prec V_\lambda$.*

Proof. Since j is the identity on V_{κ_0}, it is easy to see that $V_{\kappa_0} \prec V_{\kappa_1}$. Considering $j(j)$, since $\mathrm{crt}(j(j)) = \kappa_1$ and $j(j)(\kappa_1) = j(j)(j(\kappa_0)) = j(j(\kappa_0)) = \kappa_2$, we have also that $V_{\kappa_1} \prec V_{\kappa_2}$. We can generalize this to prove that for every $n \in \omega$, $V_{\kappa_n} \prec V_{\kappa_{n+1}}$. But then $\langle (V_{\kappa_n}, \mathrm{id}_{V_{\kappa_n}}), n \in \omega\rangle$ forms a direct system, whose direct limit is V_λ. $\square$

Let's go back to j and j^+. Theorem 0.4 proves that j^+ is a Δ_0-elementary embedding, and j^+ is a full elementary embedding iff it is a Σ_n-elementary embedding for every n. So between I3 and I1 there are many intermediate steps:

Definition 0.7 (Σ^1_n Elementary Embedding)**.** *Let $j : V_\lambda \prec V_\lambda$. Then j is Σ^1_n iff j^+ is a Σ_n-elementary embedding, i.e., iff for every Σ^1_n-formula $\varphi(X)$, for every $A \subseteq V_\lambda$,*

$$V_\lambda \vDash \varphi(A) \leftrightarrow \varphi(j^+(A)).$$

It is not clear, however, if these hypothesis are really different. In fact, this is not true. The following theorems will prove that if n is odd, then j is Σ^1_n iff it is Σ^1_{n+1}.

To prove this we need what is often called the 'descriptive set theory' on V_λ. The fact that λ has cofinality ω and is a strong limit, makes possible to develop a description of the closed subsets of V_λ as projection of trees similar to the classic one in descriptive set theory. Fix λ and a critical sequence $\langle \kappa_0, \kappa_1, \dots \rangle$.

Let $\varphi(X, Y)$ a Σ^1_0-formula, namely

$$\forall x_0 \exists y_0 \forall x_1 \exists y_1 \cdots_n \psi(X, Y, x_0, y_0, \dots).$$

We define the tree $T_{\varphi(X,Y)}$. The m-th level of $T_{\varphi(X,Y)}$ is the set of (a, b, F, P) such that:

- $a, b \subseteq V_{\kappa_m}$;
- F is a partial function : $(V_{\kappa_m})^{\leq n+1} \to V_{\kappa_m}$ such that for all $d_0, \dots, d_n$ where F is defined then

$$\psi(a, b, d_0, F(d_0), d_1, F(d_0, d_1), \dots);$$

- $P : ((V_{\kappa_m})^{\leq n+1} \setminus \mathrm{dom}(F)) \to (\omega \setminus (m + 1))$.

We say that $(a, b, F, P) < (a', b', F', P')$, with the first term in the m-th level and the second in the m'-th level, when $a \subseteq a'$, $b \subseteq b'$, $a' \cap V_{\kappa_m} = a$, $b' \cap V_{\kappa_m} = b$, $F \subseteq F'$, and if $P(\vec{d}) < m$ then $\vec{d} \in \mathrm{dom}(F)$, otherwise $P'(\vec{d}) = P(\vec{d})$.

Informally, a and b are just the attempts to construct X and Y, F is an approximation of the Skolem function that would witness the first order part of φ, and P is booking the level in which the elements of V_{κ_m} that are not yet in the dominion of F *will* be in its extension.

It is clear that if $T_{\varphi(X,Y)}$ has an infinite branch, the union of the a's and b's will give suitable X and Y, and the F's will construct a total Skolem function (thanks to the P's).

Lemma 0.8. *Let $\varphi(X, Y)$ a Σ^1_0-formula and $B \subseteq V_\lambda$. Then $V_\lambda \vDash \exists X\, \varphi(X, B)$ iff $T_{\varphi(X,B)}$ has an infinite branch.*

The Σ^1_2 case is a bit more complex. Fix a $B \subseteq V_\lambda$. For every $a \subseteq V_{\kappa_m}$ let

$$G_m(a) = \{(c, F, P) : (a, c, F, P) \in m\text{-th level of } T_{\neg\varphi(X,B,Y)}\}.$$

Then the m-th level of $\mathcal{T}_{\varphi(X,B)}$ is

$$\{(a, H) : a \subseteq V_{\kappa_m}, H : G_m(a) \to \lambda^+\}.$$

We say that $(a, H) < (a', H')$ when $a' \cap V_{\kappa_m}$ and if $(a, c, F, P) < (a', c', F', P')$, then $H(c, F, P) > H'(c', F', P')$.

Lemma 0.9. *Let $\varphi(X, Y)$ a Σ^1_0-formula and $B \subseteq V_\lambda$. Then $V_\lambda \vDash \exists X \forall Y\ \varphi(X, B)$ iff $\mathcal{T}_{\varphi(X,B)}$ has an infinite branch.*

If $\mathcal{T}_{\varphi(X,B)}$ has an infinite branch, with A the union of the a's in the branch, then the H's assure that there are no possible infinite branches in $T_{\neg\varphi(A,B,Y)}$ for every $Y \subset V_\lambda$, because otherwise it would be possible to build a descending chain in λ^+.

Lemma 0.10. *Let $j : V_\lambda \prec V_\lambda$. Then j is Σ^1_1 iff j^+ preserves the well-founded relations.*

Proof. 'Being well-founded' is a Δ^1_1 relation, so if j is Σ^1_1 it preserves well-founded relations. Vice versa, let $\varphi(X, Y)$ a Σ^1_0 formula, $B \subseteq V_\lambda$ and suppose that $V_\lambda \vDash \exists X \varphi(X, B)$. Fix an $X \subseteq V_\lambda$ such that $V_\lambda \vDash \varphi(X, B)$. Therefore by Theorem 0.4 $V_\lambda \vDash \varphi(j^+(X), j^+(B))$ and $V_\lambda \vDash \exists X\ \varphi(X, j^+(B))$. If $V_\lambda \vDash \forall X \varphi(X, B)$, then $T_{\neg\varphi(X,B)}$ is without infinite branches. Fix a well-ordering R of V_λ, and define the relative Kleene-Brouwer order. Therefore the Kleene-Brouwer order of $T_{\neg\varphi(X,B)}$ is well-founded. But then the Kleene-Brouwer order relative to $j^+(R)$ on $T_{\neg\varphi(X,j^+(B))}$ is well-founded. So $V_\lambda \vDash \forall X \varphi(X, j^+(B))$. □

Definition 0.11 (λ-Ultrafilters Tower)**.** *Fix λ and $\langle \kappa_n : n \in \omega \rangle$ a cofinal sequence of regular cardinals in λ. Then $\vec{\mathcal{U}} = \langle \mathcal{U}_n : n \in \omega \rangle$ is a* λ-ultrafilters tower *iff for every $n \in \omega$, $\mathcal{U}_n$ is a normal, fine, κ-complete ultrafilter on $[\kappa_{n+1}]^{\kappa_n}$, and if $m < n$ then for every $A \in \mathcal{U}_m$*

$$\{x \in [\kappa_{n+1}]^{\kappa_n} : x \cap \kappa_{m+1} \in A\} \in \mathcal{U}_n.$$

A λ-ultrafilters tower is complete *if for every sequence $\langle A_i \in i \in \omega \rangle$ with $A_i \in \mathcal{U}_i$ there exists $X \subseteq \lambda$ such that for every $i \in \omega$, $X \cap \kappa_{i+1} \in A_i$.*

Lemma 0.12 (Gaifman ([4]), Powell ([11]), 1974)**.** *If $\vec{\mathcal{U}}$ is a complete λ-tower, then the direct limit of the ultrapowers* $\mathrm{Ult}(V, \mathcal{U}_n)$ *is well-founded.*

When $j : V_\lambda \prec V_\lambda$, it is natural to consider the sequence $\vec{\mathcal{U}}_j = \langle \mathcal{U}_n : n \in \omega \rangle$ defined as
$$\mathcal{U}_n = \{A \subseteq [\kappa_{n+1}]^{\kappa_n} : j\text{"}\kappa_{n+1} \in j(A)\}.$$

Lemma 0.13. *If j is Σ^1_1, then $\vec{\mathcal{U}}_j$ is a complete λ-tower.*

Proof. Let $\langle A_i : i \in \omega \rangle$ such that $A_i \in \mathcal{U}_i$. Let $X = j\text{"}\lambda$. Then for every $i \in \omega$, $X \cap \kappa_{i+2} = j\text{"}\kappa_{i+1} \in j^+(A_i)$, so
$$\exists X \subset \lambda \forall i \in \omega \; X \cap \kappa_{i+2} \in j^+(A_i).$$
This is a Σ^1_1 statement, so by Σ^1_1-elementarity $\exists X \subset \lambda \forall i \in \omega \; X \cap \kappa_{i+1} \in A_i$. $\square$

Definition 0.14. *Let M be a transitive class, with $V_\lambda \subset M$. Then M is* Σ^1_n-correct in λ *if for every Σ^1_n-formula $\varphi(X)$, for every $A \subseteq V_\lambda$ and $A \in M$, $(V_\lambda \vDash \varphi(A))^M$ iff $V_\lambda \vDash \varphi(A)$.*

Theorem 0.15 (Martin ([10]))**.** *Let $j : V_\lambda \prec V_\lambda$. If j is Σ^1_1 then* $\mathrm{Ult}(V, \vec{\mathcal{U}}_j)$ *is Σ^1_2-correct, therefore j is Σ^1_2.*

Proof. Consider the direct limit $\mathrm{Ult}(V, \vec{\mathcal{U}}_j)$ of the ultrapowers $\mathrm{Ult}(V, \mathcal{U}_n)$. By Lemma 0.12 and Lemma 0.13 $\mathrm{Ult}(V, \vec{\mathcal{U}}_j)$ is well founded, and it is possible to collapse it on a model M.

$$\begin{tikzcd}
V \arrow[r, "j_0"] \arrow[dr, "j_1"'] \arrow[ddr, "j_n"'] & M_0 \cong \mathrm{Ult}(V, \mathcal{U}_0) \arrow[r, "i_0"] \arrow[d, "j_{01}"] & M \cong \mathrm{Ult}(V, \vec{\mathcal{U}}_j) \\
& M_1 \cong \mathrm{Ult}(V, \mathcal{U}_1) \arrow[d, "j_{12}"] \arrow[ur, "i_1"'] & \\
& \vdots \arrow[d] & \\
& M_n \cong \mathrm{Ult}(V, \mathcal{U}_n) \arrow[uuur, "i_n"'] \arrow[d] & \\
& \vdots &
\end{tikzcd}$$

By the usual theory of normal ultrafilters, it is possible to prove that the above diagram commute, that $\mathrm{crt}(j_n) = \kappa$, $\mathrm{crt}(j_{n,n+1}) = \kappa_n$, so $V_{\kappa_n} \subset M_n$,

$\mathrm{crt}(i_n) = \kappa_n$ and therefore $V_\lambda \subset M$. Let $i = i_n \circ j_n$ for every n. Then $i \upharpoonright V_\lambda = j$, $i(\lambda) = \lambda$ and $i((\lambda^+)^V) = (\lambda^+)^V$.

Observe that for every $B \subseteq V_\lambda$, $B \in M$ we have that $(T_{\varphi(B)})^M = T_{\varphi(B)}$, so M is Σ^1_1-correct in λ. Unfortunately, the same cannot be said to the Σ^1_2 case, since it is possible that $(\mathcal{T}_{\varphi(X,B,Y)})^M \neq \mathcal{T}_{\varphi(X,B,Y)}$. The reason for this is that while it is true that $(G_m(a))^M = G_m(a)$ for every $m \in \omega$ and $a \in V_{\kappa_m}$, it is possible that there exists $H : G_m(a) \to \lambda^+$ with $H \notin M$.

Claim 0.16. *If $c \in V_\lambda$ and $F : c \to \mathrm{Ord}$, then $i \circ F \in M$.*

Proof. Let $c \in V_\alpha$, with $\alpha < \lambda$, and pick $n \in \omega$ such that $\mathrm{crt}(i_n) > \alpha$. Then $i \circ F = i_n \circ j_n \circ F = i_n(j_n \circ F) \circ i_n \upharpoonright c = i_n(j_n \circ F) \in M$. □

By Σ^1_1 correctness in λ,

$$(V_\lambda \vDash \exists X \forall Y \varphi(X, B, Y))^M \to V_\lambda \vDash \exists X \forall Y \varphi(X, B, Y),$$

so it suffices to prove the other direction.

Suppose that $(V_\lambda \vDash \forall X \exists Y \neg\varphi(X, B, Y))^M$. Then $(\mathcal{T}_{\varphi(X,B,Y)})^M$ is well-founded. Let $L : (\mathcal{T}_{\varphi(X,B,Y)})^M \to \mathrm{Ord}$ that witnesses it. Define $\tilde{L} : \mathcal{T}_{\varphi(X,B,Y)} \to \mathrm{Ord}$ as $\tilde{L}((a, H)) = L((a, i \circ H))$. By the previous claim $i \circ H \in M$, and moreover $i \circ H$ is a function from $G_m(a)$ to $i(\lambda^+) = \lambda^+$, so $(a, i \circ H) \in (\mathcal{T}_{\varphi(X,B,Y)})^M$ and $\tilde{L}$ is well-defined. It remains to prove that $\tilde{L}$ witnesses that $\mathcal{T}_{\varphi(X,B,Y)}$ is well-founded. Suppose that $(a, H) < (a', H')$. If in $T_{\varphi(X,B,Y)}$ we have that $(a, c, F, P) < (a', c', F', P')$ then $H(c, F, P) > H'(c', F', P')$, so

$$i \circ H(c, F, P) > i \circ H'(c', F', P'),$$

therefore $(a, i \circ H) < (a', i \circ H')$. It follows that

$$L((a, i \circ H)) > L((a', i \circ H')),$$

so $\tilde{L}((a, H)) > \tilde{L}((a', H')$.

Then $\mathcal{T}_{\varphi(X,B,Y)}$ cannot have an infinite branch and $V_\lambda \vDash \forall X \exists Y \neg\varphi(X, B, Y)$, i.e., M is Σ^1_2-correct in λ.

This proves that $i \upharpoonright V_\lambda$ is Σ^1_2: if φ is Σ^1_2 in V_λ and $V \vDash \varphi(B)$, then by elementarity $\mathrm{Ult}(V, \vec{\mathcal{U}}_j) \vDash \varphi(i(B))$, therefore by Σ^1_2-correctness also $V \vDash \varphi(i(B))$. Since $i \upharpoonright V_\lambda = j$ we're done. □

This theorem proves that Σ^1_1 and Σ^1_2 elementary embeddings in V_λ are the same. It is possible, after a pair of technical lemmas, to generalize this for Σ^1_n and Σ^1_{n+1}.

Lemma 0.17. *Let n be odd. Then "j is Σ^1_n" is a Π^1_{n+1} formula in V_λ, with j as parameter.*

Proof. By definition j is Σ^1_n iff

$$\forall B \subseteq V_\lambda \; \forall \Sigma^1_0 \text{ formula } \varphi(X_1, \dots, X_n, Y)$$
$$\exists A_1 \; \forall A_2 \dots \exists A_n \; V_\lambda \vDash \varphi(\vec{A}, j^+(B)) \to \exists A_1 \; \forall A_2 \dots \exists A_n \; V_\lambda \vDash \varphi(\vec{A}, B).$$

But for every $D \subset V_\lambda$, $V_\lambda \vDash \varphi(\vec{A}, D)$ iff there exists a branch in $T_{\varphi(\vec{A})}$ whose projection is D. Using this is easy to check the Lemma. $\square$

Lemma 0.18. *Let n be odd, $n > 1$. Let j be Σ^1_n, $\mathrm{crt}(j) = \kappa$. Let $\beta < \kappa$, $A, B \subseteq V_\lambda$. Then there exists a $k : V_\lambda \prec V_\lambda$ that is Σ^1_{n-1} such that $k(B) = j(B)$, $k(A') = A$ for some $A' \subseteq V_\lambda$ and $\beta < \mathrm{crt}(k) < \kappa$.*

Proof. The formula "$\exists k, Y,\ k$ is Σ^1_{n-2}, $k(B) = j(B), k(Y) = A, \beta < \mathrm{crt}(k) < \kappa$" is Σ^1_n by Lemma 0.17. The following formula is clearly true:

$$j \text{ is } \Sigma^1_{n-2},\ j(j(B)) = j(j(B)),\ j(A) = j(A),\ j(\beta) = \beta < \mathrm{crt}(j) < j(\kappa)$$

but then, with a smart quantification of some of the parameters of the formula above, we have

$$\exists k, Y \; k \text{ is } \Sigma^1_{n-2},\ k(j(B)) = j(j(B)), k(Y) = j(A),\ j(\beta) < \mathrm{crt}(k) < j(\kappa)$$

By elementarity the lemma is proved. $\square$

Note that the Lemma holds not only if n is odd, but also when $n = 0$.

Theorem 0.19 ([9])**.** *Let n be odd. Then if j is Σ^1_n, it is also Σ^1_{n+1}.*

Proof. Theorem 0.15 is the case $n = 1$. We proceed by induction on n.

Let $B \subseteq V_\lambda$. We have to prove that for every ψ Σ_{n+1} formula and $B \subseteq V_{\lambda+1}$,

$$V_\lambda \vDash \psi(B) \text{ iff } V_\lambda \vDash \psi(j^+(B)).$$

By induction, the direction from left to right is immediate, so it suffices to show the other direction. Suppose

$$V_\lambda \vDash \exists X_1 \; \forall X_2 \; \exists X_3 \dots \forall X_{n+1} \; \varphi(X_1, \dots, X_{n+1}, j^+(B)).$$

Let $X_0 = A$ a witness. By Lemma 0.18, we can pick a k that is Σ^1_{n-2} such that $k(B) = j(B)$ and there exists $A' \subseteq V_{\lambda+1}$ such that $k(A') = A$. By induction, k is also Σ^1_{n-1}, and since

$$V_\lambda \vDash \forall X_2 \; \exists X_3 \dots \forall X_{n+1} \; \varphi(k(A'), X_2, \dots, X_{n+1}, k(B)),$$

then

$$V_\lambda \vDash \forall X_2 \; \exists X_3 \dots \forall X_{n+1} \; \varphi(A', X_2, \dots, X_{n+1}, B).$$

Therefore A' is a witness for $\psi(B)$. $\square$

Theorem 0.19 shows a peculiar asimmetry. What problems arise when n is even? The key is in the proof of Lemma 0.17. When the number of the quantifiers of a Σ_n formula is odd, then the last one is an existential quantifier. Since by Lemma 0.9 the satisfaction relation for Σ^1_1 formulae is Σ^1_1, this quantifier is absorbed by the satisfaction formula. When n is even, however, the last quantifier is an universal one, and "being Σ^1_n" is still Π^1_{n+2}. This seems an ineffectual detail, but in fact Laver proved that it is an essential one, since he showed that being a Σ^1_{n+1}-embedding, for n even, is strictly stronger than being a Σ^1_n one.

Theorem 0.20 ([9])**.** *Let h and k be Σ^1_n embeddings. Then $h \cdot k$ is a Σ^1_n-embedding.*

Proof. By Theorem 0.19 we can suppose n odd. Let φ be a Σ^1_0 formula with $n+1$ variables. Then

$$V_\lambda \vDash \forall Y \ \big(\exists X_1 \ \forall X_2 \ldots \exists X_n \ \varphi(X_1, \ldots, X_n, k(Y)) \\ \to \exists X_1 \ \forall X_2 \ldots \exists X_n \ \varphi(X_1, \ldots, X_n, Y)\big).$$

This formula is Π^1_{n+1}. Since by Theorem 0.19 h is Σ^1_{n+1}, we have

$$V_\lambda \vDash \forall Y \ \big(\exists X_1 \ \forall X_2 \ldots \exists X_n \ \varphi(X_1, \ldots, X_n, h(k)(Y)) \\ \to \exists X_1 \ \forall X_2 \ldots \exists X_n \ \varphi(X_1, \ldots, X_n, Y)\big).$$

so $h(k)$ is Σ^1_n. ∎

Note that the converse is not true. Let k be Σ^1_n. The formula "$\exists h \ h(h) = k, \ h$ is Σ^1_{n-2}" is a Σ^1_n-formula. Since $k(k) = k(k)$, then "$\exists h \ h(h) = k(k), \ h$ is Σ^1_{n-2}" is true, so by elementarity $\exists h \ h(h) = k, \ h$ is Σ^1_{n-2}. Let h be the one with minimal critical point. Then

$$\kappa = \mathrm{crt}(h) = \min\{\mathrm{crt}(h) : h \text{ is } \Sigma^1_{n-2}, \ h(h) = k\}$$

is Σ^1_n-definable, so h cannot be Σ^1_n.

However, if we switch application with composition, then also the converse is true.

Lemma 0.21. *If h and k are Σ^1_m and $h \circ k$ is Σ^1_{m+1}, then k is Σ^1_{m+1}.*

Proof. Suppose that

$$V_\lambda \vDash \forall X_1 \ \exists X_2 \ldots \forall X_{m+1} \ \varphi(X_1, \ldots, X_{m+1}, B)$$

with $B \subseteq V_\lambda$. Then

$$V_\lambda \vDash \forall X_1\ \exists X_2 \dots \forall X_{m+1}\ \varphi(X_1, \dots, X_{m+1}, h \circ k(B)).$$

Let $A \subseteq V_\lambda$. Then in particular

$$V_\lambda \vDash \exists X_2 \dots \forall X_{m+1}\ \varphi(h(A), X_2, \dots, X_{m+1}, h \circ k(B)).$$

By elementarity

$$V_\lambda \vDash \exists X_2 \dots \forall X_{m+1}\ \varphi(A, X_2, \dots, X_{m+1}, k(B)).$$

Since this is true for every $A \subseteq V_\lambda$, we have

$$V_\lambda \vDash \forall X_1\ \exists X_2 \dots \forall X_{m+1}\ \varphi(X_1, \dots, X_{m+1}, k(B)).$$

□

Theorem 0.22 ([9])**.** *Let $h, k \in \mathcal{E}_\lambda$. Then h, k are Σ^1_n iff $h \circ k$ is Σ^1_n.*

Proof. If h and k are Σ^1_n, then obviously $h \circ k$ is Σ^1_n. We prove by induction on $m \leq n$ that h and k are Σ^1_m.

The case $m = 0$ is by hypothesis. Suppose it is true for m. Then by Theorem 0.20 $h(k)$ is Σ^1_m. It is easy to calculate that $h \circ k = h(k) \circ h$, and this is Σ^1_{m+1} by hypothesis. By using Lemma 0.21 in the left side, we have that k is Σ^1_{m+1}, and using it on the right side we have that h is Σ^1_{m+1}. □

Theorem 0.22 is promising for our objective, that is proving that being Σ^1_{n+2} is strictly stronger than being Σ^1_n for an elementary embedding. The most natural idea for doing this is using some sort of reflection, to prove that if there is a $j \in \mathcal{E}_\lambda$ Σ^1_{n+2}, then there is a $k \in \mathcal{E}_{\lambda'}$ that is Σ^1_n. A common idea for similar proofs is to use a direct limit of elementary embeddings, but unfortunately this is not possible in this setting:

Theorem 0.23. *(Laver, [8]) There exists a j that is Σ^1_n that has a stabilizing direct limit of members of $\mathcal{A}_j$ that is not Σ^1_1,*

So we will consider inverse limits instead.

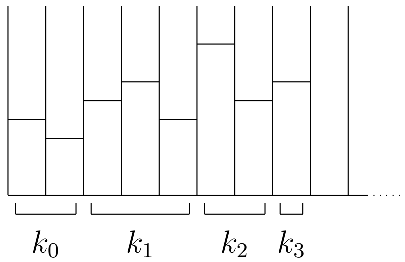
Let $\langle j_0, j_1, \dots \rangle$ be a sequence of elements of $\mathcal{E}_\lambda$, and let $J = j_0 \circ j_1 \circ \dots$ be the inverse limit of the sequence. By definition the dominion of J is $\{x \in V_\lambda : \exists n_x\ \forall i > n_x\ j_i(x) = x\}$. But we know that $j_i(x) = x$ iff $x \in V_{\mathrm{crt}(j_i)}$, so this is $\{x \in V_\lambda : \exists n_x\ \forall i > n_x\ x \in V_{\mathrm{crt}(j_i)}\}$. That is, $x \in \operatorname{dom} J$ depends only on the rank of x, and this implies that $\operatorname{dom} J = V_\alpha$ for some α. It is also possible to calculate α, since $\beta < \alpha$ iff $\exists n\ \forall m \geq n\ \beta < \mathrm{crt}(j_m)$:

$$\alpha = \sup_{n \geq 0} \inf_{m \geq n} \mathrm{crt}(j_m) = \liminf_{n \in \omega} \mathrm{crt}(j_n).$$

With some cosmetic change, we can also suppose that α as the supremum of the critical points, not only the limit inferior. This will also simplify the following proofs and notations.

So let $\lambda_n = \inf_{m \geq n} \mathrm{crt}(j_m)$. Then $\alpha = \sup_{n \in \omega} \lambda_n$, and λ_n is increasing in n. If the supremum is also a maximum, we incur in the trivial case, where J is in fact just a finite composition of elementary embeddings: if n is the first one such that $\lambda_n = \alpha$, then all $\mathrm{crt}(j_m)$ with $m > n$ are bigger than α, so they are constant in the domain of J and they don't change anything.

Suppose then that α is a proper supremum of the λ_n sequence. We can suppose $\mathrm{crt}(j_n) = \lambda_n$ by aggregating multiple elementary embeddings in just one: consider the largest n such that $\mathrm{crt}(j_n) = \lambda_0$ (there will be a largest one because α is a proper supremum of the λ_n sequence), define the new k_0 as the old $j_0 \circ \cdots \circ j_n$, and repeat this for every λ_n. The following is a grafical example:



The columns represent the behaviours of each j_n on λ, where the column on the left represent j_0, and the horizontal lines indicate the critical point.

Definition 0.24. *Let $J = j_0 \circ j_1 \circ \dots$. Then we define $J_n = j_n \circ j_{n+1} \circ \dots$ and $J_{0(n-1)} = j_0 \circ j_1 \circ \cdots \circ j_{n-1}$.*

Lemma 0.25. *Let $j_m \in \mathcal{E}_\lambda$ for every $m \in \omega$, define $\alpha_m = \mathrm{crt}(j_m)$ and suppose that for every $m \in \omega$, $\alpha_m < \alpha_{m+1}$. Let $\alpha = \sup_{m \in \omega} \alpha_m$ and $J = j_0 \circ j_1 \circ \dots$. Then*

- *$J''\alpha$ is unbounded in λ;*
- *$J : V_\alpha \prec V_\lambda$ is elementary.*

Proof.
- Let $\delta_m = \sup J_m``\alpha$. We prove that when $\delta_m < \lambda$, then $\delta_m \neq \delta_{m+1}$. Obviously $\delta_m \geq \alpha$, because otherwise $J_m(\delta_m) \in J_m``\alpha$ and $J_m(\delta_m) < \delta_m$. In particular δ_m is above α_m, i.e., the critical point of j_m, so δ_m is moved by j_m and there exists a $\mu < \delta_m$ such that $j_m(\mu) > \delta_m$. By contradiction, suppose that $\delta_m = \delta_{m+1}$. Then μ is also less than δ_{m+1}, so by definition there exists an $i \in \omega$ such that

$J_{m+1}(\alpha_i) \geq \mu$. Therefore

$$J_m(\alpha_i) = j_m \circ J_{m+1}(\alpha_i) \geq j_m(\mu) \geq \delta_m,$$

contradiction.

Suppose then that $\delta_0 < \lambda$. Therefore for any $m \in \omega$ $\delta_{m+1} < \delta_m$, but this creates a strictly descending sequence of ordinals, contradiction.

- Fix $n \in \omega$ and let $k = j_0(j_1(\dots j_n(j_n)\dots))$. Then

$$\begin{aligned}\mathrm{crt}(k) = \mathrm{crt}(j_0(j_1(\dots j_n(j_n)\dots))) = \\ = j_0(j_1(\dots j_n(\mathrm{crt}(j_n))\dots)) = j_0 \circ j_1 \circ \dots \circ j_n(\alpha_n) = J(\alpha_n).\end{aligned}$$

By Lemma 0.6, then, $V_{J(\alpha_n)} \prec V_\lambda$. But $J \upharpoonright V_{\alpha_n} : V_{\alpha_n} \to V_{J(\alpha_n)}$ is an elementary embedding, because $J \upharpoonright V_{\alpha_n} = J_{0,n} \upharpoonright V_{\alpha_n}$, so for every $n \in \omega$, $J \upharpoonright V_{\alpha_n} \prec V_\lambda$. With methods similar to those in the proof of Theorem 0.4, it is possible to prove that this implies $J : V_\alpha \prec V_\lambda$.

□

Like in the V_λ case, we can extend J to $V_{\alpha+1}$ in the expected way: when $A \subseteq V_\alpha$, $J(A) = \bigcup_{\beta<\alpha} J(A \cap V_\beta)$. Now we want to prove an equivalent of Lemma 0.18, but for inverse limits.

Lemma 0.26. *Let $J : V_\alpha \prec V_\lambda$ an inverse limit of Σ^1_{n+1} elementary embeddings. Then for all $A, B \subseteq V_\alpha$ there exist $K : V_\alpha \prec V_\lambda$ inverse limit of Σ^1_n elementary embeddings and $A' \subseteq V_\alpha$ such that $k(A') = A$ and $k(B) = J(B)$.*

Proof. We define k_m and A_m by induction, with repeated uses of Lemma 0.18. At the end, K will be the inverse limit of the k_m's, and the A_m's will be the images of A' through the inverse limit.

Let $A_0 = A$, k_0 and A_1 such that k_0 is a Σ^1_n elementary embedding, $A_1 \subseteq V_\lambda$, $k_0(A_1) = A = A_0$,

$$k_0(J_1(B)) = j_0(J_1(B)) = J(B)$$

and $\mathrm{crt}(k_0) < \mathrm{crt}(j_0)$.

More generally, k_{m+1} and A_{m+2} are such that k_{m+1} is a Σ^1_n elementary embedding, $A_{m+2} \subseteq V_\lambda$, $k_{m+1}(A_{m+2}) = A_{m+1}$,

$$k_{m+1}(J_{m+2}(B)) = j_{m+1}(J_{m+2}(B))$$

and $\mathrm{crt}(j_m) < \mathrm{crt}(k_{m+1}) < \mathrm{crt}(j_{m+1})$.

So k_m and A_m satisfy:

- k_m is Σ^1_n;
- $k_m(A_{m+1}) = A_m$;
- $k_m(J_{m+1}(B)) = J_m(B)$;
- $\mathrm{crt}(k_0) < \mathrm{crt}(j_0) < \mathrm{crt}(k_1) < \cdots < \mathrm{crt}(j_m) < \mathrm{crt}(k_m) < \mathrm{crt}(j_{m+1}) < \dots$

Let K be the inverse limit of the k's. Then $\sup_{m\in\omega} \mathrm{crt}(k_m) = \alpha$, and by Lemma 0.25 $K : V_\alpha \prec V_\lambda$ is an elementary embedding. Note that

$$\begin{aligned} K(B \cap V_{\mathrm{crt}(k_m)}) &= k_0 \circ k_1 \circ \cdots \circ k_m(K_{m+1}(B \cap V_{\mathrm{crt}(k_m)}) = \\ &= k_0 \circ \cdots \circ k_m(B \cap V_{\mathrm{crt}(k_m)}) = k_0 \circ \cdots \circ k_m(J_{m+1}(B \cap V_{\mathrm{crt}(k_m)}) = \\ &= J(B \cap V_{\mathrm{crt}(k_m)}). \end{aligned}$$

So $K(B) = J(B)$.

Finally, consider $A_{m+1} \cap V_{\mathrm{crt}(k_m)}$:

$$A_{m+2} \cap V_{\mathrm{crt}(k_m)} = k_{m+1}(A_{m+2} \cap V_{\mathrm{crt}(k_m)}) = A_{m+1} \cap V_{\mathrm{crt}(k_m)},$$

so

$$K(A_{m+1} \cap V_{\mathrm{crt}(k_m)}) = k_0 \circ \cdots \circ k_m(A_{m+1} \cap V_{\mathrm{crt}(k_m)}) = A \cap V_{K(\mathrm{crt}(k_m))}.$$

Define $A' = \bigcup_{m\in\omega}(A_{m+1} \cap V_{\mathrm{crt}(k_m)})$. Then $K(A') = A$. $\square$

We use Lemma 0.26 to calculate the strength of an inverse limit:

Theorem 0.27 ([9]). *If $J : V_\alpha \prec V_\lambda$ is an inverse limit of Σ^1_n elementary embeddings, then J is Σ^1_n.*

Proof. The case $n = 0$ is Lemma 0.25 combined with an obvious generalization of Theorem 0.4, so we proceed by induction on n.

Suppose that J is Σ^1_{n-1}, we need to prove that for every φ Π^1_{n-1}-formula, and any $B \subseteq V_\lambda$,

$$V_\lambda \vDash \exists X \varphi(X, J(B)) \to V_\alpha \vDash \exists X \varphi(X, B).$$

Suppose $V_\lambda \vDash \exists X \varphi(X, J(B))$, and fix A a witness. Using Lemma 0.26, we find K inverse limit of Σ^1_{n-2} elementary embeddings such that $K(A') = A$ and $K(B) = J(B)$ for some $A' \subseteq V_\alpha$. So $V_\lambda \vDash \varphi(K(A'), K(B))$, and by elementarity $V_\alpha \vDash \varphi(A', B)$, that is $V_\alpha \vDash \exists X\ \varphi(X, B)$. $\square$

Finally, we can prove that the existence of a Σ^1_{n+2} elementary embedding is strictly stronger than the existence of a Σ^1_n elementary embedding.

Theorem 0.28 ([9]). *Let $j : V_\lambda \prec V_\lambda$ be Σ^1_{n+2}. Then*

- *for every $B \subseteq V_\lambda$, there exist an $\alpha < \lambda$ and a $k_\alpha : V_\alpha \prec V_\lambda$ such that $k_\alpha(B_\alpha) = B$ for some $B_\alpha \subseteq V_\alpha$. In fact, we can find an ω-club $C \subseteq \lambda$ of such α's.*
- *there exist an $\alpha < \lambda$ and a $j_\alpha : V_\alpha \prec V_\alpha$ that is Σ^1_n. Moreover, we can find an ω-club $C \subseteq \lambda$ of such α's.*

Proof.

- Let

  $$G = \{\langle l_0, \dots, l_m\rangle : l_i : V_\lambda \prec V_\lambda \text{ is } \Sigma^1_n,\ \operatorname{crt} l_0 < \operatorname{crt} l_1 < \dots < \operatorname{crt} l_m < \kappa_0, \\ \exists B_0, \dots, B_m\ l_0(B_0) = B,\ \forall i\ l_{i+1}(B_{i+1}) = B_i\}.$$

  By Lemma 0.18 the set

  $$\{\theta < \kappa_0 : \exists l\ \langle l_0, \dots, l_m, l\rangle \in G, \operatorname{crt}(l) = \theta\}$$

  is unbounded in κ_0. Pick an infinite branch $\langle l_0, l_1, \dots\rangle$ of G, and let $\alpha = \sup_{i\in\omega} \operatorname{crt}(l_i)$. Let k_α the inverse limit of the l_i's. Then by Theorem 0.27 k_α is Σ^1_n. Define $B' = \bigcup_{m\in\omega}(B_{m+1} \cap V_{\operatorname{crt}(l_m)})$ as before to have $k_\alpha(B') = B$. To prove the existence of the ω-club C, note that we could have used *any* infinite branch of T, and the set of the ordinals that are the supremum of the critical points of the elementary embeddings appearing in an infinite branch of T (like α) contains an ω-club.
- Let the B above be j. Then there exists α (again, any $\alpha \in C$ works), $k_\alpha : V_\alpha \prec V_\lambda$, and $j_\alpha \subseteq V_\alpha$ such that $k_\alpha(j_\alpha) = j$. Suppose, by Theorem 0.19, that n is odd. Then by Lemma 0.17 "j is Σ^1_n" is Π^1_{n+1}. Again by Theorem 0.19 and Lemma 0.27 k_α is Σ^1_{n+1}, so by elementarity $j_\alpha : V_\alpha \prec V_\alpha$ is Σ^1_n.

□

This ends the proof that "$\exists j\ j$ is Σ^1_{n+2}" is strictly stronger than "$\exists j\ j$ is Σ^1_n": let λ minimum such that there exists $j : V_\lambda \prec V_\lambda$ that is Σ^1_{n+2}, then there exists a $\lambda' < \lambda$ and a $k : V_{\lambda'} \prec V_{\lambda'}$ that is Σ^1_n. Since λ was the minimum, k cannot be Σ^1_{n+2}.

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