I0 as an AD-like axiom

Vincenzo Dimonte

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Main Sources: [4], [1]
Main Results: The Coding Lemma under I0

The most popular fact about I0 involves the similarities it has with $\mathsf{AD}^{L(\mathbb{R})}$.

In Chapter Three we have already seen some properties that $L(V_{\lambda+1})$ and $L(\mathbb{R})$ have in common. It is time now to add AD and I0 to the picture. The following Theorem collects some classical properties of $L(\mathbb{R})$ under AD:

Theorem 0.1 ($V = L(\mathbb{R}) + \mathsf{AD}$)**.**

- *ω_1 is measurable;*
- *the Coding Lemma holds;*
- *if $\Theta = \sup\{\alpha : \exists\pi : \mathbb{R} \twoheadrightarrow \alpha, \pi \in L(\mathbb{R})\}$, then for every $\alpha < \Theta$ there exists $\pi : \mathbb{R} \twoheadrightarrow \mathcal{P}(\alpha)$, $\pi \in L(\mathbb{R})$.*

The original formulation of the Coding Lemma is in [2], but is too general for our purposes. We will prove the analogue of the following slightly weaker reformulation:

Theorem 0.2 (Coding Lemma)**.** *Suppose $L(\mathbb{R}) \vDash \mathsf{AD}$. Let $\eta < \Theta$ and $\rho : \mathbb{R} \twoheadrightarrow \eta$, $\rho \in L(\mathbb{R})$. Then there exists $\gamma_\rho < \Theta$ such that for every $A \subseteq \mathbb{R}\times\mathbb{R}$, $A \in L(\mathbb{R})$, there exists $B \subseteq \mathbb{R} \times \mathbb{R}$ such that:*

- *$B \subseteq A$;*
- *$B \in L_{\gamma_\rho}(\mathbb{R})$;*
- *for every $\alpha < \eta$ if $\exists(a, b) \in A\ \rho(a) = \alpha$ then $\exists(a, b) \in B\ \rho(a) = \alpha$.*

The most famous and used consequence of the Coding Lemma is a sort of "'strong limit-ness"' for Θ in $L(\mathbb{R})$.

Corollary 0.3. *Suppose the Coding Lemma holds in $L(\mathbb{R})$. Then for every $\eta < \Theta$ there exists a surjection $\pi : \mathbb{R} \twoheadrightarrow \mathcal{P}(\eta)$.*

These results have a correspondent in $L(V_{\lambda+1})$:

Lemma 0.4 ([4])**.** *Suppose that there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with* $\mathrm{crt}(j) < \lambda$*. Then for every $\delta < \lambda$ regular, define*

$$S_\delta^{\lambda^+} = \{\eta < \lambda^+ : \mathrm{cof}(\eta) = \delta\},$$

the stationary set of ordinals with the same cofinality and let $\mathcal{F}$ be the club filter on λ^+. Then there exists $\eta < \lambda$ and $\langle S_\alpha : \alpha < \eta\rangle \in L(V_{\lambda+1})$ a partition of $S_\delta^{\lambda^+}$ such that for every $\alpha < \eta$ $\mathcal{F} \restriction S_\alpha$ is a $L(V_{\lambda+1})$-ultrafilter in λ^+. In particular λ^+ is measurable.

Proof. Define $\kappa_0 = \mathrm{crt}(j)$. Suppose that there is a δ such that this is false, and pick the minimum one. Then δ is definable (using only λ as a parameter, that is a fixed point), so $j(\delta) = \delta$, and this means that $\delta < \kappa_0$. Moreover, note that $j \restriction \lambda^+$ is in $L(V_{\lambda+1})$, because the elements of λ^+ are well-orders of λ, so $j \restriction \lambda^+$ depends on $j \restriction V_{\lambda+1}$, that in turn is defined by $j \restriction V_\lambda$, that is in $V_{\lambda+1}$.

First, we prove that there doesn't exist a partition of $S_\delta^{\lambda^+}$ in κ_0 stationary sets. Looking for a contradiction, let $\langle S_\alpha : \alpha < \kappa_0\rangle$ be such partition. Then $j(\langle S_\alpha : \alpha < \kappa_0\rangle) = \langle T_\alpha : \alpha < \kappa_1\rangle$ is a partition of $S_\delta^{\lambda^+}$ in κ_1 stationary sets. Since $j \restriction \lambda^+$ is in $L(V_{\lambda+1})$, $D = \{\eta < \lambda^+ : j(\eta) = \eta\}$ is in $L(V_{\lambda+1})$. Now,

- D it's δ-closed: let $\vec{T}$ be a δ-sequence of elements of D. Then $\vec{T}$ can be coded in $V_{\lambda+1}$, and since $j(\eta) = \eta$ for every $\eta \in \vec{t}$ and $j(\delta) = \delta$ we have $j(\vec{t}) = \vec{t}$, so $j(\sup \vec{t}) = \sup j(\vec{t}) = \sup \vec{t}$.

D it's unbounded because λ^+ it's regular, so we have that for every $\alpha < \kappa_1$, $\bar{D} \cap T_\alpha \neq \emptyset$, where $\bar{D}$ is the closure of D. But if $\gamma \in \bar{D} \cap T_\alpha$, then $\gamma \in S_\delta^{\lambda^+}$, so $\mathrm{cof}(\gamma) = \delta$ and $\gamma \in D$. Fix a $\gamma \in T_{\kappa_0}$ such that $j(\gamma) = \gamma$. There must exist an α such that $\gamma \in S_\alpha$, but then $j(\gamma) = \gamma \in j(S_\alpha) = T_{j(\alpha)}$, and so $\gamma \in T_{j(\alpha)} \cap T_{\kappa_0}$. Since there is no α such that $j(\alpha) = \kappa_0$, we have that $T_{j(\alpha)} \neq T_{\kappa_0}$, and this a contradiction because $\langle T_\alpha : \alpha < \kappa_1\rangle$ was a partition.

Let I be the non-stationary ideal, i.e. the ideal of the non-stationary sets, and consider $\mathbb{B} = \mathcal{P}(S_\delta^{\lambda^+})/I$. We prove that $\mathbb{B}$ is atomic, and its atoms are dense. Looking for a contradiction suppose that there exists a set $S \in I^+$ that contains no atom, so that

$$\forall T \subset S\ (T \in I^+ \to (\exists T_0, T_1 \in I^+\ (T_0 \cup T_1 = T \wedge T_0 \cap T_1 = \emptyset))).$$

We construct by induction a tree $\mathcal{T}$ of subsets of S that refines reverse inclusion. Indicating $Lev_\gamma(\mathcal{T})$ as the γ-th level of $\mathcal{T}$ we define $Lev_0(\mathcal{T}) = \{S\}$ and every $T \in \mathcal{T}$ has exactly two successors $T_0, T_1 \in I^+$ such that $T_0 \cup T_1 = T$ and $T_0 \cap T_1 = \emptyset$, chosen with DC_λ. At a limit α, we put the intersections $T = \bigcap_{\beta<\alpha} T_\beta$, with $T_\beta \in Lev_\beta(\mathcal{T})$, such that $T \in I^+$. We call $Res(\alpha) = S \setminus Lev_\alpha(\mathcal{T})$ the set of the points in S that don't belong to any T in $Lev_\alpha(\mathcal{T})$. Now, let $\alpha \leq \kappa_0$ be a limit ordinal, we want to prove by induction that $Res(\alpha) \in I$ and $Lev_\alpha(\mathcal{T}) \neq \emptyset$. Let $x \in S \setminus \bigcup_{\beta<\alpha} Res(\beta)$, then for every $\beta < \alpha$ there exists $T \in Lev_\beta(\mathcal{T})$ such that $x \in T$, so we can define $b_x = \{Z \in \bigcup_{\beta<\alpha} Lev_\beta(\mathcal{T}) : x \in Z\}$ a branch of $\mathcal{T}$ of length α and $x \in \bigcap b_x$. So S is the union of $\bigcup_{\beta<\alpha} Res(\beta)$, the points already skimmed by the process, and all of the $\bigcap b_x$. By induction every $Res(\beta) \in I$, and since I is the dual of the club filter, I is λ^+-complete by DC_λ, so $\bigcup_{\beta<\alpha} Res(\beta) \in I$. We consider $\{\bigcap b_x : x \in S \setminus \bigcup_{\beta<\alpha} Res(\beta)\}$: it is a partition of $S \setminus \bigcup_{\beta<\alpha} Res(\beta) \in I^+$ and since λ is a strong limit we have that there cannot be more than $2^{|\alpha|} < \lambda$ branches, so by λ^+-completeness there must be at least one set $T \in I^+$ such that $T = \bigcap b_x$ for some branch b_x, i.e. there must exist $T \in Lev_\alpha(\mathcal{T})$. As for the branches such that $\bigcap b_x \notin I^+$, their union, again by completeness, must be in I, so

$$Res(\alpha) = \bigcup_{\beta<\alpha} Res(\beta) \cup \bigcup \{\bigcap b_x : \bigcap b_x \notin I^+\} \in I,$$

and we can carry on the induction. If we consider a branch in $\mathcal{T}$ of length κ_0, say $\langle T_\alpha : \alpha < \kappa_0 \rangle$, then

$$\{Res(\kappa_0) \cup (T_0 \setminus T_1)\} \cup \{T_\beta \setminus T_{\beta+1} : \beta < \kappa_0\}$$

is a partition of S in κ_0 stationary sets, and we've just seen that this cannot be, so we've reached a contradiction.

We've proved that the atoms of $\mathbb{B} = \mathcal{P}(S_\delta^{\lambda^+})/I$ are dense, so for every $S \subseteq S_\delta^{\lambda^+}$ there exists $T \subseteq S$ atom for $\mathbb{B}$, in other words there exists $T \subseteq S$ in I^+ such that $\mathcal{F} \restriction T$ is an ultrafilter. Since by DC_λ we have that $\mathcal{F}$ is λ^+-complete, $\mathcal{F} \restriction T$ is a measure. By induction (using DC_λ) we define $\langle S_\alpha : \alpha < \eta \rangle$:

- Let $S = S_\delta^{\lambda^+}$. Then there exists a $T \in I^+$ such that $\mathcal{F} \restriction T$ is a measure, and we choose $S_0 = T$.

- Let α ordinal, and suppose that for every $\beta < \alpha$, S_β is defined. If $S_\delta^{\lambda^+} \setminus \bigcup_{\beta<\alpha} S_\beta \in I$, then we stop the sequence and $\eta = \alpha$. Otherwise there exists $S_\alpha \subseteq S_\delta^{\lambda^+} \setminus \bigcup_{\beta<\alpha} S_\beta$ such that $S_\alpha \in I^+$ is stationary and $\mathcal{F} \restriction S_\alpha$ is a measure.

In particular, $\mathcal{F} \upharpoonright S_0$ is a measure for λ^+. $\square$

First we prove a weakening of the Coding Lemma, that will be used in the proof of the Coding Lemma itself.

Lemma 0.5 (Weak Coding Lemma, [4])**.** *Suppose that there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$. Let $\eta < \Theta$ and $\rho : V_{\lambda+1} \twoheadrightarrow \eta$, $\rho \in L(V_{\lambda+1})$. Then there exists $\gamma_\rho < \Theta$ such that for every $A \subseteq V_{\lambda+1} \times V_{\lambda+1}$, $A \in L(V_{\lambda+1})$, there exists $B \subseteq V_{\lambda+1} \times V_{\lambda+1}$ such that:*

- *$B \subseteq A$;*
- *$B \in L_{\gamma_\rho}(V_{\lambda+1})$;*
- *for cofinally $\alpha < \eta$ if $\exists (a, b) \in A\ \rho(a) = \alpha$ then $\exists (a, b) \in B\ \rho(a) = \alpha$.*

Proof. Since this lemma is quite rich in terms of quantifiers, we will use some abbreviation, calling WCL the sentence of the Lemma, WCL_η the same, but with fixed η, $WCL_{\eta,\rho}$ the same, but with fixed η and ρ, and so on...

Note that the last point can be re-written as $\{\rho(a) : \exists b\ (a, b) \in A\}$ is bounded in η and $\{\rho(a) : \exists b\ (a, b) \in B\}$ is unbounded in η.

Looking for a contradiction, suppose that WCL is false and let η be the least such that $\neg WCL_\eta$. Then η is definable, so $j(\eta) = \eta$, and η is a limit: in fact, $WCL_\eta \to WCL_{\eta+1}$, since for cofinally $\alpha < \eta + 1$ means for $\alpha = \eta$, and in that case we can just choose an element in $\{(a, b) \in A : \rho(a) = \eta\}$.

Let ρ be such that $\neg WCL_{\eta,\rho}$. We define by induction, using DC_λ, $\langle(\gamma_\xi, Z_\xi) : \xi < \lambda\rangle$:

- Suppose that we have defined (γ_ξ, Z_ξ). Since $\neg WCL_{\eta,\rho}$, for every $\gamma < \Theta$ $\neg WCL_{\eta,\rho,\gamma}$; so by $\neg WCL_{\eta,\rho,\gamma_\xi}$ there exists $Z_{\xi+1} \subseteq V_{\lambda+1} \times V_{\lambda+1}$ such that
  - $\{\rho(a) : \exists b\ (a, b) \in Z_{\xi+1}\}$ is unbounded in η;
  - for every $B \subseteq Z_{\xi+1}$, $B \in L_{\gamma_\xi}(V_{\lambda+1})$, $\{\rho(a) : \exists b\ (a, b) \in B\}$ is bounded in B.

  Let $\gamma_{\xi+1}$ be such that $Z_{\xi+1} \in L_{\gamma_{\xi+1}}(V_{\lambda+1})$ and $\gamma_\xi + \omega < \gamma_{\xi+1}$. By Lemma 0.39 in Chapter Three we can suppose that $\gamma_{\xi+1} < \Theta$.

- In the limit case, we choose γ_ξ such that $\gamma_\xi > \gamma_\zeta + \omega$ for every $\zeta < \xi$ and a $Z_\xi \subseteq V_{\lambda+1} \times V_{\lambda+1}$ such that $Z_\xi \in L_{\gamma_\xi}(V_{\lambda+1})$.

So, for every $\xi_1 < \xi_2 < \lambda$ we have:

- $\gamma_{\xi_1} + \omega < \gamma_{\xi_2}$;

- $Z_{\xi_1} \in L_{\gamma_{\xi_1}}(V_{\lambda+1})$;

- $\{\rho(a) : \exists b\ (a,b) \in Z_{\xi+1}\}$ is unbounded in η and for every $B \subseteq Z_{\xi+1}$, $B \in L_{\gamma_\xi}(V_{\lambda+1})$, $\{\rho(a) : \exists b\ (a,b) \in B\}$ is bounded in B.

Let $\rho_0 = \rho$, $\rho_{n+1} = j(\rho_n)$,

$$\langle (\gamma^0_\xi, Z^0_\xi) : \xi < \lambda \rangle = \langle (\gamma_\xi, Z_\xi) : \xi < \lambda \rangle$$

and

$$\langle (\gamma^{n+1}_\xi, Z^{n+1}_\xi) : \xi < \lambda \rangle = j(\langle (\gamma^n_\xi, Z^n_\xi) : \xi < \lambda \rangle).$$

Let n be the minimum such that $\gamma^n_{\kappa_0+1} \leq \gamma^{n+1}_{\kappa_0+1}$ (there must exist, otherwise $\langle \gamma^n_{\kappa_0} : n \in \omega \rangle$ would be a descendent chain of ordinals). By elementarity we have that:

- $Z^n_{\kappa_0+1} \in L_{\gamma^n_{\kappa_0+1}}(V_{\lambda+1})$;

- $\{\rho_{n+1}(a) : \exists b\ (a,b) \in Z^{n+1}_{\kappa_1+1}\}$ is unbounded in η and for every $B \subseteq Z^{n+1}_{\kappa_1+1}$, $B \in L_{\gamma^{n+1}_{\kappa_1}}(V_{\lambda+1})$ we have that $\{\rho_{n+1}(a) : \exists b\ (a,b) \in B\}$ is bounded in η;

- $j(Z^n_{\kappa_0+1}) = Z^{n+1}_{\kappa_1+1}$.

Let $B = \{(j(a), j(b)) : (a,b) \in Z^n_{\kappa_0+1}\}$. The parameters used in the definition of B are $j \upharpoonright V_{\lambda+1}$ (that in turn is defined by $j \upharpoonright V_\lambda$) and $Z^n_{\kappa_0+1}$, so $B \in L_{\gamma^n_{\kappa_0+1}+1}(V_{\lambda+1}) \subseteq L_{\gamma^n_{\kappa_1}}(V_{\lambda+1})$. If $(a,b) \in Z^n_{\kappa_0+1}$, then

$$j((a,b)) = (j(a), j(b)) \in j(Z^n_{\kappa_0+1}) = Z^{n+1}_{\kappa_1+1},$$

so $B \subseteq Z^{n+1}_{\kappa_1+1}$. Finally, for every $\alpha < \eta$ there exists $(a,b) \in Z^n_{\kappa_0+1}$ such that $\alpha < \rho_n(a)$, so $\alpha \leq j(\alpha) < \rho_{n+1}(j(a))$, and $\{\rho_{n+1}(a) : \exists b\ (a,b) \in B\}$ is unbounded in η. Contradiction. $\square$

Theorem 0.6 (Coding Lemma). *Suppose that there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$. Let $\eta < \Theta$ and $\rho : V_{\lambda+1} \twoheadrightarrow \eta$, $\rho \in L(V_{\lambda+1})$. Then there exists $\gamma_\rho < \Theta$ such that for every $A \subseteq V_{\lambda+1} \times V_{\lambda+1}$, $A \in L(V_{\lambda+1})$, there exists $B \subseteq V_{\lambda+1} \times V_{\lambda+1}$ such that:*

- *$B \subseteq A$;*

- *$B \in L_{\gamma_\rho}(V_{\lambda+1})$;*

- *for every $\alpha < \eta$ if $\exists (a,b) \in A\ \rho(a) = \alpha$ then $\exists (a,b) \in B\ \rho(a) = \alpha$.*

Proof. The proof is by induction on η. Let $\rho : V_{\lambda+1} \twoheadrightarrow \eta$. For every $\alpha < \eta$ we define $\rho_\alpha : V_{\lambda+1} \twoheadrightarrow \alpha$:

$$\rho_\alpha(a) = \begin{cases} \rho(a) & \text{if } \rho(a) < \alpha; \\ 0 & \text{otherwise.} \end{cases}$$

By induction, for every $\alpha < \eta$ there exists $\gamma_{\rho_\alpha} < \Theta$ that satisfies the Coding Lemma for η, ρ_α. Let $\beta_0 = \sup\{\gamma_{\rho_\alpha} : \alpha < \eta\}$. Since Θ is regular, we have that $\beta_0 < \Theta$. Let γ_ρ be the ordinal that witnesses the Weak Coding Lemma for η, ρ, and let $\beta_1 = \sup\{\beta_0, \gamma_\rho\}$. Since $\beta_1 < \Theta$, there exists $\pi : V_{\lambda+1} \twoheadrightarrow L_{\beta_1}(V_{\lambda+1})$, and this π can be codified as a subset of $V_{\lambda+1}$. We call β the ordinal $< \Theta$ such that the code of π is in $L_\beta(V_{\lambda+1})$. We prove that $\beta + 1$ witnesses the Coding Lemma for η, ρ.

Fix A and define $A_\alpha = \{(a,b) \in A : \rho(a) < \alpha\}$. We can suppose that for every $\alpha < \eta$ there exists $(a,b) \in A$ such that $\rho(a) = \alpha$. We want to code the set $\{(A_\alpha, B) : \alpha < \eta,\ B \text{ witnesses the Coding Lemma for } \eta, \rho_\alpha, A_\alpha\}$ as a subset of $V_{\lambda+1} \times V_{\lambda+1}$:

$$\begin{aligned} A^* = \{(a,b) \in V_{\lambda+1} \times V_{\lambda+1} : \rho(a) > 0, \pi(b) \subseteq A_{\rho(a)}, \\ \forall \xi < \rho(a)\ \exists (x,y) \in \pi(b)\ \rho(x) = \xi\}. \end{aligned}$$

Since β testifies the Coding Lemma for all the ρ_α, we have that $\{\rho(a) : \exists b\ (a,b) \in A^*\}$ is cofinal in η, and since the Weak Coding Lemma holds for η, ρ, β, there exists $B^* \subseteq A^*$ such that $B^* \in L_\beta(V_{\lambda+1})$ and $\{\rho(a) : \exists b\ (a,b) \in B^*\}$ is cofinal in η.

Let $B = \bigcup\{\pi(b) : (a,b) \in B^*\}$. Then $B \subseteq A$ and $B \in L_{\beta+1}(V_{\lambda+1})$. Moreover, for every $\alpha < \eta$, there exists $(a,b) \in B^*$ such that $\rho(a) > \alpha$, $\pi(b) \in A_{\rho(a)}$ and there exists $(x,y) \in \pi(b)$ such that $\rho(x) = \alpha$. But then $(x,y) \in B$, and B testifies the Coding Lemma for $\eta, \rho, \beta+1, A$. $\square$

Lemma 0.7. *Suppose that there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\operatorname{crt}(j) < \lambda$. Then in $L(V_{\lambda+1})$ for every $\alpha < \Theta$ there exists a surjection $\pi : V_{\lambda+1} \twoheadrightarrow \mathcal{P}(\alpha)$.*

Proof. Fix $\alpha < \Theta$, $\rho : V_{\lambda+1} \twoheadrightarrow \alpha$, and let γ be the maximum between the least $\beta < \Theta$ such that $\rho \in L_\beta(V_{\lambda+1})$ and the witness for the Coding Lemma for α, ρ. For every $A \subseteq \alpha$ define $A^* = \{(a,0) \in V_{\lambda+1} : \rho(a) \in A\}$. Then $A^* \in L(V_{\lambda+1})$. So there exists $B^* \subseteq A^*$, $B^* \in L_\gamma(V_{\lambda+1})$ such that $\{\rho(a) : \exists b\ (a,b) \in B^*\} = A$. But then $A \in L_{\gamma+1}(V_{\lambda+1})$. This means that $\mathcal{P}(\alpha) \subseteq L_{\gamma+1}(V_{\lambda+1})$, and using $\pi : V_{\lambda+1} \twoheadrightarrow \gamma + 1$ we're done. $\square$

In fact, the comparison carries on even for stronger results (cfr. with [3]):

Theorem 0.8 ([5]). *Suppose that there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$. Then Θ is a limit of γ such that:*

- *γ is weakly inaccessible in $L(V_{\lambda+1})$;*
- *$\gamma = \Theta^{L_\gamma(V_{\lambda+1})}$ and $j(\gamma) = \gamma$;*
- *for all $\beta < \gamma$, $\mathcal{P}(\beta) \cap L(V_{\lambda+1}) \in L_\gamma(V_{\lambda+1})$;*
- *for cofinally $\kappa < \gamma$, κ is a measurable cardinal in $L(V_{\lambda+1})$ and this is witnessed by the club filter on a stationary set;*
- *$L_\gamma(V_{\lambda+1}) \prec L_\Theta(V_{\lambda+1})$.*

In conclusion, I0 can be considered as the very first example of Higher Determinacy Axiom, i.e. an axiom that leads to similar consequences of AD, but in larger models. We will see other examples of this kind of axioms in the next Section.

References

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