I0, Generic Absoluteness and Combinatorics

Vincenzo Dimonte

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Theorem (Kunen, 1971)
If \( j : V \prec M \), then \( M \neq V \).
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Corollary
There is no \( j : V_\eta \prec V_\eta \), with \( \eta \geq \lambda + 2 \).
This leaves room for a new breed of large cardinal hypotheses:
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**Definition**

$I_3$ iff there exists $\lambda$ s.t. $\exists j : V_\lambda \prec V_\lambda$;

$I_2$ iff there exists $\lambda$ s.t. $\exists j : V_{\lambda+1} \prec V_{\lambda+1}$;

$I_1$ iff there exists $\lambda$ s.t. $\exists j : V_{\lambda+1} \prec V_{\lambda+1}$;

$I_0$ For some $\lambda$ there exists a $j$:

$L(V_{\lambda+1}) \prec L(V_{\lambda+1})$, with $\text{crt}(j) < \lambda$.

Why are they large cardinals?

The critical point of $j$ is measurable, $n$-huge, supercompact in $V_\lambda$.

$\lambda$ is a strong limit cardinal (in fact, Rowbottom).
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- **I₂** iff there exists \( \lambda \) s.t. \( \exists j : V_{\lambda+1} \prec_1 V_{\lambda+1} \);
- **I₁** iff there exists \( \lambda \) s.t. \( \exists j : V_{\lambda+1} \prec V_{\lambda+1} \);
- **I₀** For some \( \lambda \) there exists a \( j : L(V_{\lambda+1}) \prec L(V_{\lambda+1}) \), with \( \text{crt}(j) < \lambda \)

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Theorem (D., Friedman, 2013)

Suppose \( I^* \) is \( I_3, I_2, I_1 \) or \( I_0 \). Then \( I^* \) is consistent with each of the following:

- GCH
- Failure of GCH at regular cardinals
- \( V=HOD \)
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Theorem (D., Wu, 2014)

Suppose I0. Then I1, i.e., \( j : V_{\lambda+1} \prec V_{\lambda+1} \), is consistent with each of the following:

- the failure of SCH at \( \lambda \)
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- TP(\( \lambda^{++} \))
- \( \neg \text{SCH} + \neg \text{AP} \)
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- $TP(\lambda^{++})$
- $\neg SCH + \neg AP + \text{(Very good scale)}$ at $\lambda$
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Suppose I₀. Then I₁, i.e., \( j : V_{\lambda+1} \prec V_{\lambda+1} \), is consistent with each of the following:

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- etc...
The key of the proofs is the relationship between rank-into-rank embeddings and forcing. There are some easy cases:

- \((V_{\lambda+1})^{V[G]} = V_{\lambda+1}\): this case is trivial, \(j\) is still a witness in \(V[G]\);
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- \(\mathbb{P} \in V_{\text{crt}(j)}\): define the extension \(k(\tau_G) = j(\tau)_G\).
- \(\mathbb{P} \in V_\lambda\): as before, since iterating \(j\) we can have \(\text{crt}(j) < \lambda\) as large as we want.
**Theorem (Hamkins, 1994)**

Suppose $\mathbf{I_1}$ witnessed by $j$ and $\lambda$
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Suppose I₁ witnessed by J and λ. Let P be a forcing iteration of length λ, with Q_δ its stages and P_δ its initial segments. Then I₁ is preserved in the forcing extension if P is:

- a reverse Easton iteration (nontrivial forcing only at limit stages, direct limit at inaccessible stages, inverse limit otherwise)
- simple (for all δ, V_{P_δ} \models |Q_δ| \leq 2^δ)
- directed closed (for all δ, Q_δ is <δ-directed closed)
- J-coherent (for all δ, J(P_δ) = P_{J(δ)})
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- \( j \)-coherent (for all \( \delta \), \( j(P_\delta) = P_{j(\delta)} \))
**Theorem (Corazza, 2007)**

Suppose I₃ witnessed by $j$ and $\lambda$. Let $\mathbb{P}$ be a forcing iteration of length $\lambda$, with $\mathbb{Q}_\delta$ its stages and $\mathbb{P}_\delta$ its initial segments. Then I₃ is preserved in the forcing extension if $\mathbb{P}$ is:

- a reverse Easton iteration (nontrivial forcing only at limit stages, direct limit at inaccessible stages, inverse limit otherwise)
- **adequate** (for all $\delta$, $V^{\mathbb{P}_\delta} \models |\mathbb{Q}_\delta| \leq$ the smallest inaccessible bigger than $\delta$)
- directed closed (for all $\delta$, $\mathbb{Q}_\delta$ is $< \delta$-directed closed)
- $j$-coherent (for all $\delta$, $j(\mathbb{P}_\delta) = \mathbb{P}_{j(\delta)}$)
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Suppose \( I_3, I_2, I_1, I_0 \) witnessed by \( j \) and \( \lambda \). Let \( P \) be a forcing iteration of length \( \lambda \), with \( Q_\delta \) its stages and \( P_\delta \) its initial segments. Then \( I_3, I_2, I_1, I_0 \) is preserved in the forcing extension if \( P \) is:

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Suppose $I_3, I_2, I_1, I_0$ witnessed by $j$ and $\lambda$. Let $P$ be a forcing iteration of length $\lambda$, with $Q_\delta$ its stages and $P_\delta$ its initial segments. Then $I_3, I_2, I_1, I_0$ is preserved in the forcing extension if $P$ is:

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- $\lambda$-bounded (for all $\delta$, $V^{P_\delta} \models |Q_\delta| \leq \lambda$)
- directed closed (for all $\delta$, $Q_\delta$ is $<\delta$-directed closed)
- $j$-coherent (for all $\delta$, $j(P_\delta) = P_{j(\delta)}$)
Note: if $j, \lambda, \kappa$ witness I0, then $j$ is iterable and if $M_\omega$ is its $\omega$-th iteration,
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**Generic Absoluteness Theorem (Woodin, 2012)**

Suppose there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$, and let $\kappa_0 = \text{crt}(j) < \lambda$ and $\kappa_{n+1} = j(\kappa_n)$.
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So, I1($\lambda$) holds in $M_\omega[\vec{\kappa}]$, and therefore I1($\kappa$) holds in a Prikry forcing extension of $L(V_{\lambda+1})$ (of $V$).
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All the subsets of \(V_{\lambda+1}\) in \(L(R)\) are \(U(j)\)-representable (Cramer), and the “simple” ones are uniformly \(U(j)\)-representable (they behave well w.r.t \(j\)) (Woodin).

An example: \(j\) and the theory of \(V_{\lambda+1}\) are simple. therefore coded by some structure. In \(M_\omega\) the sets disappear, but the structure remains
Where is Generic Absoluteness coming from?

I0 is very similar to AD$^L(\mathbb{R})$. Woodin defined $U(j)$-representable sets in a manner similar to homogeneously Souslin sets.

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An example: $j$ and the theory of $V_{\lambda+1}$ are simple. therefore coded by some structure. In $M_\omega$ the sets disappear, but the structure remains. $\vec{\kappa}$ is the key to decrypt the code and reconstruct the sets.
The Theorem holds also with any Prikry-generic in $V$ instead of $\vec{\kappa}$.
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**Generic Absoluteness Theorem (extended)**

Suppose there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$. Let $(M_\omega, j_\omega)$ be the $\omega$-th iterate of $(L(V_{\lambda+1}), j)$. Then there exists $\pi : L(\omega(V_{\lambda+1})) \prec L(\omega(V_{\lambda+1}))$ with $\pi \upharpoonright V_\lambda = \text{id}$.
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The Theorem holds also with any Prikry-generic in $V$ instead of $\vec{\kappa}$, and with a bit of work it is possible to extend it to any generic in $V$ that adds a cofinal $\omega$-sequence.

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Suppose there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$. Let $(M_\omega, j_\omega)$ be the $\omega$-th iterate of $(L(V_{\lambda+1}), j)$. Let $G \in V$ generic for $M_\omega$ such that $(\text{cof}(\lambda) = \omega)^{M_\omega[G]}$. Then there exists $\pi : (L_\omega(V_{\lambda+1}))^{M_\omega[G]} \prec L_\omega(V_{\lambda+1})$ with $\pi \upharpoonright V_{\lambda} = \text{id}$.

Therefore assuming $\text{I}_0(j, \lambda, \kappa)$, if $\mathbb{P}$ is a forcing notion that adds a cofinal $\omega$-sequence to $\kappa$ and such that $j_{0,\omega}(\mathbb{P})$ has a generic in $V$, then $\text{I}_1(\kappa)$ holds in a generic extension of $V$.
Sufficient condition:
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**Definition**

A forcing notion $\mathbb{P}$ is $\lambda$-good iff for any $\mathcal{D}$ family of open dense sets, $|\mathcal{D}| < \lambda$, $\forall p \in \mathbb{P} \exists q \in \mathbb{P} \exists \langle \mathcal{D}_i : i \in \omega \rangle$ such that $\mathcal{D} = \bigcup_{i \in \omega} \mathcal{D}_i$ and $\mathcal{D}_i$ is dense below $q$. 
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Pikry forcing is $\lambda$-good, Gitik-Magidor extender Prikry forcing is $\lambda$-good.
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Pikry forcing is $\lambda$-good, Gitik-Magidor extender Pikry forcing is $\lambda$-good, diagonal supercompact Pikry forcing is $\lambda$-good...
Is there a Prikry-like forcing that is not $\lambda$-good?
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Is it possible to avoid generic absoluteness?
Thanks for your attention