The *-Prikry condition

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Abstract

The *-Prikry condition is a property that is similar to the Prikry condition, and states that for every $p \in \mathbb{P}$ and for every open dense $D \subseteq \mathbb{P}$, there are $n \in \omega$ and $q \leq^* p$ such that for any $r \leq q$ with $l(r) = l(q) + n$, $r \in D$. We prove this for the tree Prikry forcing and the long extender Prikry forcing. The exposition is didactic-minded, and not research-minded.

Keyword: Prikry forcing.

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1 Introduction

Well, it happened to everybody. One spends weeks to prove, write and check all the details of a new theorem, and then the referee answers, with all the possible tact: "I'm sorry, this has already been proven by something else". It is hard to accept it, and the first stage of grief is denial. But sometimes (only sometimes), after all the five stages are passed, one realizes that maybe all those weeks were not a waste of time, and such an effort can be recycled in another way. This is what happened to us, and this paper is the fruit of such recycling.

To understand why we are doing this, let us add more details. The key notion in this paper is Prikry forcing and its generalizations. Pikry forcing adds a cofinal ω sequence to a measurable cardinal, and its generalizations usually add more of such ω-sequences, in different settings. What all this

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forcings have in common is the “Prikry condition”, a property that is fundamental in proving the typical characteristics of Prikry forcings (for example the fact that they do not add bounded sets). For our paper [1], though, we needed a slight modification of that, sometimes used but never named, and we called it “*-Prikry condition”. The proof of the *-Prikry condition for the original Prikry forcing is classic, but the more sophisticated the forcing is, the more difficult it is to find a proof of the *-Prikry condition for it (and we are not alone in this, see for example [6] and [7], where they incur in the same problem). Therefore it was natural for us to write such proofs for the Gitik-Magidor extender Prikry forcing, Gitik-Sharon diagonal supercompact Prikry forcing and Neeman diagonal supercompact Prikry forcing.

The referee contested it, and rightly so: the proof for the *-Prikry condition is in fact very similar to the one for the Prikry condition, therefore the ten pages of proof had no actual new content, too many pages for a research paper. And by the way, this is why proofs for the *-Prikry condition are so elusive: for someone that is well-versed in Prikry forcing, it is very easy to mentally reconstruct a proof of the *-Prikry condition.

But not everybody is well-versed in Prikry forcing. Moreover, sometimes even the proof of the Prikry condition is obscure to the uninitiated reader. This is a problem that often happens in mathematical research: the proof for the extender Prikry forcing is a more complex form of the proof for the tree Prikry forcing, that in turn is a more complex form of the proof for the original Prikry forcing. The expert, in reading the first proof, has clearly in mind in an intuitive way the ideas used in the second and third proof, and therefore will not need a pedantic retread of them. But the beginner that tries to read such a proof will probably get completely confused.

And this is a problem that is here to stay. While mathematical practice has a well-proven way to indicate to the reader the theorems on which a paper is based, it has not a way to indicate the techniques and ideas, so papers are more and more full of “gaps” that are not inteligible to the young student that just started to research, and that is pushed by the “publish or perish” environment to go straight to the more recent papers, therefore sometimes not building the ideas that are necessary to understand them.

So while publishing a research paper on the *-Prikry-condition would be redundant, we believe there is a space for it as divulgative/educational notes. We publish here the proof of the *-Prikry condition for tree Prikry forcing and Gitik-Magidor extender Prikry forcing. In short, we do this for:

- the reader who read our paper [1] and want to know more details;
- the reader who stumbled in a proof of the Prikry condition for the

Gitik-Magidor extender Prikry forcing, and did not understand some details;

- the curious reader that wants to know more about Prikry forcing for no particular reason.

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2 Preliminaries

Trees are a typical structure that is investigated in combinatorics. Let α be an ordinal. For any $s \in [\alpha]^n$, $\mathrm{lh}(s) = n$. A tree on α is a subset of $[\alpha]^{<\omega}$ closed under initial segments. If T is a tree, for any $s \in T$, denote $T_s = \{t \in T : t \subseteq s \wedge s \subseteq t\}$, $\mathrm{Suc}_T(s) = \{\beta \in \alpha : t^\frown\langle\beta\rangle \in T\}$ and finally for any $n \in \omega$, $\mathrm{Lev}_n(T) = \{s \in T : \mathrm{lh}(s) = n\}$.

3 Prikry and tree Prikry forcing

Prikry forcing is useful because it is a very "‘delicate"’ forcing [2]: it does not add bounded subsets of κ, and is κ^+-cc, so it does not change the cardinal structure above κ. In other words, it makes κ singular while changing the universe at least as possible.

The following is instead the tree Prikry forcing:

Definition 3.1. *Let κ be a measurable cardinal. Fix U an ultrafilter on κ. The tree Prikry forcing $\mathbb{P}$ is the set of conditions $p = (s_p, T^p)$, where s_p is a finite sequence of ordinals in κ, and T^p is a tree of increasing sequences in κ with stem s_p, such that for any $t \in T^p$, $\mathrm{Suc}_{T^p}(t) \in U$. We say that $p \leq q$ if $s_p \supseteq s_q$ and $T^p \subseteq T^q$. We say that $p \leq^* q$ if $p \leq q$ and $s_p = s_q$. For any $p \in \mathbb{P}$ and $t \in T^p$, we write $p \oplus t$ for $(t, (T^p)_t)$.*

The difference between the two forcings is minimal: the only difference is that standard Prikry forcing uses a normal ultrafilter, while for tree Prikry forcing normality is not needed. As for the majority of times the ultrafilters are normal, the two forcing notions are interchangeable, and using one or the other is a matter of better clarity of the proof.

At first glance, Prikry forcing does not seem delicate at all, as it is not even ω-closed. But $\leq^*$ is actually κ-closed, and the crucial notion that makes everything work is the Prikry Condition:

Lemma 3.2 (Prikry condition)**.** *Let $\mathbb{P}$ be a Prikry forcing or a tree Prikry forcing on κ, and let σ be a statement of the forcing language. Then for any $p \in \mathbb{P}$ there exists a $q \leq^*$ such that $q \vDash \sigma$ or $q \nvDash \sigma$.*

In many cases (obviously in [1], but also in [6], for example), the Prikry condition is not that useful, but a slight variation actually it is.

Definition 3.3. *Let $(\mathbb{P}, \leq, \leq^*)$ be a forcing notion. We say that $\mathbb{P}$ satisfies the *-Prikry condition if:*

- *there exists a length measure of the conditions of $\mathbb{P}$, i.e. $l : \mathbb{P} \to \omega$ such that $l(1_{\mathbb{P}}) = 0$ and for any $p, q \in \mathbb{P}$, if $p \leq q$ then $l(p) \geq l(q)$, and $p \leq^* q$ iff $l(p) = l(q)$;*
- *for every $p \in \mathbb{P}$ and for every open dense $D \subseteq \mathbb{P}$, there are $n \in \omega$ and $q \leq^* p$ such that for any $r \leq q$ with $l(r) = l(q) + n$, $r \in D$.*

Such condition is usually satisfied by forcings that satisfy the Prikry condition, and the proof tends to be very similar.

Lemma 3.4. *Prikry forcing on κ has the *-Prikry condition.*

Proof. It is actually a very well known fact, see for example Lemma 1.13 in [2]. For completeness, we write the proof here.

Let U be the ultrafilter that generates the Prikry forcing $\mathbb{P}$. For any $p = \langle s, A \rangle \in \mathbb{P}$, $\mathrm{lh}(p) = \mathrm{lh}(s)$. Let $p = \langle s, A \rangle$ and $D \subseteq \mathbb{P}$ open dense. Let $h : [A]^{<\omega} \to 2$ the partition such that $h(t) = 1$ iff there exists a C such that $\langle s^\frown t, C \rangle \in D$. By the Rowbottom Theorem, there exists an $B \in U$ that is homogeneous, i.e., for every $n \in \omega$ there exist $s_1, s_2 \in [B]^{<\omega}$ such that $h(s_1) = h(s_2)$. Then, as D is open dense, there must exist an n such that for any $m \geq n$ and any $t \in [B]^m$ $h(t) = 1$. That is because, by openness of D, if $t \in [B]^n$ and $h(t) = 1$, then for all $t' \in [B]^{<\omega}$ $h(t^\frown t') = 1$, and by homogeneity this means that if the length of t' is bigger than n, then $h(t') = 1$. Since by density there exists at least one t such that $h(t) = 1$, we're done.

For any $t \in [B]^m$, with $m > n$, let C_t such that $\langle s^\frown t, C_t \rangle \in D$. Let C be the diagonal interesection of all the $t \in [B]^m$, with $m > n$, and let $B' = B \cap C$. Then $\langle s, B' \rangle$ witnesses the *-Prikry condition: $\square$

The following is the most basic non-immediate example, and the method used in the proof is also the base for the more sophisticated methods in the next sections:

Lemma 3.5. *Let κ be a measurable cardinal. Then the tree Prikry forcing $\mathbb{P}$ on κ has the *-Prikry condition.*

Proof. Fix U a measure on κ. For any $p \in \mathbb{P}$, we define $l(p) = \mathrm{lh}(s_p)$. The proof is in three steps:

- in the first claim, we modify the tree T^p so that if some condition $r \leq p$ is in D, then all the conditions $t \leq p$ such that $s_r = s_t$ are in D;
- in the second claim, we modify the previous tree, so that if some condition $r \leq p$ with $T^s = (T^p)_s$ is in D, then all the conditions $t \leq p$ with $T^t = (T^p)_t$ and $\mathrm{lh}(s_r) = \mathrm{lh}(s_t)$ are in D;
- in the third claim, we put the two claims together, and prove the lemma.

Claim 3.6 (First claim)**.** *For any D open dense set and for any $p \in \mathbb{P}$, there exists $q \leq^* p$ such that if there exists $r = (s_r, T^r) \leq q$ (i.e. $r \leq^* q \oplus s_r$) such that $r \in D$, then $q \oplus s_r \in D$.*

Proof of claim. We can suppose $p = 1_{\mathbb{P}}$. It is done by induction. Informally, we consider $T^{1_{\mathbb{P}}}$, and we restrict it asking at each level whether there is a possible way to shrink it to reach D: if there is, then we just shrink it; otherwise we do nothing. More formally:

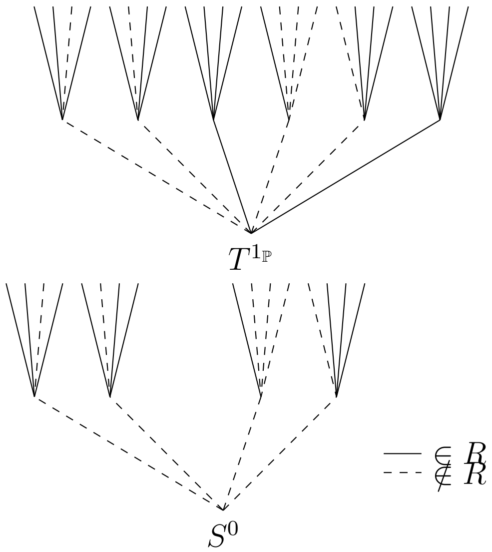
- if there exists a $r \leq^* 1_{\mathbb{P}}$ such that $r \in D$, then let S_0 be T^r; otherwise $S_0 = T^{1_{\mathbb{P}}}$ (note that in the first case $(\langle\rangle, S_0) = r \in D$);
- if there exists a $r \leq (\langle\rangle, S_n)$ such that $l(s_r) = n + 1$ and $r \in D$, then let $(S_{n+1})_{s_r} = (T^r)_{s_r}$; otherwise $(S_{n+1})_{s_r} = (S_n)_{s_r}$ (note that in the first case $(s_r, (S_{n+1})_{s_r}) = r \in D$); to complete the definition, let $S_{n+1} \restriction [\kappa]^{n+1} = S_n \restriction [\kappa]^{n+1}$.

Let $S = \cap_{n\in\omega} S_n$ and $q = (\langle\rangle, S)$. Then q is as desired: since the n-th level is changed just in the first n steps, we have that for any $s \in S$ of length $n \in \omega$, $\mathrm{Suc}_S(s) = \cap_{m<n} \mathrm{Suc}_{S_m}(s) \in U$, therefore $q \in \mathbb{P}$. Let $t \leq^* q \oplus s_t$, i.e. $s_t \in S$ and $T^t \subseteq S$, with $t \in D$. Suppose $\mathrm{lh}(s_t) = n$. Then also $s_t \in S_n$, as $S_{n+1} \restriction [\kappa]^n = S_n \restriction [\kappa]^n$, so $t \leq (\langle\rangle, S_n)$. This means that in the construction the first case was true, therefore $q \oplus s_t = (s_t, S_{s_t}) \leq (s_t, (S_{n+1})_{s_t}) \in D$. □

Claim 3.7 (Second claim)**.** *For any D open dense set and for any $p \in \mathbb{P}$, there exists $q \leq^* p$ and $n \in \omega$ such that for any $s_1, s_2 \in T^q$ such that $l(s_1) = l(s_2) = n$, $(s_1, (T^q)_{s_1}) \in D$ iff $(s_2, (T^q)_{s_2}) \in D$.*

Proof. We can still assume $p = 1_{\mathbb{P}}$. Let $R = \{s \in T^{1_{\mathbb{P}}} : p \oplus s \in D\}$.

Informally, we are climbing up level by level, deleting at each level either the branches that are in R or the ones that are not, so that the sets of successors are still in U. The first step, then, will be simple: let $B^0_{\langle\rangle} = \{\delta \in \mathrm{Suc}_{T^{1_{\mathbb{P}}}}(\langle\rangle) : \langle\delta\rangle \in R\}$. Then either $B^0_{\langle\rangle}$ or $\mathrm{Suc}_{T^{1_{\mathbb{P}}}}(\langle\rangle) \setminus B^0_{\langle\rangle}$ are in U. Call such $A^0_{\langle\rangle}$. Then let $\langle\mu_0, \dots, \mu_l\rangle \in S^0$ iff $\mu_0 \in A^0_{\langle\rangle}$.



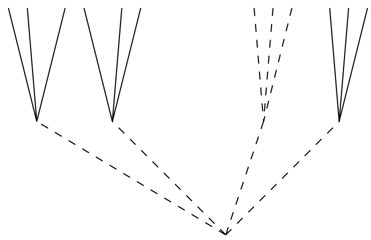
Note that in S^0, $\mathrm{Suc}_{S^0}(\langle\rangle) = \mathrm{Suc}_{T^{1_{\mathbb{P}}}}(\langle\rangle) \cap A^0_{\langle\rangle}$, while for all $s \in S^0$ of length ≥ 1, $\mathrm{Suc}_{S^0}(s) = \mathrm{Suc}_{T^{1_{\mathbb{P}}}}(s)$, and the sequences in S^0 of length 1 are either all in R or all outside.

The second step shows more complexity. First, for any $\langle\mu\rangle \in S^0$ we restrict its successors so that they are either all in R or all outside R. Therefore let

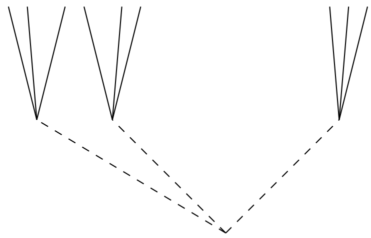
$$B^1_{\langle\mu\rangle} = \{\delta \in \mathrm{Suc}_{S^0}(\langle\mu\rangle) : \langle\mu, \delta\rangle \in R\}$$

for any $\langle\mu\rangle \in \mathrm{Suc}_{S^0}(\langle\rangle)$. Then either $B^1_{\langle\mu\rangle}$ or $\mathrm{Suc}_{S^0}(\langle\mu\rangle) \setminus B^1_{\langle\mu\rangle}$ is in U. Call it $A^1_{\langle\mu\rangle}$. Now define $S^{1,0}$ so that $\langle\mu_0, \dots, \mu_l\rangle \in S^{1,0}$ iff $\mu_0 \in A^0_{\langle\rangle}$ and $\mu_1 \in A^1_{\langle\mu_0\rangle}$. Note that for all $s \in S^{1,0}$ with $\mathrm{lh}(s) = 1$, $\mathrm{Suc}_{S^1_0}(s) = \mathrm{Suc}_{S^0}(s) \cap A^1_{\langle s(0)\rangle}$, while if $\mathrm{lh}(s) \neq 1$ then $\mathrm{Suc}_{S^1_0}(s) = \mathrm{Suc}_{S^0}(t)$.

This is not enough. Singularly, all the 2-sequences that share the same root are either all in R or all outside, but it can be that all the 2-sequences that start with μ_1 are in R, and all the 2-sequences that start with μ_2 ar not in R. Therefore we must choose only the μ's that give a consistent result.



$S^{1,0}$



S^1

Let

$$B^1_{\langle\rangle} = \{\mu \in \mathrm{Suc}_{S^0}(\langle\rangle) : \mathrm{Suc}_{S^{1,0}}(\langle\mu\rangle) = B^1_{\langle\mu\rangle}\},$$

i.e. the set of μ's such that for any $\delta \in \mathrm{Suc}_{S^{1,0}}(\langle\mu\rangle)$, $\langle\mu, \delta\rangle \in R$. Then either $B^1_{\langle\rangle}$ or $\mathrm{Suc}_{S^0}(\langle\rangle) \setminus B^1_{\langle\rangle}$ is in U. Let $A^1_{\langle\rangle}$ be it. Now define $S^{1,1} = S^1$ as $\langle\mu_0, \dots, \mu_l\rangle \in S^1$ iff $\langle\mu_0, \dots, \mu_l\rangle \in S^{1,0}$ and $\mu_0 \in A^1_{\langle\rangle}$. Note that for all $s \in S^1$, if $\mathrm{lh}(s) = 0$ then $\mathrm{Suc}_{S^1}(s) = \mathrm{Suc}_{S^{1,0}}(s) \cap A^1_{\langle\rangle}$, otherwise $\mathrm{Suc}_{S^1}(s) = \mathrm{Suc}_{S^{1,0}}(s)$. The sequences in S^1 of length 2 are either all in R or all outside it.

By induction the construction continues level-by-level, each time starting with $S^{n+1,0} \subseteq S^n$, and then going down to S^n, a tree such that all the $n+1$-branches are either all in R or all outside it. More technically, suppose S^n is defined. For all $t \in S^n$, $\mathrm{lh}(t) = n + 1$, define $B^{n+1}_t = \{\delta \in \mathrm{Suc}_{S^n}(t) : t^\frown\langle\delta\rangle \in R\}$. Then either B^{n+1}_t or $\mathrm{Suc}_{S^n}(t) \setminus B^{n+1}_t$ is in U. Let A^{n+1}_t be it. Define $\langle\mu_0, \dots, \mu_l\rangle \in S^{n+1,0}$ iff $\langle\mu_0, \dots, \mu_l\rangle \in S^n$ and $\mu_{n+1} \in A^{n+1}_{\langle\mu_0,\dots,\mu_n\rangle}$. Note that for all $s \in S^{n+1,0}$, $\mathrm{lh}(s) = n + 1$,

$$\mathrm{Suc}_{S^{n+1,0}}(s) = \mathrm{Suc}_{S^n}(s) \cap A^{n+1}_s,$$

otherwise $\mathrm{Suc}_{S^{n+1,0}}(s) = \mathrm{Suc}_{S^n}(s)$.

Let $t \in \mathrm{Lev}_m S^{n+1}$ and suppose that $S^{n+1,n-m}$, B^{n+1}_s and A^{n+1}_s are defined for all $s \in S^{n+1}$ with $\mathrm{lh}(s) = m + 1$. Let

$$B^{n+1}_t = \{\delta \in \mathrm{Suc}_{S^{n+1}}(t) : \mathrm{Suc}_{S^{n+1,n-m}}(t^\frown\langle\delta\rangle) = B^{n+1}_{t^\frown\langle\delta\rangle}\}.$$

Then either B^{n+1}_t or $\mathrm{Suc}_{S^{n+1}}(t) \setminus B^{n+1}_t$ is in U. Let A^{n+1}_t be it.

Suppose A_t^{n+1} is defined for all $t \in S^{n+1}$ of length m. Then $\langle \mu_0, \dots, \mu_l \rangle \in S^{n+1,n+1-m}$ iff $\langle \mu_0, \dots, \mu_l \rangle \in S^{n+1,n-m}$ and $\mu_i \in A^{n+1}_{\langle \mu_0, \dots, \mu_m \rangle}$. Note that for all $s \in S^{n+1,n+1-m}$ of length $n+1-m$,

$$\mathrm{Suc}_{S^{n+1,n+1-m}}(s) = \mathrm{Suc}_{S^{n+1,n-m}}(s) \cap A_s^{n+1},$$

otherwise $\mathrm{Suc}_{S^{n+1,n+1-m}}(s) = \mathrm{Suc}_{S^{n+1,n-m}}(s)$. Call $S^{n+1,n+1} = S^{n+1}$. Then all the sequences in S^{n+1} of length $n+1$ either are all in R or all outside it.

Now, let $S = \bigcap_{n \in \omega} S^n$. The last remark is sufficient to prove the claim. We prove that $(\langle\rangle, S) \in \mathbb{P}$. It suffices to prove that for any $t \in S$, $\mathrm{Suc}_S(t) \in U$. So let $t \in S$, $\mathrm{lh}(t) = n$. Then $\mathrm{Suc}_S(t)$ will be modified in the construction of S only in the stages $S^{n+1,n}$ with $i \in \omega$, therefore

$$\mathrm{Suc}_S(t) = \mathrm{Suc}_{T^{1_{\mathbb{P}}}}(t) \cap \bigcap_{i \in \omega} A_t^{n+i},$$

that is a countable intersection of elements of U, and therefore in U. $\square$

Claim 3.8 (Third claim)**.** *For any $p \in \mathbb{P}$ and for any D open dense there exists a $p \leq^* q$ and an $n \in \omega$ such that for any $t \leq q$ with $\mathrm{lh}(t) = n$, $t \in D$.*

Proof of claim. Putting the first and second claims together, we have that for any D dense set in $\mathbb{P}$ and for any $p \in \mathbb{P}$ there exists a $q \leq^* p$ and a $n \in \omega$ such that for all $s \in T^q$ with $\mathrm{lh}(s) = n$, $q \oplus s \in D$. Pick a $q \leq^* p$ as the first claim and a $q' \leq^* q$ as the second claim. By density, there exists a $r \leq q$, $r \in D$. Let $n = \mathrm{lh}(r)$. Then by the first claim $q \oplus s_r \in D$, and by the second claim we proved that all the extensions of q of the same length of r are in D. $\square$

$\square$

4 Extender-based Prikry forcing

Even more problematic to prove is the *-Prikry condition for the extender-based Prikry forcing, if only for the complexity of the forcing. It was introduced by Gitik and Magidor, and the reader can find an exhaustive description in [2]. The aim of the forcing is to add many Prikry sequences to a strong enough cardinal, blowing up its power while not changing the power function below it. This is more difficult than just having λ singular and $2^\lambda > \lambda^+$: the proof for this is to take λ measurable, forcing $2^\lambda > \lambda^+$ and then adding a Prikry sequence to λ. But Dana Scott [5] proved that if λ is measurable and $2^\lambda > \lambda^+$, then for a measure one set below λ, $2^\kappa > \kappa^+$, therefore this method would not give the first failure of GCH on λ. The solution is to exploit the extender structure of the cardinal to add many Prikry sequences, at the same time blowing up the power and changing the cofinality.

Definition 4.1. *Let κ and γ be cardinal. Then κ is γ-strong iff there is a $j : V \prec M$ such that $\mathrm{crt}(j) = \kappa$, $\gamma < j(\kappa)$ and $V_{\kappa+\gamma} \subseteq M$.*

We write the definition as it is in [2].

Suppose GCH, and let λ be a 2-strong cardinal.

For any $\alpha < \lambda^{++}$, define a λ-complete normal ultrafilter on λ as $X \in U_\alpha$ iff $\alpha \in j(X)$. For any $\alpha, \beta < \lambda^{++}$, define $\alpha \leq_E \beta$ iff $\alpha \leq \beta$ and for some $f \in^\lambda \lambda$, $j(f)(\beta) = \alpha$. Then by a result in [2], $\langle \lambda^{++}, \leq \rangle$ is a λ^{++}-directed order, and there exists $\langle \pi_{\alpha\beta} : \alpha, \beta \in \lambda^{++}, \alpha \leq_E \beta \rangle$ such that $\langle \lambda^{++}, \langle U_\alpha : \alpha < \lambda^{++} \rangle, \leq_E \rangle$ is a nice system. There is no need to define a nice system here, the term is introduced only because the extender-based Prikry forcing is built on a nice system, the full definition can be found in [2].

Fix a nice system $\langle \lambda^{++}, \langle U_\alpha : \alpha < \lambda^{++} \rangle, \leq_E \rangle$. For any $\nu < \lambda$ and $\lambda < \alpha < \lambda^{++}$, let us denote $\pi_{\alpha,0}(\nu)$ by $\nu^{\alpha,0}$. We will write just ν^0 when α is obvious. By a $^\circ$-increasing sequence of ordinals we mean a sequence $\langle \nu_0, \dots, \nu_n \rangle$ of ordinals below λ such that $\nu_0^0 < \cdots < \nu_n^0$. We say that μ is permitted for $\langle \nu_0, \dots, \nu_n \rangle$ iff $\mu^0 > \nu_i^0$ for all $i = 0 \dots n$.

Also, choose the system so that if $A \in U_\alpha$, $\mu_0, \mu_1 \in A$ and $\mu_0^0 < \mu_1^0$, then $|\{\mu \in A : \mu^0 = \mu_0^0\}| < \mu_1^0$.

Definition 4.2. *The set of forcing conditions $\mathbb{P}$ consists of all the elements p of the form*

$$\{\langle \gamma, p^\gamma \rangle | \gamma \in g \setminus \{\max(g)\}\} \cup \{\langle \max(g), p^{\max(g)}, T \rangle\},$$

where

1. *$g \subseteq \lambda^{++}$ of cardinality $\leq \lambda$ which has a maximal element according to $\leq_E$ and $0 \in g$.*

2. *for $\gamma \in g$, p^γ is a finite $^\circ$-increasing sequence of ordinals $< \lambda$.*

3. *T is a tree, with a trunk $p^{\max(g)}$, consisting of $^\circ$-increasing sequences. All the splittings in T are required to be on sets in $U_{\max(g)}$, i.e., for every $\eta \in T$, if $\eta \geq p^{\max(g)}$ then the set*

$$\mathrm{Suc}_T(\eta) = \{\mu < \lambda : \eta^\frown \langle \mu \rangle \in T\} \in U_{\max(g)}.$$

Also require that for $\eta_1 \geq_T \eta_2 \geq_T p^{mc}$, $\mathrm{Suc}_T(\eta_1) \subseteq \mathrm{Suc}_T(\eta_2)$.

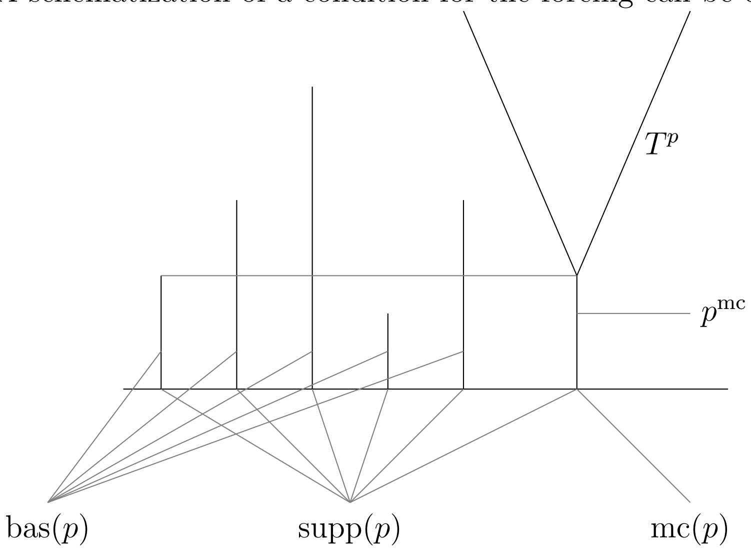
4. *For every* $\mu \in \mathrm{Suc}_T(p^{\max(g)})$, $|\{\gamma \in g : \mu$ *is permitted for* $p^\gamma\}| \leq \mu^0$.

5. *For every* $\gamma \in g$, $\pi_{\max(g),\gamma}(\max(p^{\max(g)}))$ *is not permitted for* p^γ.

6. $\pi_{\max(g),0}$ *projects* $p^{\max(g)}$ *onto* p^0 *(so* $p^{\max(g)}$ *and* p^0 *are of the same length).*

Let us denote g by $\mathrm{supp}(p)$, $\max(g)$ by $\mathrm{mc}(p)$, T by T^p, $p^{\max(g)}$ by p^{mc} and $\mathrm{bas}(p) = p \restriction (\mathrm{supp}(p) \setminus \mathrm{mc}(p))$.

A schematization of a condition for the forcing can be drawn like this:



Clearly, the picture ignores many elements (especially the π's), but it's a good approximation.

Definition 4.3. *Let* $p, q \in \mathbb{P}$. *We say that* p *extends* q *and denote this by* $p < q$ *iff*

1. $\mathrm{supp}(p) \supseteq \mathrm{supp}(q)$.

2. *For every* $\gamma \in \mathrm{supp}(q)$, p^γ *is an end-extension of* q^γ.

3. $p^{\mathrm{mc}(q)} \in T^q$.

4. *For every* $\gamma \in \mathrm{supp}(q)$,

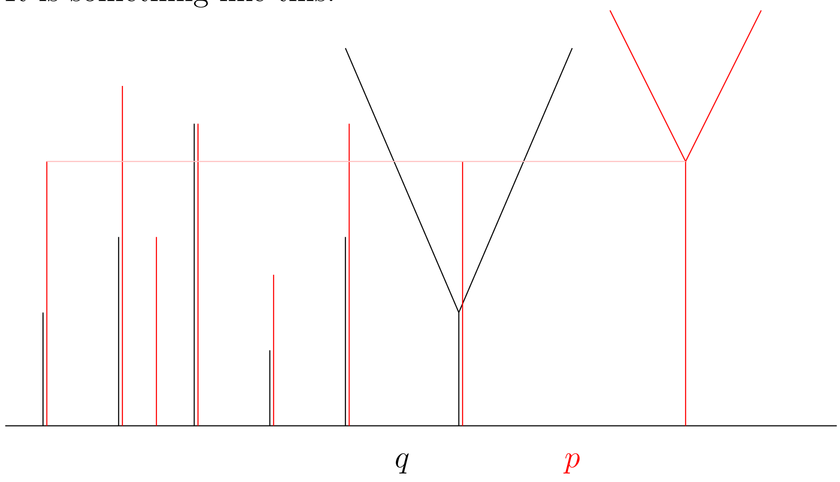
$$p^\gamma \setminus q^\gamma = \pi_{\mathrm{mc}(q),\gamma}[(p^{\mathrm{mc}(q)} \setminus q^{\mathrm{mc}(q)}) \restriction (\mathrm{lh}(p^{\mathrm{mc}}) \setminus (i+1))],$$

where $i \in \mathrm{dom}(p^{\mathrm{mc}(q)})$ *is the largest such that* $p^{\mathrm{mc}(q)}(i)$ *is not permitted for* q^γ.

5. $\pi_{\mathrm{mc}(p),\mathrm{mc}(q)}$ *projects* $T^p_{p^{\mathrm{mc}}}$ *into* $T^q_{q^{\mathrm{mc}}}$.

6. *For every* $\gamma \in \operatorname{supp}(q)$ *and* $\mu \in \operatorname{Suc}_{T^p}(p^{\mathrm{mc}})$, *if* μ *is permitted for* p^γ, *then* $\pi_{\mathrm{mc}(p),\gamma}(\mu) = \pi_{\mathrm{mc}(q),\gamma}(\pi_{\mathrm{mc}(p),\mathrm{mc}(q)}(\mu))$.

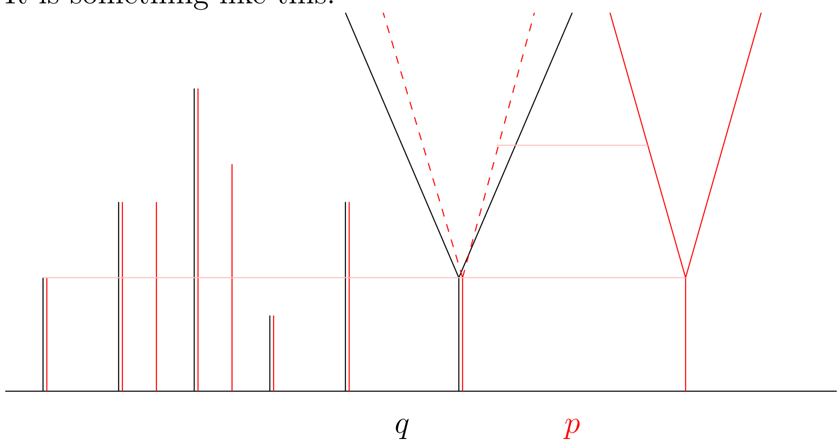
It is something like this:



Definition 4.4. *Let* $p, q \in \mathbb{P}$. *We say that* p *is a direct extension of* q *and denot this by* $p <^* q$ *iff*

1. $p < q$
2. *for every* $\gamma \in \operatorname{supp}(q)$, $p^\gamma = q^\gamma$.

It is something like this:



Definition 4.5. *Let* $p \in \mathbb{P}$ *and* $t \in T^p_{p^{\mathrm{mc}}}$. *Then* $p \oplus t$ *is defined as follows:*

1. $\operatorname{supp}(p \oplus t) = \operatorname{supp}(p)$;
2. $(p \oplus t)^{\mathrm{mc}} = p^{\mathrm{mc}} {}^\frown t$;

3. $T^{p\oplus t} = \{s \in T^p : s \subseteq (p \oplus t)^{\mathrm{mc}} \vee (p \oplus t)^{\mathrm{mc}} \subseteq s\}$;

4. *if* $\gamma \in \mathrm{supp}(p)$,

$$(p \oplus t)^\gamma = p^{\gamma \frown} \pi_{\mathrm{mc}(p),\gamma}[t \restriction (\mathrm{lh}(t) \setminus (i_\gamma + 1))],$$

where i_γ *is the largest such that* $t(i)$ *is not permitted by* p^γ.

If $s = \mathrm{bas}(p)$ *for some* $p \in \mathbb{P}$, $\alpha \geq_E \gamma$ *for all* $\gamma \in \mathrm{supp}(p) \setminus \mathrm{mc}(p)$ *and* t *is a finite* $^\circ$*-increasing sequence of ordinals* $< \lambda$, *then* $s \oplus (\alpha, t)$ *is defined as follows:*

1. $\mathrm{supp}(s \oplus t) = \mathrm{supp}(s)$;

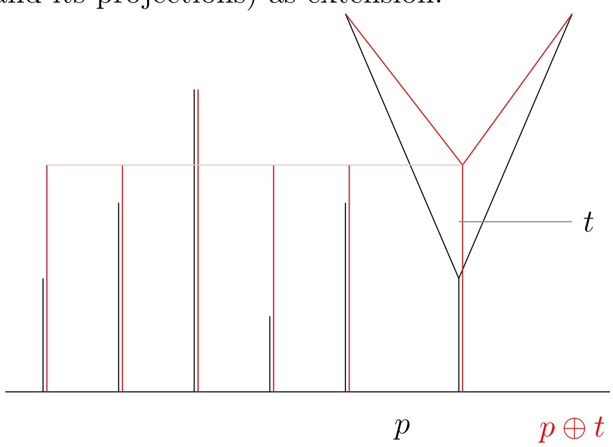
2. *if* $\gamma \in \mathrm{supp}(s)$,

$$(s \oplus t)(\gamma) = p^{\gamma \frown} \pi_{\alpha,\gamma}[t \restriction (\mathrm{lh}(t) \setminus (i_\gamma + 1))],$$

where i_γ *is the largest such that* $t(i)$ *is not permitted by* $s(\gamma)$.

Note that the previous definition is independent from p, *and* $\mathrm{bas}(p \oplus t) = \mathrm{bas}(p) \oplus (\mathrm{mc}(p), p^{\mathrm{mc} \frown} t)$.

A condition in $\mathbb{P}$ is therefore a set of finite sequences and T indicating the possible extensions not only of the last one, but, via projection, of all of them. Morally, $p \oplus t$ is the largest extension of p that we can have choosing t (and its projections) as extension:



Theorem 4.6 (Gitik, Magidor). *Let* $\mathbb{P}$ *as above. Then*

$$V^{\mathbb{P}} \vDash 2^\lambda = \lambda^{++} \ \wedge \ \forall \kappa < \lambda \ 2^\kappa = \kappa^+$$

Proposition 4.7. *Let* $\mathbb{P}$ *as above. Then* $\mathbb{P}$ *has the *-Prikry condition.*

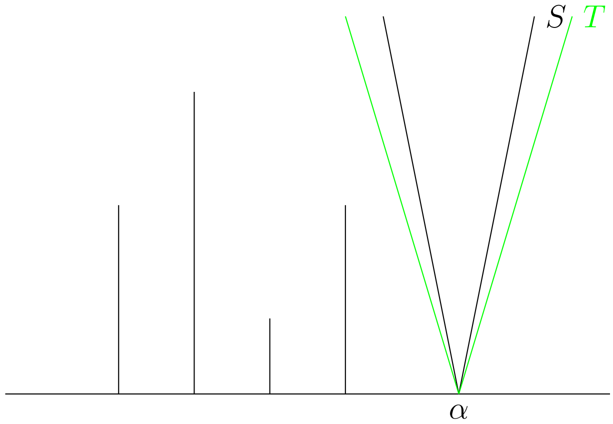
Proof. The proof goes through the same three claims as the proof for Lemma 3.5. Suppose that D is a dense open subset of $\mathbb{P}$ and $p \in \mathbb{P}$. Without loss of generality, we can assume $p = \{\langle 0, \langle\rangle, TC\rangle\}$, where TC is the complete tree of the increasing finite sequences in λ. Note that in this case any $q \in \mathbb{P}$ with $q^{\mathrm{mc}} = \langle\rangle$ is a direct extension of p. Fix an elementary submodel N of $H(\nu)$ with ν sufficiently large to contain all the relevant information of cardinality λ^+ and closed under λ-sequences of its elements. Pick $\alpha < \lambda^{++}$ above all the elements of $N \cap \lambda^{++}$.

Let T be a tree such that $\{\langle 0, \langle\rangle\rangle\} \cup \{\langle \alpha, \langle\rangle, T\rangle\}$ is in $\mathbb{P}$.

Claim 4.8 (First claim)**.** *There exists $r \cup \{\langle \alpha, \langle\rangle, S\rangle\}$, with $S \subseteq T$, such that for every $t \in S$, if for some $q, R \in N$, $q \cup \{\langle \alpha, t, R\rangle\} \leq^* (r \cup \{\langle \alpha, \langle\rangle, S\rangle\}) \oplus t$ and $q \cup \{\langle \alpha, t, R\rangle\} \in D$, then $(r \cup \{\langle \alpha, \langle\rangle, S\rangle\}) \oplus t \in D$;*

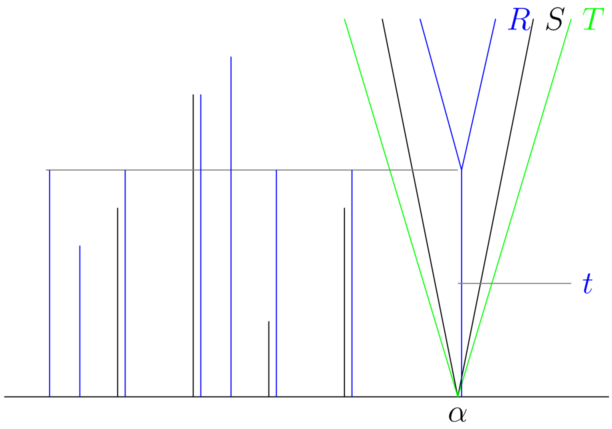
It is a way to reduce the task to simpler conditions. If there exists an extension of $r \cup \{\langle \alpha, \langle\rangle, S\rangle\}$ that satisfies certain properties and is in D, then also the immediate estension of $r \cup \{\langle \alpha, \langle\rangle, S\rangle\}$ via t is in D.

In pictures, there exists something like this:



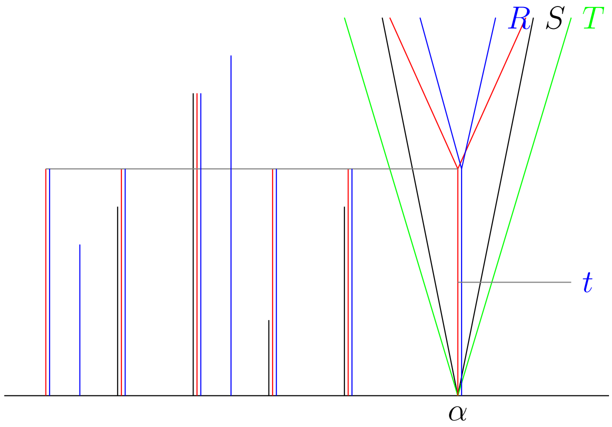
$r \cup \{\langle \alpha, \langle\rangle, S\rangle\}$

such that if there exists something like the blue condition in D



$r \cup \{\langle \alpha, \langle\rangle, S\rangle\}$ $\quad$ $q \cup \{\langle \alpha, t, R\rangle\}$

then the red condition is also in D:



$r \cup \{\langle \alpha, \langle\rangle, S\rangle\}$ $\quad$ $(r \cup \{\langle \alpha, \langle\rangle, S\rangle\}) \oplus t$ $\quad$ $q \cup \{\langle \alpha, t, R\rangle\}$

Proof. If there is a $r \in N$ and a $T' \subseteq T$ such that $r \cup \{\langle \alpha, \langle\rangle, T'\rangle\} \in D$, then this satisfies the Lemma.

If not, let $A = \mathrm{Suc}_T(\langle\rangle)$. We shall define by recursion the sequences $\langle r_\mu : \mu \in A\rangle$ and $\langle T^\mu : \mu \in A\rangle$, the first one increasing.

Let $\mu = \min(A)$. If there are an $s \in N$ and a $T' \subseteq T$ with trunk $\langle \mu \rangle$ such that $s \cup \{\langle \alpha, \langle\mu\rangle, T'\rangle\} \in D$, then set $r_\mu = s$ and $T^\mu = T'$. Otherwise do nothing, i.e., $r_\mu = \{\langle 0, \langle\rangle\rangle\}$ and $T^\mu = T$.

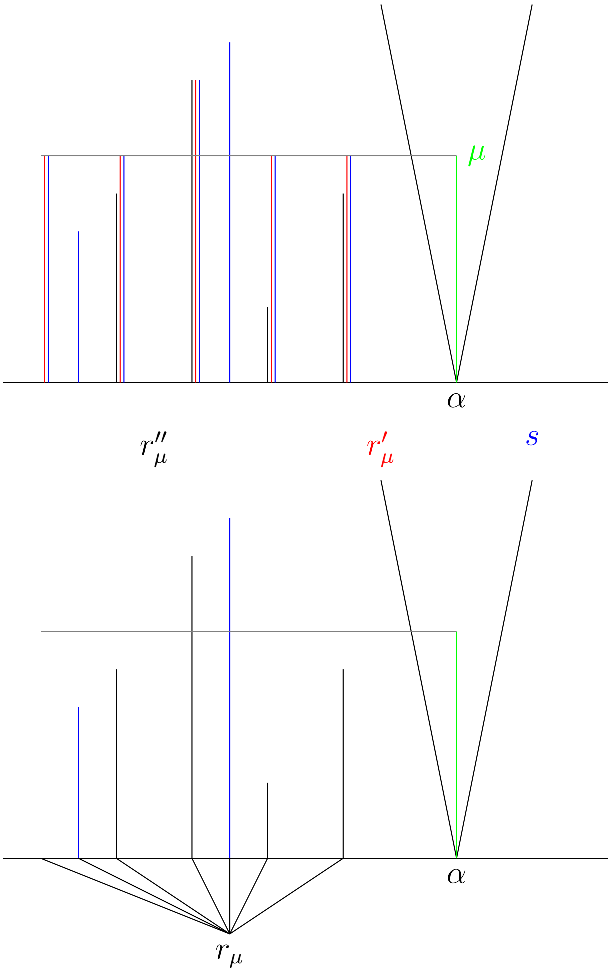
Suppose now that r_ξ and T^ξ are defined for any $\xi < \mu$ in A. Let $r''_\mu = \bigcup_{\xi \in \mu \cap A} r_\xi$ and consider $r'_\mu = r''_\mu \oplus (\alpha, \langle\mu\rangle)$. There are two cases:

1. If there are an $s \in N$ and a $T' \subseteq T$ such that

$$D \ni s \cup \{\langle \alpha, \langle\mu\rangle, T'\rangle\} <^* r'_\mu \cup \{\langle \alpha, \langle\mu\rangle, T\rangle\},$$

then set $r_\mu = r''_\mu \cup ((s \oplus (\alpha, \langle \mu \rangle)) \setminus r'_\mu)$ and $T^\mu = T'$.

2. Otherwise do nothing, i.e., $r_\mu = r''_\mu$ and $T^\mu = T$.



Subclaim 4.9. *For any $\gamma \in \operatorname{supp}(r) \setminus \operatorname{supp}(r''_\mu)$, μ is not permitted for $(r_\mu)^\gamma$.*

Proof of Subclaim. By definition, as if γ is not in $\operatorname{supp}(r''_\mu)$, it must be in $\operatorname{supp}(s \oplus (\alpha, \langle \mu \rangle))$, and μ is not permitted for $s \oplus (\alpha, \langle \mu \rangle)(\gamma)$. $\square$

Let $s_1 = \bigcup_{\mu \in A} r_\mu$. We need to trim T to some S^1 so that $s_1 \cup \{\langle \alpha, \langle\rangle, S^1 \rangle\}$ is an element of $\mathbb{P}$.

For $i < \lambda$ let

$$C_i = \begin{cases} A & \text{if there is no } \mu \in A \text{ such that } \mu^0 = i; \\ \bigcap_{\mu \in A, \mu^0 = i} \operatorname{Suc}_{T^\mu}(\langle \mu \rangle) & \text{otherwise.} \end{cases}$$

Note that $A \in U_\alpha$, and therefore by our choice of the nice system we have that for any $i \in \lambda$, if there is a $\mu_1 \in A$ such that $\mu_1^0 = i$, for any $\mu_1 < \mu_2 \in A$, $|\{\mu \in A : \mu^0 = i\}| < \mu_2$, so by λ-completeness $C_i \in U_\alpha$. Set $A^* = A \cap \Delta^*_{i<\lambda} C_i$. Then for every $\delta \in A^*$ and for every $\mu \in A$ if $\delta^0 < \mu^0$ then $\mu \in \mathrm{Suc}_{T^\mu}(\langle\delta\rangle)$. S^1 will be the tree obtained from T by eliminating all the branches that do not start with $\mu \in A^*$, replacing $T_{\langle\mu\rangle}$ with $T^\mu_{\langle\mu\rangle}$ and intersecting all the levels with A^*, i.e., $\langle\delta_0, \ldots, \delta_n\rangle \in S^1$ iff $\langle\delta_0, \ldots, \delta_n\rangle \in T^{\delta_0}$ and $\forall i \leq n$, $\delta_i \in A^*$.

Subclaim 4.10. *$s_1 \cup \{\alpha, \langle\rangle, S^1\} \in \mathbb{P}$.*

Proof of Subclaim. The only non-trivial point is to show condition (4) of the definition of $\mathbb{P}$, i.e., that for any $\delta \in \mathrm{Suc}_{S^1}(\langle\rangle) = A^*$,

$$|\{\gamma \in \mathrm{supp}(s_1) : \delta \text{ is permitted for } r^\gamma\}| \leq \delta^0.$$

Let

$$B_\delta = \{\gamma \in \mathrm{supp}(s_1) : \delta \text{ is permitted for } r^\gamma\}.$$

Since $\mathrm{supp}(s_1) = \bigcup_{\mu \in A} \mathrm{supp}(r_\mu)$, we can divide B_δ in

$$B_{\delta,\mu} = \{\gamma \in \mathrm{supp}(r_\mu) : \delta \text{ is permitted for } r^\gamma\}.$$

We can also suppose that μ is such that $r_\mu \neq r''_\mu$, i.e., μ is a stage that follows step (1). By Subclaim 4.9 if $\gamma \in \mathrm{supp}(r_\mu) \setminus \bigcup_{\xi \in A} \mathrm{supp}(r_\xi)$, then μ is not permitted for p^γ, so we can restrict the division to $B_\delta = \bigcup_{\mu \in A, \mu^0 < \delta^0} B_{\delta,\mu}$. Again, by our choice of the nice system, if $\mu^0 < \delta^0$ then there are less then δ^0 other elements $\xi \in A$ such that $\xi^0 = \mu^0$, therefore the former is a union of $\leq \delta^0$ elements.

Now fix a $B_{\mu,\delta}$. Since $\delta \in A^*$, by definition of A^* $\delta \in \mathrm{Suc}_{T^\mu}(\langle\mu\rangle)$. Since $s \cup \{\langle\alpha, \langle\mu\rangle, T^\mu\rangle\} \in \mathbb{P}$, by point (4) of the definition of $\mathbb{P}$ we have

$$|\{\gamma \in \mathrm{supp}(s) : \delta \text{ is permitted for } s^\gamma\}| \leq \delta^0.$$

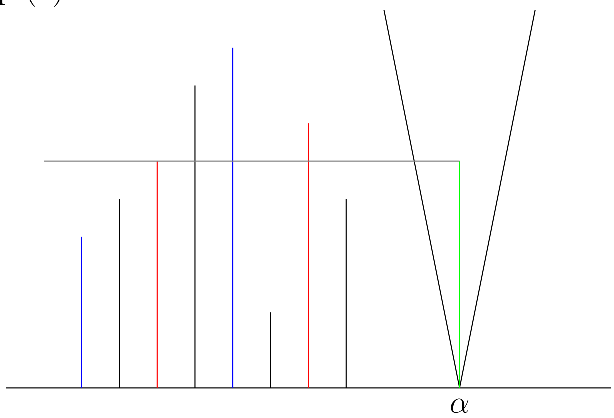
But $\mathrm{supp}(s) = \mathrm{supp}(r_\mu)$, and δ is permitted for s^γ iff δ is permitted for $(r_\mu)^\gamma$, as either $s^\gamma = (r_\mu)^\gamma$, or $s^\gamma = (r''_\mu \oplus \langle\mu\rangle)^\gamma$, $(r_\mu)^\gamma = (r''_\mu)^\gamma$ and μ is permitted for $(r''_\mu)^\gamma$, but in the second case δ is trivially permitted both for s^γ and $(r_\mu)^\gamma$. Therefore $|B_\delta| \leq \delta^0$. □

Subclaim 4.11. *For every $\delta \in \mathrm{Suc}_{S^1}(\langle\rangle)$, if for some $q, R \in N$,*

$$q \cup \{\alpha, \langle\delta\rangle, R\} \leq^* (s_1 \cup \{\alpha, \langle\rangle, S^1\}) \oplus \langle\delta\rangle$$

and $q \cup \{\alpha, \langle\delta\rangle, R\} \in D$, then $(s_1 \cup \{\alpha, \langle\rangle, S^1\}) \oplus \langle\delta\rangle \in D$.

Proof of Subclaim. Recall the construction of s_1 at the δ-th stage. Since $s_1 \oplus \langle\delta\rangle \restriction \mathrm{supp}(s) = s$, we have also that $q \cup \{\alpha, \langle\delta\rangle, R\} \leq^* r_\delta \cup \{\langle\alpha, \langle\rangle, T\rangle\}$ and $q \cup \{\alpha, \langle\delta\rangle, R\} \in D$, therefore the construction at the δ-th stage followed step (1).



(In the picture r_δ is black, s_1 is black+blue and q is black+blue+red. This implies that there exists a s such that $s \cup \{\langle\alpha, \langle\delta\rangle, T^\delta\rangle\} \in D$, with

$$(r_\delta)^\gamma = \begin{cases} (r_\xi)^\gamma & \text{if there exists } \xi \in \delta \cap A,\ \gamma \in \mathrm{supp}(r_\xi); \\ s^\gamma & \text{otherwise.} \end{cases}$$

By the fact that $S^1_{\langle\delta\rangle} \subseteq T^\delta_{\langle\delta\rangle}$ and by Subclaim 4.9, this implies that

$$r \cup \{\langle\alpha, \langle\rangle, S^1\rangle\} \oplus \langle\delta\rangle \leq^* s \cup \{\langle\alpha, \langle\delta\rangle, T^\delta\rangle\},$$

and by density we proved the claim. □

Now we climb up the tree, by induction. Suppose that the first n levels are already defined.

We define r_t and T^t for any $t \in S^n$ of length $n+1$, by induction on the lexicographical order.

Let $r''_t = s_n \cup \bigcup_{s<t} r_t$ and $r'_t = r''_t \oplus (\alpha, t)$. There are two cases:

1. If there are an $s \in N$ and a $T' \subseteq S^n$ such that

$$D \ni s \cup \{\langle\alpha, t, T'\rangle\} <^* r'_t \cup \{\langle\alpha, \langle\rangle, S^n\rangle\} \oplus t,$$

then set $r_t = r''_t \cup ((s \oplus (\alpha, t)) \setminus r'_t)$ and $T^t = T'$.

2. Otherwise do nothing, i.e., $r_t = r''_t$ and $T^t = S^n$.

Let $s_{n+1} = \bigcup_{t \in \mathrm{Lev}_n(S^n)} r_t$.

Subclaim 4.12. *For any $\gamma \in \operatorname{supp}(s_{n+1}) \setminus \operatorname{supp}(r''_\mu)$, μ is not permitted for $(r_\mu)^\gamma$.*

Proof. As before. $\square$

For $i < \lambda$ let

$$C_i = \begin{cases} \{t(n) : t \in \operatorname{Lev}_n(S^n)\} & \text{if there is no } t \in \operatorname{Lev}_n(S^n) \text{ such that } t(n)^0 = i; \\ \bigcap_{t \in \operatorname{Lev}_n(S^n), t(n)^0 = i} \operatorname{Suc}_{T^t}(t) & \text{otherwise.} \end{cases}$$

As before, $C_i \in U_\alpha$, we define $A^* = \operatorname{Suc}_{S^n}(\langle\rangle) \cap \Delta^*_{i<\lambda} C_i$. Let $\langle \mu_0, \dots, \mu_i \rangle \in S^{n+1}$ iff $\langle \mu_0, \dots, \mu_i \rangle \in S^n$, $\forall l \leq i$, $\mu_l \in A^*$ and if $i \geq n$, $\langle \mu_0, \dots, \mu_i \rangle \in T^{\langle \mu_0, \dots, \mu_{n-1} \rangle}$.

Subclaim 4.13. $s_{n+1} \cup \{\alpha, \langle\rangle, S^{n+1}\} \in \mathbb{P}$.

Proof of Subclaim. The proof is similar to the previous one. In this case, we split B_δ in the union of

$$\{\gamma \in \operatorname{supp}(s_n) : \delta \text{ is permitted for } (s_{n+1})^\gamma = (s_n)^\gamma\}$$

and

$$B_{t,\delta} = \{\gamma \in \operatorname{supp}(r_t) : \delta \text{ is permitted for } (s_{n+1})^\gamma = (r_t)^\gamma\},$$

with $t(n)^0 < \delta^0$, thanks to Subclaim 4.12. By induction the first one has $\leq \delta_0$ elements, and the rest is as Subclaim 4.10, $\square$

Subclaim 4.14. *For every $t \in S^{n+1}$, if for some $q, R \in N$,*

$$q \cup \{\alpha, t, R\} \leq^* (s_{n+1} \cup \{\alpha, \langle\rangle, S^{n+1}\}) \oplus t$$

and $q \cup \{\alpha, t, R\} \in D$, then $(s_{n+1} \cup \{\alpha, \langle\rangle, S^{n+1}\}) \oplus t \in D$.

Proof of Subclaim. As before. $\square$

Finally, let $r = \bigcup_{n\in\omega} s_n$ and $S = \bigcap_{n\in\omega} S^n$. It is in $\mathbb{P}$ and satisfies the first claim. $\square$

Claim 4.15 (Second claim)**.** *There exists $r \cup \{\langle \alpha, \langle\rangle, S^* \rangle\}$, with $S^* \subseteq S$, if $t_1, t_2 \in S$ are of the same length, then $(r \cup \{\langle \alpha, \langle\rangle, S^* \rangle\}) \oplus t_1 \in D$ iff $(r \cup \{\langle \alpha, \langle\rangle, S^* \rangle\}) \oplus t_2 \in D$.*

Proof. The r will be the same of the first claim, so we work only on the tree S. The proof follows closely the proof of the second claim in Lemma 3.5, but it needs more care because now we require for $\eta_1 \geq_T \eta_2 \geq_T p^{mc}$, $\mathrm{Suc}_T(\eta_1) \subseteq \mathrm{Suc}_T(\eta_2)$. Therefore every time we reduce a level, we reduce also all the levels above, via an intersection.

Let

$$R = \{t \in S : r \cup \{\langle \alpha, \langle\rangle, S \rangle\} \oplus t \in D\}.$$

Therefore we have to find $S^* \subseteq S$ such that for any $t_1, t_2 \in S^*$ fo the same length, $t_1 \in R$ iff $t_2 \in R$.

Let $B^0_{\langle\rangle} = \{\delta \in \mathrm{Suc}_S(\langle\rangle) : t \in R\}$. Then either $B^0_{\langle\rangle}$ or $\mathrm{Suc}_S(\langle\rangle) \setminus B^0_{\langle\rangle}$ are in U_α. Call such $A^0_{\langle\rangle}$. Then let $\langle \mu_0, \dots, \mu_l \rangle \in S^0$ iff $\forall i\ \mu_i \in A^0_{\langle\rangle}$. We are intersecting all the levels of S to $A^0_{\langle\rangle}$ so that for any $\eta_1 \leq_{S^0} \eta_2$, $\mathrm{Suc}_{S^0}(\eta_2) \subseteq \mathrm{Suc}_{S^0}(\eta_1)$, and we are going to this this repeatedly without further comment. Note that for all $s \in S^0$, $\mathrm{Suc}_{S^0}(s) = \mathrm{Suc}_S(t) \cap A^0_{\langle\rangle}$, and the sequences in S^0 of length 1 are either all in R or all outside.

By induction the construction continues level-by-level, each time starting with $S^{n+1,0} \subseteq S^n$, and then going down to S^{n+1}, a tree such that all the $n+1$-branches are either all in R or all outside it. More technically, Suppose S^n is defined. For all $t \in S^n$, $\mathrm{lh}(t) = n+1$, define $B^{n+1}_t = \{\delta \in \mathrm{Suc}_{S^n}(t) : t^\frown\langle\delta\rangle \in R\}$. Then either B^{n+1}_t or $\mathrm{Suc}_{S^n}(t) \setminus B^{n+1}_t$ is in U_α. Let A^{n+1}_t be it. Define $\langle \mu_0, \dots, \mu_l \rangle \in S^{n+1,0}$ iff $\langle \mu_0, \dots, \mu_l \rangle \in S^n$ and $\forall i > n\ \mu_i \in A^{n+1}_{\langle \mu_0, \dots, \mu_n \rangle}$. Note that for all $s \in S^{n+1,0}$, $\mathrm{lh}(s) \geq n+1$,

$$\mathrm{Suc}_{S^{n+1,0}}(s) = \mathrm{Suc}_{S^n}(s) = \cap A^{n+1}_{\langle s(0), \dots, s(n) \rangle}.$$

Let $t \in \mathrm{Lev}_m S^{n+1}$ and suppose that $S^{n+1,n-m}$, B^{n+1}_s and A^{n+1}_s are defined for all $s \in S^{n+1}$ with $\mathrm{lh}(s) = m+1$. Let

$$B^{n+1}_t = \{\delta \in \mathrm{Suc}_{S^{n+1}}(t) : \mathrm{Suc}_{S^{n+1,n-m}}(t^\frown\langle\delta\rangle) = B^{n+1}_{t^\frown\langle\delta\rangle}\}.$$

Then either B^{n+1}_t or $\mathrm{Suc}_{S^{n+1}}(t) \setminus B^{n+1}_t$ is in U_α. Let A^{n+1}_t be it.

Suppose A^{n+1}_t is defined for all $t \in S^{n+1}$ of length m. Then $\langle \mu_0, \dots, \mu_l \rangle \in S^{n+1,n+1-m}$ iff $\forall i \geq m\ \mu_i \in A^{n+1}_{\langle \mu_0, \dots, \mu_m \rangle}$. Call $S^{n+1} = S^{n+1,n+1}$. Note that for all $s \in S^{n+1,n+1-m}$ of length bigger than m,

$$\mathrm{Suc}_{S^{n+1,n+1-m}}(s) = \mathrm{Suc}_{S^{n+1,n-m}}(s) \cap A^{n+1}_{\langle s(0), \dots, s(m) \rangle}$$

and all the sequences in S^{n+1} of length $n+1$ either are all in R or all outside it.

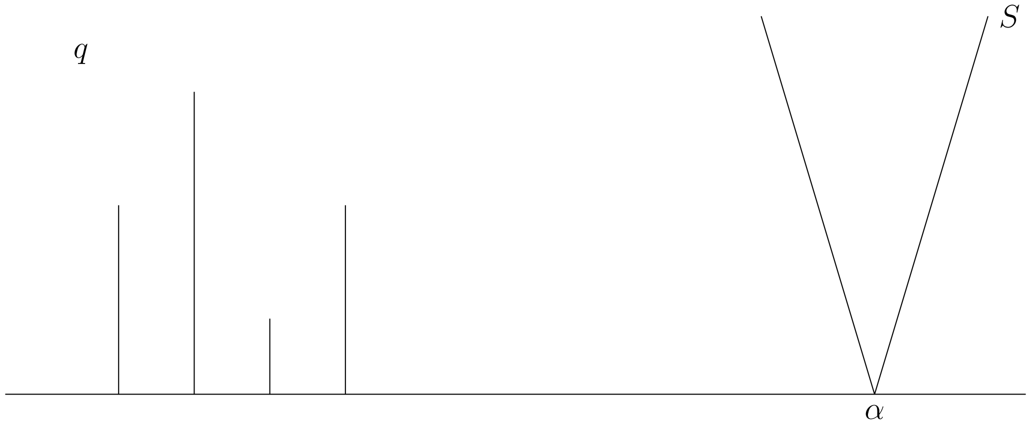
Now, let $S^* = \bigcap_{n \in \omega} S^n$. The last remark is sufficent to prove the claim. We prove that $r \cup \{\langle \alpha, \langle\rangle, S^* \rangle\} \in \mathbb{P}$. Since $S^* \subseteq S$, and we were careful to build it so that $\mathrm{Suc}_{S^*}(t) \subseteq \mathrm{Suc}_{S^*}(s)$ when $t \supseteq s$, it suffices to prove that for any $t \in S^*$, $\mathrm{Suc}_{S^*}(t) \in U_\alpha$. So let $t \in S^*$, $\mathrm{lh}(t) = m$. Then $\mathrm{Suc}_S(t)$ will be modified in the construction of S only in the stages $S^{n,n-i}$ where $i < m$, therefore

$$\mathrm{Suc}_{S^*}(t) = \mathrm{Suc}_S(t) \cap \bigcap_{i \leq m, n \in \omega} A^n_{\langle t(0),\dots,t(i)\rangle},$$

that is a countable intersection of elements of U_α, and therefore in U_α. $\square$

In the proof of Lemma 3.5, the first and second claims were enough to prove that if D is a dense open subset of $\mathbb{P}$ and $p \in \mathbb{P}$, then there is a $q <^* p$ and $k \in \omega$ such that for any $t \in T^q_{q^{\mathrm{mc}}}$ with $\mathrm{lh}(t) = k$, $q \oplus t \in D$. This needs more work.

For ease of notation, let us call the previous condition of $\mathbb{P}$, $r \cup \{\langle \alpha, \langle\rangle, S\rangle\}$. This is our q.



We just need to prove that there are $s, t, R \in N$ such that

$$s \cup \{\langle \alpha, t, R\rangle\} \leq^* (r \cup \{\langle \alpha, \langle\rangle, S\rangle\}) \oplus t = q \oplus t$$

and $s \cup \{\langle \alpha, t, R\rangle\} \in D$, and then by the two properties the Lemma is proved.

Pick some $\beta \in N \cap \lambda$ which is $\leq_E$ above every element of $\mathrm{supp}(r)$. This is possible since $\mathrm{supp}(r) \in N$. Shrink S to a tree S^* to insure that for every $\mu \in \mathrm{Suc}_{S^*}(\langle\rangle)$ and $\gamma \in \mathrm{supp}(r)$, if μ is permitted for r^γ, then $\pi_{\alpha,\gamma}(\mu) = \pi_{\beta,\gamma}(\pi_{\alpha,\beta}(\mu))$.

Subclaim 4.16. *The former is possible.*

Proof. For any $\mu \in \mathrm{Suc}_S(\langle\rangle)$, let

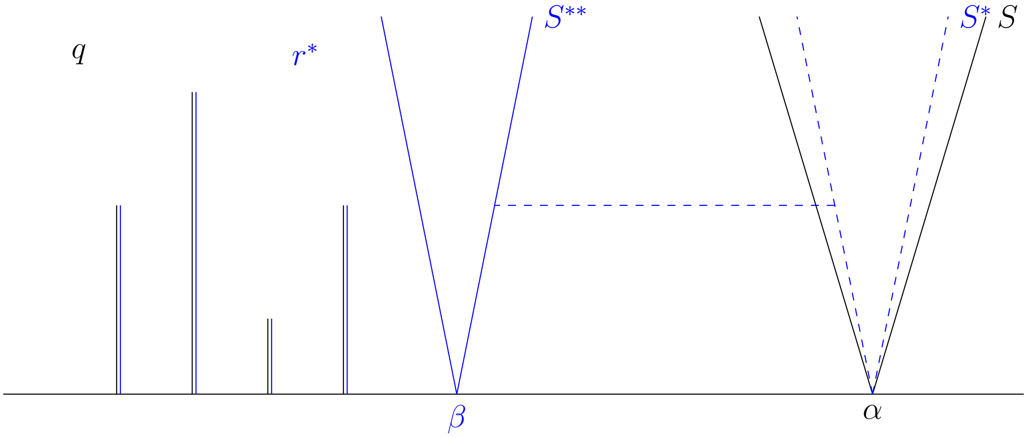
$$B_\mu = \{\gamma \in \mathrm{supp}(r) : \mu \text{ is permitted for } p^\gamma\}.$$

Then we have $|B_\mu| \leq \mu^0$. Let $\langle \xi_i : i < \lambda \rangle$ an enumeration of $\mathrm{supp}(r)$ such that for any $\mu \in \mathrm{Suc}_S(\langle\rangle)$, $B_\mu \subseteq \{\xi_i : i < \mu^0\}$. For any $i < \lambda$, let

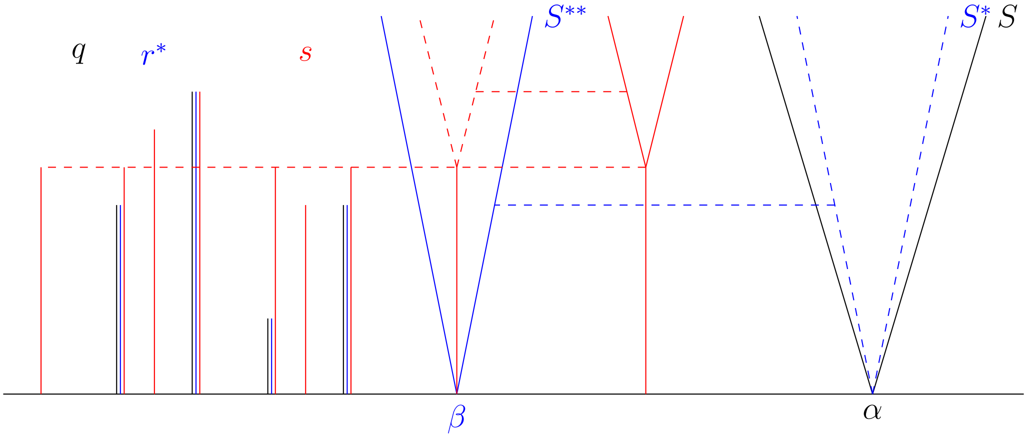
$$C_i = \{\mu \in \mathrm{Suc}_S(\langle\rangle) : \pi_{\alpha,\xi_i}(\mu) = \pi_{\beta,\xi_i}(\pi_{\alpha,\beta}(\mu))\}.$$

Let $A^* = \Delta^*_{i<\lambda} C_i$ and let S^* be the intersection of S with A^*. $\square$

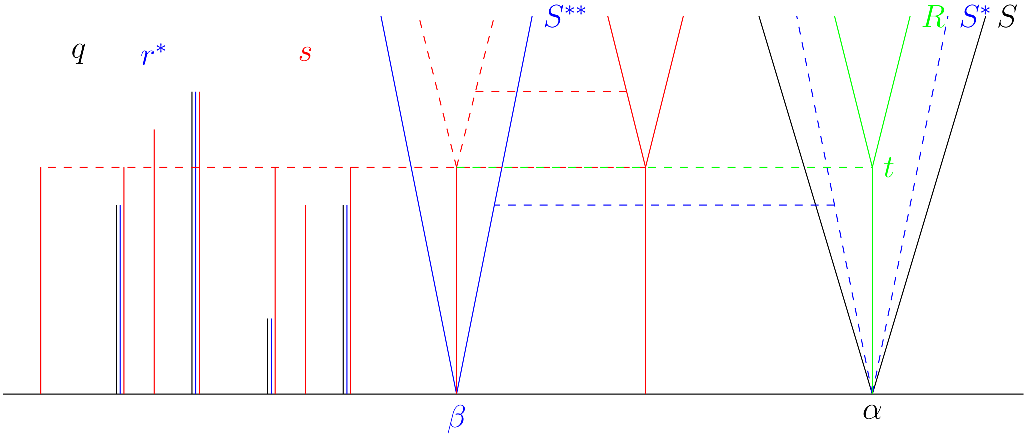
Let S^{**} be the projection of S^* to β via $\pi_{\alpha,\beta}$. Let $r^* = r \cup \{\langle \beta, \langle\rangle, S^{**}\rangle\}$.



Then $r^* \in N$, and since N is an elementary submodel there exists $s \in N$, $s < r^*$ and $s \in D$.

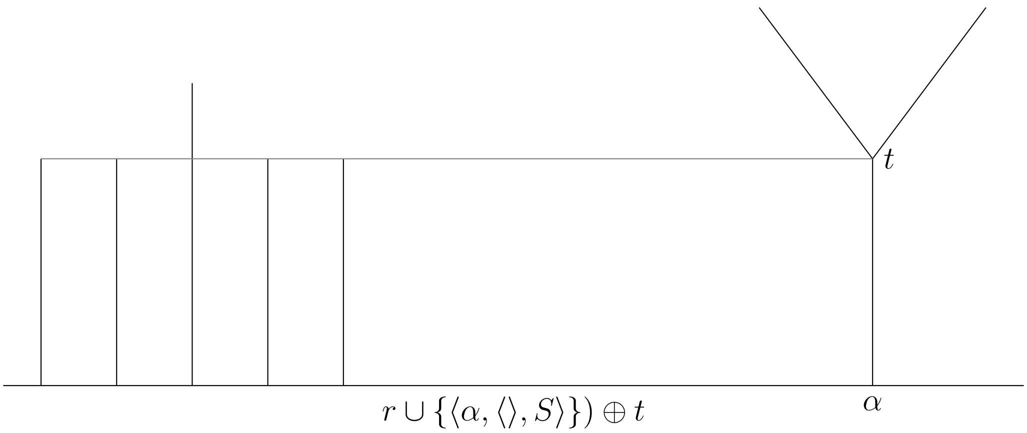


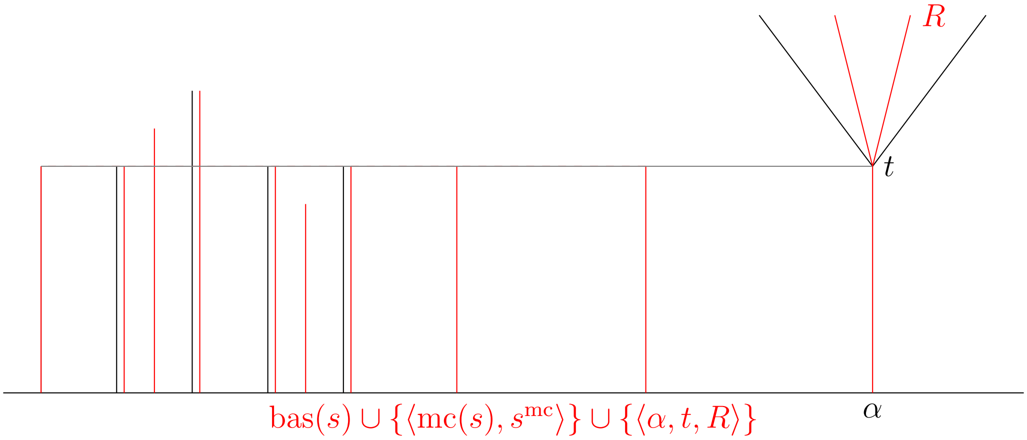
By definition of extension, $s(\beta) \in S^{**}$, therefore there exists a $t \in S^*$ such that $\pi_{\alpha,\beta}(t) = s(\beta)$. Note also that $\mathrm{mc}(s) <_E \alpha$ by the choice of N. Let R be the tree with stem t, derived intersecting S^*_t with $(\pi^{-1}_{\alpha,\mathrm{mc}(s)})''T^s$ and shrinking, if necessary, in order to insure the equality of projections $\pi_{\alpha,\gamma}$ and $\pi_{\mathrm{mc}(s),\gamma} \circ \pi_{\alpha,\mathrm{mc}(s)}$ for the relevant γ's in $\mathrm{supp}(s)$.



Then $\mathrm{bas}(s) \cup \{\langle \mathrm{mc}(s), s^{\mathrm{mc}} \rangle\} \cup \{\langle \alpha, t, R \rangle\} < s$, therefore it is in D. But we also have

$$\mathrm{bas}(s) \cup \{\langle \mathrm{mc}(s), s^{\mathrm{mc}} \rangle\} \cup \{\langle \alpha, t, R \rangle\} \leq^* (r \cup \{\langle \alpha, \langle\rangle, S \rangle\}) \oplus t,$$





and this proves that there is a $q <^* p$ and $k \in \omega$ such that for any $t \in T^q_{q^{\mathrm{mc}}}$ with $\mathrm{lh}(t) = k$, $q \oplus t \in D$. $\square$

5 Exercises and open problems

Exercise 1. *Prove that the Gitik-Sharon diagonal supercompact Prikry forcing, as defined in [3], has the *-Prikry condition.*

Exercise 2. *Prove that the Neeman diagonal supercompact Prikry forcing, as defined in [4], has the *-Prikry condition.*

Question 1. *Are *-Prikry condition and Prikry condition equivalent?*

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