Enriques surface fibrations with even index

Joint work with F. Suzuki.

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Projective and birational higher dimensional geometry
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B = a smooth curve
$f : X \to B$ a morphism

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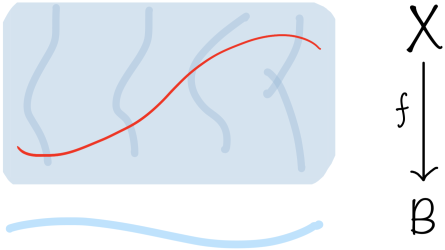


Graber–Harris–Starr theorem: If the general fiber of f is rationally connected, then f has a section.

$\therefore$ Any rationally connected variety X/K, over $K = k(B)$, has a K-point.

Serre (1958) (in a letter to Grothendieck):
Is the same conclusion true for varieties X/K with $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$?

Serre adds that it is “sans doute trop optimiste”.

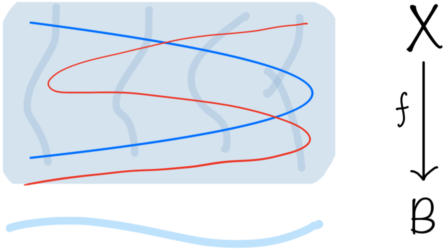
Graber–Harris–Mazur–Starr, Lafon, Starr ($\sim$ 2002**)**
No: There exist Enriques surface fibrations over curves with no section.

A question of Esnault:

If X/K is $\mathcal{O}$-acyclic. Does X/K admit a 0-cycle of degree 1?

More geometrically: If $f : X \to B$ is a fibration with general fiber X_b satisfying $H^i(X_b, \mathcal{O}_{B_b}) = 0$ for $i > 0$: Do we have

$$\gcd\Big(\deg(C/B) \mid C \subset X \text{ a curve} \Big) = 1?$$



Main result of this talk:

Theorem (O.-Suzuki)

There exists an Enriques surface fibration

$$X \to \mathbb{P}^1$$

such that the index is **even**.
In other words, every curve $C \subset X$ has even degree over $\mathbb{P}^1$.

Thus, Serre's question has a negative answer even with 'rational point' replaced by '0-cycle of degree 1'.

Other consequences

The 3-fold X gives counterexamples to other questions:

1. The Integral Hodge conjecture
2. The Hasse principle for the reciprocity obstruction for varieties over function fields of curves
3. Murre's conjecture on the universality of Abel-Jacobi maps

The Integral Hodge Conjecture

Colliot-Thélène–Voisin: For $f : X \to B$ with $\mathcal{O}$-acyclic fibers:

$$f_* : H_2(X, \mathbb{Z}) \to H_2(B, \mathbb{Z})$$

is surjective.

Thus there is a homology class $\sigma \in H_2(X, \mathbb{Z})$ which has degree 1 on a fiber.
$\therefore$ "there is no topological obstruction to the existence of sections"

This class is automatically Hodge, so we obtain a counterexample to

The integral Hodge conjecture (IHC):

$$H^{k,k}(X, \mathbb{C}) \cap H^{2k}(X, \mathbb{Z})$$

is generated by classes of algebraic subvarieties.

In our example, 4σ is algebraic, but σ is not.

Enriques surfaces

Surfaces S with

- $\pi_1(S) = \mathbb{Z}/2$
- $2K_S = 0$

There is a universal cover $\pi : Z \to S$ where Z is a K3 surface

Example

Let $S \subset \mathbb{P}^2 \times \mathbb{P}^2$ be the surface defined by the 2×2 minors of a generic matrix

$$\begin{pmatrix} p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \end{pmatrix} \qquad \begin{array}{rcl} p_i & = & p_i(x_0, x_1, x_2) \\ q_i & = & q_i(y_0, y_1, y_2) \end{array}$$

where $\deg p_i = (2, 0)$ and $\deg q_i = (0, 2)$. Then S is an Enriques surface.

Here is the K3 cover:
On $\mathbb{P}^5 = \operatorname{Proj} k[x_0, x_1, x_2, y_0, y_1, y_2]$, there is an involution

$$\iota : \mathbb{P}^5 \to \mathbb{P}^5$$

defined by $\iota^*(x_i) = x_i$, $\iota(y_i) = -y_i$.
Consider the quadrics

$$F_i = p_i + q_i \qquad \begin{array}{rcl} p_i & = & p_i(x_0, x_1, x_2) \\ q_i & = & q_i(y_0, y_1, y_2) \end{array}$$

These define a K3 surface

$$Z = \{F_0 = F_1 = F_2 = 0\} \subset \mathbb{P}^5$$

ι acts freely on Z, as Z is disjoint from

$$\operatorname{Fix}(\iota) = P_1 \cup P_2 \qquad \begin{array}{rcl} P_1 & = & V(x_0, x_1, x_2) \simeq \mathbb{P}^2 \\ P_2 & = & V(y_0, y_1, y_2) \simeq \mathbb{P}^2 \end{array}$$

Hence $S = Z/\iota$ is a smooth Enriques surface.

Two Enriques surface fibrations

- $X \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ is the threefold defined by the 2×2 minors of a generic matrix

$$\begin{pmatrix} p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \end{pmatrix} \qquad \begin{array}{rcl} p_i & = & s^2 A_i + st B_i + t^2 C_i \\ q_i & = & s^2 D_i + st E_i + t^2 F_i \end{array}$$

  where $\deg p_i = (\mathbf{2}, \mathbf{2}, \mathbf{0})$ and $\deg q_i = (\mathbf{2}, \mathbf{0}, \mathbf{2})$.

  Then X is a smooth threefold, and the first projection defines an Enriques surface fibration

$$p : X \to \mathbb{P}^1.$$

- $Y \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ is defined by

$$\begin{pmatrix} p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \end{pmatrix} \qquad \begin{array}{rcl} p_i & = & s A_i + t B_i \\ q_i & = & s C_i + t D_i \end{array}$$

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Properties of X

- X has Kodaira dimension 1
- X is simply connected and $H^*(X, \mathbb{Z})$ has no torsion.
- Hodge diamond

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 50 & & 0 & \\
0 & & 99 & & 99 & & 0
\end{array}
$$

- $CH_0(X) = \mathbb{Z}$ (as expected by the Bloch conjecture)

Properties of Y

- Y has Kodaira dimension 1
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- Hodge diamond

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 26 & & 0 & \\
0 & & 45 & & 45 & & 0
\end{array}
$$

- $CH_0(Y) = \mathbb{Z}$ (as expected by the Bloch conjecture)

Strategy

We first study the geometry of Y.

Thus $Y \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ is the codimension 2 subvariety defined by the minors of

$$\begin{pmatrix} p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \end{pmatrix} \qquad \begin{array}{rcl} p_i & = & sA_i + tB_i \\ q_i & = & sC_i + tD_i \end{array}$$

where $\deg p_i = (\mathbf{1}, \mathbf{2}, \mathbf{0})$ and $\deg q_i = (\mathbf{1}, \mathbf{0}, \mathbf{2})$.

On Y we prove certain congruence of intersection numbers between curves C and divisors E_i.

We then use this to study curves on X, using a degeneration argument.

X is the variety that will give the main counterexample.

The geometry of Y

Let $F_i = p_i + q_i$, considered as a $(1,2)$ form on $\mathbb{P}^1 \times \mathbb{P}^5$.

$$
\begin{tikzcd}
 & Z = Bl_{\mathbb{P}^1 \times (P_1 \cup P_2)} Z_0 \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\
Z_0 = \{F_0 = F_1 = F_2 = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^5 & & Y \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2
\end{tikzcd}
$$

π is the blow-up of the fixed points of ι:

- $(\mathbb{P}^1 \times P_1) \cap Z_0$ ($= 12$ points $p_{1,1}, \ldots, p_{1,12}$); and
- $(\mathbb{P}^1 \times P_2) \cap Z_0$ ($= 12$ points $p_{2,1}, \ldots, p_{2,12}$)

$\rightsquigarrow$ 24 exceptional divisors

$$
\begin{array}{ccc}
E_{1,1}, & \ldots & E_{1,12} \\
E_{2,1}, & \ldots & E_{2,12}
\end{array}
$$

p is a double cover, ramified along the $E_{i,j}$.

Out of the 24 $E_{i,j}$'s, we single out $E_{1,1}, \ldots, E_{1,12}$ (from the fixed points on P_1).

If Y is defined by $\begin{pmatrix} p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \end{pmatrix}$, the $E_{1,i}$ are the components of

$$E_1 = \{p_0 = p_1 = p_2 = 0\} \subset Y.$$

Claim

For a curve $C \subset Y$ we have

$$\deg(C/\mathbb{P}^1) \equiv C \cdot \left(\sum_{j=1}^{12} E_{1,j} \right) \mod 2.$$

$\therefore$ If $C \subset Y$ is a section of $Y \to \mathbb{P}^1$, then C has to intersect at least one of the $E_{1,j}$'s (!).

We consider a degeneration $\mathcal{Y} \to T$ with special fiber $\mathcal{Y}_0$.

If $Y = \mathcal{Y}_t$ is a very general fiber, then there is a specialization map

$$CH_1(Y) \to CH_1(\mathcal{Y}_0)$$

compatible with intersection products.

So it suffices to prove the congruence

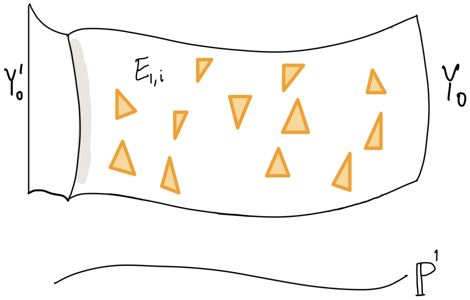
$$\deg(C/\mathbb{P}^1) \equiv C \cdot \left(\sum_{j=1}^{12} E_{1,j} \right) \mod 2.$$

on $\mathcal{Y}_0$.

The degeneration: $\mathcal{Y} \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2 \to \operatorname{Spec} k[\epsilon]$ defined by the minors of

$$M_\epsilon = \begin{pmatrix} p_0 & p_1 & p_2 \\ sy_0^2 + \epsilon r_0 & sy_1^2 + \epsilon r_1 & sy_2^2 + \epsilon r_2 \end{pmatrix}$$

Special fiber over $\epsilon = 0$: $\mathcal{Y}_0 = Y_0 \cup Y_0'$



- $Y_0 \cap Y_0' = \{s = 0\}$= an Enriques surface
- $V(p_0, p_1, p_2) = E_{1,1} \cup \cdots \cup E_{1,12}$ does not intersect Y_0' (hence lies in $(\mathcal{Y}_0)_{\text{reg}}$).

$\deg(C/\mathbb{P}^1) \equiv C \cdot \left(\sum_{j=1}^{12} E_{1,j}\right) \mod 2$

Y_0 is defined by the matrix

$$\begin{pmatrix} p_0 & p_1 & p_2 \\ y_0^2 & y_1^2 & y_2^2 \end{pmatrix}$$

Let $D_1 = \{p_0 = 0\}$; this is a divisor of type $(1, 2, 0)$.

For $C \subset Y_0$ a curve,

$$\deg(C/\mathbb{P}^1) \equiv D_1 \cdot C \mod 2$$

On the other hand,

$$D_1 = 2 \cdot V(y_0) + \sum_{j=1}^{12} E_{1,j}$$

This gives the desired congruence.

Main point: some Cartier divisor becomes double on the degeneration Y_0.

The threefold X and proof of the main theorem

Theorem

Let X be defined by a very general matrix in $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$

$$\begin{pmatrix} p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \end{pmatrix}$$

where $\deg p_i = (\mathbf{2}, \mathbf{2}, \mathbf{0})$ *and* $\deg q_i = (\mathbf{2}, \mathbf{0}, \mathbf{2})$.
Then any curve $C \subset X \to \mathbb{P}^1$ *has even degree over* $\mathbb{P}^1$.

On $X \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ there are now $24 + 24 = 48$ exceptional divisors

$$\begin{array}{ccc} E_{1,1}, & \dots & E_{1,24} \\ E_{2,1}, & \dots & E_{2,24} \end{array}$$

We focus on $E_{1,1}, \dots, E_{1,24}$; the components of

$$E_1 = \{p_0 = p_1 = p_2 = 0\}.$$

Basic strategy: Prove the following *key congruence*:

$$\deg(C/\mathbb{P}^1) \equiv C \cdot \left(\sum_{k=1}^{12} E_{1,j_k} \right) \mod 2 \tag{1}$$

for any **12-tuple** $1 \le j_1 < \ldots < j_{12} \le 24$.

This will imply the theorem: We would get that

$$C \cdot E_{1,1} \equiv \cdots \equiv C \cdot E_{1,24} \mod 2,$$

and hence that $\deg(C/\mathbb{P}^1)$ is even.

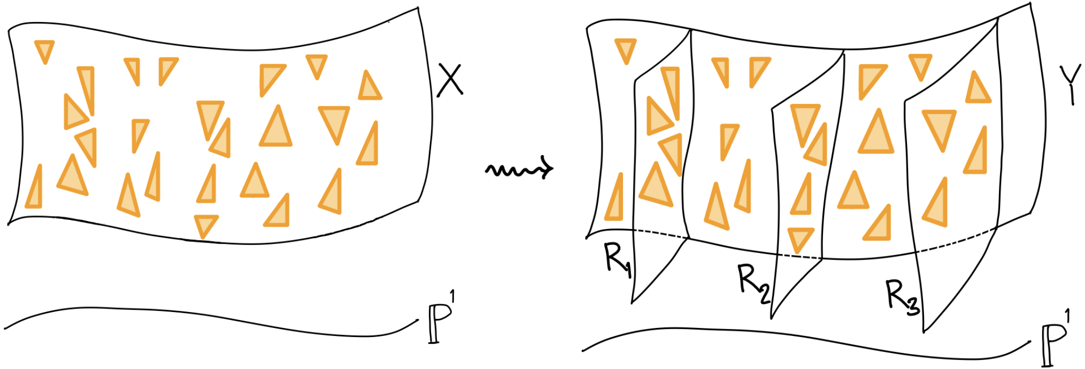
We want to prove that

$$\deg(C/\mathbb{P}^1) \equiv C \cdot \left(\sum_{j=1}^{12} E_{1,j_k} \right) \mod 2 \tag{2}$$

1. **Monodromy argument**: Reduce to proving (2) for *some* 12-tuple $j_1 < \ldots < j_{12}$.
2. **Specialization argument**: Prove (2) for some $(j_1, \ldots, j_{12})$ by analyzing a certain degeneration of X.

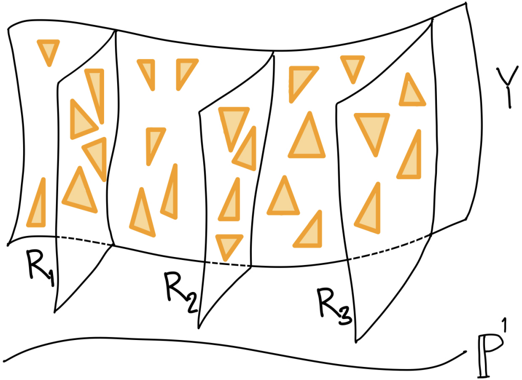
Here is the degeneration:

$$M = \begin{pmatrix} sp_0 + \epsilon r_0 & (s-t)p_1 + \epsilon r_1 & (s+t)p_2 + \epsilon r_2 \\ stq_0 + \epsilon s_0 & t(s-t)q_1 + \epsilon s_1 & t(s+t)q_2 + \epsilon s_2 \end{pmatrix}$$



The special fiber over $\epsilon = 0$ is a union

$$Y \cup R_1 \cup R_2 \cup R_3$$



- Y is the previous Enriques surface fibration with 12 planes $E_{1,j_1}, \dots, E_{1,j_{12}}$
- On Y, we know that

$$\deg(C/\mathbb{P}^1) \equiv C \cdot \left(\sum_{k=1}^{12} E_{1,j_k} \right) \mod 2 \tag{3}$$

The main congruence (2) follows from this.

Thank you for the attention!

A counterexample to a question of Murre

Let $CH^p(V)_{alg} \subset CH^p(V)$ denote the subgroup of cycle classes algebraically equivalent to 0.

$\rightsquigarrow$ Abel-Jacobi map

$$\psi^p : CH^p(V)_{alg} \to J^p(V) = \frac{H^{2p-1}(V, \mathbb{C})}{H^{2p-1}(V, \mathbb{Z}) + F^p H^{2p-1}(V, \mathbb{C})}$$

This is defined by integration:
Take $\gamma \in CH^p(V)_{alg}$
$\rightsquigarrow$ $[\gamma] = \partial\Gamma$ in $H^p(V, \mathbb{Z})$ where Γ is a $(2n-2p+1)$-chain.
$\rightsquigarrow$ define

$$\psi^p(\gamma) = \left(\omega \mapsto \int_\Gamma \omega \qquad \mod H^{2p-1}(V, \mathbb{Z})\right)$$

where $\omega \in F^{n-p+1}H^{2n-2p+1}(V, \mathbb{C})$. (Note that $H^{2p-1}(V, \mathbb{C})/F^pH^{2p-1}(V, \mathbb{C})$ is dual to this vector space).

Theorem

Let

$$J_a^p(V) := \text{ the image of } \psi^p \text{ in } J^p(V).$$

Then $J_a^p(V)$ is an abelian variety (the Lieberman jacobian).
and $\psi^p : CH^p(V)_{alg} \to J_a^p(V)$ is a regular homomorphism.

Here $\psi : CH^p(V)_{alg} \to A$ is *regular* if
$\forall$ smooth proj. S, $\forall s_0 \in S$, $\forall \Gamma \in CH^p(S \times V)$, then the composition

$$S \to CH^p(V)_{alg} \xrightarrow{\phi} A$$

given by $s \mapsto \Gamma_*(s - s_0)$, is a *morphism* of algebraic varieties.

Murre's conjecture: $J_a^p(V)$ is universal among regular homomorphisms $CH^p(V) \to A$ to an abelian variety A:

$$\begin{tikzcd} CH^p(V) \arrow[rr] \arrow[dr, "\psi^p"] & & A \\ & J_a^p(V) \arrow[ur] & \end{tikzcd}$$

The universality of ψ^p was known for
$p = 1$: Picard variety
$p = \dim X$: Albanese variety
$p = 2$: Proved by Murre (using algebraic K-theory, results by Saito, Bloch–Ogus theory, Merkurjev–Suslin, ..).

We get a counterexample for $p = 3$ for $V = X \times E$ for a very general elliptic curve E.