Left distributive algebras beyond I0

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Forget about large cardinals.

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There are some limitations:

In these cases, if j is not trivial, then some ordinals are moved. We call *critical point* of j the least ordinal (cardinal) moved.

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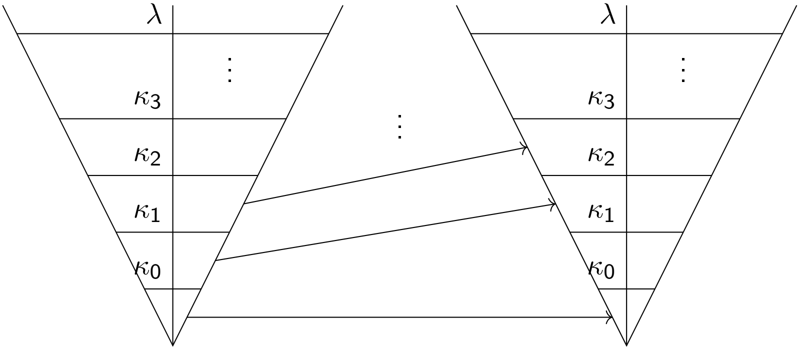
Assumption

I3: There are elementary embeddings $j : V_\lambda \prec V_\lambda$, λ limit.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κ_2
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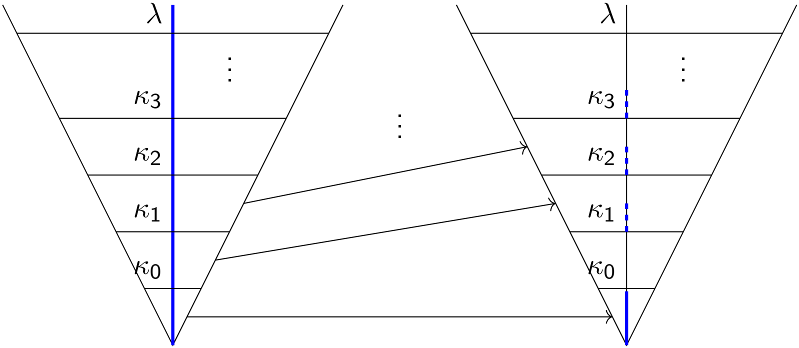

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Then $j^+ : (M, X) \prec (N, j^+(X))$.

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Keep in mind that $j(k)$ is difficult to calculate: while, for example, $j \circ k(x)$ is definable from j, k, x, this is not true for $j \cdot k(x)$, that is known only on $\mathrm{ran}(j)$.

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Let $\equiv_{LD}$ the congruence on T_n generated by all pairs of the form $t_1 \cdot (t_2 \cdot t_3), (t_1 \cdot t_2) \cdot (t_1 \cdot t_3)$

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Open problem

What about $A_{\{j,k\}}$? Can it be free?

This is a hard problem

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Theorem (Laver, Steel)

Let $\leq_L$ be the left-division, i.e., $w <_L v$ iff there are $u_1, \dots u_n$ such that $v = (\dots((w \cdot u_1) \cdot u_2) \cdot \dots \cdot u_n)$

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Then $<_L$ is irreflexive on $\mathcal{E}_\lambda$.

This proves, for example, that the associativity rule does not hold in $\mathcal{E}_\lambda$:

$$j \cdot (j \cdot j) = (j \cdot j) \cdot (j \cdot j) = ((j \cdot j) \cdot j) \cdot ((j \cdot j) \cdot j)$$

But then $(j \cdot j) \cdot j <_L j \cdot (j \cdot j)$, so $(j \cdot j) \cdot j \neq j \cdot (j \cdot j)$.

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Some examples: with (DC) are indicated inequalities asked by Dehornoy's Criterion, with (LST) inequalities that come from Laver-Steel Theorem (therefore always true). With such small words the left distributive law does not appear, but if we continue it will come up.

Some examples: $j \neq k$ (DC) $j \cdot j \neq k$ (DC) $j \cdot k \neq j$ (LST) $j \cdot k \neq k$ (DC) $k \cdot j \neq j$ (DC) $k \cdot j \neq k$ (LST) $k \cdot k \neq j$ (DC) $j \cdot j \neq j \cdot k$ (DC) $j \cdot j \neq k \cdot j$ (DC) $j \cdot j \neq k \cdot k$ (DC) $j \cdot k \neq k \cdot j$ (DC) $j \cdot k \neq k \cdot k$ (DC) $k \cdot j \neq k \cdot k$ (DC) ...

There is a whole hierarchy above I3, with larger and larger embeddings:

- I3: $j : V_\lambda \prec V_\lambda$
- I1: $j : V_{\lambda+1} \prec V_{\lambda+1}$
- I0 (or E_0): $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$, where $L(V_\lambda)$ is the smallest ZF model that contains $V_{\lambda+1}$
- I0$\sharp$ (ore E_1): $j : L(V_{\lambda+1}, (V_{\lambda+1})^\sharp) \prec L(V_{\lambda+1}, (V_{\lambda+1})^\sharp)$, where $(V_{\lambda+1})^\sharp$ is a description of the truth in $L(V_{\lambda+1})$ coded as a subset of $V_{\lambda+1}$;
- E_2: $j : L(V_{\lambda+1}, (V_{\lambda+1})^{\sharp\sharp}) \prec L(V_{\lambda+1}, (V_{\lambda+1})^{\sharp\sharp})$
- ...
- E_α: $j : L(E_\alpha) \prec L(E_\alpha)$
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First question: can we define application on these embeddings? Laver did it for I1.

The problem from I0 and beyond is that j is not amenable in $L(V_{\lambda+1})$ or $L(E_\alpha)$: there is a Θ such that $j \restriction L_\Theta(V_{\lambda+1}) \notin L(V_{\lambda+1})$.

The first step is to reduce us to embeddings that are ultrapowers, called weakly proper embeddings:

Theorem (Woodin)

Let $j : L(E_\alpha) \prec L(E_\alpha)$ with $\mathrm{crt}(j) < \lambda$. Then there are two embeddings $j_U, k_U : L(E_\alpha) \prec L(E_\alpha)$ such that $j = k_U \circ j_U$ and

- $\mathrm{crt}(j_U) < \lambda$ and it comes from an ultrafilter, so its behaviour it's definable from $j_U \restriction E_\alpha$;
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The second step is to partition $L(E_\alpha)$ in fragments on which k is amenable, called Z_s, so that $j \cdot k = \bigcup_s j(k \upharpoonright Z_s)$. Is this an embedding?

Theorem (D.)

Suppose E_α and that $L(E_\alpha) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$. Let $\mathcal{E}(E_\alpha)$ be the "set" of weakly proper elementary embeddings from E_α to itself. Then we can define an operation $\cdot$ on $\mathcal{E}(E_\alpha)$ that is a left-distributive algebra and such that $\rho_\alpha : \mathcal{E}(E_\alpha) \to \mathcal{E}_\lambda$, $\rho_\alpha(j) = j \upharpoonright V_\lambda$, is a homeomorphism.

This means that the following diagram commutes:

$$\begin{tikzcd} F_1 \arrow[r, "\pi_1"] \arrow[dr, "\pi_2"'] & \mathcal{E}(E_\alpha)_j \arrow[d, "\rho"] \\ & \mathcal{E}_{\rho_\alpha(j)} \end{tikzcd}$$

So ρ_α is an isomorphism on $\mathcal{E}(E_\alpha)_j$, and this is free.

Note: for any $j, k : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ weakly proper, $j = k$ iff $\rho_0(j) = \rho_0(k)$. So ρ_0 is an isomorphism from $\mathcal{E}(E_\alpha)_{j,k}$ to $\mathcal{E}_{\rho_\alpha(j),\rho_0(k)}$.

Second question: are there α and $j, k \in \mathcal{E}(E_\alpha)$ such that ρ_α is not an isomorphism on $\mathcal{E}(E_\alpha)_{j,k}$?

Answer negative for any α successor, or limit with cofinality $> \omega$.

Theorem (D., 2012)

If there is a ξ such that $L(E_\xi) \nvDash V = \mathrm{HOD}_{V_{\lambda+1}}$, then there is a $\alpha < \xi$ such that $L(E_\alpha) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$, and there are 2^λ different elements of $\mathcal{E}(E_\alpha)$ that coincide on V_λ.

There is a property that I am not going to define, it is called properness. Every weakly proper I0-embedding is proper, but the Theorem above says that we can find both proper and non-proper embeddings that coincide on V_λ

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This is fodder for many new inequalities, and some even meet Dehornoy's criterion!

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Open problem

Is $\mathcal{E}_{j,k}$ free?

Thanks you for your attention