Generalized descriptive set theory under I0

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Joint work with Luca Motto Ros and Xianghui Shi

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Definable subsets: *Borel* sets, *analytic* sets, *projective* sets...
Regularity properties: Perfect set property (PSP), Baire property, Lebesgue measurability...

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By the first and second points, all the other points are true in any zero-dimensional Polish space, and "partially" true in every Polish space.

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Generalized Cantor and Baire spaces: ${}^{\kappa}2$ and ${}^{\kappa}\kappa$, endowed with the *bounded topology*, i.e., the topology generated by the sets $N_s = \{x \in {}^{\kappa}2 : s \sqsupseteq \kappa\}$ with $s \in {}^{<\kappa}2$ or ${}^{<\kappa}\kappa$ respectively

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Regularity properties: κ-PSP for a set A = either $|A| \leq \kappa$ or ${}^{\kappa}2$ topologically embeds into A; κ-Baire property (sometimes)...

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The culprit here seems to be the fact that κ is regular. What if κ is singular? There is already some bibliography on that...

A.H.Stone, *Non-separable Borel sets*, 1962

Baire spaces: $\Pi_{n\in\omega} T_n$, where each T_n is discrete. In particular, the space $B(\lambda) = {}^{\omega}\lambda$ and, if $\mathrm{cof}(\lambda) = \omega$, the space $C(\lambda) = \Pi_{n\in\omega}\lambda_n$, where λ_n's are cofinal in λ

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Baire space: $V_{\lambda+1}$, where λ satisfies I0(λ), with the topology where the open sets are $O_{a,\alpha} = \{x \subseteq V_\lambda : x \cap V_\alpha = a\}$, with $\alpha < \lambda$ and $a \subseteq V_\alpha$

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Regularity properties: different definitions of PSP (the details later).

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We wanted to give some order to this variety of approaches, and define a single framework where they all live, and that is close to the "classical" approach.

Baire and Cantor spaces

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Proposition (Džamonja-Väänänen, D.-Motto Ros)

The following spaces are homeomorphic:

- $^{\lambda}2$
- $\Pi_{n\in\omega}{}^{\lambda_n}2$
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It is therefore immediate to see that when λ is strong limit, then all the spaces above are homeomorphic!
On the other hand, ${}^{\lambda}2 \not\approx {}^{\lambda}\lambda$, as ${}^{\lambda}\lambda$ has density $\lambda^{<\lambda} > \lambda$;

Universality properties

Definition

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Proposition (D.-Motto Ros)

Every λ-Polish space is continuous image of a closed subset of ${}^{\omega}\lambda$, therefore results on the generalized Cantor space partially spread to all λ-Polish spaces.

Definable subsets

On ${}^{\lambda}2$ we consider λ^+-Borel sets, as in GDST. It can be proven that these sets can be stratified in a hierarchy with exactly λ^+-many levels

On ${}^{\lambda}2$ we consider λ^+-Borel sets, as in GDST. It can be proven that these sets can be stratified in a hierarchy with exactly λ^+-many levels. Also, since λ is singular, λ^+-Borel $=$ λ-Borel.

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Classical case

In the classical case, tfae:

- A is a continuous image of a Polish space;
- $A = \emptyset$ or A is a continuous image of ${}^{\omega}\omega$;
- A is a continuous image of a closed set $F \subseteq {}^{\omega}\omega$;
- A is the continuous/Borel image of a Borel subset of ${}^{\omega}\omega$ or ${}^{\omega}2$;
- A is the projection of a closed subset of $X \times {}^{\omega}\omega$;
- A is the projection of a Borel subset of $X \times Y$, where Y is ${}^{\omega}\omega$ or ${}^{\omega}2$.

New case

If λ is a singular cardinal of cofinality ω, tfae:

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- A is the projection of a Borel subset of $X \times Y$, where Y is ${}^{\omega}\lambda$ or ${}^{\lambda}2$.

Again, this is not true if λ is regular.

Proposition (D.-Motto Ros)

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Generalized Souslin theorem (D.-Motto Ros)

A subset of a λ-Polish space is λ-bianalytic iff it is λ-Borel.

Perfect set property

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A.H.Stone

Every λ-analytic subset of a uniformly zero-dimensional λ-Polish space has the λ-PSP.

Silver dichotomy

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Corollary

Let λ be a strong limit cardinal of cofinality ω. Suppose that λ is limit of measurable cardinals. Let E be a coanalytic equivalence relation on a uniformly zero-dimensional λ-Polish space. Then exactly one of the following holds:

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Let λ be a strong limit cardinal of cofinality ω. Suppose that λ is limit of measurable cardinals. Let E be a coanalytic equivalence relation on ${}^{\omega}\lambda$. Then exactly one of the following holds:

- E has at most λ many classes:
- there is a continuous injection $\varphi : {}^{\omega}\lambda \to {}^{\omega}\lambda$ such that for distinct $x, y \in {}^{\omega}\lambda$ $\neg\varphi(x) E \varphi(y)$.

Corollary

Let λ be a strong limit cardinal of cofinality ω. Suppose that λ is limit of measurable cardinals. Let E be a coanalytic equivalence relation on a uniformly zero-dimensional λ-Polish space. Then exactly one of the following holds:

- E has at most λ many classes:
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But the Baire category argument fails. Instead of that, we have some argument that relies on the properness of the diagonal Prikry forcing..

List of things that do not generalize well in this setting:

- λ-analytic sets are not exactly those that are the continuous image of ${}^{\lambda}\lambda$: in fact, it is possible that all the λ-projective sets are the continuous image of ${}^{\lambda}\lambda$

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- λ-analytic sets are not exactly those that are the continuous image of ${}^{\lambda}\lambda$: in fact, it is possible that all the λ-projective sets are the continuous image of ${}^{\lambda}\lambda$
- it is not clear how to define the λ-meager sets: either the countable union of nowhere dense sets, but then they are really small (it is not clear even if Borel sets have the Baire property), or the λ-union of nowhere dense sets, but then the whole space is meager.

A bit further...

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If there are ω Woodin cardinals and a measurable above, then every subset of $\mathbb{R}$ in $L(\mathbb{R})$ is κ-weakly homogeneously Suslin for some κ

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Theorem

All the κ-weakly homogeneously Suslin subsets of ${}^{\omega}2$ have the PSP, the Baire property and are Lebesgue measurable.

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It induces on $L(V_{\lambda+1})$ a structure that is similar to $L(\mathbb{R})$ under AD. For example, $L(V_{\lambda+1}) \not\models AC$, and λ^+ is measurable in $L(V_{\lambda+1})$.

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Shi, *Axiom I_0 and higher degree theory*, 2015

Suppose that j witnesses $\mathrm{I0}(\lambda)$, and that every set in $L(V_{\lambda+1})$ is $U(j)$-representable. Then every set has the λ-PSP (space embedded: $C(\lambda)$).

Cramer, *Inverse limit reflection and the structure of $L(V_{\lambda+1})$*, 2015

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We are looking to generalize the second statement, again defining a unique backdrop that works for any space.

Definition

A family $\mathbb{U}$ of ultrafilters is *orderly* iff there exists a set K such that for all $U \in \mathbb{U}$ there is $n \in \omega$ such that ${}^nK \in U$. Such n is called the *level* of U.

A *tower* of ultrafilters in such a $\mathbb{U}$ is a sequence $(U_n)_{n\in\omega}$ such that for all $m < n < \omega$:

- $U_n \in \mathbb{U}$ has level n;
- U_n projects to U_m;

A tower of ultrafilters $(U_n)_{n\in\omega}$ is *well-founded* iff for every sequence $(A_n)_{n\in\omega}$ with $A_n \in U_n$ there is $z \in {}^{\omega}K$ such that $z \restriction n \in A_n$ for any $n \in \omega$.

Definition

Let $\kappa \geq \lambda$ be a cardinal, and $\mathbb{U}$ be an orderly family of κ-complete ultrafilters. A $(\mathbb{U}, \kappa)$-representation for $Z \subseteq {}^{\omega}\lambda$ is a function $\pi : \bigcup_{i \in \omega} {}^{i}\lambda \times {}^{i}\lambda \to \mathbb{U}$ such that:

- if $s, t \in {}^{i}\lambda$, then $\pi(s, t)$ has level i;
- for any $(s, t) \in {}^{n}\lambda$, if $(s', t') \sqsupseteq (s, t)$ then $\pi(s', t')$ projects to $\pi(s, t)$;
- $x \in Z$ iff there is a $y \in {}^{\omega}\lambda$ such that $(\pi(x \restriction i, y \restriction i))_{i \in \omega}$ is well-founded

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If $\lambda = \omega$ and $A \subseteq \mathbb{R}$ is κ-weakly homogeneously Suslin, then A is $(\mathbb{U}, \kappa)$-representable for some $\mathbb{U}$.
Consider the homeomorphism between $V_{\lambda+1}$ and ${}^{\omega}\lambda$. Then the image of a $U(j)$-representable set is $(\mathbb{U}, \kappa)$-representable for some $\mathbb{U}, \kappa$, and viceversa.

Definition

A $(\mathbb{U}, \kappa)$-representation π for a set $Z \subseteq {}^{\omega}\lambda$ has the *tower condition* if there exists $F : ran\pi \to \bigcup \mathbb{U}$ such that:

- $F(U) \in U$ for all $U \in ran\pi$
- for every $x, y \in {}^{\omega}\lambda$, the tower of ultrafilters $(\pi(x \restriction i, y \restriction i))_{i \in \omega}$ is well-founded iff there is $z \in {}^{\omega}K$ such that $z \restriction i \in F(\pi(x \restriction i, y \restriction i))$ for all $i \in \omega$.

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If κ is much larger than λ (e.g., $\lambda = \omega$ and κ measurable), then the tower condition is for free.

Scott Cramer proved that under I0 every representation has a tower condition.

Theorem (D.-Motto Ros)

Let λ be strong limit with $\mathrm{cof}(\lambda) = \omega$ and let $\kappa \geq \lambda$ be a cardinal. If $Z \subseteq {}^{\omega}\lambda$ admits a $(\mathbb{U}, \kappa)$-representation with the tower condition, then Z has the λ-PSP

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Corollary

Assume $\mathrm{I0}(\lambda)$, as witnessed by j. If $A \in \mathcal{P}(V_{\lambda+1}) \cap L(V_{\lambda+1})$ is $U(j)$-representable, then A has the λ-PSP

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Assume $\mathsf{I0}(\lambda)$. All λ-projective subsets of any uniformly zero-dimensional λ-Polish space have the λ-PSP

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Assume I0(λ), as witnessed by a proper j with $\mathrm{crt}(j) = \kappa$. Let $\mathbb{P}$ be the Prikry forcing on κ with respect to the measure generated by j. Then there exists a $\mathbb{P}$-generic extension $V[G]$ of V in which all κ-projective subsets of any uniformly zero-dimensional κ-Polish space have the κ-PSP.

A look into the future

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Let E be an equivalence relation on ${}^{\omega}2$. Then exactly one of the following holds:

- the classes of E are well-ordered;
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Thanks for watching.