Very Large Cardinals and Elementary Embeddings Properties

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A fundamental step in Large Cardinal Theory:

(Reinhardt, 70) κ is a Reinhardt cardinal iff it's the critical point of an elementary embedding $j : V \prec V$.

Theorem (Kunen, 1971)(AC)

If $j : V \prec M$, then $M \neq V$.

Corollary

There is no $j : V_\eta \prec V_\eta$, with $\eta \geq \sup\{\text{critical sequence}\} + 2$.

After that, two paths were available:

Daedalus Path Better to stay low and going back to cardinals weaker than supercompact (strong, Woodin, etc.).

Icarus Path Let's see how high we can get before burning our wings.

I1: There exists an elementary embedding $j : V_{\lambda+1} \prec V_{\lambda+1}$.

Woodin proposed an even stronger axiom:

Definition (Woodin, 1984)

I0: There exists an elementary embedding $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$.

This axiom is more interesting, since it produces a structure on $L(V_{\lambda+1})$ that is strikingly similar to the structure of $L(\mathbb{R})$ under AD.

Definition

We say that $X \subseteq V_{\lambda+1}$ is an *Icarus set* if there exists an elementary embedding $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$.

Examples:

- $\emptyset$, $V_{\lambda+1}$ are Icarus sets iff I0 holds.
- any well-ordering of $V_{\lambda+1}$ cannot be an Icarus set.

We define $\Theta^{L(X,V_{\lambda+1})}$ as the supremum of the α's such that in $L(X, V_{\lambda+1})$ there exists a surjection $\pi : V_{\lambda+1} \twoheadrightarrow \alpha$.It "measures the complexity" of X, in the sense that if $\Theta^{L(Y,V_{\lambda+1})} < \Theta^{L(X,V_{\lambda+1})}$ then we consider "X is an Icarus set" as something stronger than "Y is an Icarus set".

All attempts to prove deep analogies with $\mathrm{AD}^{L(X,\mathbb{R})}$ failed, without further hypotheses.

Theorem (Woodin)

It is possible to define a sequence $V_{\lambda+1} \subset E_\alpha \subset V_{\lambda+2}$ such that:

- if X is an Icarus set and $\Theta^{L(E_\alpha)} < \Theta^{L(X, V_{\lambda+1})}$, then $E_\alpha \subset L(X, V_{\lambda+1})$;
- if α is a successor, then there exists an Icarus set X such that $L(E_\alpha) = L(X, V_{\lambda+1})$;
- if α is a limit, then $L(E_\alpha) = L(\bigcup_{\beta<\alpha} E_\beta)$.

We have said that deeper analogies hold only under further hypotheses.
We define them in the more general setting of $j : L(N) \prec L(N)$, $V_{\lambda+1} \subset N \subset V_{\lambda+2}$.

Definition

j is *proper* iff for every $X \in L(N) \cap V_{\lambda+2}$ $\langle X, j(X), j(j(X)), \dots \rangle \in L(N)$.

Theorem

Suppose that $L(N) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$. Then j is proper iff the set of fixed points of j is cofinal in $\Theta^{L(N)}$.

But is properness really a property?

Theorem (Woodin)

Suppose E_α exists. If

- $\alpha = 0$, or
- α is a successor ordinal, or
- α is a limit ordinal with cofinality $> \omega$

then every elementary embedding $j : L(E_\alpha) \prec L(E_\alpha)$ is proper.

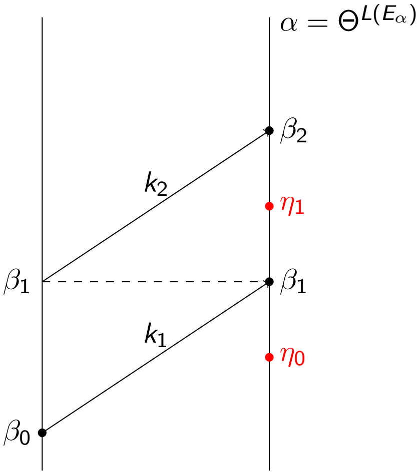
Theorem

Suppose that there exists ξ such that $L(E_\xi) \nvDash V = \mathrm{HOD}_{V_{\lambda+1}}$. Then there exists α such that every elementary embedding $j : L(E_\alpha) \prec L(E_\alpha)$ is not proper. We call α *totally non-proper ordinal*.

Does the properness of an elementary embedding completely depend on its underlying structure?
In other words, if I have a proper elementary embedding, is any attempt to find a non-proper elementary embedding on the same domain hopeless? No.

Theorem

Let α be the minimal ordinal such that $L((E_\alpha)^\sharp) \cap V_{\lambda+2} = L(E_\alpha) \cap V_{\lambda+2}$. Then there exist $j, k : L(E_\alpha) \prec L(E_\alpha)$ with j proper and k non-proper. We call α *partially non-proper ordinal*.


$\alpha = \Theta^{L(E_\alpha)}$
β_2
k_2
η_1
β_1
β_1
k_1
η_0
β_0

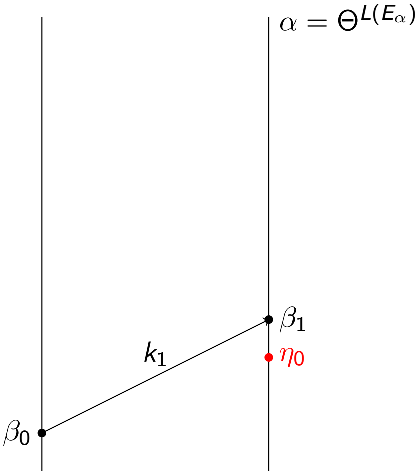
For this game I has a winning quasistrategy in $L(E_\alpha)$. So I has a winning strategy in V.
If II plays a sequence cofinal in α, the resulting j will have no fixed point between β_0 and $\Theta^{L(E_\alpha)}$.

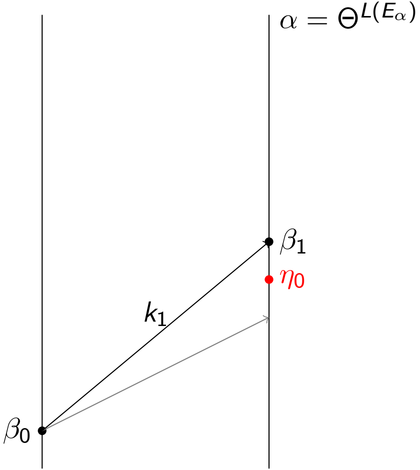
Proposition

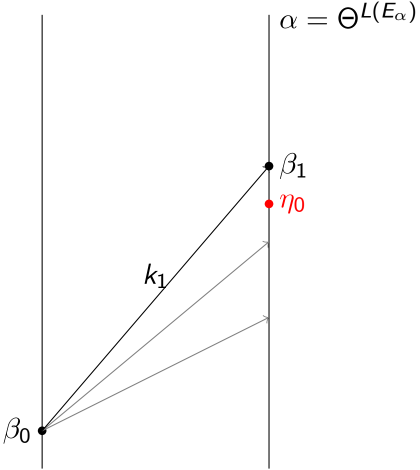
Let β be one of the totally proper ordinals we know. Then if $j, k : L(E_\beta) \prec L(E_\beta)$ and $j \upharpoonright V_\lambda = k \upharpoonright V_\lambda$, $j = k$.

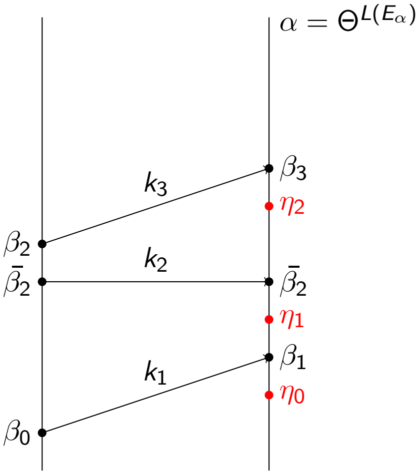
Proposition

Let α be the partially proper ordinal above, and fix j. Then there are 2^λ proper and non-proper $k : L(E_\alpha) \prec L(E_\alpha)$ such that $j \upharpoonright V_\lambda = k \upharpoonright V_\lambda$.


$\alpha = \Theta^{L(E_\alpha)}$
β_1
k_1
η_0
β_0


$\alpha = \Theta^{L(E_\alpha)}$
β_1
η_0
k_1
β_0

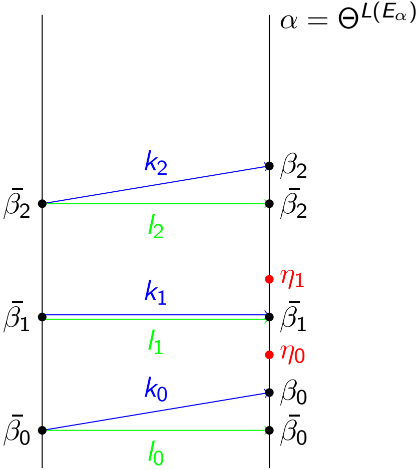

$\alpha = \Theta^{L(E_\alpha)}$
β_1
η_0
k_1
β_0


$\alpha = \Theta^{L(E_\alpha)}$
β_3
k_3
η_2
β_2
k_2
$\bar{\beta}_2$
$\bar{\beta}_2$
η_1
β_1
k_1
η_0
β_0

If j is proper, then the set C_j of fixed points under $\Theta^{L(E_\alpha)}$ is an ω-club. What can we say about $C_j \Delta C_k$?

Proposition

There exist j and k such that $C_j \Delta C_k$ is unbounded.

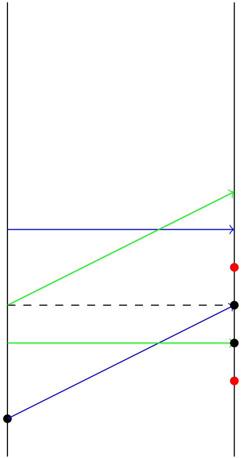


$\alpha = \Theta^{L(E_\alpha)}$
k_2
β_2
$\bar\beta_2$
$\bar\beta_2$
l_2
η_1
k_1
$\bar\beta_1$
$\bar\beta_1$
l_1
η_0
k_0
β_0
$\bar\beta_0$
$\bar\beta_0$
l_0

- There are classes that can “contain” only non-proper elementary embeddings, and the ones we know they have very few of them;
- there is one class that seems to contain all the possible zoology of elementary embeddings;
- the non-properness of an elementary embedding can or cannot depend from the underlying structure.

Suppose that E^0_∞ does not exists. Is it true that the E^0-sequence is a witness for every Icarus set? In other words, zuppose that $\alpha < \Upsilon$ is a limit ordinal such that there are no $X \subset V_{\lambda+1}$ with $L(E^0_\alpha) = L(X, V_{\lambda+1})$. Is it true that there are no Y Icarus such that $\Theta^{E^0_\alpha} = \Theta^{L(Y,V_{\lambda+1})}$?
Is there a totally non-proper ordinal smaller than the one found?
Is there a partially non-proper ordinal smaller than the one found?
Is there a totally non-proper ordinal below every partially non-proper ordinal?
Are there structural consequences of non-proper ordinals?
Is the smallest λ such that I0(λ) holds smaller than the smallest λ such that $\Upsilon_\lambda > 1$ holds?
Is the algebra of elementary embeddings from $L(E^0_\alpha)$ to itself interesting?
Suppose that j, k are proper elementary embeddings. Is $j \circ k$ proper?
Is it possible to generalize the Outer Model Program to $\Upsilon > 1$?

$\alpha = \Theta^{L(E_\alpha)}$



- Questions on the algebra of elementary embeddings in $L(E_\alpha)$.
- Questions on partially and totally non-proper ordinals.
- Is there a non-proper elementary embedding $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$?
- Questions on the validity of the Icarus sets.