Forcing and Very Large Cardinals: DOs and DON'Ts

Joint FWF-JSPS Seminar on Forcing in Set Theory

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Short History of Quite Large Cardinals.

κ is measurable iff is a κ-complete ultrafilter on κ.
(Keisler, 62) iff it's the critical point of an elementary embedding $j : V \prec M$.
(Solovay, Reinhardt, 60's) κ is γ-supercompact iff it's the critical point of an elementary embedding $j : V \prec M$ such that ${}^{\gamma}M \subseteq M$ (and $\gamma < j(\kappa)$).
(Reinhardt, 67) κ is η-extendible iff there is a ζ and a $j : V_{\kappa+\eta} \prec V_{\zeta}$, with κ critical point of j and $\eta < j(\kappa)$.
(Reinhardt, 70) κ is a Reinhardt cardinal iff it's the critical point of an elementary embedding $j : V \prec V$.

Theorem (Kunen, 1971)

If $j: V \prec M$, then $M \neq V$.

The *critical sequence* has an important role in the proof:

Definition

$\kappa_0 = crit(j)$, $\kappa_{n+1} = j(\kappa_n)$, $\lambda = \sup_{n\in\omega} \kappa_n$.

Kunen's proof uses a choice function that is in $V_{\lambda+2}$. So

Corollary

There is no $j: V_\eta \prec V_\eta$, with $\eta \geq \lambda + 2$.

After that, two paths were available:

Daedalus Path: Better to stay low and going back to cardinals weaker than supercompact (strong, Woodin, etc.).

Icarus Path: Let's see how high we can get before burning our wings.

I1: There exists an elementary embedding $j : V_{\lambda+1} \prec V_{\lambda+1}$.
Woodin proposed an even stronger axiom:

Definition

I0: There exists an elementary embedding $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$.

This axiom is more interesting, since it produces a structure on $L(V_{\lambda+1})$ that is strikingly similar to the structure of $L(\mathbb{R})$ under AD.

The rank-into-rank axioms, like all large cardinals, behave quite well after small forcings:

Theorem (Hamkins)

Let $j : V_\lambda \prec V_\lambda$ or $j : V_{\lambda+1} \prec V_{\lambda+1}$ and $\mathbb{P} \in V_{\mathrm{crt}(j)}$. Then j lifts to $\hat{\jmath} : V[G]_\lambda \prec V[G]_\lambda$ or $\hat{\jmath} : V[G]_{\lambda+1} \prec V[G]_{\lambda+1}$.

The proof is the composition of two lemmas:

- the very well known Lifting Lemma, which assures that j lifts to $V_\lambda[G]$;
- a corollary of the Name Rank Lemma, which assures that $V_\lambda[G] = V[G]_\lambda$.

Note that if we want just to maintain I1, we can pick $\mathbb{P} \in V_\lambda$, and lifting the n-th iterate of j such that $\mathbb{P} \in V_{\kappa_n}$.

With a small change, the previous theorem works also for I0:

Theorem (Woodin)

Let $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$ and $\mathbb{P} \in V_{\mathrm{crt}(j)}$ be ω-closed. Then j lifts to $\hat{\jmath} : L(V[G]_{\lambda+1}) \prec L(V[G]_{\lambda+1})$.

The proof is similar, with the only exception that we usually don't have $L(V[G]_{\lambda+1}) = L(V_{\lambda+1})[G]$. For this case we need ω-closure.

It is possible with the usual tricks to extend these results to class forcing:

Theorem

Let $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$ and $\mathbb{P}_\lambda$ be a j-coherent directed closed simple reverse Easton iteration. Then j lifts to $L(V[G]_{\lambda+1})$.

Corollary

$Con(\mathrm{I0}) \rightarrow Con(\mathrm{I0} + \mathrm{GCH})$

Corollary

$Con(\mathrm{I0}) \rightarrow Con(\mathrm{I0} + \forall\kappa \text{ regular } 2^\kappa > \kappa^+)$

Corollary

$Con(\mathrm{I0}) \rightarrow Con(\mathrm{I0} + V = \mathrm{HOD})$

But this indestructibility has also a dark side.

It is more difficult to kill a very large cardinal.

In fact, it is also difficult to create one:

Theorem (Laver)

If in V there are no elementary embeddings $j : V_{\lambda+1} \prec V_{\lambda+1}$ with $\mathrm{crt}(j) = \kappa$ and $\mathbb{P} < \kappa$, then in $V[G]$ there are no elementary embeddings $j : V[G]_{\lambda+1} \prec V[G]_{\lambda+1}$ with $\mathrm{crt}(j) = \kappa$.

To be honest, this is in line with other large cardinals. See for example Hamkins, Woodin for strong and Woodin cardinals.

The main results of relative consistency are:

Theorem (Laver)

$A(\lambda)$ strongly implies $B(\lambda)$, i.e. $A(\lambda)$ implies $B(\lambda)$ and if $A(\lambda)$ then there exists $\lambda' < \lambda$ such that $B(\lambda')$, in the following cases:

- $A =$"there exists $j : V_\lambda \prec V_\lambda$ Σ^1_{n+2}", $B =$"there exists $j : V_\lambda \prec V_\lambda$ Σ^1_n":
- $A =$"there exists $j : V_{\lambda+1} \prec V_{\lambda+1}$", $B =$"there exists $j : V_\lambda \prec V_\lambda$ Σ^1_n" for any n:
- $A =$"there exists $j : L_1(V_{\lambda+1}) \prec L_1(V_{\lambda+1})$, $B =$"there exists $j : V_{\lambda+1} \prec V_{\lambda+1}$"
- $A =$"there exists $j : L_{\lambda^+ + \omega+1}(V_{\lambda+1}) \prec j : L_{\lambda^+ + \omega+1}(V_{\lambda+1})$", $B =$"there exists $j : L_{\lambda^+}(V_{\lambda+1}) \prec j : L_{\lambda^+}(V_{\lambda+1})$"

The methods used are reflection and inverse iterations.

Sketch of Proof

We pick $j : V_\lambda \prec V_\lambda$ that is Σ^1_{n+2}. In other words, $j : V_{\lambda+1} \prec_{n+2} V_{\lambda+1}$. By reflection, find a $k : V_{\lambda+1} \prec_n V_{\lambda+1}$ such that $j \upharpoonright V_\lambda \in \mathrm{ran}(k)$. Note that k knows that $j \upharpoonright V_\lambda$ is Σ^1_n. Continue creating a chain, whose inverse limit is $K : V_{\lambda'+1} \prec_n V_{\lambda+1}$ and such that $j \upharpoonright V_\lambda \in \mathrm{ran}\, K$. Then the inverse image of j through K is an elementary embedding $j' : V_{\lambda'} \prec V_{\lambda'}$ that is Σ^1_n.

First degree analogies (without I0 and AD):

Let $\Theta^{L(V_{\lambda+1})}_{V_{\lambda+1}}$ be the supremum of the α's such that in $L(V_{\lambda+1})$ there exists a surjection $\pi : V_{\lambda+1} \twoheadrightarrow \alpha$.

$L(\mathbb{R})$	$L(V_{\lambda+1})$
Θ is regular	$\Theta^{L(V_{\lambda+1})}_{V_{\lambda+1}}$ is regular
DC holds	DC_λ holds.

In fact these analogies hold for every model of $\mathsf{HOD}_{V_{\lambda+1}}$.

Second degree analogies (under I0 and AD):

$L(\mathbb{R})$ under AD	$L(V_{\lambda+1})$ under I0
$\neg$AC	$\neg$AC
ω_1 is measurable	λ^+ is measurable
the Coding Lemma holds	the Coding Lemma holds.

The most immediate corollary for the Coding Lemma is: For every $\alpha < \Theta$ there exists a surjection $\pi : \mathbb{R} \twoheadrightarrow \mathcal{P}(\alpha)$.

Third degree analogy:

Theorem

Suppose that there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$. Then Θ is a limit of γ such that:

- γ is weakly inaccessible in $L(V_{\lambda+1})$;
- $\gamma = \Theta^{L_\gamma(V_{\lambda+1})}$ and $j(\gamma) = \gamma$;
- for all $\beta < \gamma$, $\mathcal{P}(\beta) \cap L(V_{\lambda+1}) \in L_\gamma(V_{\lambda+1})$;
- for cofinally $\kappa < \gamma$, κ is a measurable cardinal in $L(V_{\lambda+1})$ and this is witnessed by the club filter on a stationary set;
- $L_\gamma(V_{\lambda+1}) \prec L_\Theta(V_{\lambda+1})$.

Unfortunately there are also things that are not similar at all, or at least that *can* be not similar. And it is all forcing's fault.

Wadge's Lemma

Suppose $V = L(\mathbb{R})$ and AD. Then for every $X, Y \subset \mathbb{R}$ either $X \leq_W Y$ or $Y \leq \mathbb{R} \setminus X$.

Theorem(Woodin)

Let $J : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$. Suppose c is a V-generic Cohen real. Then there exist $X, Y \subset V[c]_{\lambda+1}$ such that there exists an elementary embedding $j : L(X, Y, V[c]_{\lambda+1}) \prec L(X, Y, V[c]_{\lambda+1})$ but $X \notin L_\omega(Y, V[c]_{\lambda+1})$ and $Y \notin L_\omega(X, V[c]_{\lambda+1})$.

Theorem

Suppose $\mathsf{AD}^{L(\mathbb{R})}$. Then the club filter on ω_1 is an ultrafilter.

Of course a direct generalization is not possible, since for every $\delta < \lambda^+$, $S_\delta = \mathrm{Cof}(\delta) \cap \lambda^+$ is stationary, and they are all disjoint. But maybe the club filter can be an ultrafilter on every cofinality. This is called *Ultrafilter Conjecture*. There is a result that goes towards it:

Theorem

Suppose there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$ and $\delta < \lambda$. Then there exists a partition $\langle T_\alpha : \alpha < \eta \rangle$ of S_δ in $\eta < \lambda$ stationary sets such that for every $\alpha < \eta$ the club filter of λ^+ on S_α is an ultrafilter.

The Ultrafilter Conjecture is true if it is possible to have $\eta = 1$. But it is very easy to kill the Ultrafilter Conjecture: let $\mathbb{P}$ be the collapse of ω_2 on ω_1. Since $\mathbb{P}$ is ω-closed, then I0 lifts to $V[G]$. But $(S_{\omega_1})^{V[G]} = S_{\omega_1} \sqcup S_{\omega_2}$, so the club filter cannot be an ultrafilter on $(S_{\omega_1})^{V[G]}$.

Definition

We say that $X \subseteq V_{\lambda+1}$ is an *Icarus set* if there exists an elementary embedding $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$.

Examples:

- $\emptyset$; $V_{\lambda+1}$ are Icarus sets iff I0 holds.
- any well-ordering of $V_{\lambda+1}$ cannot be an Icarus set.

We can define accordingly $\Theta^{L(X,V_{\lambda+1})}$. In fact, $\Theta^{L(X,V_{\lambda+1})}$ "'measures the complexity"' of X. The first and second degree analogies hold.
However, the third analogy resisted all attempts to be proved, without further hypotheses.

Theorem

Let $X \subseteq V_{\lambda+1}$ be an Icarus set, and let $Y \in L(X, V_{\lambda+1}) \cap V_{\lambda+2}$. Then

$$\Theta^{L(Y,V_{\lambda+1})} < \Theta^{L(X,V_{\lambda+1})} \quad \text{iff} \quad (Y, V_{\lambda+1})^\sharp \in L(X, V_{\lambda+1})$$

We can build an absolute standard sequence:

$$L(E_0) = L(V_{\lambda+1}); L(E_1) = L((V_{\lambda+1})^\sharp, V_{\lambda+1});$$
$$L(E_2) = L((V_{\lambda+1})^{\sharp\sharp}, V_{\lambda+1}), \dots$$
$$L(E_\omega) = L(\bigcup_{n\in\omega} L(E_n))$$

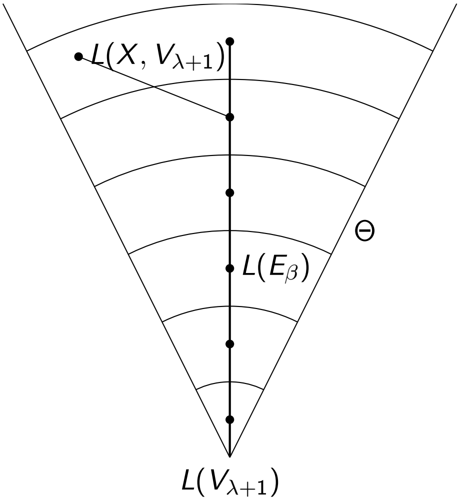
It is possible that there are no Icarus sets associated to $L(E_\omega)$.

However, we insist that $L(E_{\omega+1})\vDash V=\mathrm{HOD}_{\{X\}\cup V_{\lambda+1}}$.
For $L(E_{\omega+1})$, we add a bit less of $(E_\omega)^\sharp$, so that $L(E_{\omega+1})=L(X,V_{\lambda+1})$ for some $X\subseteq V_{\lambda+1}$.

Teorema (Woodin)

It is possible to define a sequence $V_{\lambda+1}\subset E_\alpha\subset V_{\lambda+2}$ such that:

- if X is an Icarus set and $\Theta^{L(E_\alpha)}<\Theta^{L(X,V_{\lambda+1})}$, then $E_\alpha\subset L(X,V_{\lambda+1})$;
- if α is a successor ordinal, then there exists an Icarus set X such that $L(E_\alpha)=L(X,V_{\lambda+1})$;
- if α is a limit ordinal, then $L(E_\alpha)=L(\bigcup_{\beta<\alpha}E_\beta)$.


$L(X, V_{\lambda+1})$
Θ
$L(E_\beta)$
$L(V_{\lambda+1})$

We've seen that a small ω-closed forcing doesn't change I0. If the forcing is not ω-closed, however, trouble can come. Just as an example:

Theorem (Woodin)

Suppose that $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ is proper. Let $\mathbb{P} \in V_\lambda$ and G generic such that $(\lambda^\omega)^V \neq (\lambda^\omega)^{V[G]}$. Then $V_{\lambda+1} \notin L_\lambda(V_{\lambda+1})$.

This seems an obstacle for the following problem:

Is properness really a property?

Theorem(Woodin)

Suppose $\alpha < \Upsilon$. If

- $\alpha = 0$, or
- α is a successor ordinal, or
- α is a limit ordinal with cofinality $> \omega$

then every weakly proper elementary embedding $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ is proper.

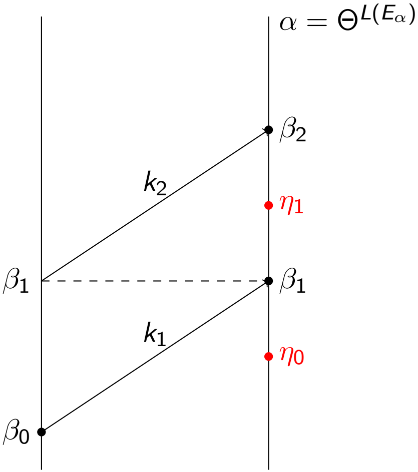
Theorem

Suppose that there exists ξ such that $L(E_\xi) \not\models V = \mathrm{HOD}_{V_{\lambda+1}}$. Then there exists α such that every elementary embedding $j : L(E_\alpha) \prec L(E_\alpha)$ is not proper. We call α *totally non-proper ordinal*.

Does the properness of an elementary embedding completely depend on its underlying structure?
In other words, if I have a proper elementary embedding, is any attempt to find a non-proper elementary embedding on the same domain hopeless?No.

Theorem

Let α be the minimal ordinal such that $L((E_\alpha)^\sharp)\cap V_{\lambda+2} = L(E_\alpha)$. Then there exist $j, k : L(E_\alpha) \prec L(E_alpha)$ with j proper and k non-proper. We call α *partially non-proper ordinal*.


$\alpha = \Theta^{L(E_\alpha)}$
β_2
k_2
η_1
β_1
β_1
k_1
η_0
β_0

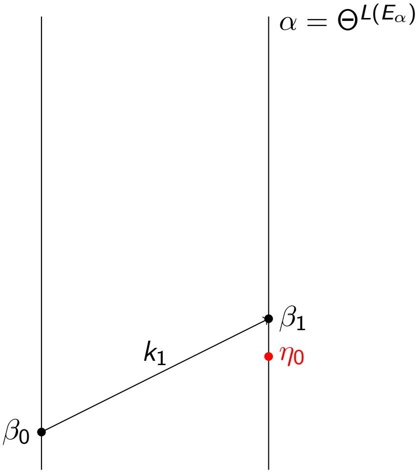
For this game I has a winning quasistrategy in $L(E_\alpha)$. So I has a winning strategy in V.
If II plays a sequence cofinal in α, the resulting j will have no fixed point between β_0 and $\Theta^{L(E_\alpha)}$.

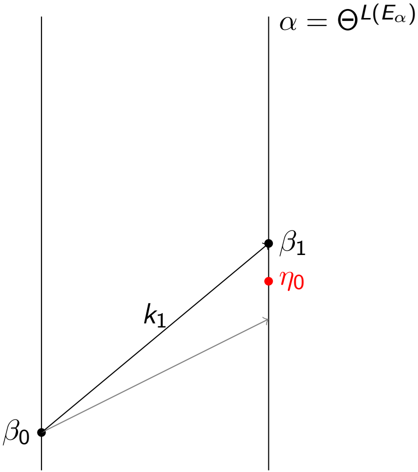
Proposition

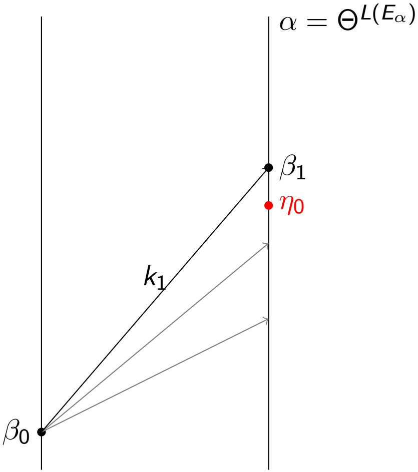
Let β be one of the totally proper elementary embeddings we know. Then if $j, k : L(E_\beta) \prec L(E_\beta)$ and $j \upharpoonright V_\lambda = k \upharpoonright V_\lambda$, $j = k$.

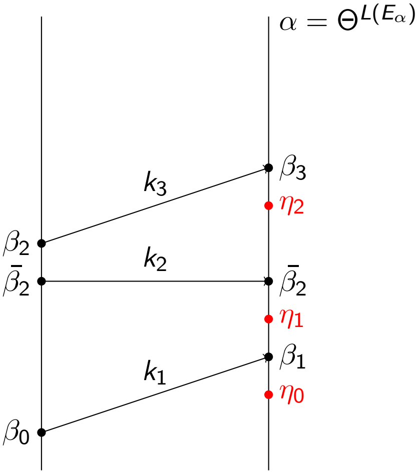
Proposition

Let α be the partially proper ordinal above, and fix j. Then there are 2^λ proper and non-proper $k : L(E_\alpha) \prec L(E_\alpha)$ such that $j \upharpoonright V_\lambda = k \upharpoonright V_\lambda$.


$\alpha = \Theta^{L(E_\alpha)}$
β_1
η_0
k_1
β_0


$\alpha = \Theta^{L(E_\alpha)}$
β_1
η_0
k_1
β_0

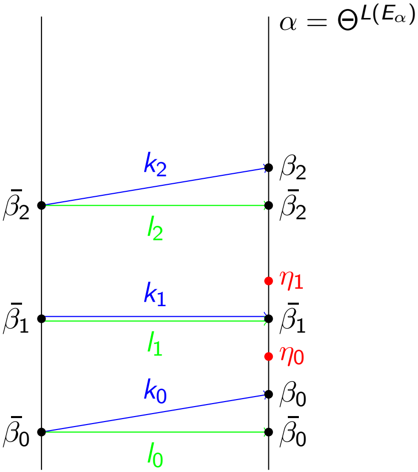

$\alpha = \Theta^{L(E_\alpha)}$
β_1
η_0
k_1
β_0


$\alpha = \Theta^{L(E_\alpha)}$
β_3
η_2
k_3
β_2
k_2
$\bar{\beta}_2$
$\bar{\beta}_2$
η_1
β_1
k_1
η_0
β_0

If j is proper, then the set C_j of fixed points under $\Theta^{L(E_\alpha)}$ is an ω-club. What can we say about $C_j \Delta C_k$?

Proposition

There exist j and k such that $C_j \Delta C_k$ is unbounded.


$\alpha = \Theta^{L(E_\alpha)}$
k_2
β_2
$\bar{\beta}_2$
$\bar{\beta}_2$
l_2
η_1
k_1
$\bar{\beta}_1$
$\bar{\beta}_1$
l_1
η_0
k_0
β_0
$\bar{\beta}_0$
$\bar{\beta}_0$
l_0

- Are there strong implications between hypothesis stronger than I0?
- Are there other analogies between $AD^{L(\mathbb{R})}$ and I0 (e.g. $\lambda^+ \to (\lambda^+)^{<\omega_1}_{\lambda}$?
- Is the Ultrafilter Conjecture consistent?
- Are there totally or partially non-proper Icarus sets?
- Are there mathematical problems equivalent to rank-into-rank axioms?
- Is there an α such that the existence of E^0_α is inconsistent?

I have never been certain whether the moral of the Icarus story should only be, as is generally accepted, 'don't try to fly too high,' or whether it might also be thought of as 'forget the wax and feathers, and do a better job on the wings.'
Stanley Kubrick

Thanks for your attention