Left distributive algebras beyond I0

Vincenzo Dimonte

6 November 2018

Forget about large cardinals.

Question

Let V_κ the cumulative hierarchy of sets. Is there a non-trivial elementary embedding $j : V_\eta \prec V_\eta$?

We are going to see that there are limitations on which η's we can consider.

If j is not trivial, then some ordinals are moved. We call critical point of j the least ordinal (cardinal) moved.

Let $\kappa_0 = \mathrm{crt}(j)$. We can define $\kappa_{n+1} = j(\kappa_n)$, and $\lambda = \sup_{n\in\omega} \kappa_n$ (this is called the critical sequence).

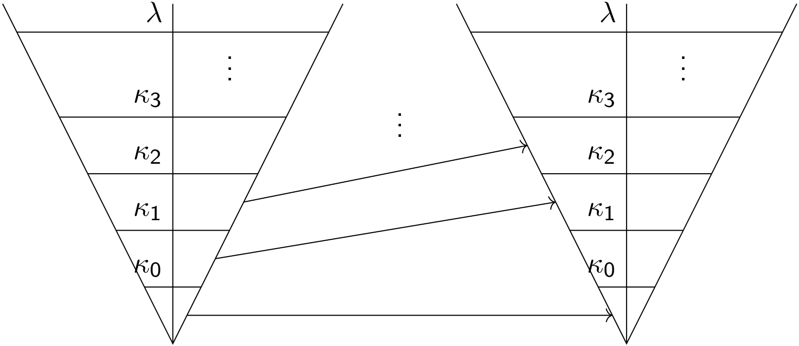
Theorem (Kunen)

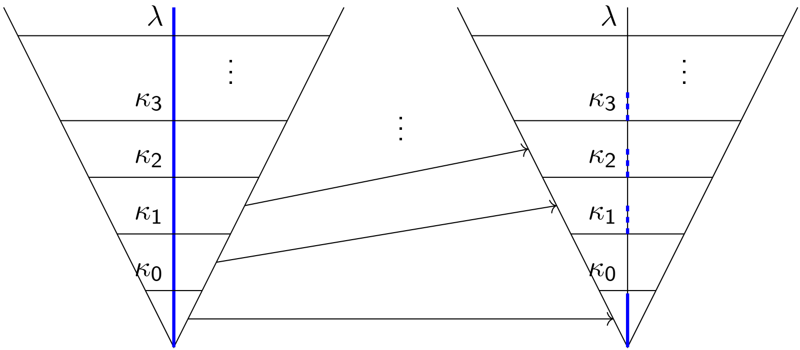
If $j : V_\eta \prec V_\eta$ and there is a well-ordering of $V_{\lambda+1}$ in V_η, then $1 = 0$.

So η can only be limit or successor of limit.

Assumption

I3: There are elementary embeddings $j : V_\lambda \prec V_\lambda$, λ limit.


λ
$\vdots$
κ_3
κ_2
κ_1
κ_0
$\vdots$
λ
$\vdots$
κ_3
κ_2
κ_1
κ_0


λ
κ_3
κ_2
κ_1
κ_0
λ
κ_3
κ_2
κ_1
κ_0

We can actually extend j to larger sets.

Picture: slicing a subset of V_λ.

Lemma

Let $j : M \prec N$, with M, N transitive. Let $X \subseteq M$. Suppose that:

- $M \cap Ord$ is singular, and $M^{<\mathrm{cof}(M \cap Ord)} \subseteq M$;
- j is cofinal;
- X is amenable, i.e., rank-fragments of X are in M.

Then $j^+ : (M, X) \prec (N, j^+(X))$.

Special case: $X = k : V_\lambda \prec V_\lambda$. Therefore $j^+(k) : V_\lambda \prec V_\lambda$. We write $j \cdot k$.

This is not to be confused with $j \circ k$! For example:

- critical sequence of $j \circ j$: $\kappa_0, \kappa_2, \kappa_4, \dots$
- critical sequence of $j \cdot j$: by elementarity $\mathrm{crt}(j^+(j)) = j(\mathrm{crt}(j))$, so $\kappa_1, \kappa_2, \kappa_3 \dots$

This is an operation on the space $\mathcal{E}_\lambda = \{j : V_\lambda \prec V_\lambda\}$, called application. What is its algebra? What are the rules?

Keep in mind thatm contrary to $j \circ k$, $j(k)$ is difficult to calculate: it is explicitly known only on $\mathrm{ran}(j)$.

One rule is left-distributivity:

$$j \cdot (k \cdot l) = (j \cdot k) \cdot (j \cdot l)$$

so $(\mathcal{E}_\lambda, \cdot)$ is a left distributive algebra. Are there other rules?

Let T_n be the sets of words constructed using generators $x_1, \dots, x_n$ and the binary operator $\cdot$.

Let $\equiv_{LD}$ be the congruence on T_n generated by all pairs of the form $t_1 \cdot (t_2 \cdot t_3), (t_1 \cdot t_2) \cdot (t_1 \cdot t_3)$. Then $T_n / \equiv_{LD}$ is the universal free LD-algebra with n generators. We call it F_n.

Given an LD-algebra A, we can consider its subalgebra A_X generated by the elements in a finite subset X. There is always a surjective homomorphism from $F_{|X|}$ to A_X. We say that A_X is free if it is an isomorphism.

In other words, A_X is free iff if two elements of A_X are equal, it must be because of left-distributivity.

Theorem (Laver)

Let $j : V_\lambda \prec V_\lambda$. Then $\mathcal{E}_j = A_{\{j\}}$ is free.

Open problem

What about $A_{\{j,k\}}$? Can it be free?

This is a hard problem. We have to prove many inequalities at the same time, and since an embedding can be represented by many words there is no clear order to use induction. For example:

$$j \cdot (k \cdot j) = (j \cdot k) \cdot (j \cdot j) = ((j \cdot k) \cdot j) \cdot ((j \cdot k) \cdot j) = \dots$$

So the challenge is in finding some “order” in all this mess. Let us see how it works for the one generator case. The key concept here is left divisibility:

Definition

In any LD-algebra, we say that $w <_L v$ iff there are $u_1, \dots u_n$ such that $v = (\dots((w \cdot u_1) \cdot u_2) \cdot \dots \cdot u_n)$.

One of the main points is the following algebraic result:

Proposition

In any LD-algebra with one generator $<_L$ is total, i.e., for any a, b we have $a = b$ or $a <_L b$ or $b <_L a$.

Then we have the following result, that holds for I3-embeddings:

Theorem (Laver, Steel)

$<_L$ is irreflexive on $\mathcal{E}_\lambda$.

This proves, for example, that the associativity rule does not hold in $\mathcal{E}_\lambda$:

$$j \cdot (j \cdot j) = (j \cdot j) \cdot (j \cdot j) = ((j \cdot j) \cdot j) \cdot ((j \cdot j) \cdot j)$$

But then $(j \cdot j) \cdot j <_L j \cdot (j \cdot j)$, so $(j \cdot j) \cdot j \neq j \cdot (j \cdot j)$.

And finally Laver's Criterion proves the freeness for one generator:

Theorem (Laver's Criterion)

Any LD-algebra with one generator is free iff $<_L$ is irreflexive.

So how does it work for the many-finite-generators case? Not so... linearly. Because of course, in a free LD-algebra the generators should be incomparable by left-divisibility.

In general, we should not expect compatibility in cases where there is a variable clash:

Definition

We say that there is a variable clash between w and u iff there are c, a's, b's and two different generators x, y such that $w = (\dots((c \cdot x) \cdot a_1) \dots) \cdot a_p$ and $u = (\dots((c \cdot y) \cdot b_1) \dots) \cdot b_q$. We write $w \not\sim u$.

Again, algebraists come to the rescue and prove that these are the only possibilities in a finitely generated LD-algebra:

Theorem (Dehornoy's quadrichotomy)

For any finitely generated LD-algebra and two of its elements w and u exactly one of the following holds: $w = u$, $w <_L u$, $u <_L w$ or $w \not\sim u$.

And finally, this is the criterion that comes from that:

Theorem (Dehornoy's Criterion)

Let $\mathcal{E}$ be a LD-algebra with n generators. Then $\mathcal{E}$ is free iff $<_L$ is irreflexive, and if $w \nsim u$ then $w \neq u$.

So, if we want to find j, k such that $\mathcal{A}_{\{j,k\}}$ is free, since we already know that $<_L$ is irreflexive, we just need to find j and k so that the "variable clash embeddings" are different. Just.

In the following, we start listing all the inequalities we need to prove for freeness. We label with (LST) those we know are true because of Laver-Steel Theorem, and we leave the ones with the variable clash...

Some examples:

$j \neq k$
$j \cdot j \neq k$
$j \cdot k \neq j$
$j \cdot k \neq k$
$k \cdot j \neq j$
$k \cdot j \neq k$ (LST)
$k \cdot k \neq j$
$j \cdot j \neq j \cdot k$
$j \cdot j \neq k \cdot j$
$j \cdot j \neq k \cdot k$
$j \cdot k \neq k \cdot j$
$j \cdot k \neq k \cdot k$
$k \cdot j \neq k \cdot k$ ...

$j \cdot (j \cdot j) \neq k$
$k \cdot (j \cdot j) \neq j$
$k \cdot (j \cdot j) \neq k$ (LST)
$j \cdot (k \cdot j) \neq j$ (LST)
$j \cdot (k \cdot j) \neq k$
$j \cdot (j \cdot k) \neq j$ (LST)
$j \cdot (j \cdot k) \neq k$
$k \cdot (k \cdot j) \neq j$
$k \cdot (k \cdot j) \neq k$ (LST)
$k \cdot (j \cdot k) \neq j$
$k \cdot (j \cdot k) \neq k$ (LST)
$j \cdot (k \cdot k) \neq j$ (LST)
$j \cdot (k \cdot k) \neq k$ ...

There is a whole hierarchy above I3, with larger and larger embeddings:

- I3: $j : V_\lambda \prec V_\lambda$
- I1: $j : V_{\lambda+1} \prec V_{\lambda+1}$
- I0 (or E_0): $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$, where $L(V_\lambda)$ is the smallest ZF model that contains $V_{\lambda+1}$
- I0$\sharp$ (or E_1): $j : L(V_{\lambda+1}, (V_{\lambda+1})^\sharp) \prec L(V_{\lambda+1}, (V_{\lambda+1})^\sharp)$, where $(V_{\lambda+1})^\sharp$ is a description of the truth in $L(V_{\lambda+1})$ coded as a subset of $V_{\lambda+1}$;
- E_2: $j : L(V_{\lambda+1}, (V_{\lambda+1})^{\sharp\sharp}) \prec L(V_{\lambda+1}, (V_{\lambda+1})^{\sharp\sharp})$
- ...
- E_α: $j : L(E_\alpha) \prec L(E_\alpha)$, where $V_{\lambda+1} \subset E_\alpha \subset V_{\lambda+2}$
- ...

First question: can we define application on these embeddings? Laver did it for I1.

The problem from I0 and beyond is that j is not amenable in $L(V_{\lambda+1})$ or $L(E_\alpha)$: there is a Θ such that $j \upharpoonright L_\Theta(V_{\lambda+1}) \notin L(V_{\lambda+1})$, where Θ is the smallest such that all the subsets of $V_{\lambda+1}$ are in $L_\Theta(V_{\lambda+1})$. In general, $j \upharpoonright E_\alpha \notin L(E_\alpha)$.

The first step is to reduce us to embeddings that are ultrapowers, called weakly proper embeddings:

Theorem (Woodin)

Let $j : L(E_\alpha) \prec L(E_\alpha)$ with $\mathrm{crt}(j) < \lambda$. Then there are two embeddings $j_U, k_U : L(E_\alpha) \prec L(E_\alpha)$ such that $j = k_U \circ j_U$ and

- $\mathrm{crt}(j_U) < \lambda$ and it comes from an ultrafilter, so its behaviour it's definable from $j_U \upharpoonright E_\alpha$;
- $k_U(X) = X$ for any $X \in E_\alpha$.

The strategy is still to divide the domain in simple pieces on which the embeddings are amenable, but these cannot be rank-pieces.

Ultrapowers embedding have something desirable: a proper class of fixed points. It is actually provable that if $j : L(E_\alpha) \prec L(E_\alpha)$ is weakly proper, and I_j it's the class of its fixed points, then every element of $L(E_\alpha)$ is definable with parameters from $E_\alpha \cup I_j$.

If we add that $V = \mathrm{HOD}_{V_{\lambda+1}}$, then we have actually that every element of $L(E_\alpha)$ is definable with parameters from $V_{\lambda+1} \cup \{V_{\lambda+1}\} \cup \{E_\alpha\} \cup \Theta \cup I_j$.

Now, if we have two different embeddings j and k, then $I_j \cap I_k$ is still a proper class, and the class of elements of $L(E_\alpha)$ definable with parameters from $V_{\lambda+1} \cup \{V_{\lambda+1}\} \cup \{E_\alpha\} \cup \Theta \cup (I_j \cap I_k)$ is an elementary substructure of $L(E_\alpha)$ whose transitive collapse is $L(E_\alpha)$.

We cut this substructure in pieces of the form $Z_{s,\beta}$, where s is a finite sequence of elements of $I_j \cap I_k$, $\beta < \Theta$, and $Z_{s,\beta}$ is the set of elements of $L(E_\alpha)$ definable from $V_{\lambda+1} \cup \{V_{\lambda+1}\} \cup \{E_\alpha\} \cup \beta \cup s$. It is clear that the $Z'_{s,\beta}s$ cover the substructure and that $k \upharpoonright Z_{s,\beta}$ is in $L(E_\alpha)$. So we can do $j(k \upharpoonright Z_{s,\beta})$.

Finally, $j(k)$ is the composition of the inverse of the collapse, $\bigcup j(k \upharpoonright Z_{s,\alpha})$ and the collapse. Is this an embedding?

Theorem (D.)

Suppose E_α and that $L(E_\alpha) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$. Let $\mathcal{E}(E_\alpha)$ be the "set" of weakly proper elementary embeddings from $L(E_\alpha)$ to itself. Then we can define an operation $\cdot$ on $\mathcal{E}(E_\alpha)$ that is a left-distributive algebra and such that $\rho_\alpha : \mathcal{E}(E_\alpha) \to \mathcal{E}_\lambda$, $\rho_\alpha(j) = j \upharpoonright V_\lambda$, is a surjective homomorphism.

So, for any α (including $\alpha = 0$) we have a LD-algebra of elementary embeddings on $L(E_\alpha)$. Is this a new algebra, or is it isomorphic to the algebra on $\mathcal{E}_\lambda$?

First, we see the one-generator case. Since $\rho_\alpha(j) = j \upharpoonright V_\lambda$ is a surjective homomorphism, and F_1 is the universal free algebra, the following diagram commutes:

$$\begin{tikzcd} F_1 \arrow[r, "\pi_1"] \arrow[dr, "\pi_2"'] & \mathcal{E}(E_\alpha)_j \arrow[d, "\rho_\alpha"] \\ & \mathcal{E}_{\rho_\alpha(j)} \end{tikzcd}$$

But since Laver proved that π_2 is an isomorphism, also ρ_α is an isomorphism. Therefore $\mathcal{E}(E_\alpha)_j$, and this is free. So the one-generator case brings nothing to the table.

Theorem (Woodin)

Let $j, k : L(E_\alpha) \prec L(E_\alpha)$ with $\alpha = 0$, successor, or limit with cofinality $> \omega$. Then $j = k$ iff $j \restriction V_\lambda = k \restriction V_\lambda$.

Therefore ρ_α is injective for all the cases above, i.e., it is an isomorphism. So $\mathcal{E}(\mathcal{E}_\alpha)_{j,k} \equiv \mathcal{E}_{j,k}$. Nothing new here. But there is a slight hope: the case $\alpha = \omega$...

Theorem (D., 2012)

If there is a ξ such that $L(E_\xi) \nVdash V = \mathrm{HOD}_{V_{\lambda+1}}$, then there is a $\alpha < \xi$ such that $L(E_\alpha) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$, and there are 2^λ different elements of $\mathcal{E}(E_\alpha)$ that coincide on V_λ.

This is it! This is finally a different algebra! Now ρ_α is still a homomorphism, but it is not an isomorphism.

So let j and k be two different embeddings such that $j \restriction V_\lambda = k \restriction V_\lambda$. Do they form a free algebra?

One thing is that the Laver-Steel Theorem still holds, so the inequalities like $j \neq j \cdot k$, $j \cdot k \neq (j \cdot k) \cdot j$ hold:

Some examples:

$\cancel{j \neq k}$

$j \cdot j \neq k$

$\cancel{j \cdot k \neq j}$

$j \cdot k \neq k$

$k \cdot j \neq j$

$\cancel{k \cdot j \neq k}$

$k \cdot k \neq j$

$j \cdot j \neq j \cdot k$

$j \cdot j \neq k \cdot j$

$j \cdot j \neq k \cdot k$

$j \cdot k \neq k \cdot j$

$j \cdot k \neq k \cdot k$

$k \cdot j \neq k \cdot k$ ...

$j \cdot (j \cdot j) \neq k$

$k \cdot (j \cdot j) \neq j$

$\cancel{k \cdot (j \cdot j) \neq k}$

$\cancel{j \cdot (k \cdot j) \neq j}$

$j \cdot (k \cdot j) \neq k$

$\cancel{j \cdot (j \cdot k) \neq j}$

$j \cdot (j \cdot k) \neq k$

$k \cdot (k \cdot j) \neq j$

$\cancel{k \cdot (k \cdot j) \neq k}$

$k \cdot (j \cdot k) \neq j$

$\cancel{k \cdot (j \cdot k) \neq k}$

$\cancel{j \cdot (k \cdot k) \neq j}$

$j \cdot (k \cdot k) \neq k$ ...

But wait, there's more! We have $j \restriction V_\lambda = k \restriction V_\lambda$, i.e., $\rho_\alpha(j) = \rho_\alpha(k)$.

But then $\rho_\alpha : \mathcal{E}_{j,k} \to \mathcal{E}_{\rho_\alpha(j)}$, and the codomain is free, therefore also inequalities like $j \neq k \cdot j$, $j \cdot j \neq (j \cdot k) \cdot j$ hold. It is like $\mathcal{E}_{j,k}$ is almost linear.

Some examples:

$\cancel{j \neq k}$

$\cancel{j \cdot j \neq k}$

$\cancel{j \cdot k \neq j}$

$\cancel{j \cdot k \neq k}$

$\cancel{k \cdot j \neq j}$

$\cancel{k \cdot j \neq k}$

$\cancel{k \cdot k \neq j}$

$j \cdot j \neq j \cdot k$

$j \cdot j \neq k \cdot j$

$j \cdot j \neq k \cdot k$

$j \cdot k \neq k \cdot j$

$j \cdot k \neq k \cdot k$

$k \cdot j \neq k \cdot k$ ...

$\cancel{j \cdot (j \cdot j) \neq k}$

$\cancel{k \cdot (j \cdot j) \neq j}$

$\cancel{k \cdot (j \cdot j) \neq k}$

$\cancel{j \cdot (k \cdot j) \neq j}$

$\cancel{j \cdot (k \cdot j) \neq k}$

$\cancel{j \cdot (j \cdot k) \neq j}$

$\cancel{j \cdot (j \cdot k) \neq k}$

$\cancel{k \cdot (k \cdot j) \neq j}$

$\cancel{k \cdot (k \cdot j) \neq k}$

$\cancel{k \cdot (j \cdot k) \neq j}$

$\cancel{k \cdot (j \cdot k) \neq k}$

$\cancel{j \cdot (k \cdot k) \neq j}$

$\cancel{j \cdot (k \cdot k) \neq k}$ ...

But wait, there is still more!

Definition

A weakly proper elementary embedding $j : L(E_\alpha) \prec L(E_\alpha)$ is <u>proper</u> if for any $X \in E_\alpha$, $\langle X, j(X), j^2(X), \dots \rangle \in L(E_\alpha)$.

Theorem (D., 2012)

If there is a ξ such that $L(E_\xi) \nvDash V = \mathrm{HOD}_{V_{\lambda+1}}$, then there is a $\alpha < \xi$ such that $L(E_\alpha) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$, and there are 2^λ different elements of $\mathcal{E}(E_\alpha)$ that are proper, and 2^λ different elements of $\mathcal{E}(E_\alpha)$ that are not proper, all coinciding on V_λ.

And the good news is that k is proper iff $j \cdot k$ is proper, so if we choose j proper and k not proper we have a third wave of inequalities, like $j \cdot k \neq k \cdot j$, $(j \cdot k) \cdot j \neq (j \cdot k) \cdot k$:

Some examples:

$\cancel{j \neq k}$
$\cancel{j \cdot j \neq k}$
$\cancel{j \cdot k \neq j}$
$\cancel{j \cdot k \neq k}$
$\cancel{k \cdot j \neq j}$
$\cancel{k \cdot j \neq k}$
$\cancel{k \cdot k \neq j}$
$\cancel{j \cdot j \neq j \cdot k}$
$j \cdot j \neq k \cdot j$
$\cancel{j \cdot j \neq k \cdot k}$
$\cancel{j \cdot k \neq k \cdot j}$
$j \cdot k \neq k \cdot k$
$\cancel{k \cdot j \neq k \cdot k}$ ...

$\cancel{j \cdot (j \cdot j) \neq k}$
$\cancel{k \cdot (j \cdot j) \neq j}$
$\cancel{k \cdot (j \cdot j) \neq k}$
$\cancel{j \cdot (k \cdot j) \neq j}$
$\cancel{j \cdot (k \cdot j) \neq k}$
$\cancel{j \cdot (j \cdot k) \neq j}$
$\cancel{j \cdot (j \cdot k) \neq k}$
$\cancel{k \cdot (k \cdot j) \neq j}$
$\cancel{k \cdot (k \cdot j) \neq k}$
$\cancel{k \cdot (j \cdot k) \neq j}$
$\cancel{k \cdot (j \cdot k) \neq k}$
$\cancel{j \cdot (k \cdot k) \neq j}$
$\cancel{j \cdot (k \cdot k) \neq k}$ ...

Unfortunately, this is still not enough. There are some inequalities that are not covered from the three cases, like $j \cdot j \neq k \cdot j$ and $j \cdot k \neq k \cdot k$.

So this leaves us with the question:

Open problem

Is $\mathcal{E}_{j,k}$ free?

Thanks you for your attention