I0, Generic Absoluteness and Combinatorics

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Corollary

There is no $j : V_\eta \prec V_\eta$, with $\eta \geq \lambda + 2$.

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λ is a strong limit cardinal (in fact, Rowbottom).

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- $(V_{\lambda+1})^{V[G]} = V_{\lambda+1}$: this case is trivial, j is still a witness in $V[G]$;
- $\mathbb{P} \in V_{\mathrm{crt}(j)}$: define the extension $k(\tau_G) = j(\tau)_G$.
- $\mathbb{P} \in V_\lambda$: as before, since iterating j we can have $\mathrm{crt}(j) < \lambda$ as large as we want.

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Theorem (Corazza, 2007)

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- a reverse Easton iteration (nontrivial forcing only at limit stages, direct limit at inaccessible stages, inverse limit otherwise)
- λ-bounded (for all δ, $V^{\mathbb{P}_\delta} \vDash |\mathbb{Q}_\delta| \leq \lambda$)
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Note: if j, λ, κ witness I0, then j is iterable and if M_ω is its ω-th iteration, Note: if j, λ, κ witness I0, then j is iterable and if M_ω is its ω-th iteration, then $j_{0,\omega}(\kappa) = \lambda$, and λ is measurable, huge, etc... in M_ω

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Generic Absoluteness Theorem (Woodin, 2012)

Suppose there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$, and let $\kappa_0 = \mathrm{crt}(j) < \lambda$ and $\kappa_{n+1} = j(\kappa_n)$

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An example: j and the theory of $V_{\lambda+1}$ are simple. therefore coded by some structure. In M_ω the sets disappear, but the structure remains. $\vec{\kappa}$ is the key to decrypt the code and reconstruct the sets.

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Therefore assuming $\mathrm{I0}(j, \lambda, \kappa)$, if $\mathbb{P}$ is a forcing notion that adds a cofinal ω-sequence to κ and such that $j_{0,\omega}(\mathbb{P})$ has a generic in V, then $\mathrm{I1}(\kappa)$ holds in a generic extension of V.

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Pikry forcing is λ-good, Gitik-Magidor extender Prikry forcing is λ-good

Sufficient condition:

Definition

A forcing notion $\mathbb{P}$ is λ-*good* iff for any $\mathcal{D}$ family of open dense sets, $|\mathcal{D}| < \lambda$, $\forall p \in \mathbb{P}\ \exists q \in \mathbb{P}\ \exists \langle \mathcal{D}_i : i \in \omega \rangle$ such that $\mathcal{D} = \bigcup_{i \in \omega} \mathcal{D}_i$ and $\mathcal{D}_i$ is dense below q, i.e., $\forall r \leq q\ \exists r^* \leq r$ such that $\mathcal{F}_{r^*}$ is $\mathcal{D}_i$-generic.

Pikry forcing is λ-good, Gitik-Magidor extender Prikry forcing is λ-good, diagonal supercompact Prikry forcing is λ-good...

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Is it possible to avoid generic absoluteness?

Thanks for your attention