Descriptive set theory and large cardinals

1. Vorlesung

The aim of these lectures is to prove some results about the connection between Descriptive Set Theory and Large Cardinals, more specifically about how the Axiom of Determinacy implies the consistency of large cardinals. The structure of the lectures is narrative: there is the main plot, that goes straight to the theorems that we want to proof, and there are going to be flashbacks, i.e., introductions to other topics (for example analytic and co-analytic sets, inner models...) that are going to be useful for the main plot but that can be also read by themselves.

1 AD implies large cardinals

First a quick notation. In the following, $(a_0, \dots)$ will denote a sequence (finite or infinite), while $\langle a_0, \dots \rangle$ will denote the recursive coding, so for example if $m, n \in \omega$ then $(m, n) \in \omega^2$ and $\langle m, n \rangle \in \omega$. When it is not obvious, we write at the bottom of the parenthesis what kind of object is coded (for example $\langle s, t \rangle_\omega \in \omega$).

Let $X \subseteq \omega^\omega$. We associate with X the following game, G_X:

$$\begin{array}{lccccc} \mathrm{I} & a_0 & & a_2 & & \\ & & & & & \dots \\ \mathrm{II} & & a_1 & & a_3 & \end{array}$$

The idea is that in this game there are two players, Player I and Player II, and they take turns in playing elements of ω, so I plays $a_0 \in \omega$, II plays $a_1 \in \omega$ and so on. We say that I wins iff $(a_0, a_1, \dots) \in X$.

Note that, provided a suitable codification in natural numbers, every game in "real life" that does not use randomness is such a game. Typical example: chess. One can label every legal dispositions of pieces with a number, and then X is the set of all the successions of moves that end with the victory of white, or where black makes an illegal move.

A *strategy* for I is a map $\varphi : \omega^{<\omega} \to \omega$. Its interpretation is that it suggests to Player I what to play, given a partial run of II, so in the example above $\varphi(\emptyset) = a_0$, $\varphi(a_1) = a_2$, $\varphi(a_1, a_3) = a_4$ and so on. Similarly one can define a strategy for II. We can already note that strategies can be coded as elements of ω^ω: the code for φ is $\varphi'(\langle n, \langle a_1, \dots, a_n \rangle \rangle) = \varphi(a_1, \dots, a_n)$, so can always think of strategies as reals (this is to be remembered, as in the future we are going to do a lot of complexity calculations).

A strategy for I is *winning* iff for every run of the game where I follows the strategy, I wins. We denote a run of such play as $\varphi * x$, where φ is the strategy for I and $x \in \omega^\omega$ is $(a_1, a_3, \dots)$. In other words, φ is winning for I iff $\forall x \in \omega^\omega\ \varphi * x \in X$. Similarly for a winning strategy for II. A set X is called *determined* if one of the two players has a winning strategy. Note that it is not possible for both to have a winning strategy.

We are going to ask for some flexibility in the definition of a game, permitting the introduction of rules. For example, we can ask for all the moves for I to be even, or for I to win just for $(a_0, a_2, \dots)$ to be in X, etc. The idea is that is always possible to reduce these games to game above, with the definition of an appropriate X. We are going to take this for granted for the rest of the lectures.

Proposition 1.1 (ZFC)**.** *There is a set that is not determined.*

Proof. Let $\langle \sigma_\alpha : \alpha < 2^{\aleph_0} \rangle$ be an enumeration of the strategies for I, and let $\langle \tau_\alpha : \alpha < 2^{\aleph_0} \rangle$ be an enumeration of the strategies for II. We build by induction $X = \{x_\alpha : \alpha < 2^{\aleph_0}\}$ and $Y = \{y_\alpha : \alpha < 2^{\aleph_0}\}$: Suppose we have already defined $\{x_\xi : \xi < \alpha\}$ and $\{y_\xi : \xi < \alpha\}$. Then $|\{x_\xi : \xi < \alpha\}| < 2^{\aleph_0}$ and $|\{\sigma_\alpha * b : b \in \omega^\omega\}| = 2^{\aleph_0}$, so there exists y_α that is in the second set and not in the first, and viceversa for x_α.

Consider the game G_X. Now, X and Y are disjoint, and for any $\alpha < 2^{\aleph_0}$ there exists $b \in \omega^\omega$ such that $\sigma_\alpha * b = y_\alpha \in Y$, so for any $\alpha < 2^{\aleph_0}$ $\sigma_\alpha * b \notin X$, and therefore no strategy is winning for I. Viceversa, for all $\alpha < 2^{\aleph_0}$ there exists $a \in \omega^\omega$ such that $a * \tau_\alpha = x_\alpha \in X$, and therefore no strategy is winning for I. $\square$

This proof makes heavy use of the "non-costructive" Axiom of Choice, and therefore it provides a "non-definable" non-determined set. On the other hand, "simply definable" sets are determined.

Theorem 1.2 (ZFC, Gale-Stewart)**.** *Every closed set is determined.*

Proof. Let X be a closed set, we play the game G_X, and assume that II has no winning strategy.

Giving a position $p = (a_0, a_1, \dots, a_{2n-1})$ that ends with a move from II, we say that p is not losing for I if II has no winning strategy from then on. So $\emptyset$ is not losing for I. So if p is not losing for I, there is a a_{2n} such that for any a_{2n+1} the position $p \frown (a_{2n}, a_{2n+1})$ is not losing for I. With this we build a strategy for I: every time the strategy suggests a move that will guarantee I to pass from a not losing position to another.

We claim that the strategy is winning. Let $(a_0, a_1, \dots)$ be a run of the game where I follows the strategy. If it is not in X, then it is in $\omega^\omega \setminus X$, that is open in ω^ω, so there is a $k \in \omega$ such that $N_{(a_0,\dots,a_{2k-1})} \subseteq \omega^\omega \setminus X$. But then $(a_0, \dots, a_{2k-1})$ is a losing position for I, contradiction. $\square$

Most of the "real life" games, if not all, are closed, because they are are of the kind: if the game reaches this state then the game is over and I or II win. This means that the payoff set is the set of all the runs that start with a winning state, for which there are finite ones, so the payoff is a finite union of clopen sets, so it is clopen.

It is easy to see that if X is determined, then $\omega^\omega \setminus X$ is determined.

Theorem 1.3 (Martin)**.** *Every Borel set is determined.*

We are now going to introduce another game. the *Perfect Set game*, to show that determined games have "nice" properties.

Definition 1.4. *We say that a set $X \subseteq \omega^\omega$ contains a perfect set iff there exists a continuous embedding from 2^ω to X.*

An embedding is an injective map that is continuous whose inverse is also continuous. Note that 2^ω is compact, therefore every injective map whose inverse is continuous is an embedding.

Note that if X contains a perfect set, then $|X| = 2^{\aleph_0}$. So if the continuum hypothesis does not hold, then the set that goes against it cannot contain a perfect set. On the other hand, if every set is countable or contains a perfect set, then the continuum hypothesis holds.

Given $X \subseteq \omega^\omega$, consider the following game:

I	(s_0^0, s_0^1)		(s_1^0, s_1^1)		
				$\dots$	
II		i_0		i_1	

where $s_n^i \in \omega^{<\omega}$, $s_n^i \neq \emptyset$, $N_{s_n^0} \cap N_{s_n^1} = \emptyset$ and $i_n \in \{0,1\}$. Given a run of this game, consider $x = s_0^{i_0} \frown s_1^{i_1} \frown \dots$. Then I wins iff $x \in X$.

To understand better this game, $\omega^{<\omega}$ should be seen as a tree: the root of the tree is the empty sequence, that then branch into ω-th different successors (one for any 1-sequence), and each one of them branch into other ω-th successors (one for any 2-sequence that start with a fixed number), etc. Branches of this tree are then reals, elements of ω^ω. The game asks I to propose two finite branches of this tree, and then II choses one of the two. I then propose two different extensions of the branch chosen by II, and II once again choses one of the two extensions. In the end an infinite branch will appear, and I wins if it is in X.

It is clear that if X contains a perfect set, then I has a winning strategy in this game: let $f : 2^{\aleph_0} \to X$ be a continuous embedding. Let $s \in 2^{<\omega}$. As f is continuous, there is a $t \in \omega^{<\omega}$ such that $f'' N_s \subseteq N_t$. Let $F(s) = t$, with t minimal. Then the strategy is as this: I as first move plays $(F(0), F(1))$. Suppose that II plays 0. Then $F(00), F(01) \sqsupseteq F(0)$, and let I play (t_1^0, t_1^1)

such that $F(0) \frown t_1^0 = F(00)$ and $F(0) \frown t_1^1 = F(01)$. And so on. The moves on 1 then never leave X, and the final run will be in X. On the other hand if I has a winning strategy σ, this clearly induces a continuous embedding from $2^{\aleph_0}$ to X: just let $f(i_0, \dots, i_n, \dots) = \sigma(i_0) \frown \sigma(i_0, i_1) \frown \dots$. The initial segments of the image depend only on the initial segments of the element in the domain, therefore f is continuous, and since σ is winning $f[2^{\aleph_0}] \subseteq X$.

In other words, X contains a perfect set iff I has a winning strategy in the game.

Theorem 1.5. *Every uncountable determined set contains a perfect set.*

Proof. It remains to prove that if there is a winning strategy for II, then X is countable. Let σ be a winning strategy for II. Given $x \in X$, call a position p of length $2n$ good for x if it has been played according to σ and $x \in N_{s_0^{i_0} \frown \dots \frown s_{n-1}^{i_{n-1}}}$. By convention, $\emptyset$ is always good for x. If every good position for x had a proper extension that is good for x, then there would be a run that follows σ but that produces x, that is in X, and therefore I would win, contradiction. So for any $x \in X$ there is a p_x that is maximal and good for x. Let $x \in X$, define A_{p_x} as the set of y such that $p_x = p_y$. Thus $X \subseteq \bigcup_{x \in X} A_{p_x}$. Now, suppose that $x \neq y \in A_{p_x}$. Then I can play (s_n^0, s_n^1) such that $s_0^{i_0} \frown \dots \frown s_{n-1}^{i_{n-1}} \frown s_n^0 \sqsubseteq x$ and $s_0^{i_0} \frown \dots \frown s_{n-1}^{i_{n-1}} \frown s_n^1 \sqsubseteq y$, and II is forced to decide one of the two, contradicting the fact that p_x was maximal for both, so the strategy would make II play something out of x and y. So $A_{p_x} = \{x\}$ and $|X| \leq |\{\text{positions}\}| = \aleph_0$. $\square$

We are now introducing the main axiom of this series of lectures:

Definition 1.6. *Axiom of Determinacy: (AD) Every set is determined.*

By Proposition **??** AD cannot be added to ZFC, as it will produce an inconsistency. But we can consider it in addition to ZF: without the Axiom of Choice, there is less possibility to create "complicated" and "non-constructive" sets, therefore it becomes a feasible axiom, that can bring order to the universe:

Theorem 1.7 (ZF+AD)**.** *All sets are Lebesgue Measurable, have the Baire property and contain a perfect set.*

Therefore this theory is as close as possible to solve the continuum hypothesis: the only obstacle is that AD is incompatible with a well-ordering of the reals, therefore $2^{\aleph_0}$ is not a cardinal number. Also, some amount of choice remains:

Definition 1.8 (Weak forms of choice). *DC:* $\forall X\ \forall R \subseteq X \times X\ \exists f : \omega \to X\ \forall n \in \omega\ f(n) R f(n+1)$.

AC_ω*: For any* $\langle A_n : n \in \omega \rangle$ *such that* $A_n \neq \emptyset$ *there exists* $f : \omega \to \omega^\omega$ *such that* $f(n) \in A_n$.

DC$(\mathbb{R})$*:* $\forall X \subseteq \omega^\omega\ \forall R \subseteq X \times X\ \exists f : \omega \to X\ \forall n \in \omega\ f(n) R f(n+1)$.

$AC_\omega(\mathbb{R})$*: For any* $\langle A_n : n \in \omega \rangle$ *such that* $A_n \subseteq \omega^\omega \setminus \{\emptyset\}$ *there exists* $f : \omega \to \omega^\omega$ *such that* $f(n) \in A_n$.

The first one is called Principle of Dependent Choice. DC implies AC_ω and DC($\mathbb{R}$), and of course AC_ω implies $\mathrm{AC}_\omega(\mathbb{R})$.

Theorem 1.9 (ZF+AD). $AC_\omega(\mathbb{R})$ *holds.*

Proof. Let $\langle A_n : n \in \omega \rangle$ be such that $A_n \subseteq \omega^\omega$, all non-empty. Consider the game

$$\begin{array}{ccccccc} \mathrm{I} & x_0 & & x_1 & & & \\ & & & & & \dots & \\ \mathrm{II} & & y_0 & & y_1 & & \end{array}$$

where I wins iff $y = (y_0, y_1, \dots) \notin A_{x_0}$. Of course, as the A_n's are not empty, Player I cannot win, so II has a winning strategy. Let τ be it. Then the function $f(n) = (n, 0, 0, \dots) * \tau$ is a choice function for $\langle A_n : n \in \omega \rangle$. $\square$

One can ask if ZF+AD implies stronger forms of choice. It turns out that DC is independent from AD:

Theorem 1.10 (Kechris). *Con(ZF+AD) implies Con(ZF+AD+DC)*

Theorem 1.11 (Woodin). *Con(ZF+AD) implies Con(ZF+AD+*$\neg AC_\omega$*)*

It is still an open question whether AD implies a local form of DC with subsets of reals.

The fact that a basic form of choice is compatible with AD suggests us that some basic results of ZFC are going to hold also on ZF+AD. For example: is ω_1 regular? In ZFC the proof is like this: Let $\langle \alpha_n : n < \omega \rangle$ such that $\alpha_n < \omega_1$ for any $n < \omega$. Use AC to fix for any $n < \omega$ a $f_n : \omega \to \alpha_n$ bijection, and then define $F : \omega \times \omega \to \omega_1$ as $F(n, m) = f_n(m)$. It is a surjection, contradiction. Here choice is used on a countable sequence of "small" sets. If only there could be a way to code countable ordinals as subsets of reals, maybe we could use $\mathrm{AC}_\omega(\mathbb{R})$ instead...

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2 First flashback: Analytic and Co-analytic sets

We remind the "classic" topology on ω^ω: A set $A \subseteq \omega^\omega$ is *clopen* iff there exists $s \in \omega^{<\omega}$ such that $A = N_s = \{x \in \omega^\omega : s \sqsubseteq x\}$. Then open sets are all the arbitrary unions of clopen sets, and closed sets are complementaries of open sets. The Borel sets are those sets in the σ-algebra generated by the open sets. In current times Descriptive Set Theory has taken two pretty distinct directions: on one hand the study of Borel sets, therefore of "simply definable" sets, very close to other branch of mathematics, and many times worried about the exact complexity of sets and problems. On the other hand the study of projective sets, therefore of "complexely definable" sets, very connected to the Axiom of Determinacy and inner model theory. We are going to focus on this second direction, but without going too far on the complexity, so we are dealing mostly with analytic and co-analytic sets.

Definition 2.1. *A set $X \subseteq \omega^\omega$ is Σ^1_1 if there exists a recursive set $R \subseteq \bigcup_{n\in\omega}(\omega^n \times \omega^n)$ (or, equivalently, defined in V_ω) such that for all $x \in \omega^\omega$ $x \in X$ iff $\exists y \in \omega^\omega \forall n \in \omega R(x \restriction n, y \restriction n)$.*

For any $a \in \omega^\omega$, a set $X \subseteq \omega^\omega$ is $\Sigma^1_1(a)$ if there exists a set $R \subseteq \bigcup_{n\in\omega}(\omega^n \times \omega^n)$ recursive in a (or, equivalently, defined in (V_ω, a)) such that for all $x \in \omega^\omega$ $x \in X$ iff $\exists y \in \omega^\omega \forall n \in \omega R(x \restriction n, y \restriction n)$.

For any $a \in \omega^\omega$, a set $X \subseteq \omega^\omega$ is $\Pi^1_1(a)$ if it is the complement of a $\Sigma^1_1(a)$ set.

A set $X \subseteq \omega^\omega$ is analytic*, or $\boldsymbol{\Sigma}^1_1$, if it is $\Sigma^1_1(a)$ for some $a \in \omega^\omega$. It is* co-analytic*, or $\boldsymbol{\Pi}^1_1$, if it is $\Pi^1_1(a)$ for some $a \in \omega^\omega$.*

So Σ^1_1 sets are those that are projections of recursive sets, and Π^1_1 are their complement. Also, $\boldsymbol{\Sigma}^1_1$ are those that are projections of closed sets, and $\boldsymbol{\Pi}^1_1$ are their complement. This is the start of another hierarchy, the *projective hierarchy*, where at each step we alternately take projections and complements. This hierarchy is above the Borel hierarchy:

Theorem 2.2. *Borel sets are exactly those that are analytic and co-analytic.*

Lemma 2.3. *If A, B are $\Sigma^1_1(a)$, then so are $\exists x A$, $A \wedge B$, $A \vee B$, $\exists m A$, $\forall m A$.*

If A, B are $\Pi^1_1(a)$, then so are $\forall x A$, $A \wedge B$, $A \vee B$, $\exists m A$, $\forall m A$.

Proof. The second part is immediate from the first. The main point is that it is possible to define a recursive coupling $\langle \cdot, \cdot \rangle : \omega^\omega \times \omega^\omega \to \omega^\omega$ and also a recursive coupling $\langle \cdot, \cdot \rangle : \omega \times \omega^\omega \to \omega^\omega$, and this will permit to contract the quantifiers in the definitions. We see the first case, and leave the other for exercise.

So let A be $\Sigma^1_1(a)$, defined via a relation R recursive in a. So we have that $(x, y) \in A$ iff $\exists z \forall n (x \restriction n, y \restriction n, z \restriction n) \in R$. So $y \in \exists A$ iff $\exists x \exists z \forall n (x \restriction n, y \restriction n, z \restriction n) \in R$. Define $(b)_0 = (\langle c, d \rangle)_0 = c$ and $(b)_1 = (\langle c, d \rangle)_1 = d$. Then $y \in \exists A$ iff $\exists u \forall n ((u)_0 \restriction n, y \restriction n, (u)_1 \restriction n) \in R$. Now define R' as $(s, t) \in R'$ iff $((s)_0, t, (s)_1) \in R$. Then R' is recursive, and it is a viable witness for the analiticity of $\exists A$.

The case for conjuctions and disjunctions uses the same coupling trick, and the fact that conjunctions and disjunctions of recursive relations is recursive. The case $\exists m \exists x$ is still done using the recursive coupling trick. □

There is a handy equivalent definition for co-analytic sets:

Theorem 2.4. *Every $\Sigma^1_1(a)$ a set is the projection of $[T]$, where T is a tree recursive in a. Therefore a set C is $\Pi^1_1(a)$ iff there exists a tree $T \subseteq (\omega \times \omega)^{<\omega}$ recursive in a such that for any $x \in \omega^\omega$, $x \in C$ iff $T(x) = \{s \in \omega^{<\omega} : \exists n \in \omega (x \restriction n, s) \in T\}$ is well-founded.*

Proof. We clarify the definitions of the objects above. A tree on $\omega \times \omega$ is a subset of $(\omega \times \omega)^{<\omega}$ that is closed under initial segments. We say that a tree is well-founded if it does not have infinite branches.

Now, $\omega^\omega \setminus C$ is $\Sigma^1_1(a)$, so let R be the relation recursive in a that defines $\omega^\omega \setminus C$. Define $T = \{(s, t) : \exists x, y \in \omega^\omega \ s \sqsubseteq x, t \sqsubseteq y, \ \forall n \in \omega \ R(x \restriction n, y \restriction n)\}$. It is recursive in a. Now, $\omega^\omega \setminus C = \{x : \exists y (x, y) \in [T]\}$, and if $(x, y) \in [T]$ then y is a branch of $[T(x)]$. So $x \in \omega^\omega \setminus C$ iff $T(x)$ is a branch iff $T(x)$ is not well-founded. □

Lemma 2.5. *The counter-image by a continuous function of a $\boldsymbol{\Sigma}^1_1$ set is a $\boldsymbol{\Sigma}^1_1$ set.*

Proof. Let $f : \omega^\omega \to \omega^\omega$ a continuous function. Then for any $s \in \omega^{<\omega}$ there is a $t \in \omega^{<\omega}$ such that $f[N_t] \subseteq N_s$. Let $F(s)$ be such a t. Note that if $s_1 \sqsupseteq s_2$ then $F(s_1) \sqsupseteq F(s_2)$, and if for all $n \in \omega$, $F(s_n) = z \restriction n$, then $f^{-1}(\bigcup_{n \in \omega} s_n) = z$.

Let A be an analytic set, the projection of $[T]$, where T is a tree on $\omega \times \omega$. So $A = \{x : \exists y \ \forall n \ (x \restriction n, y \restriction n) \in T\}$. Consider now $F[T] = \{(F(s), F(t)) : (s, t) \in T\}$. Then $\{z : \exists y \ \forall n \ (z \restriction n, y \restriction n) \in F[T]\} = \{z : \exists y \ \forall n \ \exists (s, t) \in T \ F(s) = z \restriction n \ F(t) = y \restriction n\} = \{f^{-1}(x) : \exists y \ \forall n \ (x \restriction n, y \restriction n) \in T\} = f^{-1}[A]$, so $f^{-1}[A]$ is analytic. □

Lemma 2.6. *There is an analytic set that is not co-analytic. And therefore there is a co-analytic set that is not analytic.*

Proof. Let U be the tree on $\omega \times \omega \times \omega$ defined as $(s,t,u) \in U$ iff $\mathrm{lh}(s) = \mathrm{lh}(t) = \mathrm{lh}(u)$, $\{(s',t') \in \omega^{<\omega} \times \omega^{<\omega} : u(\langle s',t' \rangle_\omega) = 0\}$ is a tree and for any $i < \omega$ such that $\langle s \restriction i, t \restriction i \rangle_\omega < \mathrm{lh}(u)$, then $u(\langle s \restriction i, t \restriction i \rangle_\omega) = 0$.

We claim that for all $A \subseteq \omega^\omega$, A is $\mathbf{\Sigma}^1_1$ iff there is a $z \in \omega^\omega$ such that A is the projection of $[U(z)]$, where z is the third coordinate. Let A be $\mathbf{\Sigma^1_1}$. Then there is a T tree on $\omega \times \omega$ such that A is the projection of $[T]$. Let $z \in \omega^\omega$, where $z(n) = 0$ iff there is $(s,t) \in T$ such that $\langle s,t \rangle_\omega = n$. Then $[U(z)] = \{(x,y) : \forall n\ (x \restriction n, y \restriction n, z \restriction n) \in U\} = \{(x,y) : \forall n \in \omega\ \forall i < n\ z(\langle x \restriction i, y \restriction i \rangle_\omega) = 0\} = \{(x,y) : \forall n \in \omega \forall i < n\ (x \restriction i, y \restriction i) \in T\} = [T]$, so A is the projection of $[U(z)]$. The other direction is trivial.

Let $B = \{(x,z) : \exists y\ (x,y,z) \in [U]\}$. Then B is $\mathbf{\Sigma}^1_1$. If B were a $\mathbf{\Pi}^1_1$, then $\{(x,z) : (x,z) \notin B\}$ would be $\mathbf{\Sigma}^1_1$, and therefore $A = \{x : (x,x) \notin B\}$ would be $\mathbf{\Sigma}^1_1$. Then let $z \in \omega^\omega$ be such that A is the projection of $[U(z)]$. Then $A = \{x : (x,x) \notin B\} = \{x : \exists y\ (x,y,x) \in [U]\} = \{x : \exists y\ (x,y,z) \in [U]\} = \{x : (x,z) \in B\}$. But then $(z,z) \notin B$ iff $(z,z) \in B$, contradiction. $\square$

We are now trying to code countable ordinals as (definable) sets of reals. Let $\langle \cdot, \cdot \rangle : \omega \times \omega \to \omega$ be a recursive pairing function. For any $x \in \omega^\omega$, consider the binary relation E_x on ω defined as $m\ E_x\ n$ iff $x(\langle m,n \rangle) = 0$. We define

$$WF = \{x \in \omega^\omega : x \text{ codes a well-founded relation}\}$$
$$WO = \{x \in \omega^\omega : x \text{ codes a well-ordering of } \omega\}.$$

Lemma 2.7. *The sets WF and WO are* Π^1_1.

Proof. For any $x \in \omega^\omega$, E_x is well-founded iff there is no $z : \omega \to \omega$ such that $z(k+1) E_x z(k)$ for any $k \in \omega$. So $x \in WF$ iff $\forall z \in \omega^\omega\ \exists k \in \omega\ \neg(z(k+1) E_x z(k))$, that is $\neg \exists z \in \omega^\omega\ \forall k \in \omega\ x(\langle z(k+1), z(k) \rangle) = 0$. But $\forall k \in \omega\ x(\langle z(k+1), z(k) \rangle) = 0$ iff $\forall n, m, j, k\ (i = (z \restriction n)(k+1) \vee j = (z \restriction n)(k) \vee m = \langle i,j \rangle \vee (x \restriction n)(m) = 0)$. Therefore WF is Π^1_1.

Now, "E_x is a linear order" iff $\forall m, n\ (m E_x n \vee n E_x m)$, so $\forall m, n\ (x(\langle m,n \rangle) = 0 \vee x(\langle n,m \rangle) = 0)$. This is almost trivially Π^1_1, therefore WO is Π^1_1. $\square$

For any $x \in WO$, we define $\|x\|$ as the order-type of the well-order E_x. Then for any $x \in WO$, $\|x\|$ is a countable ordinal, and for any α countable ordinal there is a $x \in WO$ such that $\|x\| = \alpha$. Define also $WO_\alpha = \{x \in WO : \|x\| = \alpha\}$ and $WO_{\leq \alpha} = \{x \in WO : \|x\| \leq \alpha\}$.

Lemma 2.8. *For any $\alpha < \omega_1$, $WO_{\leq \alpha}$ is Borel.*

Proof. For any $\alpha < \omega_1$, let $B_\alpha = \{(x,n) : E_x \text{ restricted to } \{m : m E_x n\} \text{ is a well-ordering of type} \leq \alpha\}$. We prove by induction that B_α is Borel. It is easy to see that B_0 is Borel. Let $\alpha < \omega_1$ and suppose that for any $\beta < \alpha$, B_β is Borel. Then $\bigcup_{\beta<\alpha} B_\beta$ is Borel, and $(x, n) \in B_\alpha$ iff $\forall m(mE_x n \to (x, m) \in \bigcup_{\beta<\alpha} B_\beta)$. Therefore B_α is Borel. Now, $x \in WO_{\leq\alpha}$ iff $\forall n \in \omega(x, n) \in \bigcup_{\beta<\alpha} B_\beta$, so also $WO_{\leq\alpha}$ is Borel. $\square$

Theorem 2.9. *If c is a $\mathbf{\Pi}^1_1$ set, then there exists a continuous function $f : \omega^\omega \to \omega^\omega$ such that $C = f^{-1}(WO)$. In other words, WO is $\mathbf{\Pi}^1_1$-universal.*

Proof. Let T be a tree such that $x \in C$ iff $T(x)$ is well-founded. We write in more detail what this means: in $\omega^{<\omega}$ there is a natural order $\sqsubseteq$, "is an initial segment of". Then to say that $T(x)$ is well-founded means that $\sqsupseteq$ is well-founded on $T(x)$. We can also define a linear order on $\omega^{<\omega}$, $\sqsubseteq^*$, that is the lexicographical order. Then $(T(x), \sqsupseteq)$ is well-founded iff $(T(x), \sqsupseteq^*)$ is well-ordered.

Let $\{t_0, t_1, \dots\}$ be an enumeration of $\omega^{<\omega}$. For each $x \in \omega^\omega$, we define $f(x) = y$ such that $y(\langle m, n\rangle) = 0$ iff $t_m, t_n \in T(x)$ and $t_m \sqsupseteq^* t_n$. Then $m\ E_{f(x)}\ n$ iff $t_m, t_n \in T(x)$ and $t_m \sqsupseteq^* t_n$, therefore $f(x) \in WO$ iff $E_{f(x)}$ is a well-ordering iff $(T(x), \sqsupseteq^*)$ is a well-ordering iff $(T(x), \sqsupseteq)$ is well-founded iff $x \in C$. Thus $C = f^{-1}[WO]$.

It remains to see that f is continuous, but this is obvious: $f(x)$ is the code of the relation $(T(x), \sqsupseteq^*)$, and if $s \sqsubseteq x$, then $T(s) \subseteq T(x)$, so $f(x)$ is fixed in finite coordinates. $\square$

Then WO cannot be $\mathbf{\Sigma}^1_1$, because the inverse image by a continuous function of an analytic set is analytic, but then that would mean that all co-analytic sets are analytic, contradiction.

Lemma 2.10 (Boundedness Lemma)**.** *If $B \subseteq WO$ is $\mathbf{\Sigma}^1_1$, then there is an $\alpha < \omega_1$ such that $B \subseteq WO_{\leq\alpha}$.*

Proof. Suppose not. Then $WO = \{x \in \omega^\omega : \exists z\ z \in B \wedge \|x\| \leq \|z\|\}$. But consider $A = \{(x, z) : z \notin WO \vee \|x\| \leq \|z\|\}$: then $(x, z) \in A$ iff $z \notin WO \vee (\exists h : \omega \to \omega \forall m \forall n\ mE_x n \to h(m)E_x h(n))$, therefore A is Σ^1_1. But then $WO = \{x \in \omega^\omega : \exists z\ z \in B \wedge (x, z) \in A\}$ is Σ^1_1, contradiction. $\square$

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3 Back at the main plot

Therefore under ZF+AD ω_1 is still regular: let $(\alpha_n : n < \omega)$ a countable set of countable ordinals, and consider $(WO_{\alpha_n} : n < \omega)$. Let f be a choice function for this sequence. Then consider the relation on ω $\langle n_1, m_1\rangle R \langle n_2, m_2\rangle$

iff $n_1 < n_2$ or $n_1 = n_2$ and $m_1 E_{f(n_1)} m_2$. Such relation is a well-order on ω of order-type ω_1, contradiction.

The theory of ZF+AD is therefore very elegant, without pathological sets and with a minimum of choice so not to render its models too bizarre. Can we say something about its consistency? If only there was a way to measure in an objective way the strength of the consistency of a theory...

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4 Second flashback: Large Cardinals

In 1908 Hausdorff asked: can there be a limit cardinal that is also regular? That would mean that κ is a limit cardinal, and yet there is no $<\kappa$-sequence that is cofinal in it (so, for example, κ is a limit of κ cardinals). ω surely satisfies this property, as every finite sequence has a finite supremum, but can there be an uncountable one? Seems an innocuous property, but with a moment of thinking one realizes that this cardinal must be very big: as it is a limit of κ cardinals, it must be that $\kappa = \aleph_\kappa$, so there is the same "number" of ordinals and cardinals below it. Confront it with $\aleph_\omega$, that is very large, and yet has only $\aleph_0$ cardinals below it. The smallest cardinal with this property is $\aleph_{\aleph_{\aleph_{\dots}}}$, but this has cofinality ω. In fact, such a cardinal should have the same "number" of *regular* cardinal and ordinals below it. Because of its size, this cardinal is called *weakly inaccessible.* We can ask even more closure to a cardinal:

Definition 4.1. *A cardinal κ is a* strong limit *cardinal iff for any $\delta, \gamma < \kappa$, $\delta^\gamma < \kappa$.*

A cardinal κ is inaccessible *iff it is an uncountable regular strong limit cardinal.*

Note that this definition does not make sense without AC.

The name inaccessible is very evocative: given any structure of cardinals less than κ, it is just impossible to "reach" κ. Again, ω has this property: a finite sequence of finite numbers has a finite maximum and the sum, product and exponentiation of finite numbers is finite. So, in a certain sense, an inaccessible cardinal is a watershed for a stronger kind of infinity: if there is an inaccessible cardinal, then the sets can be divided in finite, small infinite and large infinite, and each step is inaccessible to the ones below. The existence of more inaccessible cardinals will yield more of these steps. The idea that the universe below an inaccessible cardinal is a world of its own is reinforced by the following theorem:

Theorem 4.2. *If κ is inaccessible, then $(V_\kappa, \in) \vDash ZFC$.*

Proof. Extensionality is obvious, Pairing, Separation, Union and Power Set hold because κ is a limit ordinal, Infinity holds because κ is uncountable, it remains Replacement. Let $x \in V_\kappa$. By induction, for any $\alpha < \kappa$, $|V_\alpha| < \kappa$: $|V_{\alpha+1}| = |\mathcal{P}(V_\alpha)| = 2^{|V_\alpha|} < \kappa$ by strong limitness and for $\gamma < \kappa$ limit $|V_\gamma| = \sup_{\alpha<\gamma} |V_\alpha| < \kappa$ by regularity. Therefore $|x| < \kappa$. Let $F : x \to V_\kappa$. Then $|F''x| \leq |x| < \kappa$. Since κ is regular, $\{rnk(y) : y \in F''x\}$ cannot be cofinal in V_κ, there is a α such that $F''x \in V_{\alpha+1}$. $\square$

Therefore ZFC+exists an inaccessible cardinal proves the existence of a model of ZFC, so it proves Con(ZFC). By the second theorem of incompleteness, this means that ZFC cannot prove the existence of an inaccessible cardinal, so the theory "ZFC+exists an inaccessible cardinal" is *strictly stronger* than ZFC. So if in a theory one can prove the existence, or the consistency, of an inaccessible cardinal, also this theory will be strictly stronger than ZFC. We have a way to "measure" the strength of a theory.

Inaccessible cardinals are, historically, the first notion of large cardinal. In the last century many new and stronger notions of large cardinals were introduced (and the quest is continuing daily), expanding the possibilities of measuring the consistency strength of a theory. A curious pattern appeared: even if such notions came from wildly diverse settings, they were actually all implying each other (at least their consistency), therefore forming a linear order (!) of consistency strengths, and becoming the standard tool for measuring the consistency strength of any theory stronger than ZFC.

There is no real definition for "large cardinal", it is an umbrella term for different things. The most classical form is as a property of a cardinal (a cardinal is A iff ...), but sometimes they are axioms that imply the existence of other kind of objects. Myself, I like to divide large cardinals in four different groups: from inaccessible to $0^\sharp$ (excluded), from $0^\sharp$ to measurable (included), from measurable to supercompact, and from supercompact on. These groups have some characteristics in common:

"Height" cardinals. Consider V with an inaccessible cardinal κ. Then we have already seen that V_κ is a model of ZFC. So in a certain sense we can consider V as a "vertical extension" of a model of ZFC. In the same way, a model with two inaccessible cardinals can be seen as a vertical extension of a model with one inaccessible cardinal, and so on. Large cardinals up to $0^\sharp$ (excluded) have often this characteristic: they seem to measure the "height" of a model, the more there are and the stronger they are, the "higher" is the model.

How "high", then, are this cardinals? We have already seen that if κ is inaccessible, then under κ there is the same amount of ordinals and regular cardinals. A cardinal κ is *Mahlo* iff it is inaccessible and the regular cardinals below κ form a stationary subset of κ. A cardinal κ is *1-Mahlo* iff the set of Mahlo cardinals below κ form a stationary subset of of κ, and so under a 1-Mahlo there is the same amount of ordinals and of Mahlo cardinals. One can go on, up to κ-Mahlo, and then define cardinals that are even larger. This is somewhat typical of large cardinals: many times the set of weaker cardinals under a large cardinal is "large". The cardinals of the first group are of this kind.

Reflection cardinals: The large cardinals in the first two groups can often be expressed via reflection principles. The Reflection Principle as it is usually called is actually the first order reflection principle. It says that for any formula $\varphi(v_1, \dots v_n)$ and any β, there is a limit ordinal $\alpha > \beta$ such that for any $x_1, \dots x_n \in V_\alpha$, $V \vDash \varphi(x_1, \dots, x_n)$ iff $V_\alpha \vDash \varphi(x_1, \dots, x_n)$. This is a theorem in ZFC, and actually equivalent to Infinity+Replacement.

Proposition 4.3. *κ is inaccessible iff for any $R \subseteq V_\kappa$ there is an $\alpha < \kappa$ such that $(V_\alpha, \in, R \cap V_\alpha) \prec (V_\kappa, \in, R)$.*

κ is Mahlo iff for any $R \subseteq V_\kappa$ there is an inaccessible $\alpha < \kappa$ such that $(V_\alpha, \in, R \cap V_\alpha) \prec (V_\kappa, \in, R)$.

Second-order reflection principles generate large cardinals that belong in the first and second group. Third-order reflection principles are inconsistent.

Inner model cardinals. These belong to the second and third group, and are those that are important in inner model theory. Ideally, those are all the cardinals for which there is a canonical inner model, that is a definable and constructible inner model that contains them. Whether this exists for a supercompact cardinal is one of the most important problems in set theory. Woodin recently proved that if there exists a canonical inner model for a supercompact cardinal, this is a model for all the large cardinals above.

Elementary embeddings cardinals. These belong to the third and fourth group. They are cardinals defined via elementary embeddings.

We are going now to introduce now measurable cardinals, the ones at the border between "small" cardinals and "large" cardinals.

We remind that a filter on a nonempty set X is a collection F of subsets of X such that $X \in F$, $\emptyset \notin F$, if $A \in F$ and $B \in F$, then $A \cap B \in F$, and if $A \in F$ and $A \subseteq B$, then $B \in F$. A filter F is principal iff there exists $A_0 \subseteq X$ such that $F = \{A \subseteq X : A_0 \subseteq A\}$. A filter U on a set X is an *ultrafilter* if for every $A \subseteq X$, either $A \in U$ or $X \setminus A \in U$, and an ultrafilter is a maximal filter. A filter F is *κ-complete* if whenever $\{A_\alpha : \alpha < \kappa\} \subseteq F$, then $\bigcap_{\alpha<\kappa} A_\alpha \in F$.

Definition 4.4. *An uncountable cardinal κ is* measurable *iff there exists a κ-complete nonprincipal ultrafilter U on κ.*

We note this essential remark for κ-complete ultrafilters:

Lemma 4.5. *Let U be a κ-complete filter on κ, let $X \in U$. Then if $(A_\beta : \beta < \gamma)$ is a partition of X, with $\gamma < \kappa$, then there is one and only one β such that $A_\beta \in U$.*

Proof. Suppose that for all $\beta < \gamma$, $A_\beta \notin U$. Then $\kappa \setminus A_\beta \in U$ for all $\beta < \gamma$. But also $\bigcap_{\beta<\gamma}(\kappa \setminus A_\beta) = \kappa \setminus \bigcup_{\beta<\gamma} A_\beta = \kappa \setminus X \in U$, by κ-completeness. But then also $\kappa \setminus X \cap X = \emptyset \in U$, contradiction. $\square$

This is a large cardinal:

Lemma 4.6. *Every measurable cardinal is inaccessible.*

Proof. Let κ be a measurable cardinal and U a κ-complete non principal ultrafilter on it. Suppose that κ is singular, and let $(\alpha_\beta : \beta < \gamma)$ be a cofinal sequence in κ with $\gamma < \kappa$. Let $A_\beta = \alpha_{\beta+1} \setminus \alpha_\beta$. Then $(A_\beta : \beta < \gamma)$ is a partition of κ, so there is one β such that $A_\beta \in U$. But $|A_\beta| < \kappa$, so A_β can be partitioned by $(\{\delta\} : \delta \in A_\beta)$, and therefore there is a single δ such that $\{\delta\} \in U$, contradiction because U is non principal.

Suppose that there is a $\lambda < \kappa$ such that $2^\lambda \geq \kappa$ (note that this includes the case κ successor). Let $S \subseteq {}^\lambda 2$ such that $|S| = \kappa$ and transfer the ultrafilter U on S. For each $\alpha < \lambda$ either $\{f \in S : f(\alpha) = 0\}$ or $\{f \in S : f(\alpha) = 1\}$ is in U. Let ϵ_α be such that $\{f \in S : f(\alpha) = \epsilon_\alpha\} \in U$ and call $X_\alpha = \{f \in S : f(\alpha) = \epsilon_\alpha\}$. Since U is complete, then $X = \bigcap_{\alpha<\lambda} X_\alpha \in U$. Let $f \in X$. Then $f(\alpha) = \epsilon_\alpha$ for any $\alpha < \lambda$, so $X = \{f\}$ and U is principal, contradiction. $\square$

Let's go deeper in what we can prove with a measurable cardinal. For this, we open a parentheses in the parentheses:

5 Third flashback: The Club Filter

We remind that for any κ cardinal such that $\mathrm{cof}(\kappa) > \omega$, a club in κ is a subset of κ that is closed and unbounded, i.e., it contains all its limit points and it is cofinal in κ.

Lemma 5.1. *Let κ be a cardinal such that $\mathrm{cof}(\kappa) > \omega$. If $C, D \subseteq \kappa$ are clubs in κ, then $C \cap D$ is a club in κ.*

Proof. If α is a limit point of $C \cap D$, then it is a limit point both of C and D, and since they are both clubs $\alpha \in C \cap D$. Let $\gamma < \kappa$. Since C is unbounded, there is $\gamma < \alpha_1 \in C$. Since D is unbounded, there is a $\alpha_1 < \beta_1 \in D$. Define by induction a sequence $\gamma < \alpha_1 < \beta_1 < \alpha_2 < \dots$ such that the α's are in

C and the β's are in D. Since $\operatorname{cof}(\kappa) > \omega$, $\sup_{n\in\omega} \alpha_n = \sup_{n\in\omega} \beta_n < \kappa$. By closure of C and D, $\sup_{n\in\omega} \alpha_n \in C$ and $\sup_{n\in\omega} \beta_n \in D$, but they are the same, so that is an element of $C \cap D$ bigger than γ. As γ was any, $C \cap D$ is unbounded. $\square$

Lemma 5.2. *Let κ be a cardinal such that $\operatorname{cof}(\kappa) > \omega$. If $(C_\alpha : \alpha < \gamma)$ is such that $\gamma < \operatorname{cof}(\kappa)$ and for any $\alpha < \gamma$ C_α is a club in κ, then $\bigcap_{\alpha<\gamma} C_\alpha$ is a club in κ.*

Proof. A limit point for $\bigcap_{\alpha<\gamma} C_\alpha$ is a limit point for all the C_α, therefore it is immediate that $\bigcap_{\alpha<\gamma} C_\alpha$ is closed.

The proof that $\bigcap_{\alpha<\gamma} C_\alpha$ is unbounded in κ (and therefore also nonempty) is by induction on γ. The successor step is as in the previous lemma, so fix $(C_\alpha : \alpha < \gamma)$ and suppose γ is limit and that the lemma holds for any $\alpha < \gamma$. For any $\alpha < \gamma$, let $C'_\alpha = \bigcap_{\xi<\alpha} C_\xi$. Then by induction $(C'_\alpha : \alpha < \gamma)$ is a sequence of clubs, with the added property that $C'_0 \supseteq C'_1 \supseteq \cdots \supseteq C'_\alpha \supseteq \ldots$. Let $\delta < \kappa$, let $\beta_0 \in C'_0$ such that $\delta < \beta_0 < \kappa$, let $\beta_1 \in C'_1$ such that $\beta_0 < \beta_1 < \kappa$ and so on. Since $\operatorname{cof}(\kappa) > \gamma$, the sequence $(\beta_\xi : \xi < \gamma)$ has a supremum, say β, less then κ. But the sequence $(\beta_\xi : \alpha < \xi < \gamma)$ is a sequence in C'_α for any $\alpha < \gamma$, therefore $\beta \in \bigcap_{\alpha<\gamma} C_\alpha$. $\square$

It is tempting therefore to say that the set of clubs in κ is $\operatorname{cof}(\kappa)$-complete, and yet it is not a filter: Let C be a club in κ, and suppose that $\omega \notin C$. Then $C \subseteq C \cup \omega$ is not a club.

Definition 5.3. *Let κ be a cardinal such that $\operatorname{cof}(\kappa) > \omega$. The* club filter *on κ is the filter generated by clubs, i.e., $\{X \subseteq \kappa : \exists C$ club in κ $C \subseteq X\}$.*

By Lemma **??** the club filter on κ is $\operatorname{cof}(\kappa)$-complete. We remind the definition of diagonal intersection, and we add it to the definition of measurable cardinal.

Definition 5.4. *Let κ be a cardinal, and $(X_\alpha : \alpha < \kappa)$ be a sequence of subsets of κ. The* diagonal intersection *of $(X_\alpha : \alpha < \kappa)$ is defined as follows: $\triangle_{\alpha<\kappa} X_\alpha = \{\xi < \kappa : \xi \in \bigcap_{\alpha<\xi} X_\alpha\}$.*

We call a filter that is closed under diagonal intersections of κ-sequences normal*.*

Lemma 5.5. *Whrn κ is regular and uncountable, the diagonal intersection of a κ-sequence of clubs in κ is a club in κ. So the club filter is normal.*

Proof. Let $(C_\alpha : \alpha < \kappa)$. As before, we can suppose that $C_0 \supseteq C_1 \supseteq \ldots$, and let $C = \triangle_{\alpha<\kappa} C_\alpha$.

Let α be a limit point for C. If $\xi < \alpha$, let $X = \{\nu \in C : \xi < \nu < \alpha\}$. If $\nu \in X$ then $\nu \in \bigcap_{\beta<\nu} C_\beta$, and therefore $\nu \in C_\xi$. So $X \subseteq C_\xi$ and $\alpha = \sup X$, and so $\alpha \in C_\xi$. In other words, $\alpha \in \bigcap_{\xi<\alpha} C_\xi$, so $\alpha \in C$.

Let $\alpha < \kappa$. By induction, let $\beta_0 > \alpha$ such that $\beta_0 \in C_0$ and let $\beta_{n+1} > \beta_n$ such that $\beta_{n+1} \in C_{\beta_n}$. Let $\beta = \sup_{n\in\omega} \beta_n$, we want to prove that $\beta \in C$, so that $\beta \in \bigcap_{\xi<\beta} C_\xi$. So let $\xi < \beta$: then there is a $n \in \omega$ such that $\xi < \beta_n$. But $\beta_{n+1} \in C_{\beta_n} \subseteq C_\xi$, $\beta_{n+2} \in C_{\beta_{n+1}} \subseteq C_\xi$, $\ldots$ so the whole sequence $(\beta_k : k > n)$ is in C_ξ, and by closure $\beta \in C_\xi$. □

In fact, the club filter is the minimal normal filter.

Lemma 5.6. *If κ is regular and uncountable and if F is a normal filter on κ that contains all final segments $X_{\alpha_0} = \{\alpha < \kappa : \alpha > \alpha_0\}$, then F contains all the clubs in κ, so also the club filter.*

Proof. Let $Lim(\kappa)$ be the set of limit ordinals under κ. Then $Lim(\kappa) = \triangle_{\alpha<\kappa} X_{\alpha+1}$, so $Lim(\kappa) \in F$. If $C \subseteq \kappa$ is a club, let $C = \{\alpha_\beta : \beta < \kappa\}$ be its increasing enumeration. Then $C \supseteq Lim(\kappa) \cap \triangle_{\beta<\kappa} X_{\alpha_\beta}$. □

We remind that a stationary set in κ is a set that intersects all the clubs. Therefore stationary sets intersect all the members of the club filter, and they are exactly those that are not in the dual ideal, that is therefore called *nonstationary ideal*.

Definition 5.7. *Let F be a filter on κ. We say that $S \subseteq \kappa$ is F-positive if it intersects all elements of F (equivalently, if its complement does not contain an element of F).*

So stationary sets are those that are positive for the club filter. The following is a useful equivalence:

Lemma 5.8. *A κ-complete filter F on κ is normal iff for every S F-positive and any $f : S \to \kappa$ such that for any $\alpha < \kappa$, $f(\alpha) < \alpha$, there are a $T \subseteq S$ F-positive and a $\gamma < \kappa$ such that $f(\alpha) = \gamma$ for any $\alpha \in T$.*

Proof. Let F be a κ-complete normal filter, let S be F-positive and let $f : S \to \kappa$ be as above. Towards a contradiction, suppose that for any $\gamma < \kappa$, $T_\gamma = \{\alpha \in S : f(\alpha) = \gamma\}$ is not F-positive, so there is a $C_\gamma \in F$ such that for any $\alpha \in C_\gamma \cap S$, $f(\alpha) \neq \gamma$. Let $C = \triangle_{\gamma<\kappa} C_\gamma$, and let $\alpha \in S \cap C$. Then $f(\alpha) \neq \gamma$ for any $\gamma < \alpha$, but this means that $f(\alpha) \geq \alpha$, contradiction.

Let F be a κ-complete filter on κ, suppose the above holds, let $(X_\alpha : \alpha < \kappa)$ be a collection of sets in F. Suppose that $\triangle_{\alpha<\kappa} X_\alpha \notin F$. Then $\kappa \setminus \triangle_{\alpha<\kappa} X_\alpha = S$ is F-positive and let $f : S \to \kappa$ any such that if $f(\alpha) = \xi < \alpha$ then $\alpha \notin X_\xi$. Let T be F-positive and γ such that $f(\alpha) = \gamma$ for any $\alpha \in T$.

Then for any $\alpha \in T$, $\alpha \notin X_\gamma$, so $X_\gamma \cap T = \emptyset$, contradiction because T was F-positive and $X_\gamma \in F$. □

We end this flashback trying to find an answer to the question: can the club filter be an ultrafilter? That would mean that there are no real stationary sets, just clubs or nonstationary sets, and that every normal filter that contains the final segments is an ultrafilter.

For $\kappa > \omega_1$ regular it is easily disproved: Consider $E^\kappa_\omega = \{\alpha < \kappa : \mathrm{cof}(\alpha) = \omega\}$ and $E^\kappa_{\omega_1} = \{\alpha < \kappa : \mathrm{cof}(\alpha) = \omega_1\}$. They are both stationary subsets: if C is a club in κ, then the supremum of any ω-sequence in C will be in $C \cap E^\kappa_\omega$ and the supremum of any ω_1-sequence in C will be in $C \cap E^\kappa_{\omega_1}$. They are disjoint, so it is not possible that the club filter is an ultrafilter: if E^κ_ω is in the filter, then $E^\kappa_{\omega_1}$ cannot be stationary, but then $\kappa \setminus E^\kappa_\omega$ should be in the filter, against the fact that E^κ_ω is stationary.

The next question is therefore whether it is possible for the club filter on the ordinals of a certain cofinality to be an ultrafilter, or more generally for the club filter on a stationary set to be an ultrafilter, included the case with $\kappa = \omega_1$. Of course, if S is stationary and there are $T_1, T_2 \subseteq S$ that are stationary and disjoint, then the club filter is not an ultrafilter on S. But then under ZFC the club filter is never an ultrafilter.

Theorem 5.9 (Solovay, ZFC)**.** *Let κ be a regular uncountable cardinal. Then every stationary set on κ is the disjoint union of κ stationary subsets.*

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6 Back at the large cardinals

Definition 6.1. *Let κ be a regular cardinal. Then a normal κ-complete nonprincipal ultrafilter on κ is a* normal measure *on κ.*

Theorem 6.2. *Let κ be a measurable cardinal. Then there is a normal measure on κ.*

Proof. We are going to focus on a fragment of an ultrapower via a κ-complete nonprincipal ultrafilter on κ, U. It is not necessary to do the whole ultrapower to reach the conclusion of the theorem. So, given $f, g : \kappa \to \kappa$, we say that $f =_U g$ iff $\{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in U$ and $f <_U g$ iff $\{\alpha < \kappa : f(\alpha) < g(\alpha)\} \in U$. The order $<_U$ on κ^κ modulo $=_U$ is well-founded: Assume toward a contradiction that $f_0, f_1, \ldots$ such that $f_0 >_U f_1 >_U \ldots$. Then $X_n = \{\alpha < \kappa : f_n(\alpha) > f_{n+1}(\alpha)\} \in U$. By completeness of U, let $\alpha \in \bigcap_{n\in\omega} X_n$. Then

$f_0(\alpha) > f_1(\alpha) > \ldots$, contradiction because there are no infinite descending sequence in V.

Then $<_U$ is a well-ordering of $\kappa^\kappa / =_U$. Note that we proved in Lemma **??** that for any $\gamma < \kappa$, $\gamma \notin U$, therefore for any $\gamma < \kappa$, $\{\alpha < \kappa : \alpha > \gamma\} \in U$. So let f be the $<_U$-minimal such that for any $\gamma < \kappa$, $\{\alpha < \kappa : f(\alpha) > \gamma\} \in U$ (in particular $f <_U id$). Let $D = \{X \subseteq \kappa : f^{-1}[X] \in U\}$. We prove that D is a normal measure.

- $\kappa \in D$: $f^{-1}[\kappa] = \{\alpha < \kappa : f(\alpha) \in \kappa\} = \kappa \in U$; $\emptyset \notin D$ iff $\{\alpha < \kappa : f(\alpha) \in \emptyset\} = \emptyset \in U$;
- if $X, Y \in D$, then $f^{-1}[X], f^{-1}[Y] \in U$, and $f^{-1}[X \cap Y] = f^{-1}[X] \cap f^{-1}[Y] \in U$; also $f^{-1}[\bigcap_{\alpha<\gamma} X_\alpha] = \bigcap_{\alpha<\gamma} f^{-1}[X_\alpha]$;
- if $X \subseteq Y$, $X \in D$, then $f^{-1}[X] \subseteq f^{-1}[Y] \in U$;
- if $X \notin D$, then $f^{-1}[X] \notin U$, so $\kappa \setminus f^{-1}[X] = f^{-1}[\kappa \setminus X] \in U$;
- by the way we have chosen f, $\{\alpha < \kappa : f(\alpha) = \gamma\} \notin U$ for any $\gamma < \kappa$, therefore $f^{-1}[\{\gamma\}] \notin U$, so $\{\gamma\} \notin D$;
- to prove that D is normal, we use Lemma **??** (note that in the case of ultrafilters being D-positive and in D is the same). Let $X \in D$, and let $h : X \to \kappa$ be such that $f(\alpha) < \alpha$ for any $\alpha < \kappa$. Let $g = h \circ f$. Then for any $\alpha < \kappa$, $g(\alpha) = h(f(\alpha)) < f(\alpha)$, so $g <_U f$, and since f was minimal there is a $\gamma < \kappa$ such that $\{\alpha < \kappa : g(\alpha) \leq \gamma\} \in U$. As $\gamma < \kappa$, there is a $\delta \leq \gamma$ such that $Y = \{\alpha < \kappa : g(\alpha) = \delta\} \in U$. Then $f[Y] = \{f(\alpha) < \kappa : g(\alpha) = \delta\}$, and $h[f[Y]] = \{g(\alpha) : g(\alpha) = \delta\} = \{\delta\}$, so h is constant on $f[Y]$. But $f^{-1}[f[Y]] = Y \in U$, so $f[Y] \in D$.

$\square$

Note that if D is a normal measure on κ, then it contains the club filter, so every element of D intersects all clubs, and therefore is stationary.

Lemma 6.3. *Every measurable cardinal is a Mahlo cardinal.*

Proof. Let κ be a measurable cardinal, we need to prove that the set of the inaccessible cardinals under κ is stationary. Let D be a normal measure on κ.

Consider $SLim(\kappa) = \{\alpha < \kappa : \alpha \text{ is strong limit}\}$. Since κ is strong limit, for any $\beta < \kappa$ the supremum of 2^β, 2^{2^β}, ecc... is below κ and it is strong limit, therefore $SLim(\kappa)$ is unbounded, and it is obviously closed, so $SLim(\kappa)$ is a club.

We want to prove that the set of regular cardinals is in D, and therefore stationary. Assume that $\{\alpha : \mathrm{cof}(\alpha) < \alpha\} \in D$. Consider the function $\mathrm{cof} : \{\alpha : \mathrm{cof}(\alpha) < \alpha\} \to \kappa$: then, by normality, there are a $\lambda < \kappa$ and a set $E_\lambda \in D$ such that for any $\alpha \in E_\lambda$, $\mathrm{cof}(\alpha) = \lambda$. Fix for any $\alpha \in E_\lambda$ a cofinal sequence $(\beta_{\alpha,\xi} : \xi < \lambda)$, and now for any $\xi < \lambda$ consider the function $f_\xi : E_\lambda \to \kappa$, $f_\xi(\alpha) = \beta_{\alpha,\xi}$. Then there are by normality a β_ξ and an $A_\xi \in D$ such that for any $\alpha \in A_\xi$, $f_\xi(\alpha) = \beta_\xi$. Now let $A = \bigcap_{\xi<\lambda} A_\xi \in D$. Then for any $\alpha \in A$ and any $\xi < \lambda$, $f_\xi(\alpha) = \beta_\xi$. But then $\alpha = \sup_{\xi<\lambda} \beta_\xi$, and $A = \{\alpha\}$, contradiction because D is nonprincipal.

The intersection of a stationary set and a club is stationary, and as the set of regular cardinals is stationary and the set of strong limits is a club, the set of inaccessible cardinals is stationary, therefore κ is Mahlo. □

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7 Back to the main topic

Let $\mathcal{D}_T$ be the set of Turing degrees. For any $x \in \omega^\omega$, we denote with $[x]_T$ its Turing degree. A *cone of Turing degrees* is a set of the form $\{[y]_T : y \geq_T x_0\}$, with $x_0 \in \omega^\omega$. A *Turing cone of reals* is a set of the form $\{y : y \geq_T x_0\}$. In both cases we call x_0 the *base* of the cone.

So, let $\mathcal{C}$ the set of cones of Turing degrees, and for any $x_0 \in \omega^\omega$, let C_{x_0} the cone with base x_0.

- The whole $\mathcal{D}_T$ is the cone with base 0, so $\mathcal{D}_T \in \mathcal{C}$;
- there is no empty cone, so $\emptyset \notin \mathcal{C}$;
- let $x, y \in \omega^\omega$; then $C_x \cap C_y = C_{\langle x,y\rangle} \in \mathcal{C}$.

Therefore cones can generate a filter:

Definition 7.1. *Let $\mathcal{F}_C = \{A \subseteq \mathcal{D}_T : \exists x \in \omega^\omega\ C_x \subseteq A\}$. It is called the* cone filter*.*

How much is $\mathcal{F}_C$ complete? The countable intersections of cones contains a cone: If we have a countable family of cones, each of which has base x_n, then we can code the sequence $(x_n : n \in \omega)$ as a single real, and the cone with $\langle x_n : n \in \omega\rangle$ as a base contains the intersection of all cones, as if a real can compute $\langle x_n : n \in \omega\rangle$, then it can compute the single x_n's. Let $(A_n : n \in \omega)$ a countable collections of sets in the cone filter. For any $n \in \omega$, let $B_n = \{x \in \omega^\omega : C_x \subseteq A_n\}$. Every $B_n \neq \emptyset$, therefore by $AC_\omega(\mathbb{R})$ there is a choice function f for $(B_n : n \in \omega)$. Then, the intersection of $C_{f(n)}$ contains a cone, and it is contained in $\bigcap_{n\in\omega} A_n$, so $\bigcap_{n\in\omega} A_n \in \mathcal{F}_C$. Then the cone filter is an ω_1-complete filter.

Theorem 7.2 (ZF+AD)**.** *The cone filter on $\mathcal{D}_T$ is an ultrafilter.*

Proof. Let $A \subseteq \mathcal{D}_T$. Consider the game

$$\begin{array}{lccccc} \mathrm{I} & x_0 & & x_1 & & \\ & & & & & \dots \\ \mathrm{II} & & y_0 & & y_1 & \end{array}$$

where I wins iff $[(x_0, y_0, x_1 \dots)]_T \in A$. Either I or II have a winning strategy. If I has a winning strategy σ, then for any $y \in \omega^\omega$, $[\sigma * y]_T \in A$, so if $y \geq_T \sigma$, $[y]_T \in A$, therefore A contains the cone with base σ and is in the cone filter. On the other hand, if II has a winning strategy τ, then for any $x \in \omega^\omega$, $[x * \tau]_T \in \mathcal{D}_T \setminus A$, so if $x \geq_T \tau$, $[x]_T \in \mathcal{D}_T \setminus A$, therefore $\mathcal{D}_T \setminus A$ contains the cone with base τ and is in the cone filter. □

Finally, $\mathcal{F}_C$ is not principal: if it were, then there would be a fixed cone C_x inside every cone, but for example $[x]_T \in C_x$ and $x \notin C_{x'}$. In other words, we have something similar to a "measure" on the Turing degrees. We can actually project it to ω_1, so that it generates a proper measure on ω_1, but we need a remark on principal ultrafilters:

Lemma 7.3.

- *Let $A_0 \subseteq \kappa$. Then $\{X \subseteq \kappa : A_0 \subseteq X\}$ is a κ-complete filter on κ.*
- *If U is a principal ultrafilter on κ, then there exists $\delta < \kappa$ such that $U = \{X \subseteq \kappa : \delta \in X\}$.*

Proof. The first point is just calculations. For the second point, let U ultrafilter on κ and A_0 witness that U is principal. Suppose there are $\gamma_0 \neq \gamma_1$ in A_0. Then neither $\kappa \setminus \{\gamma_0\}$ nor $\{\gamma_0\}$ are in U, contradiction. □

For any $x \in \omega^\omega$, let $f(x) = \omega_{CK}[x]$ be the least ordinal that is not computable with x as an oracle. Then define $U = \{X \subseteq \omega_1 : f^{-1}[X] \in \mathcal{F}_C\}$.

- for any $x \in \omega^\omega$, $\omega_{CK}[x] < \omega_1$, therefore $f^{-1}[\omega_1] = \mathcal{D}_T \in \mathcal{F}_C$, so $\omega_1 \in U$;
- $f^{-1}[\emptyset] = \emptyset \notin \mathcal{F}_C$, so $\emptyset \notin U$;
- let $(X_n : n \in \omega)$ a countable collection of sets in U, so that for any $n \in \omega$, $f^{-1}[X_n] \in \mathcal{F}_C$. Then $f^{-1}[\bigcap_{n\in\omega} X_n] = \bigcap_{n\in\omega} f^{-1}[X_n] \in \mathcal{F}_C$, so $\bigcap_{n\in\omega} X_n \in U$;
- suppose that $X \notin U$; then $f^{-1}[X] \in \mathcal{F}_C$, and since $\mathcal{F}_C$ is an ultrafilter, then $f^{-1}[\omega_1 \setminus X] = \mathcal{D}_T \setminus f^{-1}[X] \in \mathcal{F}_C$, so $\omega_1 \setminus X \in U$;

- suppose that there is a $\delta < \omega_1$ such that $U = \{X \subseteq \omega_1 : \delta \in X\}$; in particular $f^{-1}(\delta) \in \mathcal{F}_C$, so there is a cone such that for any x in the cone, $\omega_{CK}[x] = \delta$; but there is a $y \in C_x$ such that $\omega_{CK}[y] > \omega_{CK}[x]$, contradiction.

We have therefore proved the following:

Theorem 7.4 (ZF+AD)**.** *There is a measure on ω_1.*

So ω_1 is "measurable". The quotes are there because we defined the measurable cardinal in the context of ZFC. We can use the same definition in the context of ZF, but does it have the same consistency strength? Does the fact that ω_1 is ZF-measurable give some insight on its consistency strength measured with ZFC-large cardinals? If only there was a way to transfer large cardinal strength to models of ZFC...

8 Fourth flashback: The Constructible Universe

Definition 8.1. *An inner model M is a transitive class such that $Ord \subseteq M$ and $M \vDash ZFC$. We can specify the theory, so for example an inner model M for ZF is a transitive class such that $Ord \subseteq M$ and $M \vDash ZF$.*

A set X is definable over a model M if there exists a formula φ and some $a_1, \dots, a_m \in M$ such that $X = \{x \in M : M \vDash \varphi(x, a_1, \dots, a_n)\}$. We define $def(M) = \{X \subseteq M : X \text{ is definable over } M\}$. Of course, if M is a set, then $def(M)$ must be a set.

Definition 8.2. *We define by transfinite induction:*

- $L_0 = \emptyset$, $L_{\alpha+1} = def(L_\alpha)$;
- *if α limit,* $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$;
- $L = \bigcup_{\alpha\in Ord} L_\alpha$.

By the way it is defined, L is transitive and contains all ordinals.

Theorem 8.3. $L \vDash ZF$

Proof.

- L is transitive and therefore extensional;
- given $a, b \in L$, let α be such that $a, b \in L_\alpha$. Then $\{a, b\} = \{x \in L_\alpha : L_\alpha \vDash x = a \vee x = b\}$, so $\{a, b\} \in def(L_\alpha) = L_{\alpha+1} \subseteq L$;

- Let φ be a formula, $X, p \in L$. Let $Y = \{u \in X : L \vDash \varphi(u, p)\}$. By the Reflection Principle, there is a α such that $X, p \in L_\alpha$ and $Y = \{u \in X : L_\alpha \vDash \varphi(u, p)\}$. Thus $Y = \{u \in L_\alpha : L_\alpha \vDash u \in X \wedge \varphi(u, p)\}$;
- let $X \in L$, and let $Y = \bigcup X$. As L is transitive, $Y \subseteq L$. Let α be such that $X \in L_\alpha$ and $Y \subseteq L_\alpha$. Then $Y = \{x \in L_\alpha : \exists y \in X\ x \in y\}$;
- let $X \in L$, let $Y = \mathcal{P}(X) \cap L$. Let α be such that $Y \subseteq L_\alpha$. Then $Y = \{x \in L_\alpha : x \subseteq X\}$;
- since $\omega \in L$, this satisfies Infinity
- Let $a_1, \ldots, a_n, A \in L$ such that for all $x \in A$ there is a unique $y \in L$ such that $\varphi(x, y, a_1, \ldots, a_n, A)$. Let α such that $a_1, \ldots, a_n, A \in L_\alpha$ and $\{y \in L : \exists x \in A \varphi(x, y, a_1, \ldots, a_n, A)\} \subseteq L_\alpha$; then it is as before.

□

The Axiom of Separation says that for any φ formula, there exists the set $Y = \{u \in X : \varphi(u)\}$. Gödel discovered that if the formula is Δ_0, then Y can be constructed by means of a finite number of elementary operations.

Theorem 8.4 (Gödel Normal Form Theorem)**.** *There exist operations $G_1, \ldots, G_{10}$ such that if $\varphi(u_1, \ldots, u_n)$ is a Δ_0-formula, then there is a composition G of $G_1, \ldots, G_{10}$ such that for all $X_1, \ldots, X_n$, $G(X_1, \ldots, X_n) = \{(u_1, \ldots, u_n) \in X_1 \times \cdots \times X_n : \varphi(u_1, \ldots, u_n)\}$.*

The $G_1, \ldots, G_n$ are called *Gödel operations.* This is not the place to introduce them, we just need to know that they exists. Suffices to say that they are very basic, like $G_1(X, Y) = \{X, Y\}$ or $G_6(X) = \bigcup X$, and they are all Δ_0. Now if φ is any formula, its interpretation inside a model M is a Δ_0 formula, so for any formula φ there exists a composition of Gödel operations G such that $\{x \in M : M \vDash \varphi(a_1, \ldots, a_n)\} = G(M, a_1, \ldots, a_n)$. On the other hand, one can prove that a composition of Gödel operations will always generate a definable set, so $def(M)$ is the set of subsets of M that are in the closure of $M \cup \{M\}$ under Gödel operations. Also, Δ_0-formulas are absolute between transitive sets, and therefore also Gödel operations are.

Theorem 8.5. *A transitive class M is a model of ZF iff it is closed under Gödel operations and every subset $X \subseteq M$ is included in some $Y \in M$.*

Proof. Let M transitive model of ZF. Then it must be closed under Gödel operations: if $X_1, \ldots, X_n \in M$, then $G(X_1, \ldots, X_n) \in def(X_1 \cup \cdots \cup X_n) \in M$. If $X \subseteq M$, then there is an α such that $X \subseteq V_\alpha \cap M$, and since $\alpha \in M$, $V_\alpha \cap M = (V_\alpha)^M \in M$.

For the opposite direction, look at the proof of $L \vDash$ZF. For almost every axiom the m.o. is the same: there is a Y we want to prove it is in L, then there is a $X \in L$ such that $Y \subseteq X$, and finally such an Y is definable with a Δ_0-formula. Substituting L with M it works the same way. The only exception is Separation, that in L uses the Reflection Principle and here it must be proved separately.

So let $X \in M$ and $Y = \{u \in X : M \vDash \varphi(u)\}$. For simplicity, we disregard the parameter.

Let $\varphi(u_1, \ldots, u_n)$ be a formula with k quantifiers. We modify that in $\bar{\varphi}(u_1, \ldots, u_n, Y_1, \ldots, Y_k)$, a Δ_0-formula obtained bounding every quantifier with a Y_i. We prove by induction on k that for every $\varphi(u_1, \ldots, u_n)$ with k quantifiers and for every $X \in M$ there exist $Y_1, \ldots, Y_k \in M$ such that $M \vDash \varphi(u_1, \ldots, u_n)$ iff $\bar{\varphi}(u_1, \ldots, u_n, Y_1, \ldots, Y_k)$ for all $u_1, \ldots, u_n \in X$. If we can do this, we are then, because then $Y = \{u \in X : \bar{\varphi}(u, Y_1, \ldots, Y_k)\}$, and so Y is in the closure under Gödel operations.

If $k = 0$ then $\bar{\varphi}$ is φ itself, so there is nothing to prove. Suppose now that $\varphi(u)$ is $\exists v \psi(u, v)$, and that $\psi(u, v)$ has k quantifiers. So $\bar{\varphi}$ is $\exists v \in Y_{k+1} \bar{\psi}(u, v, Y_1, \ldots, Y_k)$. We now that there is a set M_1 (in V, by Reflection in V) such that $X \subseteq M_1 \subseteq M$ and for all $u \in X$, $\exists v \in M\ M \vDash \psi(u, v)$ iff $\exists v \in M_1\ M\psi(u, v)$. But then there is a $Z \in M$ such that $M_1 \subseteq Z$. So for all $u \in X$, $\exists v \in M\ M \vDash \psi(u, v)$ iff $\exists v \in Z\ M\psi(u, v)$. Now use the induction hypothesis, so that $M \vDash \psi(u, v)$ iff $\bar{\psi}(u, v, Y_1, \ldots, Y_k)$ and define $\bar{\varphi}$ bounding the only remaining quantifier with Z. □

Lemma 8.6. *The function $\alpha \mapsto L_\alpha$ is definable with a Σ_1 formula.*

Proof. It is known that if a function is defined by induction, and the induction step is Σ_1-definable, then the function itself is Σ_1-definable. So we need to verify that def is Σ_1-definable, or, for the Gödel normal form Theorem, that the closure under Gödel operations is Σ_1-definable. It is, because $Y = cl(M)$ iff there exists a function f with domain ω such that Y is the range of f, $f(0) = M$ and for any $n \in \omega$ $f(n+1) = f(n) \cup \{G_i(x, y) : x, y \in f(n)\}$. □

Corollary 8.7. *The property "x is constructible" is absolute for inner models of ZF.*

Proof. Let M be an inner model of ZF. Then if x is constructible in M, it means that there is an $\alpha \in M$ such that $(x \in L_\alpha)^M$, but then since this is Σ_1, $x \in L_\alpha$, i.e., x is constructible.

Suppose that $x \in L_\alpha$. Since M is a model of ZF, there is $(L_\alpha)^M$, but if $(y = L_\alpha)^M$ by upward absoluteness $y = L_\alpha$, so $L_\alpha = (L_\alpha)^M$, and $x \in (L_\alpha)^M$, i.e., x is constructible in M. □

Now the following is immediate:

Proposition 8.8. *Let $V = L$ be the formula $\forall x\ \exists \alpha\ x \in L_\alpha$. Then:*

- *if M is a transitive set, then $M \vDash ZF + V = L$ implies that $M = L_\alpha$ for some α ordinal;*
- *if M is an inner model of ZF and $M \vDash V = L$, then $M = L$.*

Also, if M is an inner model of ZF then $L \subseteq M$.

Actually, asking for M to be a model of ZF is too much. In the end, if M is closed under Gödel functions, if for any $U \in M$, $\{G_i(x, y) : x, y \in U\} \in M$ and if $(L_\beta : \beta < \alpha) \in M$ for any $\alpha \in M$, then being constructible is absolute for such an M. It is easy to see that L_δ, for δ limit, satisfies this properties. So the following holds:

Lemma 8.9 (Gödel Condensation Lemma)**.** *For every limit ordinal δ, if $M \prec L_\delta$ then the transitive collapse of M is L_γ for some $\gamma \leq \delta$.*

Proof. If $M \prec L_\delta$ in particular it satisfies both what is needed to "understand" constructibility and $V = L$, so the lemma holds. □

Theorem 8.10. *If M is well-ordered, then also the closure of M under Gödel operations is well-ordered. With this one can build a definable well-ordering of L, therefore $L \vDash AC$.*

Proof. We are going to skip the details of this highly technical proof. The idea is that any element in the closure under Gödel operations of M is the application of a certain operation G_i to some elements, that are applications of certain operations to other elements, and so on. Therefore for any element in the closure we can build a labeled finite tree that defines it, with the leaves labeled with members of M (that are well-ordered) and each node labeled with a Gödel operation. We can order such trees in a lexicographical way, so that there is a linear order in the set of this trees, and this will induce a well-order in the closure of M.

This can propagate by induction to all L. $\emptyset$ is trivially well-ordered, and if L_α is well-ordered, then $L_{\alpha+1}$ is well-ordered. If γ is a limit ordinal, then if for any $\alpha < \gamma$, L_α is well-ordered, one can build a well-order of L_γ stacking the wellorders of $L_{\alpha+1}$ on $L_{\alpha+1} \setminus L_\alpha$. This is definable because the hierarchy of the L_α's is definable (it is Σ_1). □

Theorem 8.11. $L \vDash ZFC$

Therefore L is minimal model of ZFC, and all models of ZF contains it. It is a handy way to prove consistency results of the type Con(ZF+...) implies Con(ZFC+...). Heuristically, in L one can solve all the problems. L is also *very* regular, it satisfies many combinatorial properties (GCH, $\Diamond$, $\Box$, universal well-ordering...) and it has many absoluteness characteristics, like the condensation properties, that says that a model "knows" when it is L.

There are also models of relative constructibility. One permits the use of an "oracle" in the construction of the universe: define $def_A(M) = \{X \subseteq M : X \text{ is definable over } (M, \in, A \cap M)\}$ and then $L_{\alpha+1}[A] = def_A(L_\alpha[A])$. The model $L[A]$ is the smallest model M of ZFC such that $A \cap M \in M$. The most common models of this kind in the research around AD are those of the form $L[a]$, with $a \in \omega^\omega$. They are useful to prove measurability (and other large cardinal properties) of cardinals inside the theory of ZF+AD. We decided not to focus on this in this seminars.

One can also permit the use of "urelements" in the construction of the universe: define $L_0(A) = tr(\{A\})$ and the rest as before. Then $L(A)$ is the smallest model of ZF that contains A. It is a model of ZFC only if $L(A)$ contains a well-ordering of A.

The most important example in this case is $L(\mathbb{R})$. We already know that there is no definable well-ordering of the reals, so if there is also no constructible well-ordering of the reals then $L(\mathbb{R})$ will be a model of ZF without Axiom of Choice. This is actually the case when in the universe there are enough large cardinals. We will come back to this later.

Lemma 8.12. *If $M \vDash \mathsf{ZFC}$, $Ord \subseteq M$ and $M \vDash \kappa$ is inaccessible, then $L \vDash \kappa$ is inaccessible. If $M \vDash \mathsf{ZFC}$, $Ord \subseteq M$ and $M \vDash \kappa$ is Mahlo, then $L \vDash \kappa$ is Mahlo.*

Proof. Let $\eta, \delta < \kappa$, and suppose that $L \vDash \eta^\delta \geq \kappa$. Then, in L, there exists a surjection from $\{f \in L : f : \delta \to \eta\}$ to κ. But then this surjection is in M, so it is extendable to η^δ, and $\eta^\delta > \kappa$ also in M, contradiction.

Therefore $\{\eta < \kappa : M \vDash \kappa \text{ is inaccessible }\} \subseteq \{\eta < \kappa : L \vDash \kappa \text{ is inaccessible }\}$. If $C \subseteq \kappa$ is a club in L, then it is a club also in M, as the concept of "unbounded" is absolute. So by Mahlo-ness in M it intersects $\{\eta < \kappa : M \vDash \kappa \text{ is inaccessible }\}$, so $\{\eta < \kappa : L \vDash \kappa \text{ is inaccessible }\}$ is stationary in L. $\square$

Lemma 8.13. *If $M \vDash \mathsf{ZF}$, $Ord \subseteq M$ and $M \vDash \kappa$ is measurable, then $L \vDash \kappa$ is inaccessible.*

Proof. Since κ is regular in M, then it is regular also in L. Suppose that in L there is a surjection π from 2^λ to κ with $\lambda < \kappa$. Then for any α, let X_α

be the only set between $\{\pi(f) : f(\alpha) = 0\}$ and $\{\pi(f) : f(\alpha) = 1\}$ that is of measure one. Let $X = \bigcap X_\alpha$. As before, $X = \{\pi(f)\}$, contradiction because measures are nonprincipal. $\square$

9 Very quickly back at the main topic

Since in ZF+AD ω_1 is measurable, then in L ω_1 is inaccessible, therefore Con(ZF+AD) implies Con(ZFC+inaccessible). In fact:

Theorem 9.1. *Suppose that every co-analytic set is either countable or contains a perfect set. Then ω_1^V is inaccessible in $L[x]$ for any $x \in \omega^\omega$.*

Maybe we can prove that ω_1 can be Mahlo, but we cannot go much far, because L cannot contain many large cardinals.

Theorem 9.2. *Suppose that κ is a measurable cardinal. Then $V \neq L$.*

We remind the ultrapower construction. If U is an ultrafilter on κ, then we can define two relations on the functions with domain κ: $f =_U g$ iff $\{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in U$, and $f \in_U g$ iff $\{\alpha < \kappa : f(\alpha) \in g(\alpha)\} \in U$. The first one is an equivalence relation, and the second one induces a relation between classes of the first equivalence relation. Now, let $(Ult(V,U), \in_U)$ be the class of all $[f]_U$, with f function with domain κ, and $[f]_U \in_U [g]_U$ iff $f \in_U g$. When U is σ-complete, then $(Ult(V,U), \in_U)$ is well-founded. Define $j_U : V \to Ult(V,U)$ as $j_U(x) = [c_x]_U$, where c_x is the function such that $c_x(\alpha) = x$ for any $\alpha < \kappa$. Then j_U is an elementary embedding from V to $Ult(V,U)$, and the transitive collapse of $Ult(V,U)$, M, is an inner model of V that satisfies ZFC.

If α is an ordinal, then $j_U(\alpha)$ is an ordinal, and if $\alpha < \beta$ then $j_U(\alpha) < j_U(\beta)$, thus $\alpha \leq j(\alpha)$ for any α ordinal. It is immediate to see that $j(\alpha+1) = j(\alpha)+1$, $j(n) = n$ for any $n \in \omega$. If U is countably complete, then $j(\omega) = \omega$: If $[f] < \omega$, then $\{\alpha < \kappa : f(\alpha) < \omega\} \in U$, and since U is ω_1-complete there must be an $n \in \omega$ such that $\{\alpha < \kappa : f(\alpha) = n\} \in U$, but then $[f] = n$. By the same argument, if U is λ-complete, then $j(\gamma) = \gamma$ for all $\gamma < \lambda$.

So if κ is measurable, and U is a measure on κ, then $j_U(\gamma) = \gamma$ for any $\gamma < \kappa$. Consider now $id : \kappa \to \kappa$, the identity function. Since U is κ-complete, every bounded set is not in U, therefore $\{\alpha < \kappa : id(\alpha) \geq \gamma\} \in U$ for any $\gamma < \kappa$, so $[id]_U > \gamma$ for any $\gamma < \kappa$, and therefore $[id]_U \geq \kappa$. However, $\{\alpha < \kappa : id(\alpha) < c_\kappa(\alpha)\} = \kappa \in U$, so $[id]_U < j_U(\kappa)$. But then $j_U(\kappa) > \kappa$.

Proof. Let κ be the least measurable cardinal, let U be a measure on κ. Let $j_U : V \to M \subseteq V$. If $V = L$, then also $M = L$, since L is minimal. In L, κ

is the least measurable cardinal, therefore by elementarity $j_U(\kappa)$ is the least measurable cardinal in $M = L$. Contradiction because $j_U(\kappa) > \kappa$. □

If only there was an inner model that could contain large cardinals like measurables...

10 Fifth flashback: HOD

We say that a set X is *ordinal-definable* if there is a formula φ such that $X = \{u : \varphi(u, \alpha_1, \dots, \alpha_n)\}$. How to express this in the language of set theory? Gödel approached the problem in this way: we define OD as the Gödel closure of $\{V_\alpha : \alpha \in Ord\}$. We want to prove now that the two definitions coincide.

Lemma 10.1. *There exists a definable well-ordering of the class OD. Therefore every element of OD is ordinal-definable.*

Proof. We already noticed that the closure under Gödel operations of something that is well-orderable is well-orderable. Now, there is an obvious well-ordering of $\{V_\alpha : \alpha \in Ord\}$, and therefore there is a well-ordering of OD. Looking back at the proof, it is also clear that it is definable, therefore there is a definable injection $F : Ord \to OD$. But then if $X \in OD$ there is an α such that $X = F(\alpha)$, so $X = \{u : u \in F(\alpha)\}$. □

Lemma 10.2. *If X is ordinal definable, then $X \in OD$.*

Proof. The key is the Reflection Principle, that says that if $V \vDash \varphi(a_1, \dots, a_n)$, then there is a β such that $V_\beta \vDash \varphi(a_1, \dots, a_n)$. So let $X = \{u : \varphi(u, \alpha_1, \dots, \alpha_n)\}$, and let β such that V_β reflects φ. Then $X = \{u \in V_\beta : V_\beta \vDash \varphi(u, \alpha_1, \dots, \alpha_n)\}$. Now, by Gödel normal form theorem we can find a composition of Gödel operations G such that $X = G(V_\beta, a_1, \dots, a_n)$. Now, every ordinal α can also be constructed with Gödel operations from V_α, as $\alpha = \{x \in V_\alpha : x \text{ is an ordinal}\}$, so $X \in OD$. □

So OD is the class of ordinal-definable sets, and satisfies AC. To satisfy all ZFC, we would need that every subset $X \subseteq OD$ is included in some $Y \in OD$, and that it is transitive. This is not true, but it holds true in the next model:

Definition 10.3. *The class of* hereditarily ordinal-definable *sets is defined as HOD=$\{x : tc(\{x\}) \subseteq OD\}$.*

In other words, HOD is the class of the ordinal-definable sets whose elements in the transitive closure are all ordinal-definable (and therefore hereditarily ordinal-definable). So HOD is transitive and it contains all ordinals.

Theorem 10.4. *HOD is a transitive model of ZFC*

Proof. To prove that HOD is a model of ZF, it suffices to show that if $X \subseteq HOD$, then there is a $Y \in HOD$ such that $X \subseteq Y$. It suffices to show that $V_\alpha \cap HOD \in HOD$ for any α ordinal. But $V_\alpha \cap HOD$ is ordinal-definable, because $V_\alpha \cap HOD = \{u : u \in V_\alpha \wedge (\forall z \in tc(\{u\})\ \exists \beta\ z \in cl(\{V_\gamma : \gamma < \beta\})\}$. Now, $V_\alpha \cap HOD \subseteq HOD$ therefore it is in HOD.

Since there is a definable well-ordering of OD, its restriction to HOD will be a well-ordering in OD whose elements are in HOD, and therefore will be definable in HOD. So HOD satisfies also the Axiom of Choice.

Finally, it is a simple exercise to see that the image of HOD sets under Gödel operations is HOD, so HOD is closed under Gödel operations. □

In conclusion, HOD is *another* model of ZFC that is inside every model of ZF. There are some fundamental differences, however, with L:

- L is absolute, HOD is relative: While $L^M = L$ for any M that contains all the ordinals, this does not hold in HOD. In fact it is also consistent that HOD^{HOD} is not HOD.

- L can be vary far from V, while HOD is always somewhat close to V (see Vopenka's Theorem). How much close, is a critical branch of research in set theory.

- L can just have small large cardinals, those in the first group, while HOD can have very large cardinals. How many, it is still a critical open question.

As an example of the last point:

Proposition 10.5 (ZF)**.** *Let κ be a cardinal, $U \in OD$ a measure on κ. Then $HOD \vDash \kappa$ is measurable.*

Proof. The key point is that $U \cap HOD \in HOD$, because it is in OD and all its elements are in HOD, and that $U \cap HOD$ is still a measure:

- $\emptyset \notin U$, so $\emptyset \notin U \cap HOD$;

- $\kappa \in U$, $\kappa \in HOD$, so $\kappa \in U \cap HOD$;

- let $(A_\alpha : \alpha < \eta)$, with $\eta < \kappa$ and $A_\alpha \in U \cap HOD$; then $\bigcap_{\alpha\in\eta}$ is in U because U is κ-complete, and is in HOD because HOD is a model of ZFC, so $\bigcap_{\alpha\in\eta} \in U \cap HOD$;

- let $A \in U \cap HOD$, $A \subseteq B \subseteq \kappa$, $B \in HOD$. Since U is a filter, $B \in U$;

- suppose $A \in HOD$, $A \subseteq \kappa$, $A \notin U \cap HOD$; then $A \notin U$, so $\kappa \setminus A \in U$; but also $\kappa \setminus A$ is definable from A and κ, so it is in HOD;

- suppose there exists $A_0 \subseteq \kappa$, $A_0 \in HOD$ such that $U \cap HOD = \{A \subseteq X : A \in HOD,\ A_0 \subseteq A\}$; as we have seen before, $A_0 = \{\delta\}$ for some $\delta < \kappa$; consider now $\{X \subseteq \kappa : \delta \in X\}$: on one hand, if $\delta \in X$, then $X \in U$, because $\{\delta\} \in U \cap HOD$, so $\{X \subseteq \kappa : \delta \in X\} \subseteq U$; on the other hand, $\{X \subseteq \kappa : \delta \in X\}$ is an ultrafilter, so $U = \{X \subseteq \kappa : A_0 \subseteq X\}$, contradiction because U is not principal.

□

Here some theorems that reflect this, without proofs:

Lemma 10.6. *If $\mathbb{P}$ is homogeneous and G is V-generic, then $HOD^{V[G]} \subseteq V$. Actually, HOD is a generic extension of $HOD^{V[G]}$.*

Theorem 10.7 (McAloon)**.** *It is consistent that $HOD^{HOD} \neq HOD$.*

Sketch of proof. Start with L, add a Cohen real x. In $L[x]$ force so that in $L[x,G]$ $2^{\aleph_n} > \aleph_{n+1}$ iff $x(n) = 0$. Then x is OD in $L[x,G]$, so $HOD^{L[x,G]} = L[x]$. By homogeneity of Cohen forcing, $HOD^{L[x]} = (HOD^{HOD})^{L[x,G]} = L \neq HOD^{L[x,G]}$. □

Theorem 10.8 (Vopenka)**.** *There is a forcing $\mathbb{P} \in HOD$ such that for any set of ordinals A, there is a G $\mathbb{P}$-generic in HOD such that $A \in HOD[G]$.*

Theorem 10.9 (Cheng, Friedman, Hamkins)**.** *It is consistent that there are supercompact cardinals in V, but not in HOD. In fact, that all the supercompact cardinals in V are not even weakly compact in HOD.*

Theorem 10.10 (Woodin)**.** *Assume that δ is an extendible cardinal. Then either*

- *For every singular cardinal $\gamma > \delta$, γ is singular in HOD and $(\gamma^+)^{HOD} = \gamma^+$ or*

- *every regular cardinal greater than δ is measurable in HOD.*

Even for OD and HOD there are relative constructions. In this case, though, they are less codified, and their definition and notation are not standard. In this paper we are going to indicate OD_x as the class of sets that are definable using as parameters ordinals and x, that is the Gödel closure of $\{V_\alpha : \alpha \in Ord\} \cup \{x\}$. There exists therefore a definable well-ordering of the class OD_x.

11 Back at the main topic

So we are trying to pull down to HOD the measurability of ω_1 that AD assures. Let U be the cone measure. It is not clear whether $U \cap HOD \in HOD$, therefore the answer is not immediate. It is better to change approach, and find another measure on ω_1 that is ordinal-definable.

Theorem 11.1 (ZF+AD)**.** *The club filter on ω_1 is an ω_1-complete ultrafilter.*

Proof. The proof will have a lot of coding, so it is worth to understand better *what* we are coding. We start with a different game.

Let $X \subseteq \omega_1$, and consider the game:

$$\begin{array}{lccccc} \mathrm{I} & \alpha_0 & & \alpha_1 & & \\ & & & & & \dots \\ \mathrm{II} & & \beta_0 & & \beta_1 & \end{array}$$

where $\alpha_0 < \beta_0 < \alpha_1 < \cdots < \omega_1$. The first player that fails to follow the rules loses, otherwise I wins iff $\sup_{n\in\omega} \alpha_n \in X$. We want to prove that I has a winning strategy iff X is in the club filter.

Suppose then that X contains a club C. The winning strategy for I will be just to play every time an element of C: since C is unbounded I can always have a legal move, and since C is closed then I will win, so this strategy is winning for I. Suppose now that I has a winning strategy σ. Fix $\gamma_0 < \omega_1$, and let $\gamma_1 = \sup \sigma'' \gamma_0^{<\omega}$. Since ω_1 is regular, we have that $\gamma_1 < \omega_1$. Define by induction $\gamma_{n+1} = \sup \sigma'' \gamma_n^{<\omega}$, and then $\gamma_\omega = \sup_{n\in\omega} \gamma_n$. We have that γ_ω is closed under σ, i.e., $s \in \gamma_\omega^{<\omega}$, $\sigma(s) < \gamma_\omega$. Let C therefore be the set of limit points closed under σ. Since for any $\gamma < \omega_1$, $\gamma < \gamma_\omega \in C$, then C is unbounded, and it is obviously closed, therefore C is a club. Let γ be a limit point of C, and let $(\gamma_i : i < \omega)$ be a a cofinal sequence in γ of elements of C. This sequence is a legal play for II. Then X contains all the limit points of C, that is still a club. So we're done, we proved that I has a winning strategy iff X is in the club filter. If this game was determined, then we would have proved the theorem, but this game is not included in the Axiom of Determinacy, so we need to find another game that is determined and that is similar to this one.

We have seen that each countable ordinal can be coded by a real. Also, ω reals can be coded by one real. Then I is going to play $x(0), x(1), \dots$ such that $(x(0), x(1), \dots)$ codes $(\alpha_0, \alpha_1, \dots)$. Given $x \in \omega^\omega$, we call $(x)_i$ the i-th real of the decomposition of x into ω reals. We consider now the following game:

$$\begin{array}{lccccc} \mathrm{I} & x(0) & & x(1) & & \\ & & & & & \dots \\ \mathrm{II} & & y(0) & & y(1) & \end{array}$$

where the rules are:

- $(x)_i, (y)_i \in WO$ for any $i \in \omega$. If this rule is violated, let i be the least such that $(x)_i \notin WO$ or $(y)_i \notin WO$. If $(x)_i \in WO$, then I wins, otherwise II wins.

- $\|(x)_0\| < \|(y)_0\| < \|(x)_1\| < \dots$. The first failure of this rule determines who wins.

- If both rules are satisfied, then I wins iff $\sup_{n\in\omega} \|(x)_n\| \in X$.

We want to prove that if I has a winning strategy then X contains a club. Let σ be a winning strategy for I. For any $\alpha < \omega_1$, let $X_\alpha = \{(((\sigma * y)_I)_n : n \in \omega,\ y \in \omega^\omega,\ \forall i < n\ (y)_i \in WO_{<\alpha}\}$. Note that if for all $i < n$, $(y)_i \in WO$, then it must be that $((\sigma * y)_I)_n \in WO$, otherwise I would lose, so $X_\alpha \subseteq WO$. We have seen that $WO_{<\alpha}$ is Σ^1_1, and so X_α is Σ^1_1. By the Boundedness Lemma, there is an α' such that $X_\alpha \subseteq WO_{<\alpha'}$. Let $f : \omega_1 \to \omega_1$ be the function that associates for any $\alpha < \omega_1$ the least α' such that $X_\alpha \subseteq WO_{<\alpha'}$. Let C be the set of closure points of f, i.e., all the limit points γ such that if $\xi < \gamma$ then $f(\xi) < \gamma$. If C' is the set of limit points of C, then C' is a club, because ω_1 is regular. Let $\gamma \in C'$ and $(\gamma_i : i < \omega)$ a sequence of elements in C cofinal in γ. Using $\mathrm{AC}_\omega(\mathbb{R})$, let $y \in \omega^\omega$ such that $\|(y)_i\| = \gamma_i$. We want to prove that y satisfies the first two rules for II, so that $\sup_{n\in\omega} \|((\sigma * y)_I)_n\| = \sup_{n\in\omega} \|(y)_n\| = \sup_{n\in\omega} \gamma_n = \gamma \in X$ and $C' \subseteq X$. We chose y so that $(y)_n \in WO$, so the first rule is satisfied. Consider $\|((\sigma * y)_I)_0\|$. Trivially $((\sigma * y)_I)_0 \in X_0$, and as $X_0 \subseteq WO_{f(0)} \subseteq WO_{<\gamma_0}$, then $\|((\sigma * y)_I)_0\| < \gamma_0 = \|(y)_0\|$, and the same is true for any $n \in \omega$.

The same argument shows that if II has a winning strategy then $\omega_1 \setminus X$ contains a club. This means that the club filter on ω_1 is an ultrafilter.

We need to prove now that it is ω_1-complete. With AC it was immediate: given X_n in the club filter, each of which had a $C_n \subseteq X_n$ club inside, and the intersection of ω club is still a club. This last statement is still true, the problem is that without choice we cannot choose one C_n for each X_n, so we need to use again the same game and $AC_\omega(\mathbb{R})$ will be enough. We are going to prove that if $(X_n : n \in \omega)$ are all sets for which I has a winning strategy, then I has a winning strategy for $\bigcap_{n\in\omega} X_n$. For any X_n, *choose* a σ_n, winning strategy for I. Assume toward a contradiction that I has no winning strategy for $\bigcap_{n\in\omega} X_n$, so let σ be a winning strategy for I for $\omega_1 \setminus \bigcap_{n\in\omega} X_n$. We want to build a play y for II that is legal against all the σ_n's and against σ, and then we will have that $\sup_{i\in\omega} \|(y)_i\| \in X_n$ for every $n \in \omega$, but it is not in $\bigcap_{n\in\omega} X_n$, contradiction.

We build $(y)_n$ by recursion. Let $X(\sigma, 0) = \{((\sigma * y)_I)_0 : y \in \omega^\omega\}$, and define $X(0) = \bigcap_{n\in\omega} X(\sigma_n, 0) \cap X(\sigma, 0)$. Then $X(0)$ is $\mathbf{\Sigma}^1_1$ (we code all strategies as just one real), so any $X(0) \subseteq WO_{<\beta_0}$ for some β_0, by the Boundedness Lemma. By induction $X(\sigma, n) = \{(\sigma * y)_n : y \in \omega^\omega \wedge \forall i < n\ (y)_i = z_i\}$, and $X(n) = \bigcup_{i<\omega} X(\sigma_i, n+1) \cap X(\sigma, n+1)$. Again this is $\mathbf{\Sigma}^1_1$, therefore by boundedness there is a β_n such that $X(n) \subseteq WO_\beta$, and choose $(y)_n \in WO_{\beta_1}$. We can suppose $\beta_{n+1} > \beta_n$. Then $y\omega^\omega$ is a move for II that satisfies the first two rules: surely all $(y)_n \in WO$; we have $\|((\sigma_n * y)_I)_0\| < (y)_0 < \|((\sigma_n * y)_I)_1\| < \ldots$, the odd inequalities because for any $i \in \omega$, $((\sigma_n * y)_I)_i \in X(i) \subseteq WO_{<\beta_i}$, and therefore $\|((\sigma_n * y)_I)_i\| < \beta_i = \|(y_i)\|$, the even inequalities because σ_n is a winning strategy for I. The same holds for σ, therefore y is a legal move for II, and this is a contradiction. □

Now, the club filter is OD. So we proved the following:

Theorem 11.2 (ZF+AD)**.** *HOD$\vDash \omega_1$ is measurable.*

In particular, Con(ZF+AD) implies Con(ZFC+there exists a measurable cardinal).

We would like to prove now that there are even more measurable cardinal. Fortunately, the proof above is fairly flexible: to prove that the club filter in δ is a measure, we just (heh...) need:

- δ to be regular (for the sup game);
- a way to map cofinally reals into δ (to code the sup game in ω^ω);
- this way should be so that a boundedness lemma holds;
- there should be a way to "choose" strategies from winning payoff sets (for completeness).

Our best bet is to find a cardinal that is "similar" to ω_1. Very vaguely, the fact that WO is Σ^1_1 seems to indicate that there is an "analytic" connection between the reals and ω_1. This is also reinforced by the following observation:

Proposition 11.3. $\omega_1 = \sup\{\alpha :$ *there exists a* $\mathbf{\Delta}^1_1$ *surjection* $\pi : \omega^\omega \to \alpha\}$.

The idea is to lift everything we have done in the second order to the third order, therefore admitting some cautious quantification on subsets of the reals.

Definition 11.4. *A set $X \subseteq \omega^\omega$ is Σ^2_1 iff there exists a formula φ that is in the language of $V_{\omega+1}$ plus a unary predicate A, and can be written as a series of quantifiers on the reals and a part that is recursive in A, and such that $X = \{x \in \omega^\omega : \exists A \subseteq \omega^\omega (V_{\omega+1}, \in, A) \vDash \varphi(x)\}$.*

A set $X \subseteq \omega^\omega$ is $\mathbf{\Sigma}^2_1$ iff as before, but recursive in some real.

A set is Π^2_1 and $\mathbf{\Pi}^2_1$ if it is the complement of a Σ^2_1 and $\mathbf{\Sigma}^2_1$ set.

A set is $\mathbf{\Delta}^2_1$ if it is both $\mathbf{\Sigma}^2_1$ and $\mathbf{\Pi}^2_1$.

This will have the role that previously $\mathbf{\Sigma}^2_1$ had. Even ω_1 is solidly tied to the second order theory:

Proposition 11.5. $\omega_1 = \sup\{\alpha :$ *there exists a* $\mathbf{\Delta}^1_1$ *surjection* $\pi : \omega^\omega \to \alpha\}$.

The problem of the third order theory of arithmetic is that it can be very wild, subsets of reals can be very pathological. So, from now on, we will work in $L(\mathbb{R})$. The reason is that in this case the third order theory can not be much distant from the second order theory, all the subsets of $\mathbb{R}$ are constructible from $\mathbb{R}$, so everything will be very regular and results on the second order theory will carry on nicely.

12 $L(\mathbb{R})$

From now on, the basic theory will be ZF+AC$_\omega(\mathbb{R})$+$V = L(\mathbb{R})$, so we are working inside $L(\mathbb{R})$. We should therefore getting acquainted with it. We can actually borrow much of what we have already done for L. So, for example, $L(\mathbb{R}) \vDash ZF$.

This new structure forces us to think about definability in a different way. We call $\mathcal{L}_{\dot{\mathbb{R}}}$ the language of set theory with an additional 1-ary relation $\dot{\mathbb{R}}$, and it is intended that $\dot{\mathbb{R}}$ will always be interpreted as $\mathbb{R}$. We can redefine the Levy hierarchy in this way: $\Sigma_1(\mathbb{R})$ formulas, for example, are Σ_1 formulas in the language $\mathcal{L}_{\dot{\mathbb{R}}}$. The following, then, is proved in the same way as in L:

Theorem 12.1.
1. $L(\mathbb{R}) \vDash ZF$;
2. *The function* $\alpha \mapsto L_\alpha$ *is definable with a* $\Sigma_1(\mathbb{R})$ *formula;*
3. *The property "x is constructible relative to* $\mathbb{R}$*" is absolute for inner models of ZF with the same* $\mathbb{R}$*;*
4. *There exists a* $\Pi_2(\mathbb{R})$ *formula, "*$V = L(\mathbb{R})$*", such that if* M *is an inner model of ZF and* $M \vDash V = L(\mathbb{R})$*, then* $M = L(\mathbb{R}^M)$*;*
5. *If* M *is a transitive set, then if* $M \vDash ZF-P+V = L$ *then* $M = L_\alpha(\mathbb{R}^M)$ *for some ordinal* α*;*
6. *If* M *is an inner model of ZF, then* $L(\mathbb{R}^M) \subseteq M$*;*
7. $L(\mathbb{R})$ *is well-ordered iff there exists a well-order of* ω^ω *in* $L(\mathbb{R})$*;*

Note that if we are looking for consistency results from ZF+AD, we can assume $V = L(\mathbb{R})$, as:

Remark 12.2 (ZF+AD). *$L(\mathbb{R}) \vDash AD$.*

Proof. We can think of AD as "for any subset X of $\mathbb{R}$ there exists a strategy $\sigma(X)$ that is winning for I or there exists a strategy $\tau(X)$ that is winning for II". If $X \in L(\mathbb{R})$, then there exist $\sigma(X)$ or $\tau(X)$, but since σ and τ can be coded by a real, those are strategies in $L(\mathbb{R}$. Being a winning strategy is absolute, so AD holds also in $L(\mathbb{R})$. $\square$

Corollary 12.3. *Con(ZF+AD)$\leftrightarrow$Con(ZF+AD+$V = L(\mathbb{R})$).*

Lemma 12.4. *There exists $\Phi : Ord \times \mathbb{R} \twoheadrightarrow L(\mathbb{R})$ $\Sigma_1(\mathbb{R})$-definable surjection in $L(\mathbb{R})$ in the language $\mathcal{L}_{\mathbb{R}}$; in fact, for any α limit ordinal there exists $\Phi_\alpha : \alpha \times \mathbb{R} \twoheadrightarrow L_\alpha(\mathbb{R})$ that is $\mathcal{L}_{\mathbb{R}}$-definable in $L_\alpha(\mathbb{R})$.*

Proof. The proof is the same as before: this time, every element of $L(\mathbb{R})$ can be coded by a finite tree labeled with ordinals in the nodes, and reals in the leaves. Code every finite sequence of reals with one reals, code every such tree with an ordinal, et voilà, you have your definable surjection. $\square$

This lemma is used to build partial Skolem functions and substructures in $L(\mathbb{R})$: in a sort of Lowenheim-Skolem way:

Suppose that $L(\mathbb{R}) \vDash \exists\varphi(x, a)$, where φ is Δ_0 in the language $\mathcal{L}_{\mathbb{R}}$ and $a \in \mathbb{R}$. Then there exists a $b \in L(\mathbb{R})$ such that $L(\mathbb{R}) \vDash \varphi(b, a)$. Since Φ is a surjection, there exists a $c \in \mathbb{R}$ and an ordinal α such that $L(\mathbb{R}) \vDash \varphi(\Phi(c, \alpha), a)$, so if we fix c, we can minimize α and choose a witness for $\exists x\varphi(x, a)$.

Definition 12.5. *For any $\varphi(x, a)$ Δ_0 formula in the language $\mathcal{L}_{\mathbb{R}}$ with one free variable, $c, a \in \mathbb{R}$, we define $h(\langle\varphi, a, c\rangle)$ as $\Phi(c, \alpha)$, if $L(\mathbb{R}) \vDash \varphi(\Phi(c, \beta), a)$ for some β, where α is the least, otherwise h is not defined. We call $H_1^{L(\mathbb{R})}(\emptyset) = H_1^{L(\mathbb{R})} = h''\mathbb{R}$.*

Suppose that $L(\mathbb{R}) \vDash \exists x\ \varphi(x, a)$, where φ is Δ_0 in the language $\mathcal{L}_{\mathbb{R}}$ and $a \in \mathbb{R}$. Then there is a c such that $L(\mathbb{R}) \vDash \varphi(h(\langle\varphi, a, c\rangle), a)$, and $\mathbb{R} \subseteq H_1^{L(\mathbb{R})}$, therefore $H_1^{L(\mathbb{R})} \vDash \exists\varphi(x, a)$, so formulas of this kind are reflected by $H_1^{L(\mathbb{R})}$. In these cases, we write $H_1^{L(\mathbb{R})} \prec_1^{\mathbb{R}\cup\{\mathbb{R}\}} L(\mathbb{R})$. Now, $V = L(\mathbb{R})$ is a $\Pi_2(\mathbb{R})$ statement, therefore it is also reflected and $H_1^{L(\mathbb{R})} \vDash V = L(\mathbb{R})$. But then, the collapse of $H_1^{L(\mathbb{R})}$, that we identify with $H_1^{L(\mathbb{R})}$, is a transitive set that satisfy $V = L(\mathbb{R})$, therefore there is a δ such that $H_1^{L(\mathbb{R})} = L_\delta(\mathbb{R})$.

Definition 12.6. *We call $\delta_{\mathbb{R}}$ the least δ such that $L_\delta(\mathbb{R}) \prec_1^{\mathbb{R}\cup\{\mathbb{R}\}} L(\mathbb{R})$.*

Therefore $H_1^{L(\mathbb{R})} = L_{\delta_{\mathbb{R}}}(\mathbb{R})$, and h is a surjection from a subset of $\mathbb{R}$ and $L_{\delta_{\mathbb{R}}}(\mathbb{R})$. What is its definability rank? Its domain is the set of φ, a, c such that there is a α such that $\varphi(\Phi(c,\alpha), a)$ holds, so it is $\Sigma_1(\mathbb{R})$. The calculation of $h(\langle \varphi, a, c\rangle)$ is similarly $\Sigma_1(\mathbb{R})$, so the whole h is $\Sigma_1(\mathbb{R})$. We proved the following:

Lemma 12.7 (ZF+$V = L(\mathbb{R})$)**.** *There is a $\Sigma_1(\mathbb{R})$-definable partial surjection $h : \mathbb{R} \twoheadrightarrow L_{\delta_{\mathbb{R}}}(\mathbb{R})$.*

We introduce two variants for this construction.

Let $X \in L(\mathbb{R})$. For any $\varphi(x, a, d)$ Δ_0 formula in the language $\mathcal{L}_{\mathbb{R}}$ with one free variable, $c, a \in \mathbb{R}$, $d, \in tc(\{X\})$, we define $h_X(\langle \varphi, a, c, d\rangle)$ as $\Phi(c, \alpha)$, if $L(\mathbb{R}) \vDash \varphi(\Phi(c, \beta), a, d)$ for some β, where α is the least, otherwise h_X is not defined. We call $H_1^{L(\mathbb{R})}(X) = h_X''\mathbb{R}$.

Again, $H_1^{L(\mathbb{R})}(X) \prec_1^{\mathbb{R}\cup\{\mathbb{R}\}} L(\mathbb{R})$. If we identify $H_1^{L(\mathbb{R})}(X)$ with its collapse, we have that $H_1^{L(\mathbb{R})}(X) = L_\delta(\mathbb{R})$. Since the whole transitive closure of X and X itself are in $H_1^{L(\mathbb{R})}(X)$, we have that X is not collapsed, therefore $X \in L_\delta(\mathbb{R})$. Now, h_X is a partial surjection from $\omega^\omega \times tc(\{X\})$ to $L_\delta(\mathbb{R})$. We have proved the following:

Lemma 12.8 (ZF+$V = L(\mathbb{R})$)**.** *For any $X \in L(\mathbb{R})$, there exist a δ such that $X \in L_\delta(\mathbb{R}) \prec_1^{\mathbb{R}\cup\{\mathbb{R}\}}$ and a $h_X : \omega^\omega \times tc(\{X\}) \twoheadrightarrow L_\delta(\mathbb{R})$ $\Sigma_1(\mathbb{R})$-definable partial surjection.*

Let λ limit ordinal. For any $\varphi(x, a)$ formula in the language $\mathcal{L}_{\mathbb{R}}$ with one free variable, $c, a \in \mathbb{R}$, we define $h^{L_\lambda(\mathbb{R})}(\langle \varphi, a, c\rangle)$ as $\Phi_\lambda(c, \alpha)$, if $L_\lambda(\mathbb{R}) \vDash \varphi(\Phi_\lambda(c, \beta), a)$ for some β, where α is the least, otherwise h is not defined. We call $H_c^{L_\lambda(\mathbb{R})}$ the closure of $\mathbb{R}$ under $h^{L_\lambda(\mathbb{R})}$.

Now, since we are using all possible formulas and we are closing under $h^{L_\lambda(\mathbb{R})}$, it is possible to prove by induction that $H_c^{L_\lambda(\mathbb{R})} \prec L_\lambda(\mathbb{R})$, i.e., that every formula with parameters in $H_c^{L_\lambda(\mathbb{R})}$ that holds in $L_\lambda(\mathbb{R})$ holds also in $H_c^{L_\lambda(\mathbb{R})}$. Again, we can identify $H_c^{L_\lambda(\mathbb{R})}$ with its collapse, so there is a $\delta \leq \lambda$ such that $H_c^{L_\lambda(\mathbb{R})} = L_\delta(\mathbb{R})$.

If $x \in H_c^{L_\lambda(\mathbb{R})}$, then there exists a finite sequence of applications of $h^{L_\lambda(\mathbb{R})}$ that defines it, and since $h^{L_\lambda(\mathbb{R})}$ is definable in $L_\lambda(\mathbb{R})$ using $\mathbb{R}$ as parameter, we have that x is definable in $L_\lambda(\mathbb{R})$ with parameters in $\mathbb{R} \cup \{\mathbb{R}\}$. Also, this constructions will give a partial surjection from ω^ω to $H_c^{L_\lambda(\mathbb{R})}$. On the other hand, if x is definable in $L_\lambda(\mathbb{R})$ with parameters in $\mathbb{R} \cup \{\mathbb{R}\}$, then it is of course in $H_c^{L_\lambda(\mathbb{R})}$. We proved the following:

Lemma 12.9 (ZF+$V = L(\mathbb{R})$)**.** *For any λ limit ordinal, $H_c^{L_\lambda(\mathbb{R})} = \{x \in L(\mathbb{R}) : x$ is definable in $L_\lambda(\mathbb{R})$ with parameters in $\mathbb{R} \cup \{\mathbb{R}\}\} \prec L_\lambda(\mathbb{R})$, and there exists a surjection from ω^ω to $H_c^{L_\lambda(\mathbb{R})}$ definable in $L_\lambda(\mathbb{R})$*

In L, we have that all the subsets of ω^ω were in $L_{\aleph_2}$. Here it is more complicated. We are trying now to see where the subsets of $\mathbb{R}$ in $L(\mathbb{R})$ live, and whether knowing that they are definable gives us more information to pinpoint their location.

As we have already noted, in $L(\mathbb{R})$ the Axiom of Choice could not hold. It does not make sense, therefore, to ask for the cardinality of $\mathbb{R}$. There is a way, though, to gauge its largeness, through surjections:

Definition 12.10. $\Theta = \sup\{\alpha : \exists\pi : \omega^\omega \twoheadrightarrow \alpha,\ \pi \in L(\mathbb{R})\}$

This ordinal is *key* in the descriptive set theory $L(\mathbb{R})$, because it is the ordinal that encompasses the whole third-order theory on the reals (i.e., the theory of the sets of reals). We explain better what does this mean:

Lemma 12.11. *$L_\Theta(\mathbb{R}) = \{x \in L(\mathbb{R}) : \exists\pi : \omega^\omega \twoheadrightarrow tc(x)\}$. Therefore $\mathcal{P}(\mathbb{R}) \subseteq L_\Theta(\mathbb{R})$.*

Proof. Suppose $x \in L_\Theta(\mathbb{R})$. Then there is a limit ordinal $\lambda < \Theta$ such that $x \in L_\lambda(\mathbb{R})$. In particular $tc(x) \subseteq L_\lambda(\mathbb{R})$. But there is a surjection $\pi_1 : \omega^\omega \to \lambda$, and a surjection $\Phi_\lambda : \lambda \times \omega^\omega \to L_\lambda(\mathbb{R})$, so combining the two there is a surjection from ω^ω to $L_\lambda(\mathbb{R})$, and therefore also to $tc(x)$.

Suppose $x \in L(\mathbb{R})$ and that there exists $\pi : \omega^\omega \twoheadrightarrow tc(x)$. Let λ be a limit ordinal such that $x \in L_\lambda(\mathbb{R})$. Let $X = H_1^{L(\mathbb{R})}(x)$. But then there is a γ such that the transitive collapse of X is $L_\gamma(\mathbb{R})$, and $x \in tc(x) \subseteq L_\gamma(\mathbb{R})$. The surjection h_x to $H_1^{L(\mathbb{R})}(x)$ proves that $\gamma < \Theta$. $\square$

So in particular $\delta_\mathbb{R} < \Theta$. There is also a sort of viceversa of $\mathcal{P}(\mathbb{R}) \subseteq L_\Theta(\mathbb{R})$:

Lemma 12.12 (ZF+$V = L(\mathbb{R})$)**.** *Let $X \in L_\Theta(\mathbb{R})$. Then there exist $E, E' \subseteq \mathbb{R}$ such that $M = (\mathbb{R}, E, E') \equiv_1 (X, =, \in)$ and the transitive collapse of $\mathbb{R}^M$ is $\mathbb{R}$.*

Proof. Since $X \in L_\Theta(\mathbb{R})$, there exists $\pi : \omega^\omega \twoheadrightarrow tc(X)$. Just define $(a, b) \in E$ iff $\pi(a) = \pi(b)$ and $(a, b) \in E'$ iff $\pi(a) \in \pi(b)$. $\square$

So, we can think of $L_\Theta(\mathbb{R})$ as the set of subsets of $\mathbb{R}$. What about $\mathbf{\Sigma}_1^2$ sets?

Lemma 12.13 (ZF+$V = L(\mathbb{R})$)**.** *For any $A \subseteq \mathbb{R}$, A is $\mathbf{\Sigma}_1^2$ iff it is Σ_1 definable over $L_{\delta_\mathbb{R}}(\mathbb{R})$.*

Proof. Let A be $\mathbf{\Sigma}^2_1$. So there is a formula φ with quantifiers bounded by $\mathbb{R}$, and $a \in \mathbb{R}$ such that $x \in A$ iff $\exists X \subseteq \mathbb{R}\varphi(x, X, a)$. But if $x \in A$ then there should be a witness X for it in $L_{\delta_{\mathbb{R}}}(\mathbb{R})$, so $x \in A$ iff $\exists X \in L_{\delta_{\mathbb{R}}}(\mathbb{R}),\ X \subseteq \mathbb{R}\varphi(x, X, a)$.

Viceversa, if A is Σ_1 definable over $L_{\delta_{\mathbb{R}}}(\mathbb{R})$, there is a Δ_0 formula φ and $a \in L_{\delta_{\mathbb{R}}}(\mathbb{R})$ such that $x \in A$ iff $L_{\delta_{\mathbb{R}}}(\mathbb{R}) \vDash \exists X\varphi(x, a, X)$. Since $a \in L_{\delta_{\mathbb{R}}}(\mathbb{R})$, there is a $c \in \mathbb{R}$ such that $h(c) = a$, and this formula is $\Sigma_1(\mathbb{R})$, so we can suppose that $a \in \mathbb{R}$. Now, if X witnesses that $x \in A$, then $X \in L_{\delta_{\mathbb{R}}}(\mathbb{R})$, so there exists an $E \subseteq \mathbb{R}$ such that $M = (\mathbb{R}, E) \equiv_1 X$ and the transitive collapse of $\mathbb{R}^M$ is $\mathbb{R}$

We can then write $x \in A$ iff $\exists E \subseteq \mathbb{R}\ M = (\mathbb{R}, E)$ is a well-founded extensional model of $V = L(\mathbb{R})$, $\mathbb{R}^M$ collapses to $\mathbb{R}$ and $M \vDash \varphi(x^M, a^M, X^M)$. □

Remark 12.14. *Let $\varphi(x, X, a)$ be a $\Delta_0(\mathbb{R})$-formula with one first-order and one second-order parameter. Then $\forall x \in \mathbb{R}\ \exists X \subseteq \mathbb{R}\ \varphi(x, X, a)\}$ is equivalent to a $\Sigma_1(\mathbb{R}$ formula with real parameters.*

Proof. For any $x \in \mathbb{R}$, $X \subseteq \mathbb{R}$, define $x \oplus X = \{\langle x, y\rangle : y \in X\}$. Let $\mathcal{A} = \{x \oplus X : \varphi(x, X, a)\}$, and suppose that $\forall x \in \mathbb{R}\ \exists X \subseteq \mathbb{R}\ \varphi(x, X, a)\}$. So $\forall x \in \mathbb{R}\ \exists Y \in \mathcal{A}_x\varphi(x, Y, a)$. In other words, $\exists \mathcal{A}\forall x \in \mathbb{R}\ \exists Y \in \mathcal{A}_x\varphi(x, Y, a)$, and this is $\Sigma_1(\mathbb{R})$. The other direction is immediate. □

Lemma 12.15 (ZF+$V = L(\mathbb{R})$)**.** *For any $A \subseteq \mathbb{R}$, A is $\mathbf{\Delta}^2_1$ iff it is in $L_{\delta_{\mathbb{R}}}(\mathbb{R})$.*

Proof. Suppose that $A \in L_{\delta_{\mathbb{R}}}(\mathbb{R})$. Then, almost vacuously, A is Σ_1 definable over $L_{\delta_{\mathbb{R}}}(\mathbb{R})$, therefore it is $\mathbf{\Sigma}^2_1$. But also $\omega^\omega \setminus A$ is in $L_{\delta_{\mathbb{R}}}(\mathbb{R})$, and so it is $\mathbf{\Sigma}^2_1$. Therefore A is $\mathbf{\Delta}^2_1$.

On the other hand, let A be $\mathbf{\Delta}^2_1$. So there are $\varphi(x, X)$ and $\psi(x, X)$ $\Delta_0(\mathbb{R})$ formulas, possibly with parameters in $\mathbb{R}$, such that for any $x \in \mathbb{R}$, $x \in A$ iff $\exists X \subseteq \mathbb{R}\varphi(x, X)$ and $x \notin A$ iff $\exists X \subseteq \mathbb{R}\psi(x, X)$. So $L(\mathbb{R}) \vDash \forall x \in \mathbb{R}\ \exists X\ \varphi(x, X) \vee\ \exists Y\ \psi(x, Y)$. By the remark above, this is equivalent to a $\Sigma_1(\mathbb{R})$ formula with real parameters, therefore also $L_{\delta_{\mathbb{R}}}(\mathbb{R})$ satisfies it. Since it is $\Sigma_1(\mathbb{R})$, there is a witness for it. Let $\gamma < \delta_{\mathbb{R}}$ such that the witness for this sentence is in $L_\gamma(\mathbb{R})$. Then $L_\gamma(\mathbb{R}) \vDash \forall x \in \mathbb{R}\ \exists X, Y \subseteq \mathbb{R}\varphi(x, X) \vee \psi(x, Y)$. Consider $A' = \{x \in \mathbb{R} : L_\gamma(\mathbb{R}) \vDash \exists X\varphi(x, X)\} \in L_{\gamma+1}(\mathbb{R})$, $A' \subseteq A$. If $x \notin A'$, then $L_\gamma(\mathbb{R}) \vDash \exists Y \subseteq \mathbb{R}\ \psi(x, Y)$, so $x \notin A$. But then $A = A' \in L_{\delta_{\mathbb{R}}}(\mathbb{R})$. □

We have seen that every set in $L_\Theta(\mathbb{R})$ can be pulled down as a structure in $\mathbb{R}$ that is very similar. When we do this with an ordinal, we have a very interesting structure: the prewellorder.

Let $\alpha < \Theta$ and let $\pi : \omega^\omega \twoheadrightarrow \alpha$. Then define the relation on the reals $x \preccurlyeq_\pi y$ iff $\pi(x) \leq \pi(y)$. Which properties does this relation have? It is reflexive and transitive. It is linear. It is well-founded, as a descending sequence of $\preccurlyeq$ will induce a descending sequence of $\leq$ in α. The only property that is missing for it to be a well-order is anti-simmetry: if $x \neq y$ but $\pi(x) = \pi(y)$, then $x \preccurlyeq_\pi y$, $y \preccurlyeq_\pi x$, but they are different. This is what is called a prewellorder, pwo for short. So now $A_\pi = \{\langle x, y\rangle : x \preccurlyeq_\pi y\}$ is a set of reals that codes α.

We can now introduce the cardinal we want to be measurable:

Definition 12.16. $\delta^1_2 = \sup\{\alpha < \Theta : \exists \pi : \omega^\omega \twoheadrightarrow \alpha,\ A_\pi \textit{ is } \mathbf{\Delta}^2_1\}$.

It is easy to see that $\delta^2_1 \leq \delta_{\mathbb{R}}$:

Lemma 12.17 (ZF+$V = L(\mathbb{R})$)**.** $\delta^2_1 \leq \delta_{\mathbb{R}}$.

Proof. Let $\alpha < \delta^2_1$, $\pi : \omega^\omega \twoheadrightarrow \alpha$ with A_π $\mathbf{\Delta}^2_1$. Then $A_\pi \in L_{\delta_{\mathbb{R}}}(\mathbb{R})$. The sentence "$ot(A_\pi) = \delta$" is "there is an morphism from $(\bigcup A_\pi, A_\pi)$ to $(\delta, \leq)$, so it is a $\Sigma_1(\mathbb{R})$ sentence. Since $L(\mathbb{R}) \vDash \exists \alpha\ ot(A_\pi) = \alpha$, then this is true also in $L_{\delta_{\mathbb{R}}}(\mathbb{R})$, so $\alpha < \delta_{\mathbb{R}}$. □

Lemma 12.18 (ZF+$V = L(\mathbb{R})$)**.** *Let $\varphi(a)$ be a formula, $a \in \mathbb{R}$. Let λ be the least such that $L_\lambda(\mathbb{R}) \vDash \varphi(a)$. Then $H^{L_\lambda}_c(\mathbb{R}) = L_\lambda(\mathbb{R})$, and there exists $\pi : \omega^\omega \twoheadrightarrow L_\lambda(\mathbb{R})$, $\pi \in L_{\lambda+1}(\mathbb{R})$.*

Proof. Well, $H^{L_\lambda}_c(\mathbb{R}) \prec L_\lambda(\mathbb{R})$, so there exists a $\bar\lambda$ such that the collapse of $H^{L_\lambda}_c(\mathbb{R})$ is $L_{\bar\lambda}(\mathbb{R})$. But then $L_{\bar\lambda}(\mathbb{R}) \vDash \varphi(a)$, and since λ was the least we have that $\bar\lambda = \lambda$. We have already seen, then, that there is a surjection to $H^{L_\lambda}_c(\mathbb{R}) = L_\lambda(\mathbb{R})$ that is definable over $L_\lambda(\mathbb{R})$, and therefore in $L_{\lambda+1}(\mathbb{R})$. □

Lemma 12.19 (ZF+$V = L(\mathbb{R})$)**.** *$\{\lambda : H^{L_\lambda}_c(\mathbb{R}) = L_\lambda(\mathbb{R})\}$ is cofinal in $\delta_{\mathbb{R}}$.*

Proof. We are going to prove that the set of λ's such that λ is the least that satisfies a $\Sigma_1(\mathbb{R})$ formula with real parameters is cofinal in $\delta_{\mathbb{R}}$, and the lemma will follow.

Suppose not. Then there exists an $\alpha < \delta_{\mathbb{R}}$ such that for any $\delta_0(\mathbb{R})$ formula $\varphi(x, a)$, with x free variable and $a \in \mathbb{R}$, either for all $\lambda < \delta_{\mathbb{R}}$ $L_\lambda(\mathbb{R}) \nvDash \exists x\ \varphi(x, a)$, or there exists $\lambda < \alpha$ such that $L_\lambda(\mathbb{R}) \vDash \exists x\ \varphi(x, a)$. In the first case, since $L_{\delta_{\mathbb{R}}}(\mathbb{R}) \prec^{\mathbb{R}\cup\{\mathbb{R}\}}_1 L(\mathbb{R})$, simply $L(\mathbb{R}) \nvDash \exists x\ \varphi(x, a)$. But then if $L(\mathbb{R}) \vDash \exists x\ \varphi(x, a)$, there must be a $\lambda < \alpha$ such that $L_\lambda(\mathbb{R}) \vDash \exists x\ \varphi(x, a)$. This means that $L_\alpha(\mathbb{R}) \prec^{\mathbb{R}\cup\{\mathbb{R}\}}_1 L(\mathbb{R})$, contradiction. □

Lemma 12.20 (ZF+$V = L(\mathbb{R})$)**.** $\delta^2_1 = \delta_{\mathbb{R}}$.

Proof. Suppose by contradiction that $\delta^2_1 < \delta_{\mathbb{R}}$. Then there is an ordinal α between δ^2_1 and $\delta_{\mathbb{R}}$ such that there is a $\pi : \omega^\omega \twoheadrightarrow L_\alpha(\mathbb{R})$ with $\pi \in L_{\alpha+1}(\mathbb{R})$. Since $\alpha > \delta^2_1$, we can restrict π to $\pi' : \omega^\omega \twoheadrightarrow \delta^2_1$, again with $\pi' \in L_{\alpha+1}(\mathbb{R})$. But then $A_{\pi'} \in L_{\delta_{\mathbb{R}}}(\mathbb{R})$, and so it is a $\mathbf{\Delta}^2_1$ pwo of length δ^2_1; contradiction. □

Now we are going to define a partial map from ω^ω to δ^2_1, so that its domain is Σ^2_1 and all the counterimages of singletons are $\mathbf{\Delta}^2_1$, with the objective to prove that δ^2_1 is measurable.

Definition 12.21. *We say that* Γ *is a* pointclass *iff it is a collection of subsets of perfect product spaces which is closed under continuous inverse image.*

A pointclass is non-selfdual *if there exists an* $A \in \Gamma$ *such that its complement is not in* Γ*.*

Lemma 12.22 (ZF+AD)**.** *Let* Γ *be a non-selfdual pointclass of sets of reals. Then there is a set that is* ω^ω*-universal for* Γ*.*

Proof. We have already noticed that any $\bar{f} : \omega^{<\omega} \to \omega^{<\omega}$ uniquely defines a continuous function $f : \omega^\omega \to \omega^\omega$, and that every continuous function f is defined by some $\bar{f}$ as above. But we can code $\omega^{<\omega}$ as ω, and therefore every $\bar{f}$ can be considered a real. Therefore, every real $y \in \omega^\omega$ defines a continuous function $f_y : \omega^\omega \to \omega^\omega$, and every continuous function $f : \omega^\omega \to \omega^\omega$ is defined by (at least) a real.

Consider the function $F : \omega^\omega \times \omega^\omega \to \omega^\omega$ defined as $F(y, x) = f_y(x)$. This is actually a continuous function: if $f_y(x) \in N_s$ for some $s \in \omega^{<\omega}$, then by continuity of f_y there is a $t \in \omega^{<\omega}$ such that $f_y'' N_t \subseteq N_s$. By the definition of f_y, there exists a $u \in \omega^{<\omega}$ such that for any $y \in N_u$ $f_y'' N_t \subseteq N_s$, therefore $F'' N_u \times N_t \subseteq N_s$.

We use Wadge's Lemma, that says that if Γ is a class of sets of reals that are determined (so, in our case, any class), then for any $A, B \in \Gamma$ either $B \leq_W A$ or $A \leq_W \tilde{B}$. Let $A \in \Gamma \setminus \tilde{\Gamma}$. For any $y \in \omega^\omega$, let f_y be the continuous function coded by y. Define $U = \{(y, x) \in \omega^\omega \times \omega^\omega : f_y(x) \in A\}$. Then $U \in \Gamma$, because it is the inverse image of A under the continuous function F. Let $B \in \Gamma$. By Wadge's Lemma, either $B \leq_W A$ or $A \leq_W \tilde{B}$, but in the second case we would have that $A \in \tilde{\Gamma}$, contradiction, so $B \leq_W A$. This means that there is a continuous function $f : \omega^\omega \to \omega^\omega$ such that $x \in B$ iff $f(x) \in A$. Let $y \in \omega^\omega$ that codes f. Then $x \in B$ iff $f_y(x) \in A$ iff $x \in U_y$, so $B = U_y$ and U is ω^ω-universal. □

The class $\mathbf{\Sigma}^2_1$ is closed under counterimages of continuous functions, so under AD there is a ω^ω-universal set for it. Let us call it U'. Now consider $U = \{\langle x, y\rangle : (x, y) \in U'\}$. Since U' is a Σ^2_1 set, then also U is a Σ^2_1 set, so there is a formula $\exists X \subseteq \omega^\omega \psi_U(x, X)$ such that ψ_U is $\delta_0(\mathbb{R})$ and $x \in U$ iff $\exists X \subseteq \omega^\omega \psi_U(x, X)$. For any $\alpha < \delta^2_1$, define $U^\alpha = \{x \in U : L_\alpha(\mathbb{R}) \vDash \exists X \subseteq \omega^\omega\ \psi_U(x, X)\}$. If $x \in U$ then there exists an $X \subseteq \omega^\omega$ such that $\psi_U(x, X)$ holds, but then there exists $\alpha < \delta^2_1$ such that $X \in L_\alpha(\mathbb{R})$, and so $x \in U^\alpha$.

In other words, $U = \bigcup_{\alpha<\delta^2_1} U^\alpha$, and it is an increasing union. Moreover, $U^\alpha \in L_{\delta_\mathbb{R}}(\mathbb{R})$, therefore it is $\boldsymbol{\Delta}^2_1$.

Define $\pi : U \to \delta^2_1$ as: $\pi(x)$ is the smallest such that $x \in U^\alpha$. Now, U is Σ^2_1, and $U^\alpha = \{x \in U : \pi(x) \leq \alpha\}$ is $\boldsymbol{\Delta}^2_1$. Also, π is $\Sigma_1(\mathbb{R})$-definable: $\pi(x) = \alpha$ iff $x \in U$ and there exists $X \in L_\alpha(\mathbb{R})$ $\varphi_U(x, X)$ and for any $\beta < \alpha$ $\forall X \in L_\beta(\mathbb{R})$ $\neg\varphi_U(x, X)$. For $x \in U$, the proposition $\pi(x) = \alpha$ is actually $\Delta_1(\mathbb{R})$-definable, as π is linear. So, once we have $x, y \in U$, the proposition "$\pi(x) \leq \pi(y)$" is $\Delta_1(\mathbb{R})$.

We want to prove that δ^2_1 is measurable with the same proof as ω_1, so we need:

- δ^2_1 regular;
- π is bounded on every $\boldsymbol{\Delta}^2_1$ subset of U;
- some way to choose strategies out of a collection of $< \delta^2_1$ sets.

Actually, the answer for the third problem will provide the answer for the other two.

13 Side plot: Coding Lemmata in $L(\mathbb{R})$

We are now trying to understand what kind of choice we need to prove that δ^2_1 is regular. We first see why $AC_\omega(\mathbb{R})$ suffices to prove that $\mathrm{cof}(\delta^2_1) > \omega$, and then try to see how to generalize this proof.

Lemma 13.1. $\mathrm{cof}(\delta^2_1) > \omega$

Proof. Suppose $(\alpha_n)_{n\in\omega}$ is a cofinal sequence in δ^2_1. Then for any $n \in \omega$, there is at least a Δ^2_1 pwo of ordertype α_n. Consider $\{A \subseteq \mathbb{R} : A \text{ is a } \Delta^2_1 \text{ pwo of ordertype } \alpha_n\}$. If we had the full AC_ω, we could choose one for each $n \in \omega$. But since they are all Δ^2_1 sets, they are all in $L_{\delta_\mathbb{R}}(\mathbb{R})$ sets, so they are image of some real under h. So now consider the collection of sets $\{x \in \mathbb{R} : h(x) \text{ is a pwo of length } \alpha_n\}$. Choose one each, and call it x_n. Then define $\langle n, t\rangle \prec \langle n', t'\rangle$ iff $n < n'$ or $n = n'$ and $(t, t') \in h(x_n)$.

This is a Σ^2_1 pwo of ordertype δ^2_1, because it is defined by a $\Sigma_1(\mathbb{R})$ formula. Now, each $h(x_n)$ is Δ^2_1, so there is a y such that $h(y) = \omega^\omega \setminus h(x_n)$. Consider the collection of sets $\{y \in \mathbb{R} : h(y) = \omega^\omega \setminus h(x_n)\}$, and choose one y_n each. then $\neg(\langle n, t\rangle \prec \langle n', t'\rangle)$ iff $n' > n$ or $n = n$ and $(y, y') \in h(y_n)$, that is again a Σ_1 formula, so $\prec$ is actually a Δ^2_1 pwo of ordertype δ^2_1, contradiction. □

How to generalize this? Suppose that we have $(\alpha_\eta)_{\eta<\gamma}$ a cofinal sequence in δ^2_1, with $\gamma < \delta^2_1$. Then there is a pwo A_π of ordertype γ. If we could choose one x_η for any $\eta < \gamma$ so that $h(x_\eta)$ is a Δ^2_1 pwo of ordertype α_η, then we could define $\langle x, t\rangle \prec \langle x', t'\rangle$ iff $x \prec_\pi x'$ or $(x, x') \in A_\pi$ and $(t, t') \in h(x_{\pi(x)})$. This would be a pwo of length δ^2_1, but it would be Σ^2_1 only if $(x_\eta)_{\eta<\gamma} \in L_{\delta_\mathbb{R}}(\mathbb{R})$. So we do not want just to choose the pwos, we want to choose them in a "simple" way.

But why asking for just one pwo for every η? Suppose that for any $\eta < \gamma$ there is a $Z_\eta \subseteq \mathbb{R}$, Δ^2_1, such that for any $x \in Z_\eta$ $h(x)$ is a pwo of ordertype α_η. Also, suppose that for any α, $\{Z_\eta : \eta < \alpha\}$ is $\Delta_1(\mathbb{R})$. Now define $\langle a, x, t\rangle \prec \langle a', x', t'\rangle$ iff $\pi(a) < \pi(a')$ or $\pi(a) = \pi(a')$, $x, x' \in Z_{\pi(a)}$ and the rank of t in the pwo $h(x)$ is less than the rank of t' in the pwo $h(x')$. If we couls prove that comparing the two ranks is Δ^2_1, then $\prec$ would also be Δ^2_1, and we could reach a contradiction. So the full choice is not needed, we just need to find, for avery collection of sets of length $< \delta$, a collection of subsets whose initial segments are $\Delta_1(\mathbb{R})$.

We are going first to prove the Coding Lemma for WO, so that we get familiar with the proof and we can generalize it to the third order.

Lemma 13.2 (ZF+AD)**.** *Suppose $Z \subseteq WO \times \omega^\omega$. Then there exists a $\boldsymbol{\Sigma}^1_2$ set Z^* such that $Z^* \subseteq Z$ and for all $\alpha < \omega_1$, $Z^* \cap (WO_\alpha \times \omega^\omega) \neq \emptyset$ iff $Z \cap (WO_\alpha \times \omega^\omega)$.*

As a basic example, consider a set Z inside WO (any set, any complexity). Then we can imagine it sliced in a partition, where every set in the partition has elements of the same norm. Then we can find a Z^* that is a subset of Z, it is $\boldsymbol{\Sigma}^1_2$ (and therefore "simple"), and yet it touches all the sets in the partition.

Proof. Consider the game

$$\begin{array}{llllll} \mathrm{I} & x(0) & & x(1) & & \\ & & & & & \dots \\ \mathrm{II} & & y(0) & & y(1) & \end{array}$$

where II wins iff whenever $x \in WO$ then y codes a countable set Y such that $Y \subseteq Z$ and for all $\alpha \leq \|x\|$, $Y \cap (WO_\alpha \times \omega^\omega) \neq \emptyset$ iff $Z \cap (WO_\alpha \times \omega^\omega) \neq \emptyset$. The idea is that I challenges II playing a countable ordinal, $\|x\|$, and II meets this challenge providing a selector Y up until $\|x\|$.

The claim is that there can be no winning strategy for player I in this game. Assume by contradiction that σ is a winning strategy for I. The set $X = \{(\sigma * y)_I : y \in \omega^\omega\}$ is $\Sigma^1_1(\sigma)$ and, since σ is winning for I, $X \subseteq WO$. But then by $\boldsymbol{\Sigma}^1_1$-boundedness there is a $\beta < \omega_1$ such that $X \subseteq WO_{<\beta}$. Using $AC_\omega(\mathbb{R})$, choose for any $Z \cap (WO_\alpha \times \omega^\omega)$, with $\alpha < \beta$, an element, and let

Y be this selector. Let y code Y and play y against σ. Then the resulting play is a win for II, contradiction.

Then II has a winning strategy τ. For $x \in WO$, let Y^x be the countable subset of Z that emerges from the play $x * \tau$. Let $Z^* = \bigcup\{Y^x : x \in WO\}$. Then Z^* is $\Sigma^1_2(\tau)$, $Z^* \subseteq Z$, and for any $\alpha < \omega_1$ $Z^* \cap (WO_\alpha \times \omega^\omega) \neq \emptyset$ iff $Z \cap (WO_\alpha \times \omega^\omega) \neq \emptyset$. $\square$

In fact, it is possible to slightly improve this result, and find a Z^* that is of the form $X \cap (WO \times \omega^\omega)$, where $X \subseteq \omega^\omega \times \omega^\omega$ is $\mathbf{\Sigma}^1_1$. With this generalization, it is possible to prove in ZF+AD that the club filter on ω_1 is normal. The details are in Koellner's paper.

To bring this result to the third order, we need to introduce some notation. Given $P \subseteq \omega^\omega$, the notion of $\mathbf{\Sigma}^1_1(P)$ is defined exactly as that of a $\mathbf{\Sigma}^1_1$, but allowing reference to P and to $\omega^\omega \setminus P$. This notation is flexible: we can allow for example $P, P' \subseteq \omega^\omega$, or $P \subseteq (\omega^\omega)^n$.

This does not change anything for the construction of a universal set: every $\mathbf{\Sigma}^1_1$ set is the projection of the body of a tree recursive in P, and therefore we can code all the trees recursive in P with one tree recursive in P, whose projection will be universal. We want also universal sets for n-uples, with the following property:

Theorem 13.3 (s-m-n Theorem, or the Good Parametrization Lemma)**.** *Let U universal set for $\mathbf{\Sigma}^1_1(P)$ sets in ω^ω. Then for any n there is a $U^{(n)} \subseteq (\omega^\omega)^{n+1}$ universal for $\mathbf{\Sigma}^1_1(P)$ sets in $(\omega^\omega)^n$ such that for any $n < m \in \omega$ there is a continuous function $s^m_n : (\omega^\omega)^{n+1} \to \omega^\omega$ such that $(y, x_1, \dots, x_m) \in U^{(m)}$ iff $(s^m_n(y, x_1, \dots, x_n), x_{n+1}, \dots, x_m) \in U^{(m-n)}$.*

Proof. Fix U the ω^ω-universal set for $\mathbf{\Sigma}^1_1(P)$ sets in ω^ω. Note that the formula "$x \in U$" is $\Sigma^1_1(P)$. Now define

$U^{(n)} = \{(y, x_1, \dots, x_n) \in (\omega^\omega)^{n+1} : (y_0, \langle y_1, x_1, \dots, x_n\rangle) \in U\}$, with $\langle y_0, y_1\rangle = y$.

So fix $m > n$. Consider

$A = \{\langle\langle y, x_1, \dots, x_n\rangle, x_{n+1}, \dots, x_m\rangle : (y_0, \langle y_1, x_1, \dots, x_n\rangle) \in U$, with $y = \langle y_0, y_1\rangle\}$.

Then A is $\Sigma^1_1(P)$, therefore there exists an $\epsilon \in \omega^\omega$ such that $A = U_\epsilon$. Fix it. Now define $s^m_n(y, x_1, \dots, x_n) = \langle \epsilon, \langle y, x_1, \dots, x_n\rangle\rangle$.

But now $(y, x_1, \dots, x_m) \in U^{(m)}$ iff $(y_0, \langle y_1, x_1, \dots, x_m\rangle) \in U$ iff

$$\langle\langle y, x_1, \dots, x_n\rangle, x_{n+1}, \dots, x_m\rangle \in A = U_\epsilon$$

iff $(\epsilon, \langle\langle y, x_1, \dots, x_n\rangle, x_{n+1}, \dots, x_m\rangle) \in U$. On the other hand,

$$(s^m_n(y, x_1, \dots, x_n), x_{n+1}, \dots, x_m) \in U^{(m-n)}$$

iff $(\langle \epsilon, \langle y, x_1, \ldots, x_n \rangle \rangle, x_{n+1}, \ldots, x_m) \in U^{(m-n)}$ iff $(\epsilon, \langle \langle y, x_1, \ldots, x_n \rangle, x_{n+1}, \ldots, x_m \rangle) \in U$. $\square$

We write $U^{(n)}(P)$ to indicate the (good) universal set for $\mathbf{\Sigma}^1_1(P)$ subsets of $(\omega^\omega)^n$. For $e \in \omega^\omega$, we indicate $U^{(n)}_e(P)$ for the projection of $U^{(n)}(P)$ on the branch e (therefore it is a $\mathbf{\Sigma}^1_1(P)$ set, and for every $\mathbf{\Sigma}^1_1(P)$ set A there is a $e \in \omega^\omega$ such that $A = U^{(n)}_e(P)$.

The s-m-n Theorem is helpful to choose indexes for $\Sigma^1_1(P)$ defined in a uniform way. For example if U_{e_1} an U_{e_2} are two $\Sigma^1_1(P)$ sets, $U_{e_1} \cap U_{e_2}$ is also a $\Sigma^1_1(P)$ set, that can have many indexes. With the theorem above, we can find a continuous function that associates to e_1 and e_2 an index for $U_{e_1} \cap U_{e_2}$:

Theorem 13.4 (Uniform Closure Theorem). *If the class $\Sigma^1_1(P)$ is closed under an operation, then there exists a continuous function on the indexes of the universal sets that witnesses it. For example:*

- *there exists a continuous function $h : \omega^\omega \times \omega^\omega \to \omega^\omega$ such that for any $e_1, e_2 \in \omega^\omega$, $U_{h(e_1,e_2)} = U_{e_1} \cap U_{e_2}$;*
- *let $\psi(x, a, U_e)$ be a formula that defines a $\Sigma^1_1(P)$ set, $A_{a,e}$.; then there exists a continuous $f : \omega^\omega \to \omega^\omega$ such that $U_{f(a.e)} = A_{a,e}$;*
- *let $\psi(x, a, U_e)$ and $A_{a,e}$ as above; then there is a continuous $h : \omega^\omega \to \omega^\omega$ such that $U_{h(e)} = \bigcup_{a \in \omega^\omega} A_{a,e}$.*

Proof. We prove only the first example, and from such proof the general method will be clear. Note also that we are going to prove it for U, but it is easy to generalize it for any $U^{(n)}$.

Let $\psi_U(e, x)$ be the $\Sigma^1_1(P)$ formula that defines U. Then for any $e_1, e_2 \in \omega^\omega$, $x \in U_{e_1} \cap U_{e_2}$ iff $x \in U_{e_1}$ and $x \in U_{e_2}$ iff $\psi_U(e_1, x) \wedge \psi_U(e_2, x)$ holds. Let $A = \{(e_1, e_2, x) \in (\omega^\omega)^3 : \psi_U(e_1, x) \wedge \psi_U(e_2, x)\}$. Then A is $\Sigma^1_1(P)$. We choose an $\epsilon \in \omega^\omega$ such that $A = U^{(3)}_\epsilon$. Now consider s^3_1. Then $x \in U_{e_1} \cap U_{e_2}$ iff $(e_1, e_2, x) \in U^{(3)}_\epsilon$ iff $(\epsilon, e_1, e_2, x) \in U^{(3)}$ iff $(s^3_1(\epsilon, e_1, e_2), x) \in U$ iff $x \in U_{s^3_1(\epsilon,e_1,e_2)}$. So define $h(e_1, e_2) = s^2_1(\epsilon, e_1, e_2)$. $\square$

Note that the continuous functions that witnesses can be many (one for every choice of ϵ), but we can choose just one.

Theorem 13.5 (Kleene's Recursion Theorem). *Suppose $f : \omega^\omega \to \omega^\omega$ is $\mathbf{\Sigma}^1_1(P)$. Then there is an $e \in \omega^\omega$ such that $U^{(2)}_e(P) = U^{(2)}_{f(e)}(P)$.*

Proof. For $a \in \omega^\omega$, let $T_a = \{\langle b, c \rangle : (a, a, b, c) \in U^{(3)}(P)\}$. Consider $d : \omega^\omega \to \omega^\omega$ defined as $d(a) = s^3_1(a, a)$, so $(a, a, b, c) \in U^{(3)}(P)$ iff $(d(a), b, c) \in U^{(2)}(P)$, in other words $T_a = U^{(2)}_{d(a)}(P)$. Let $Y = \{(a,b,c) : (b,c) \in U^{(2)}_{f(d(a))}(P)\}$; this is a $\mathbf{\Sigma}^1_1(P)$ set on $(\omega^\omega)^3$, so there is a $a_0 \in \omega^\omega$ such that $Y = U^{(3)}_{a_0}(P)$. But then $(b,c) \in U^{(2)}_{d(a_0)}(P)$ iff $(d(a_0),b,c) \in U^{(2)}(P)$ iff $(a_0,a_0,b,c) \in U^{(3)}(P)$ iff $(a_0,b,c) \in U^{(3)}_{a_0}(P) = Y$ iff $(b,c) \in U^{(2)}_{f(d(a_0))}(P)$, so $d(a_0)$ is as desired. $\square$

Theorem 13.6 (Moschovakis' Coding Lemma)**.** *Assume ZF+AD. Suppose $X \subseteq \omega^\omega$ and $\pi : X \to Ord$. Suppose $Z \subseteq X \times \omega^\omega$. Let $Q = \{\langle a,b\rangle : \pi(a) \leq \pi(b)\}$ and $Q_a = \{b : \pi(b) = \pi(a)\}$. Then there is an $e \in \omega^\omega$ such that $U^{(2)}_e(Q) \subseteq Z$ and for all $a \in X$, $U^{(2)}_e(Q) \cap (Q_a \times \omega^\omega) \neq \emptyset$ iff $Z \cap (Q_a \times \omega^\omega) \neq \emptyset$.*

Proof. Assume toward a contradiction that there is no such e. Consider $G = \{e \in \omega^\omega : U^{(2)}_e(Q) \subseteq Z\}$. So for each $e \in G$, there should be an $a \in X$ such that $U^{(2)}_e(Q)$ "misses" $Z \cap (Q_a \times \omega^\omega)$. Let α_e be the least section "missed", i.e., $\alpha_e = \min\{\alpha : \exists a \in X\ \pi(a) = \alpha \wedge U^{(2)}_e(Q) \cap (Q_a \times \omega^\omega) = \emptyset \wedge Z \cap (Q_a \times \omega^\omega) \neq \emptyset\}$.

Now play the game

I	$x(0)$		$x(1)$		$\dots$
II		$y(0)$		$y(1)$	

where I wins if $x \in G$ and either $y \notin G$ or $\alpha_x \geq \alpha_y$. By our initial assumption, α_x always exists for $x \in G$, and $U^{(2)}_e(Q) \cap (Q_{<\alpha_x} \times \omega^\omega)$ is a selector for $Z \cap (Q_{<\alpha_x} \times \omega^\omega)$.

We claim that I does not have a winning strategy. Suppose that σ is a winning strategy for I. Since σ is winning, then for any $y \in \omega^\omega$, $U^{(2)}_{(\sigma * y)_I}(Q) \subseteq Z$. Note that $\bigcup_{y \in \omega^\omega} U^{(2)}_{(\sigma * y)_I}(Q)$ is a $\mathbf{\Sigma}^1_1(Q)$ set, so it is $U^{(2)}_{e_\sigma}(Q)$ for some $e_\sigma \in \omega^\omega$. But then $\alpha_{e_\sigma} \geq \alpha_{(\sigma * y)_I}$ for any $y \in \omega^\omega$. Choose $a \in X$ such that $\pi(a) = \alpha_{e_\sigma}$, and pick $(x_1,x_2) \in Z \cap (Q_a \times \omega^\omega)$. Now still $U^{(2)}_{e_\sigma} \cup \{(x_1,x_2)\}$ is $\mathbf{\Sigma}^1_1(Q)$, so let e^* index that. Then $e^* \in G$, and $\alpha_{e_\sigma} < \alpha_{e^*}$. Let II play e^*. Then $\alpha_{(\sigma * e^*)_I} \leq \alpha_{e_\sigma} < \alpha_{e^*}$, so II wins, contradiction.

Now we claim that also II does not have a winning strategy. This will lead to a contradiction under AD. Assume then that τ is a winning strategy for II.

Given $e \in \omega^\omega$ and $a \in X$, then $U^{(2)}_e(Q) \cap (Q_{<a} \times \omega^\omega)$ is $\mathbf{\Sigma}^1_1(Q)$, so let $h_0 : \omega^\omega \times X \to \omega^\omega$ be a function that associate to e, a as before the index for that set (from the Uniform Closure Theorem), so $U^{(2)}_{h_0(e,a)}(Q) = U^{(2)}_e(Q) \cap (Q_{<a} \times \omega^\omega)$. Such h_0 is $\mathbf{\Sigma}^1_1(Q)$. So the set coded by $h_0(e,a)$ is the first $\pi(a)$ slices of the set coded by e, that therefore cannot select anything in the $\pi(a)$-th slice.

For any $e \in \omega^\omega$, also this set is $\mathbf{\Sigma}^1_1(Q)$: $\bigcup_{a \in X}(U^{(2)}_{(h_0(e,a)*\tau)_{II}}(Q) \cap (Q_a \times \omega^\omega))$, so let $h_1 : \omega^\omega \to \omega^\omega$ be a function that associate to e the code of this set (from the Uniform Closure Theorem). In other words, for any $a \in X$, we let I play the first $\pi(a)$ slices of the set $U^{(2)}_e(Q)$ against the strategy τ, and since τ is winning it will be that either the set played by I is not contained in Z, or II plays something that select the $\pi(a)$-th slice. For any $a \in X$ we consider the intersection of what II played with the $\pi(a)$-th slice, and we unite everything.

By the Recursion Theorem there is a fixed point for h_1, so an e^* such that $U^{(2)}_{e^*}(Q) = U^{(2)}_{h_1(e^*)}(Q)$. So if I plays an initial segment of $U^{(2)}_{e^*}(Q)$, then II will play something whose next slice is the same. It would be great if $e^* \in G$...

Suppose for contradiction that $U^{(2)}_{e^*}(Q) \setminus Z \neq \emptyset$. Choose $(x_1, x_2) \in U^{(2)}_{e^*}(Q) \setminus Z$ with $\pi(x_1)$ minimal. so $(x_1, x_2) \in U^{(2)}_{e^*}(Q) = U^{(2)}_{h_1(e^*)}(Q) = \bigcup_{a \in X}(U^{(2)}_{(h_0(e^*,a)*\tau)_{II}}(Q) \cap (Q_a \times \omega^\omega))$. So let $a \in X$ such that $(x_1, x_2) \in (U^{(2)}_{(h_0(e^*,a)*\tau)_{II}}(Q) \cap (Q_a \times \omega^\omega))$. Now, $\pi(x_1) = \pi(a)$ was minimal, so the initial segment of $U^{(2)}_{e^*}(Q)$, coded by $h_0(e^*, a)$, is contained in Z, but then $h_0(e^*, a) \in G$. Since τ is a winning strategy, $(h_0(e^*, a) * \tau)_{II} \in G$, and so $(x_1, x_2) \in Z$, contradiction.

Suppose now, for contradiction, that α_{e^*} exists. Let $a \in X$ such that $\pi(a) = \alpha_{e^*}$. Thus $h_0(e^*, a) \in G$ and $\alpha_{h_0(e^*,a)} = \alpha_{e^*}$. Since τ is winning, $\alpha_{(h_0(e^*,a)*\tau)_{II}} > \alpha_{h_0(e^*,a)} = \alpha_{e^*}$, which is impossible, because $U^{(2)}_{(h_0(e^*,a)*\tau)_{II}}(Q) \subseteq U^{(2)}_{e^*}(Q)$.

But then e^* codes a selector, and we assumed that there was no selector, so II cannot have a winning strategy. So both I and II do not have a winning strategy, contradiction with AD. $\square$

We are also remarking that there exists a Uniform version of the Coding Lemma, but we are not providing the proof, that is in Koellner-Woodin paper:

Theorem 13.7 (Uniform Coding Lemma). *Assume ZF+AD. Suppose $X \subseteq \omega^\omega$ and $\pi : X \to \mathrm{Ord}$. Suppose $Z \subseteq X \times \omega^\omega$. For any $a \in X$, let $A_{<a} = \{b \in X : \pi(b) < \pi(a)\}$ and $A_a = \{b \in X : \pi(b) = \pi(a)\}$. Then there exists an $e \in \omega^\omega$ such that for all $a \in X$ $U^{(2)}_e(A_{<a}, A_a) \subseteq Z \cap (A_a \times \omega^\omega$, and $U^{(2)}_e(A_{<a}, A_a) \neq \emptyset$ iff $Z \cap (A_a \times \omega^\omega) \neq \emptyset$.*

In other words, the Coding Lemma proves that if we have a Z partitioned via a π function to the ordinals, there is a selector for Z that has the same complexity as π (or it is $\mathbf{\Sigma}^1_1$ if Z is less complex than that). So we can extrapolate the following corollary:

Corollary 13.8 (ZF+AD). *Let $X \subseteq \omega^\omega$, $\pi : X \to \mathrm{Ord}$ such that A_π is $\mathbf{\Delta}^2_1$*

(therefore also X must be $\mathbf{\Delta}^2_1$), $Z \subseteq X \times \omega^\omega$. Then there is a selector $Z^* \subseteq Z$ that is $\mathbf{\Delta}^2_1$.

The Uniform Coding Lemma permits a bit more:

Corollary 13.9 (ZF+AD)**.** *Let $X \subseteq \omega^\omega$, $\pi : X \to \mathrm{Ord}$ such that for any $a \in X$ $A_{<a}$ and A_a are Δ^2_1 (therefore A_π and X are Σ^2_1), $Z \subseteq X \times \omega^\omega$. Then there is a selector $Z^* \subseteq Z$ such that for any $a \in X$ $Z^* \cap (A_{<a} \times \omega^\omega)$ is uniformly Δ^2_1, that is, there is a single Δ^2_1 formula with parameter a that defines $Z^* \cap (A_{<a} \times \omega^\omega)$ (and therefore Z^* is Σ^2_1).*

14 For the last time, back at the main plot

The Coding Lemma now is the choice principle that we need to prove for δ^2_1 what we proved for ω_1, for example that δ^2_1 is regular and that the club filter on δ^2_1 is δ^2_1-complete. The idea is that $AC_\omega(\mathbb{R})$ gives already some result up to ω_1, and the Coding Lemma will push the result ro δ^2_1. See for example this:

Lemma 14.1 (ZF+AD+V=$L(\mathbb{R})$)**.** *δ^2_1 is regular.*

Proof. This is an example of how the Coding Lemma works as a choice principle. It is also one of those proof that have an easy idea, but that need tons of formulas to implement it. Therefore we look first at the idea, then we go through the details, but not all the way down the end.

So let $\lambda < \delta^2_1$, and suppose that there exists $g : \lambda \to \delta^2_1$ cofinal. Then there is a $\mathbf{\Delta}^2_1$ pwo π of ordertype λ, and for any $\alpha < \lambda$ there is a $\mathbf{\Delta}^2_1$ pwo of ordertype $g(\alpha)$. If we could choose one pwo π_α of ordertype $g(\alpha)$ for any α we will be done, as we could put all this ordertype together to find a $\mathbf{\Delta}^2_1$ pwo of ordertype δ^2_1, i.e., $\pi'(\langle x, y\rangle) = \pi_{\pi(x)}(y)$, contradiction. But we cannot choose just one. We are therefore going to select a bunch of them in a $\mathbf{\Delta}^2_1$ way, and put them together in a $\mathbf{\Delta}^2_1$ way, so that the contradiction is reached all the same.

Let U be the universal set for $\mathbf{\Sigma}^2_1$ sets in $\omega^\omega \times \omega^\omega$. Let $\pi_0 : \omega^\omega \twoheadrightarrow \lambda$. Then consider the set $Z = \{(\langle x, y\rangle, z) : U(x) = \omega^\omega \times \omega^\omega \setminus U(y),\ U(x)$ is a pwo of length $g(\pi_0(z))\}$. Let $\pi : U \to \mathrm{Ord}$, $\pi(\langle x, y\rangle)$ being the ordertype of $U(x)$. Then π satisfies the hypotheses for the Uniform Coding Lemma. Let Z^* be the selector of Z, π via the Uniform Coding Lemma. So if there are x, y, z such that $\langle x, y\rangle$ code a pwo of length $g(\pi_0(z))$, there are $x', y', z' \in Z^*$ that satisfies the same, and for any $x \in U$, $Z^* \cap (A_{\leq\pi(x)} \times \omega^\omega)$ is $\mathbf{\Delta}^2_1$. Now for any $\langle x, y\rangle$ that defines a $\mathbf{\Delta}^2_1$ pwo, define $\xi_{x,y}$ the ordertype of the pwo, and $\rho(x, y, t)$ as the rank of t in the pwo defined by $\langle x, y\rangle$.

We can now define the pwo of rank δ^2_1. Let $\langle u, x, y, t\rangle <^* (u', x', y', t')$ iff $\pi_0(u) < \pi_0(u')$ or there are v, v' such that $\pi_0(u) = \pi_0(v) = \pi_0(u') = \pi_0(v')$, $(\langle x, y\rangle, v) \in Z^* \cap (A_{\leq\pi(x)} \times \omega^\omega)$, $(\langle x', y'\rangle, v') \in Z^* \cap (A_{<\leq\pi(x')} \times \omega^\omega)$, and $\rho(x, y, t) \leq \rho(x', y', t')$ (in all the other cases they are all related to each other at the bottom, for example). We want to calculate the ordertype of $<^*$. Let us fix the first coordinate. Then $\pi_0(u) \in \lambda$ and $g(\pi_0(u)) \in \delta^2_1$, therefore there are $\langle x, y\rangle$ that code a $\mathbf{\Delta}^2_1$ of length $g(\pi_0(u))$, so $(\langle x, y\rangle, u) \in Z$. Pick $(\langle x', y'\rangle, u') \in Z^*$ that satisfies the same. Then $\langle u', x', y', t\rangle <^* \langle u', x', y', t'\rangle$ iff the rank of t in $U(x')$ is less or equal than the rank of t' in $U(x')$, so we recreated exactly $U(x')$ in $<^*$, that is of length $g(\pi_0(u)$. Therefore the pwo $<^*$ is a pwo with length $g(0) + g(1) + \cdots + g(\beta) + \ldots$ for $\beta \in \lambda$, and as g is cofinal $<^*$ has length δ^2_1.

To reach a contradiction, we have to prove that $<^*$ is $\mathbf{\Delta}^2_1$. The equations via π are $\mathbf{\Delta}^2_1$, as A_π is $\mathbf{\Delta}^2_1$, $Z^* \cap (A_{\leq\pi(x)} \times \omega^\omega)$ is again $\mathbf{\Delta}^2_1$, so it remains to prove that $\rho(x, y, t) \leq \rho(x', y', t')$ is expressible in a $\mathbf{\Delta}^2_1$ way.

This is "there is a set $A \subseteq \xi_{x,y} \times \xi_{x',y'}$ which is an order-preserving map of the initial segment of $U(x)$ up to t onto the initial segment of $U(x')$ up to t'. This is still not $\mathbf{\Delta}^2_1$, but we can choose the set A in a $\mathbf{\Delta}^2_1$ way again with the Coding Lemma. The details are left to the reader, at this point. □

Theorem 14.2 ((ZF+AD+$V = L(\mathbb{R})$) Moschovakis' Boundedness Lemma)**.** *For any $X \subseteq U$, if X is Δ^2_1 then there is a $\gamma < \delta^2_1$ such that $X \subseteq U^{L_\gamma(\mathbb{R})}$.*

Proof. Consider $\pi : U \to \delta^2_1$, the function that associates to x the least α such that $x \in U^\alpha$. Now define $\pi'(x)$ as $\pi(x)$ is $x \in X$, and as 0 if $x \notin X$. Suppose that $\pi[X]$ is unbounded in δ^2_1. Then also $\pi'[X]$ is unbounded in δ^2_1.

Now consider $A = \{(x, y) \in \omega^\omega \times \omega^\omega : \pi'(x) \leq \pi'(y)\}$. This is a pwo, with length the ordertype of $\pi[X]$. Since δ^2_1 is regular, it must be that A has length δ^2_1.

But we can write A like this: we say that $(x, y) \in A$ iff $x \notin X$, or $x \in X$, $y \notin X$ and $\pi(x) = 0$, or $x, y \in X$ and $\pi(x) \leq \pi(y)$. SInce π is defined with a $\Delta_1(\mathbb{R})$ formula, A is Δ^2_1. Contradiction. □

Theorem 14.3 (ZF+AD+$V = L(\mathbb{R})$)**.** *The club filter on δ^2_1 is a δ^2_1-complete ultrafilter.*

Proof. The structure of the proof is the same as in the proof that the club filter on ω_1 is a measure, so we are only going to pinpoint the problematic passages.

The game is the same, with U instead of WO and π instead of $\|\|$.

Consider $X_\alpha = \{(((\sigma * y)_I)_n : n \in \omega,\ y \in \omega^\omega,\ \forall i < n\ (y)_i \in U^\alpha\}$, with σ winning strategy for I. Since U^α is Δ^2_1, then X_α is also Δ^2_1. Since σ is a winning strategy, $X_\alpha \subseteq U$. By Moschovakis Boundedness Lemma, then, there is an α' such that $X_\alpha \subseteq U^{\alpha'}$. The rest follows in the same way, and the club filter on δ^2_1 is an ultrafilter.

Now, let X_η, with $\eta < \gamma < \delta^2_1$, be a $< \delta^2_1$-collection of subsets of ω^ω such that I has a winning strategy for the game G_{X_η}. Instead of choosing one strategy for any X_η, we are going to find a Δ^2_1 set of strategies for it.

So let $\pi_0 : \omega^\omega \twoheadrightarrow \gamma$, and let A_{π_0} be the relative Δ^2_1 pwo. Let $Z = \{(x, \sigma) \in \omega^\omega \times \omega^\omega : \sigma$ is a winning strategy for I for the game $G_{X_{\pi_0(x)}}\}$. Since A_{π_0} is Δ^2_1, by the (normal) Coding Lemma there is a $Z^* \subseteq Z$ selector for Z that is Δ^2_1. So for any $\eta < \gamma$ there is a $(x, \sigma) \in Z^*$ such that σ is a winning strategy for I in G_{X_η}.

Let $Y = \{\sigma \in \omega^\omega : \exists x \in \omega^\omega\ (x, \sigma) \in Z^*\}$. Then Y is still Δ^2_1. Now, fix a winning strategy σ' for I for the game $G_{\bigcap_{\eta<\gamma} X_\eta}$. For any σ, define $X(\sigma, 0) = \{((\sigma * y)_I)_0 : y \in \omega^\omega\}$ and let $X(0) = \bigcup_{\sigma \in Y} X(\sigma, 0) \cup X(\sigma, 0)$. Then $X(0)$ is Δ^2_1, so it is bounded by π, and the rest follows. □

Corollary 14.4 (ZF+AD+$V = L(\mathbb{R})$)**.** *HOD*$\models$ *δ^2_1 is measurable*

Corollary 14.5. *Con(ZF+AD)$\to$Con(ZFC+there exist 2 measurable cardinals).*

We can actually do something more, using the Coding Lemma. We focus now on Θ:

Lemma 14.6 (ZF+$V = L(\mathbb{R})$)**.** *If $\alpha < \Theta$, then there is an OD surjection $\pi : \omega^\omega \to \alpha$.*

Proof. If there exists a surjection $\pi : \omega^\omega \twoheadrightarrow \alpha$, then there exist $\beta \in Ord$ and $c \in \omega^\omega$ such that $\pi = \Phi(\beta, c)$. For each $c \in \omega^\omega$, let β be the smallest such that $\Phi(\beta, c)$ is a surjection from ω^ω to α, and let $\pi_c = \Phi(\beta, c)$.

Now define $\pi : \omega^\omega \to \alpha$:

$$\pi(\langle c, x\rangle) = \begin{cases} \pi_c(x) & \text{if } \pi_c \text{ is defined} \\ 0 & \textit{otherwise.} \end{cases}$$

This is an OD surjection. □

Lemma 14.7 (ZF+$V = L(\mathbb{R})$)**.** *Θ is regular.*

Proof. For any $\alpha < \Theta$, let π_α be an OD surjection from ω^ω to α. Assume for contradiction that Θ is singular, and let $f : \alpha \to \Theta$ be a cofinal map witnessing this. Let $g : \omega^\omega \to \alpha$ be a surjection. Then the map $\pi(\langle x, y\rangle) = \pi_{f(g(x))}(y)$ is a surjection from ω^ω to Θ, contradiction. □

Since Θ is regular in $L(\mathbb{R})$, it is regular in $HOD^{L(\mathbb{R})}$

Theorem 14.8 (ZF+AD+$V = L(\mathbb{R})$)**.** *$HOD \vDash \Theta$ is inaccessible.*

Proof. We just need to prove that Θ is strong limit in HOD, i.e., that for any $\eta < \Theta$, $|\mathcal{P}(\eta)|^{HOD} < \Theta$. We start by proving that there is a surjection $\pi : \omega^\omega \twoheadrightarrow \mathcal{P}(\eta)^{HOD}$.

Consider $\pi_\eta : \omega^\omega \twoheadrightarrow \eta$ that is $\mathbf{\Delta}^2_1$. For any $\alpha < \eta$, let $A_\alpha = \{x \in \omega^\omega : \pi_\eta(x) = \alpha\}$. For any $S \in \mathcal{P}(\eta)^{HOD}$, define $Z_S = \bigcup_{\alpha \in S}(A_\alpha \times \omega^\omega)$. Then we can use the Coding Lemma, and there exists an e such that $U^{(2)}_e(A_{\pi_\eta})$ is a selector for Z. Now consider $\{\beta < \eta : U^{(2)}_e(A_{\pi_\eta}) \cap (A_\beta \times \omega^\omega) \neq \emptyset\}$. We have that $U^{(2)}_e(A_{\pi_\eta}) \cap (A_\beta \times \omega^\omega) \neq \emptyset$ iff $Z \cap (A_\beta \times \omega^\omega) \neq \emptyset$ iff $\beta \in S$, therefore this is exactly S.

For any $e \in \omega^\omega$, let $f(e) = \{\beta < \eta : U^{(2)}_e(A_{\pi_\eta}) \cap (A_\beta \times \omega^\omega) \neq \emptyset\}$. Then we just proved that for any $S \in \mathcal{P}(\eta)^{HOD}$, there is an e such that $f(e) = S$, so f is a surjection from ω^ω to $\mathcal{P}(\eta)^{HOD}$.

Suppose now that $|\mathcal{P}(\eta)|^{HOD} \geq \Theta$. Then there would be in HOD surjection $\rho : \mathcal{P}(\eta) \twoheadrightarrow \Theta$, and $\rho \circ f : \omega^\omega \twoheadrightarrow \Theta$ would be in $L(\mathbb{R}$, contradiction. $\square$

So for now we have the consistency of two measurable cardinals and an inaccessible above. But with a trick we can do much better. Instead of considering $\mathbf{\Sigma}^2_1$ pwos, we can consider $\mathbf{\Sigma}^2_1(P)$ pwos, with $P \subseteq \omega^\omega$, and doing everything again. So we define $\delta^2_1(P)$ has the supremum of the lengths of pwos that are $\mathbf{\Delta}^2_1(P)$, $\delta_{\mathbb{R},P}$ the least such that $L_{\delta_{\mathbb{R},P}}(\mathbb{R}) \prec_1^{P,\mathbb{R}\cup\{\mathbb{R}\}} L(\mathbb{R})$, they are the same, there is a universal set $U(P)$, and a map from it to $\delta^2_1(P)$. We prove $\mathbf{\Delta}^2_1(P)$ boundedness, and actually the old Uniform Coding Lemma is enough is enough to prove everything we need. So also the club filter in $\delta^2_1(P)$ is a measure in HOD.

But let $\alpha < \Theta$ and A_α a pwo of length α. Then A_α is surely $\mathbf{\Delta}^2_1(A_\alpha)$, therefore $\delta^2_1(P) \geq \alpha$. But then the $\delta^2_1(P)$'s are cofinal in Θ. So we proved this:

Theorem 14.9. *Con(ZF+AD+$V = L(\mathbb{R})$) implies Con(ZFC+exists an inaccessible limit of measurable cardinals).*

Where do we go from here? The first step would be to weaken ZF+AD+$V = L(\mathbb{R})$, but in fact the proof of the consistency of ZF+AD gives that the consistency of the former is a consequence of the consistency of the latter. Fyi:

Theorem 14.10 (ZFC)**.** *Suppose that there are ω Woodin cardinals and a measurable cardinal above them. Then $L(\mathbb{R}) \vDash AD$. Also, Con(ZFC+there are ω Woodin cardinals) implies Con(ZF+AD).*

On the other hand, there is a long proof, that passes from the fact that δ^2_1 is λ-strong for any $\lambda < \Theta$ and that Θ is Woodin in HOD, that:

Theorem 14.11. *Con(ZF+AD) implies Con(ZFC+there are ω Woodin cardinals).*

This is all. Thanks for reading.

15 Epilogue

We can prove some small partial results about the consistency of ZF+AD. We need first to highlight a combinatorial property of the measurable cardinals.

Theorem 15.1. *Let κ be a measurable cardinal, let D be a normal measure on κ, and let $F_n : [\kappa]^n \to I$, with $n \in \omega$ and $|I| < \kappa$. Then there exists a set $H \in D$ such that $|F_n'' H| = 1$.*

Proof. We prove this by induction on n. We have already seen the case $n = 1$. Suppose it is true for n. Let $F : [\kappa]^{n+1} \to I$, where $|I| < \kappa$. For each $\alpha < \kappa$, we define F_α on $[\kappa \setminus \{\alpha\}]^n$ as $F_\alpha(x) = F(\{\alpha\} \cup x)$. By induction hypothesis, there exists for each $\alpha < \kappa$ a set $X_\alpha \in D$ such that F_α is constant on $[X_\alpha]^n$. Let i_α be such constant. Let $X = \triangle_{\alpha<\kappa} X_\alpha$. Since D is normal, $X \in D$. If $\gamma < \alpha_1 < \cdots < \alpha_n$ are in X, then $\{\alpha_1, \ldots, \alpha_n\}$ then $\{\alpha_1, \ldots, \alpha_n\} \in [X_\gamma]^n$ and so $F(\{\gamma, \alpha_1, \ldots, \alpha_n\}) = F_\gamma(\{\alpha_1, \ldots, \alpha_n\}) = i_\gamma$. Now, there exist $i \in I$ and $H \in D$ such that $i_\gamma = i$ for all $\gamma \in H$. It follows that $F(x) = i$ for all $x \in [H]^{n+1}$. $\square$

Theorem 15.2 (Martin)**.** *If κ is a measurable cardinal, then all $\mathbf{\Sigma}^1_1$ sets of reals are determined.*

Proof. Let A be $\mathbf{\Sigma}^1_1$, and let T be a tree such that $x \in A$ iff $T(x)$ is ill-founded. Let $\preccurlyeq$ be a linear ordering in $\omega^{<\omega}$ that extends $\supset$ (it can be done in many ways: for example, $s \preccurlyeq t$ iff $s \supset t$, or s and t are incompatible and $s(n) < t(n)$, where n is the least n such that $s(n) \neq t(n)$). Then $x \in A$ iff $T(x)$ is not well-ordered by $\preccurlyeq$.

For any $s \in \omega^{<\omega}$, we define $T_s = \{t : (u, t) \in T, u \sqsubseteq s\}$. We also fix an enumeration of $\omega^{<\omega}$ $t_0, t_1, \ldots$, and we define $K_s = \{t_0, \ldots, t_{n-1}\} \cap T_s$, where $\text{lh}(s) = 2n$, and $k_s = |K_s|$.

We define an auxiliary game G^*:

$$\begin{array}{lccccc} \text{I} & a_0 & & a_1 & & \\ & & & & & \ldots \\ \text{II} & & (b_0, h_0) & & (b_1, h_1) & \end{array}$$

where the a's and b's are natural numbers, and the h_n is an order-preserving mapping from $(K_s, \preccurlyeq)$ into κ, where $s = (a_0, b_0, a_1, b_1, \dots, a_n, b_n)$ and are such that $h_0 \subseteq h_1 \subseteq \dots$. If the rules are always followed, II wins, otherwise I wins. So the game is open, and therefore determined.

If II wins, she has constructed an order-preserving mapping $h = \bigcup_{n\in\omega} h_n$ of $(T(x), \preccurlyeq)$ into κ, with $x = (a_0, b_0, a_1, \dots)$, so that $T(x)$ is well-founded and $x \notin A$. In a certain sense, the game G^* is more difficult for II: not only must II play so that $X \notin A$, but she has also to build a map that witnesses it. So if II has a winning strategy for G^*, then she has a winning strategy for the original game. It remains to prove that if I has a winning strategy for G^*, it has a winning strategy for the original game.

Let σ^* be a winning strategy for I in G^*. Suppose we are at the point of a play where the two players have produced $s = \langle a_0, b_0, \dots, a_n, b_n \rangle$ and II has constructed $h_0 \subseteq \dots \subseteq h_n$ order-preserving. Then h_n is an order-preserving mapping from $(K_s, \preccurlyeq)$ into κ. Let E be the range of h_n: then $|E| = k_s$. Notice that if we fix E, then h_n is unique, it is the only order-preserving map from two finite sets, so σ^* for the next move depends only on s and E.

For each $s \in \omega^{<\omega}$, let $F_s : [\kappa]^{k_s} \to \omega$, $F_s(E) = \sigma^*(s, E)$. Since κ is measurable, there is a set $H \subseteq \kappa$ such that $|F_s'' H| = 1$ for any s. Let $\sigma(s)$ be such value.We want to prove that this is a winning strategy for I.

Let $x = (a_0, b_0, a_1, \dots)$ be a play following the strategy σ. Assume towards a contradiction that $x \notin A$. Then $(T(x), \preccurlyeq)$ is well-ordered, with order-type less than ω_1. Since H is uncountable (much more than that), there exists an embedding h of $(T(x), \preccurlyeq)$ into H. Now consider the play of the game G^* where they both play x, and each time II plays h_n as the restriction of h to K_s. Actually this is a play where I follows σ^*: a_0 is the first step for both σ and σ^*. Now, $a_1 = \sigma((a_0, b_0))$, and $\sigma((a_0, b_0))$ is $\sigma^*(s, E)$ for any $E \in [H]^{k_s}$, in particular for $E = h(K_{(a_0,b_0)})$. And so on. But if $x \notin A$, then II has won a play of G^* where I was playing the winning strategy σ^*, contradiction. Therefore σ is a winning strategy for I.

□