Large cardinals in mathematics and infinite combinatorics

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A point of view: the development of mathematics is driven by a search for completion.

The integers are developed for completing the natural numbers under substraction.

The rationals are developed for completing the integers under division.

The reals are developed for completing the rationals under Cauchy sequences.

The complex numbers are developed for completing the reals under square roots.

What about counting?

$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \dots \quad \infty \; \infty + 1$

There are some psychological studies that indicates that the concept of "infinity plus one" is natural for children

Monaghan, John (2001). "Young Peoples' Ideas of Infinity". *Educational Studies in Mathematics* **48** (2): 239–257

In mathematics: uniqueness of an expansion of a function in a trigonometric series.

Theorem (Cantor, 1870)

Suppose

$$a_0/2 + \sum_{n=1}^{+\infty}(a_n \cos nx + b_n \sin nx) = 0 \text{ for any } x \in \mathbb{R}.$$

Then $a_n = b_n = 0$.

In trying to extend this results (weakening the hypothesis from $\forall x$), Cantor arrived to this definition:

Definition (Cantor, 1872)

Let S be a set of reals. Then $S' = \{x \in S : x \text{ is a limit point of } S\}$.
Define by induction:

- $S^{(0)} = S'$;
- $S^{(n+1)} = S^{(n)'}$;
- $S^{(\infty)} = \bigcap_{n \in \mathbb{N}} S^{(n)}$.

But maybe $S^{(\infty)}$ has some isolated points...

- $S^{(\infty+1)} = S^{(\infty)'} \ldots$

Definition (Cantor, 1883)

Two ordered sets $(S, \leq_S)$ and $(T, \leq_T)$ have the same *order type* if there is an order isomorphism between them, i.e., $\exists f : S \to T$ bijective such that $x \leq_S y$ iff $f(x) \leq_T f(y)$.

α is an *ordinal number* if it's the order type of a well-ordered set (i.e., linear without infinite descending chains).

-

1

2

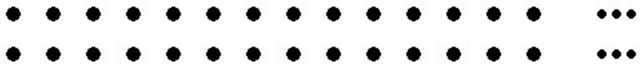
3

∞

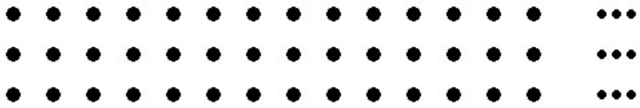
ω

$\omega + 1$

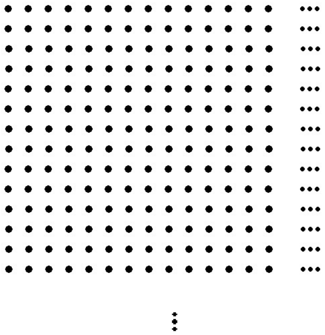
$$\omega + 2$$



$$\omega + \omega = \omega \cdot 2$$



$$\omega + \omega + \omega = \omega \cdot 3$$



$\omega \cdot \omega$

(the order type of the Sieve of Eratosthenes)

$$\omega \cdot \omega$$

But wait a minute...

$\omega + 1$ is after ω, but it's not bigger!

Definition(Cantor, 1874-1884)

Two sets have the same *cardinality* if there is a bijection between them.
κ is a *cardinal number* if it is the cardinality of an ordinal number.

ω is both a cardinal and an ordinal number. When we use it as a cardinal, we call it $\aleph_0$.

There is a bijection between $\omega + 1$ and ω (Hilbert's Paradox of the Grand Hotel). Is there an ordinal really bigger?

Theorem (Cantor, 1874)

$|\mathcal{P}(\omega)| > \aleph_0$.

The smallest cardinal bigger than $\aleph_0$ is $\aleph_1$, then $\aleph_2$, $\aleph_3$, $\ldots \aleph_\omega$, $\aleph_{\omega+1}$, $\ldots \aleph_{\omega^\omega}$ $\ldots$

Operations are defined, like sum, multiplications...

Definition

$\kappa^\gamma = |\{f : \gamma \to \kappa\}|$.

For example $2^\kappa = |\mathcal{P}(\kappa)|$.

Main Problems in Set Theory #2

Suppose for any n, $2^{\aleph_n} < \aleph_\omega$. How big is $2^{\aleph_\omega}$?
Best result: $2^{\aleph_\omega} < \aleph_{\omega_4}$.

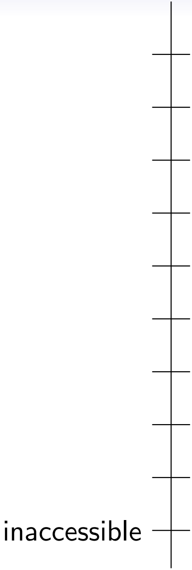
But wait a minute...

$\aleph_1$ is still too close to $\aleph_0$: $2^{\aleph_0} \geq \aleph_1$, but $2^n < \aleph_0$ for all n!

Definition(Sierpiński, Tarski, Zermelo, 1930)

κ is an inaccessible cardinal iff

- $\kappa > \aleph_0$;
- for any $\gamma, \eta < \kappa$, $\gamma^\eta < \kappa$;
- for any $A \subseteq \kappa$, $|A| < \kappa \to \sup(A) < \kappa$.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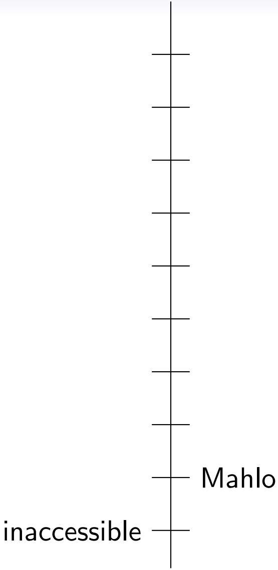
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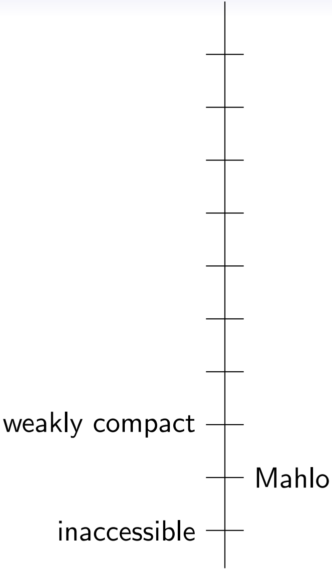
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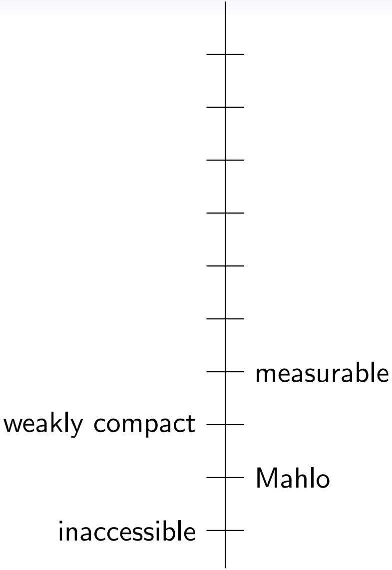
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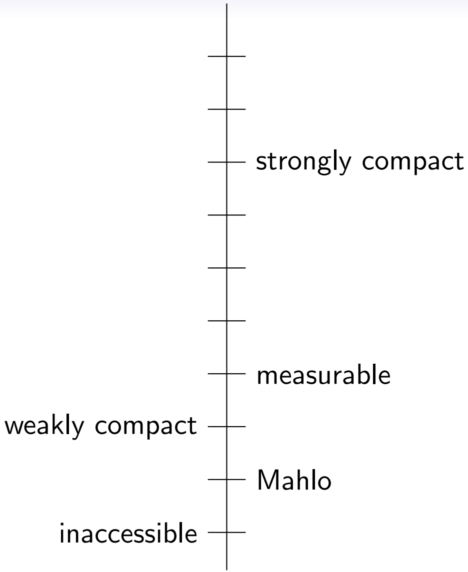
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Mathematics works through *theorems*.

They are logical derivations of the form if. . . then. . . .

It is clear that there needs to be a starting point, i.e., an axiomatic system.

ZFC is now the favourite axiomatic system for mathematics. We can say it's *the* mathematics.

Theorem(Gödel, 1931)

Any effectively generated theory capable of expressing elementary arithmetic cannot be both consistent and complete.

For any formal effectively generated theory T including basic arithmetical truths and also certain truths about formal provability, if T includes a statement of its own consistency then T is inconsistent.

A statement is *independent from ZFC* if ZFC cannot prove it or disprove it.

If there is an inaccessible cardinal, then one can prove that ZFC is consistent. Then ZFC cannot prove that there exists an inaccessible cardinal, so it's independent from ZFC.

Theorem

The existence of an inaccessible cardinal is equiconsistent to

- the measurability of the projective sets in $\mathbb{R}$;
- the existence of Kurepa trees.

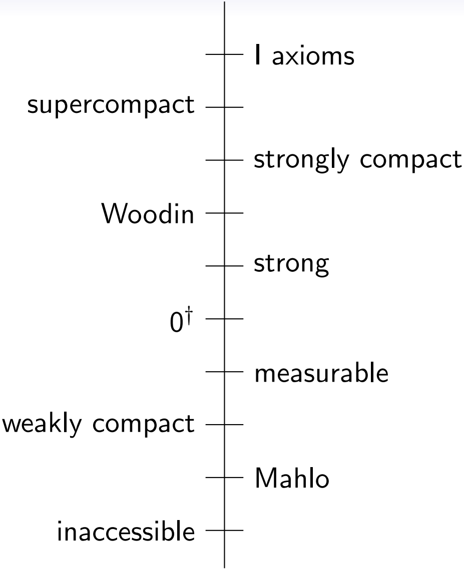
Theorem

The existence of a measurable cardinal is equiconsistent to

- every Borelian measure on $\mathcal{B}([0,1])$ can be extended on a measure on $\mathcal{P}([0,1])$;
- there exists a cardinal κ and a non-trivial homomorphism $h : \mathbb{Z}^\kappa \setminus \mathbb{Z}^{<\kappa} \to \mathbb{Z}$.

Theorem(Nyikos, Fleissner 1982)

The consistency of the normal Moore conjecture is between measurable and strongly compact.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Theorem (Wiles, 1995)

Suppose there are unboundedly inaccessible cardinals. Then for any $n > 2$ there are no a, b, c integers such that $a^n + b^n = c^n$.

In 1983 Pitowsky constructed hidden variable models for spin-1/2 and spin-1 particles in quantum mechanics. Pitowsky's functions calculate in this model the probabilities of spin values.

Theorem (Farah, Magidor, 2012)

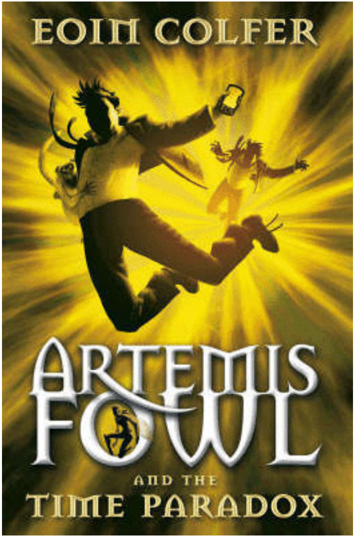
If there exists a measurable cardinal, then Pitowski functions do not exist.

There are also very debatable results...

Theorem (H. Friedman, 2012)

The existence of a measurable cardinal is close to equiconsistent to the existence of God.

...and large cardinals even appear in pop culture!


EOIN COLFER
ARTEMIS
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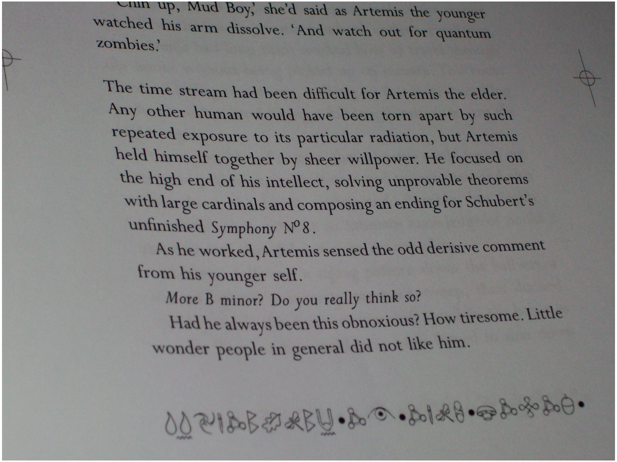


'Chin up, Mud Boy,' she'd said as Artemis the younger watched his arm dissolve. 'And watch out for quantum zombies.'

The time stream had been difficult for Artemis the elder. Any other human would have been torn apart by such repeated exposure to its particular radiation, but Artemis held himself together by sheer willpower. He focused on the high end of his intellect, solving unprovable theorems with large cardinals and composing an ending for Schubert's unfinished *Symphony N°8*.

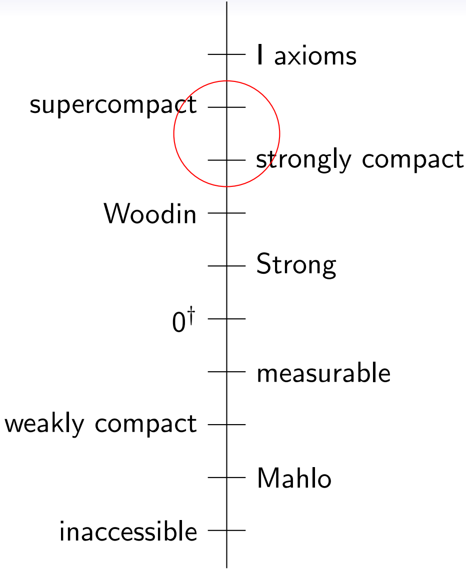
As he worked, Artemis sensed the odd derisive comment from his younger self.

More B minor? Do you really think so?

Had he always been this obnoxious? How tiresome. Little wonder people in general did not like him.

Main questions when dealing with a large cardinal:

- What is the relationship between it and other large cardinals? E.g. is it really different? Is it really stronger (or weaker)?
- What are its consequences on set theory? And mathematics?
- Which theorems *needs* it to be proven?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Main Problems in Set Theory #3

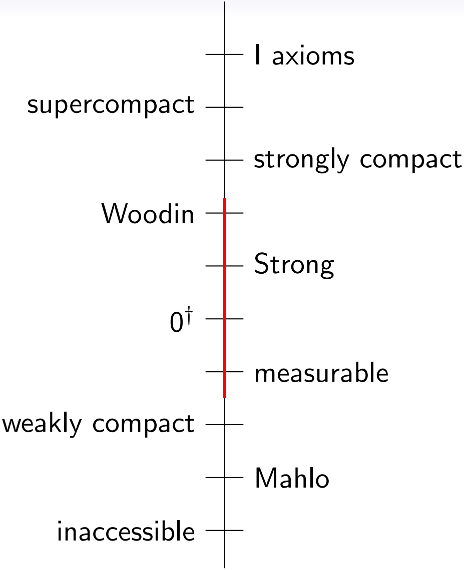
Is supercompact equiconsistent to strongly compact?

Main Problems in Set Theory #5

Suppose κ is strongly compact. Is it true that if for any $\eta < \kappa$ $2^\eta = \eta^+$, then this is true for every η?

Main Problems in Set Theory #1

Is there an inner model for supercompact?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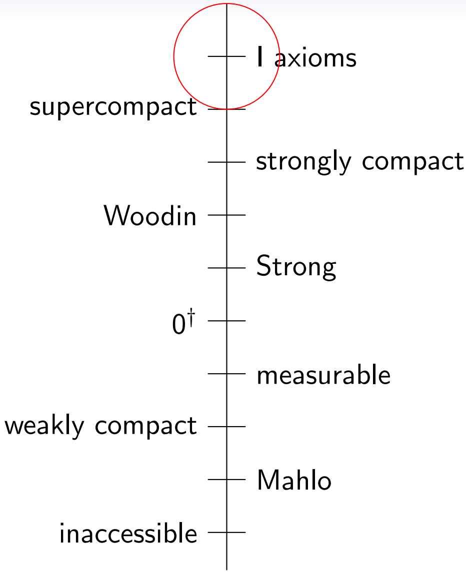
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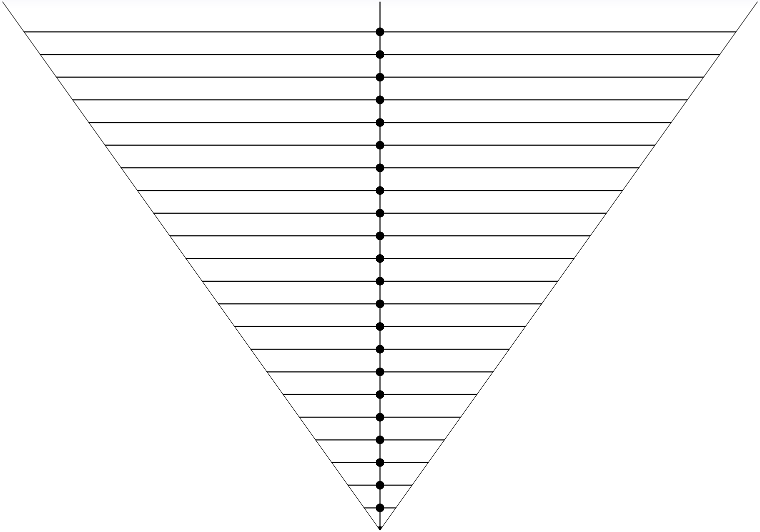
Definition

The rank of $\emptyset$ is 0.The rank of a set S is the supremum of the ranks of all $s \in S$.V_α is the set of the sets of rank $< \alpha$.$V = \bigcup_\alpha V_\alpha$ is the universe of sets.

Examples: $V_0 = \emptyset$. $\emptyset \in V_1$, $\{\emptyset\} \in V_2$, $\{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\} \in V_3$, ...



Definition

A function $j : M \to N$ is an *elementary embedding* if it is injective and for any $x \in M$ and any formula φ, $M \vDash \varphi(x)$ iff $N \vDash \varphi(j(x))$. We write $j : M \prec N$.

It's a morphism for the logical structure.

If $x, y \in M$ and $x \in y$, then $j(x) \in j(y)$. If $\exists x \in M$ that satisfies something, then $\exists y \in N$ that satisfies the same thing. If all $x \in M$ satisfy something relative to a parameter p, then all $y \in N$ satisfy the same thing relative to the parameter $j(p)$.

$j(0) = 0,\ j(1) = 1,\ j(2) = 2,\ \dots\ j(n) = n,\ \dots,\ j(\omega) = \omega,\ j(\aleph_\omega) = \aleph_\omega, \dots$

The critical point of j is the smallest ordinal α such that $j(\alpha) \neq \alpha$ (it's easy to see that $j(\alpha) \geq \alpha$).

Theorem (Scott, Keisler 1962)

The following are equivalent:

- there is a κ-additive measure on κ (κ is measurable);
- there exists $j : V \prec M \subseteq V$, with κ critical point of j.

Can $V = M$? It would be a very strong hypothesis...

Theorem (Kunen, 1971)

There is no $j : V \prec V$.

The proof uses greatly the Axiom of Choice.

Main Problems in Set Theory #4

Is there a $j : V \prec V$ when $\neg AC$?.

Let's define a local version of such hypothesis:

Definition

- I3: There exists $j : V_\lambda \prec V_\lambda$;
- I1: There exists $j : V_{\lambda+1} \prec V_{\lambda+1}$.

Theorem (Laver, 1989)

Suppose I3. Then the word problem for the left-distributive algebra is decidable.

The gist is that the set of elementary embeddings on V_λ is an acyclic left-distributive algebra.

This was later proved in ZFC. But there are similar (more technical) results who are still proven only with I3.

Let X a set. Then $L(X)$ is the smallest model of ZF that contains X.

- $L_0(X) = X$;
- $L_1(X) =$ the set of subsets of $L_0(X)$ that are definable;
- $L_2(X) =$ the set of subsets of $L_1(X)$ that are definable;
- $\dots$
- $L_\omega(X) = \bigcup_{n\in\omega} L_n(X)$;
- $\dots$

Definition

I0: there exists $j; L(V_{\lambda+1}) \prec L(V_{\lambda+1})$.

Does it have interesting consequences?

Let X be a set of infinite strings of natural numbers. Then consider the game

$$\begin{array}{lcccccc} I & a_0 & & a_1 & & & \\ & & & & & & \dots \\ II & & b_0 & & b_1 & & \end{array}$$

where I wins iff $(a_0, b_0, \dots) \in X$.

A winning strategy for I (for II) is a function that tells the right moves to I (II), so that I always wins (loses). X is determined iff there is a winning trategy for I or for II.

Axiom of Determinacy: every X is determined.

AD is in contradiction with AC, but maybe a local version can hold.

Theorem (Woodin, 80's)

I0 implies the consistency of AD.

But...

Theorem

AD is equiconsistent to infinitely many Woodin cardinals.

So the question is still open...

As it is now, finding a theorem that *needs* I0 seems hopeless (see Main Problem #1).

But there are good chances to find consequences of I0.

For some still unknown reason, I0 has a very rich structure compared to I1 and I3, and this structure has some striking similarities to AD in $L(\mathbb{R})$.

Let me give you a possible future development.

We introduced recently *generic I0*, a variant of I0. Generic I0 gives a positive answer to Main Problem #5 (the proper subalgebra of $\aleph_\omega$). It would not be surprising if I0 would imply generic I0, therefore solving the problem.

Other things that generic I0 proves:

Theorem (D. 2015)

Suppose generic I0 at $\aleph_\omega$. Then

- $L(V_{\aleph_\omega}) \not\models$ AC;
- $L(V_{\aleph_\omega}) \models \aleph_{\omega+1}$ is a measurable cardinal;
- $L(V_{\aleph_\omega}) \models 2^{\aleph_\omega}$ is very, very large (an inaccessible limit of measurables).

Note that this goes against our results for Main Problem #2, but that's because $L(V_{\aleph_\omega})$ does not satisfy choice.

Can we go *beyond* I0? Of course!

Woodin defined $L(E_\alpha)$, a strengthening of I0. $L(E_0)$ is just I0, and the higher is α the stronger the axiom is.

Question: where does the sequence stop? It must stop somewhere!

Possible indication:

Theorem (D. 2012)

Suppose that there exists ξ such that $L(E_\xi) \nvDash V = HOD_{V_{\lambda+1}}$. Then there exists an $\alpha < \xi$ such that there are many proper and many non proper elementary embeddings from $L(E_\alpha)$ to itself.

This result seems highly suspicious, so maybe the assumption is inconsistent.

To recap:

- Completing the natural numbers under the operation of "'counting"', opens up a very rich universe;
- large cardinal are seemingly innocuous infinite combinatorial properties;
- yet they function very well as a measure for calculating the strength of many mathematical propositions;
- for unknown reason, they are ordered in a linear way;
- large cardinals above Woodin are more misterious, because we don't have inner model theory there;
- yet they are also the more potentially productive.

Thanks for your attention!