Very Large Cardinals and Combinatorics

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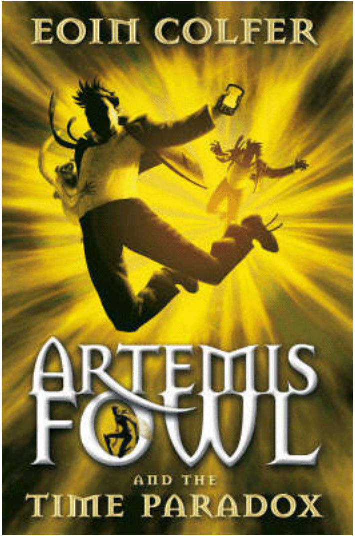
This project started in Kobe.


EOIN COLFER
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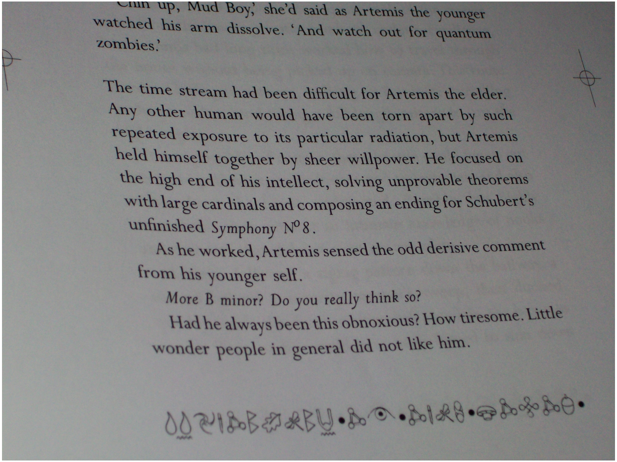

Chin up, Mud Boy,' she'd said as Artemis the younger watched his arm dissolve. 'And watch out for quantum zombies.'

The time stream had been difficult for Artemis the elder. Any other human would have been torn apart by such repeated exposure to its particular radiation, but Artemis held himself together by sheer willpower. He focused on the high end of his intellect, solving unprovable theorems with large cardinals and composing an ending for Schubert's unfinished Symphony Nº8.

As he worked, Artemis sensed the odd derisive comment from his younger self.

More B minor? Do you really think so?

Had he always been this obnoxious? How tiresome. Little wonder people in general did not like him.

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These are all question non-answerable in ZFC.

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- Large cardinals hypotheses enlarge our multiverse (more universes!)
- $V = L$ has very nice structural properties;
- it is also interesting to go the other way, and investigating properties opposed to those in $V = L$;
- combinatorial properties can be local (regarding one cardinal) or global (regarding all cardinals, or at least a class).

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- TP_κ (Tree Property) is König's Lemma for κ. $TP_{\kappa^{++}}$ is both a stronger failure of the local GCH and a failure of $\square$.

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- I0 For some λ there exists a $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$, with $\mathrm{crt}(j) < \lambda$

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- Special case: local case exactly at the large cardinal.

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Theorem (Levy, Solovay, 1967)

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Con(inaccessible)$\rightarrow$Con(inaccessible+GCH).

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Easton's Theorem uses a class forcing to force many different behaviours of the power function on regular cardinals. The trick is to use the same method (reverse Easton iteration), changing the forcing but preserving the large cardinal.

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Theorem (Corazza, 2007)

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- a reverse Easton iteration (nontrivial forcing only at limit stages, direct limit at inaccessible stages, inverse limit otherwise)
- adequate (for all δ, $V^{\mathbb{P}_\delta} \vDash |\mathbb{Q}_\delta| \leq$ the smallest inaccessible bigger than δ)
- directed closed (for all δ, $\mathbb{Q}_\delta$ is $< \delta$-directed closed)
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- λ-bounded (for all δ, $V^{\mathbb{P}_\delta} \vDash |\mathbb{Q}_\delta| \leq \lambda$)
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If the forcing is large and closed, then it is trivial: if $I^*(\lambda)$ and $\mathbb{P}$ is λ-closed, then $I^*(\lambda)^{V^{\mathbb{P}}}$.
So the interesting case is local combinatoric properties in λ.

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Generic Absoluteness Theorem (extended)

Suppose there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$. Let (M_ω, j_ω) be the ω-th iterate of $(L(V_{\lambda+1}), j)$

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Corollary

Suppose there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) = \kappa$. Let $\mathbb{Q}$ be a "'Prikry-like"' forcing in κ (κ-good). Then in the generic extension there exists $k : V_{\kappa+1} \prec V_{\kappa+1}$.

Can we lower the hypotheses of the last Theorem to I1? Can we improve the Theorem to I0?

Is there a combinatorial property that is non-trivially inconsistent with I*?

Or some that is equiconsistent?

From the European Charter for Researchers:
"Researchers should ensure that their research activities are made known to society at large in such a way that they can be understood by non-specialists, thereby improving the public's understanding of science"