The Largest Cardinal

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Main Source: [2]
Main Result: Under AC there are no elementary embeddings $j : V \prec V$. Introduction of I3 and I1.

Toward the end of the Sixties, elementary embeddings were the "next big thing" in the large cardinals setting. For example:

Definition 0.1. *κ is γ-supercompact iff there exists $j : V \prec M$ with $\operatorname{crt}(j) = \kappa$ such that $\gamma < j(\kappa)$, ${}^{\gamma}M \subseteq M$.*

κ is γ-strong iff there exists $j : V \prec M$ with $\operatorname{crt}(j) = \kappa$ such that $\gamma < j(\kappa)$, $V_{\kappa+\gamma} \subseteq M$.

These new cardinals are all critical points of elementary embeddings between V and an inner model M, and generally more complete is M, the stronger is the cardinal. Following this thread, William Reinhardt in 1970 asked if the existence of an elementary embedding $j : V \prec V$ was possible. In that case, the critical point of j would be the greatest large cardinal ever hypothesized, since there is nothing more complete than V. Unfortunately, the so-called 'Reinhardt Cardinal' was proved to be inconsistent with ZFC few months after its appearance:

Theorem 0.2 (Kunen, 1971)**.** *If $j : V \prec M$, then $M \neq V$.*

This theorem, while clear in its enunciation, brings some difficulties in a formal level, since it is referring to classes and it quantifies on classes. Many work with the implicit assumption that all classes are definable, but in this case the proof of Kunen Theorem is trivial. Kunen, in his [2], proved the theorem in Morse-Kelley theory, but this is too much. Nowadays, the most accepted way to interpret Kunen Theorem is to consider it a *metatheorem*, and for each j to prove this in $\mathsf{ZFC}(j)$, that is ZFC with j as an added relation symbol, plus instancies of ZFC axioms with j inside the formula (the most important example is Replacement).

There are various different proofs of Kunen's Theorem. The following is the original proof by Kunen, in [2], and it uses a combinatorial theorem:

Theorem 0.3 (Erdös-Hajnal)**.** *Assume* AC*. For any λ cardinal, there exists a function $f : [\lambda]^\omega \to \lambda$ such that for every $y \subseteq \lambda$ with $|y| = \lambda$, $f''[y]^\omega = \lambda$.*

For a proof of this see for example ([1], pag. 319–320).

Let M be an inner model and let $j : V \prec M$ not trivial: we will prove that there exists a set in V that is not in M, so $M \neq V$. Let κ be the critical point of j, i.e., the smallest ordinal moved by j. Then we define the critical sequence:

Definition 0.4 (Critical Sequence)**.** *Let $N \subseteq M$ and let $j : M \prec N$ be such that for every $a \in M$, $j \restriction a \in M$. Let κ be the critical point of j. Then the following is the critical sequence:*

- $\kappa_0 = \kappa$;
- $\kappa_{n+1} = j(\kappa_n)$;
- $\lambda = \sup_{n \in \omega} \kappa_n$.

Note that λ is the smallest fixed point of j bigger than κ, because if $\kappa < \alpha < \lambda$, then there exists $n \in \omega$ such that $\kappa_n < \alpha \leq \kappa_{n+1}$, so $\alpha \leq \kappa_{n+1} = j(\kappa_n) < j(\alpha)$.

Proposition 0.5. *$j"\lambda \notin M$.*

Proof. Towards a contradiction, suppose $j"\lambda \in M$ and pick an f as the Erdös-Hajnal Theorem. So by elementarity $j(f)$ is a function from $[j(\lambda)]^\omega = [\lambda]^\omega$ to λ with the same property. Now, $j"\lambda \subseteq \lambda$ and $|j"\lambda| = \lambda$, so it should be the case that $j(f)"[j"\lambda]^\omega = \lambda$. Since $\kappa \in \lambda$, then $\kappa = j(f)(a)$ for some $a = \langle a_n : n \in \omega \rangle \in [j"\lambda]^\omega$, but

$$j(f)(\langle j(\alpha_n) : n \in \omega \rangle) = j(f(\langle \alpha_n : n \in \omega \rangle)),$$

so $\kappa \in \mathrm{ran}(f)$, contradiction. □

Since $j"\lambda \notin M$, M cannot possibly be V, so Kunen's Theorem is proved. Two considerations rise from this proof:

- AC is necessary for the proof;
- one key object in the proof is the Jonsson function, that is in $V_{\lambda+2}$ and witnesses $j"\lambda \notin M$. What if this function does not exist?

Following these considerations, one corollary of Kunen's Theorem is direct:

Corollary 0.6. *Let* η *such that there exists* $j : V_\eta \prec V_\eta$. *Define* λ *the supremum of the critical sequence of* j. *Then* $\lambda \leq \eta < \lambda + 2$.

This Corollary (in fact the space left void by this Corollary) lead to the definition of new large cardinal hypotheses:

Definition 0.7. I3 *There exists* $j : V_\lambda \prec V_\lambda$.

I1 *There exists* $j : V_{\lambda+1} \prec V_{\lambda+1}$.

Initially these axioms gained popularity because of their connection with Determinacy. Martin, in 1980, proved that a hypothesis between I3 and I1 implied the determinacy of $\mathbf{\Pi}^1_2$ sets, and Woodin in 1984 proved that a hypothesis stronger than I1 implied $\mathsf{AD}_\mathbb{R}$. However, further work managed to weaken these hypotheses: $Det(\mathbf{\Pi}^1_2)$ is in fact equiconsistent with one Woodin cardinal, amd $\mathsf{AD}_\mathbb{R}$ is equiconsistent with a δ Woodin limit cardinal, limit of γ-strong for every $\gamma < \delta$. Nowadays there are no known hypotheses equiconsistent with rank-into-rank axioms.

References

[1] A. Kanamori, *The Higher Infinite*. Springer, Berlin (1994).

[2] K. Kunen, *Elementary embeddings and infinite combinatorics*. Journal of Symbolic Logic **36** (1971), 407–413.