Stuctures in cyclic linear logic

Pietro Di Gianantonio

Dipartimento di Matematica e Informatica, Università di Udine
via delle Scienze 206 I-33100 Udine Italy
E-mail: digianantonio@dimi.uniud.it

Abstract. The aim of this work is to give an alternative presentation for the multiplicative fragment of cyclic linear logic of Yetter. The new presentation uses, as formalism, the calculus of structures, and has the interesting feature of avoiding the cycling rule. The main point in this work is to show how cyclicity can be substituted by deepness, ie the possibility of applying a proof rule in any point of a formula. We present a proof of cut elimination for the new calculus.

1 Introduction

In this work, we formulate in the calculus of structure [Gug02] the multiplicative fragment of Cyclic Linear Logic CyLL [Yet90]. What we obtain after accomplishing this goal is a logic where the rules have a more intuitive explanation than the ones used by CyLL, so so we claim that this new formulation improves our understanding of CyLL, in particular, it gives an account for the most peculiar rule in CyLL, the "Cycling" rule.

2 Structures for MLL

In this section we give a short presentation of the calculus of structures, the formalism used in the rest of the paper. In particular, we present multiplicative linear logic [Gir87] in the calculus of structures.

We need to remark two facts. Firstly, in classical linear logic negation can be moved to the atom level. This way all non-atomic sub-formulas appear in a positive contest, that is, they do not appear inside negations. Logical implication is closed by positive context, i.e., if A can be derived from B and $S[\ ]$ is a positive context then $S[A]$ can be derived from $S[B]$. This is an important fact that can be exploited in defining a logical calculus, and in particular it can be used to define extensions of sequent calculi. The second fact that we want to remark is that sequent calculus reduces the derivability of a formula A to the derivability of a set of sequents. A set of sequents $\vdash B_{1,1}, \ldots, B_{1,n_1}$ $\vdash B_{2,1}, \ldots, B_{2,n_2}$ $\ldots$ $\vdash B_{m,1}, \ldots, B_{m,n_m}$ is derivable if and only if the formula $(B_{1,1} \parr \ldots \parr B_{1,n_1}) \otimes (B_{2,1} \parr \ldots \parr B_{2,n_2}) \otimes \ldots \otimes (B_{m,1} \parr \ldots \parr B_{m,n_m})$ is derivable, or one can see the above set of sequents as a different writing of the previous formula. This alternative writing of a formula is a way of marking the points where logical rules can be applied.

The calculus of structures generalizes the sequent calculus and decomposes a formula not only in a set of sequents, but in a complex structure and works on any level of this structure. This way it is possible not even to mention formulas and use only "structures." From another point of view, one does not use sequents but only formulas written in a different way.

MLL can be defined in the calculus of structures as follows. Structures, denoted by A, B, C, and D, are generated by

$$A ::= a \mid \overline{a} \mid [A, A] \mid (A, A) \mid [\,] \mid (\,)$$

Structures can be seen as an alternative writing of formulas. Atomic structures are denoted by a, b, c. $[A, B]$ is called a *par structure* and can be thought as the formula $A \parr B$; (A, B) is called a *times structure* and can be thought as the formula $A \otimes B$. We use the empty par and times structures, $[\,]$ and $(\,)$, to denote the unit of the par and times structures: $\perp$, $\mathbf{1}$. We denote by $\overline{a}$ the *negation* of the atomic structure a. Observe that negation can be applied only to atoms. Sometimes it is convenient to write also expressions of the form $\overline{A}$ with A a non-atomic structure. This has to be interpreted as the structures recursively defined by the following (De Morgan) rules.

$$\begin{aligned}
\overline{[A, B]} &\triangleq (\overline{A}, \overline{B}) \\
\overline{(A, B)} &\triangleq [\overline{A}, \overline{B}] \\
\overline{\overline{A}} &\triangleq A \\
\overline{[\,]} &\triangleq (\,) \\
\overline{(\,)} &\triangleq [\,]
\end{aligned}$$

By convention parenthesis associate to the left, that is with $[A, B, C]$ we indicate the structure $[[A, B], C]$. Within this convention, the formula $a \parr b \parr (b^\perp \otimes ((a^\perp \otimes c) \parr c^\perp))$ is represented by the structure $[a, b, (\overline{b}, [(\overline{a}, c), \overline{c}])]$

Formally the translation $\underline{\quad}_{\mathsf{S}}$ from formulae of MLL to structures is recursively defined by:

$$\begin{aligned}
\underline{a}_{\mathsf{S}} &\triangleq a \\
\underline{\mathbf{1}}_{\mathsf{S}} &\triangleq (\,) \\
\underline{\perp}_{\mathsf{S}} &\triangleq [\,] \\
\underline{M \otimes N}_{\mathsf{S}} &\triangleq (\underline{M}_{\mathsf{S}}, \underline{N}_{\mathsf{S}}) \\
\underline{M \parr N}_{\mathsf{S}} &\triangleq [\underline{M}_{\mathsf{S}}, \underline{N}_{\mathsf{S}}] \\
\underline{M^\perp}_{\mathsf{S}} &\triangleq \overline{\underline{M}_{\mathsf{S}}}
\end{aligned}$$

The vice-versa translation $\underline{\quad}_{\mathsf{F}}$ from structures to MLL formulae is recursively defined by:

$$\underline{a}_{\mathsf{F}} \triangleq a$$

$$
\begin{aligned}
\underline{\bar{a}}_{\mathsf{F}} &\triangleq a^{\perp} \\
\underline{(\,)}_{\mathsf{F}} &\triangleq \mathbf{1} \\
\underline{[\,]}_{\mathsf{F}} &\triangleq \perp \\
\underline{(A,B)}_{\mathsf{F}} &\triangleq \underline{A}_{\mathsf{F}} \otimes \underline{B}_{\mathsf{F}} \\
\underline{[A,B]}_{\mathsf{F}} &\triangleq \underline{A}_{\mathsf{F}} \parr \underline{B}_{\mathsf{F}}
\end{aligned}
$$

One should remark that the introduction of the structures is not strictly necessary and one can present a calculus working directly on formulae. We prefer to make the extra syntactical work of introducing the new concept of structure for two reasons. The first one is that we found that proofs can be written more elegantly using structures, the second reason is that the use of structures makes the presentation of our calculus more similar to the sequent calculus, and in particular to the presentation given in [Gug02,Gb01,Yet90]

Par and times structures which differ just for the association of the parenthesis or for the permutation of their substructures are considered logically equivalent. We formalise this fact by the introduction of a relation, $\sim$, on structures. We can see the transitive and contextual closure of $\sim$ as the equivalence relation that equates structures that can be proved logically equivalent using only simple arguments. Notice that the relation $\sim$ itself is not an equivalence relation, and is defined by the following set of schemata.

The commutative and associative laws are given by:

$$
\begin{aligned}
[A,B] &\sim [B,A] && \text{par commutative} \\
(A,B) &\sim (B,A) && \text{times commutative} \\
[A,[B,C]] &\sim [[A,B],C] && \text{par associative} \\
(A,(B,C)) &\sim ((A,B),C) && \text{times associative}
\end{aligned}
$$

The laws for the unit constants are given by:

$$
\begin{aligned}
A &\sim [A,[\,]] \\
[A,[\,]] &\sim A && \text{par unit} \\
A &\sim (A,(\,)) \\
(A,(\,)) &\sim A && \text{times unit}
\end{aligned}
$$

As mentioned previously we take full advantage, in presenting our calculus, of the fact that logical rules are closed by positive context. Structural contexts, denoted by S, R and T, are generated by the grammar:

$$
S ::= \circ \mid [A,S] \mid [S,A] \mid (A,S) \mid (S,A)
$$

we denote by (SA) the structure obtained by replacing, in the structural context S, $\circ$ by the structure A. When this will not cause ambiguity, we will abbreviate the expression (SA) with SA.

On structures, we define the following proof rules.

$$\frac{}{\vdash (\,)}\ \textit{Empty}$$

$$\frac{\vdash SB}{\vdash SA}\ \text{if } A \sim B \qquad \textit{Equivalence}$$

$$\frac{\vdash S(\,)}{\vdash S[\,A, \overline{A}\,]}\ \textit{Axiom}$$

$$\frac{\vdash S(\,A, \overline{A}\,)}{\vdash S[\,]}\ \textit{Cut}$$

$$\frac{\vdash S(\,[\,A, B\,], C\,)}{\vdash S[\,A, (\,B, C\,)\,]}\ \textit{Switch}$$

In [Gb01] a calculus quite similar to the previous one has been presented. The main difference is that in [Gb01] also the exponential connective are included. Similarly to what has been done in [Gb01] we can prove that our calculus satisfies the cut-elimination property and to be equivalent to multiplicative linear logic.

Proposition 1. **(i)** *For any MLL formula M, M is provable in MLL if and only if the structure $\underline{M}_S$ is provable.*
(ii) *For any structure A, if A is provable then A can be proved without using the cut-rule.*

We omit the proof since it can be obtained by copying the proof for the corresponding proposition for cyclic linear logic. This proof can be found in the next section.

3 Multiplicative Cyclic Linear Logic

The non commutative version of the calculus is obtained by simply removing the commutative rules for the par and times, which are considered not any more valid. The new set of schemata defining the $\sim$ relation is:

$$
\begin{array}{rcll}
[[A,B],C] & \sim & [A,[B,C]] & \\
[A,[B,C]] & \sim & [[A,B],C] & \text{par associative} \\
\\
((A,B),C) & \sim & (A,(B,C)) & \\
(A,(B,C)) & \sim & ((A,B),C) & \text{times associative} \\
\\
A & \sim & [A,[\,]] & \\
A & \sim & [[\,],A] & \\
[[\,],A] & \sim & A & \\
[A,[\,]] & \sim & A & \text{par unit} \\
\\
A & \sim & (A,(\,)) & \\
A & \sim & ((\,),A) & \\
(A,(\,)) & \sim & A & \\
((\,),A) & \sim & A & \text{times unit}
\end{array}
$$

notice that we need to introduce extra schemata for $\sim$. In the commutative calculus these extra schemata are not necessary, in fact they relate structures that are related by the transitive closure of $\sim$. It follows that, in the commutative calculus, what can be derived using the Equivalence rule with the extra schemata, can also be derived by several application of the original Equivalence rule. By similar reasons we need to add a symmetric version of the Switch rule:

$$\frac{\vdash S(A,[B,C])}{\vdash S[(A,B),C]}\ SwitchR$$

which is not present in the original system since is there derivable.

Definition 1. *The system* MW *is the inference calculus on non-commutative structures formed by the rules: Empty, Equivalence, Switch, SwitchR, Axiom, Cut.*

The system MW turns out to be equivalent to multiplicative CyLL.

Theorem 1. *The system* MW *is equivalent to multiplicative* CyLL.

Proof. In order to prove this theorem we consider the presentation of CyLL given in [Yet90].

We start by proving that CyLL is as powerful as the MW system. This fact follows immediately from two properties of CyLL. The first one is that implication is closed by positive context. That is, if C is a formula context not containing negation, and the formulas $C(M)$ and $\overline{M}\parr N$ are derivable, then also the formula $C(N)$ is derivable. The above property can be proved in the following way. Let Π be a proof, in CyLL, for $C(M)$, since Π cannot examine the structure of M until the formula M appears as an element of a sequent, looking at the proof bottom-up-wise, the proof Π is “independent” of M until it builds a sequent

$\Phi(M)$ containing the formula M. From the sequent $\Phi(M)$, using the Cycling rule it is then possible to derive a sequent Φ', M, having M has last formula, from which, by the Cut rule, one derives Φ', N and, by the Cycling rule, one derives $\Phi(N)$. From $\Phi(N)$, by following the pattern in Π, one can finally derive $C(N)$.

The second property is that for any rule

$$\frac{SB}{SA}$$

contained in system MW, the formula $\underline{[\overline{A}; B]}_{\mathsf{F}}$ is derivable in CyLL. This fact can be checked straightforwardly.

The other way round is also simple. First we define a translation from sequents and sets of sequents, in the CyLL calculus, into structures, by the rules:

$$\begin{aligned} \underline{M_1, \ldots, M_m}_{\mathsf{S}} &\triangleq [\,\underline{M_1}_{\mathsf{S}}, \ldots, \underline{M_m}_{\mathsf{S}}\,] \\ \underline{\{\Gamma_1, \ldots, \Gamma_n\}}_{\mathsf{S}} &\triangleq (\,\underline{\Gamma_1}_{\mathsf{S}}, \ldots, \underline{\Gamma_n}_{\mathsf{S}}\,) \end{aligned}$$

It is then easy to check, for any CyLL rule, different from the Cycling rule, that if the rule has form:

$$\frac{\vdash \Gamma_1 \ldots \vdash \Gamma_n}{\vdash \Delta} \quad ,$$

then from the structure $\underline{\{\Gamma_1, \ldots, \Gamma_n\}}_{\mathsf{S}}$ one can derive, using the rules in MW, the structure $\underline{\Delta}_{\mathsf{S}}$. For the Cycling rule, that has form:

$$\frac{\vdash \Delta, \Gamma}{\vdash \Gamma, \Delta}\ \mathit{Cycling}$$

we prove something weaker, namely that the Cycling rule is admissible in system MW, that is, if $\vdash [A, B]$ then $\vdash [B, A]$. The proof works as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash (\,)}\ \mathit{Empty}}{\vdash [B, \overline{B}]}\ \mathit{Axiom}}{\vdash [B, ((\,), \overline{B})]}\ \mathit{Equivalence}}{\begin{array}{c}\vdots\ \mathit{Hypothesis}\\ \vdash [B, ([A, B], \overline{B})]\end{array}}}{\vdash [B, [A, (B, \overline{B})]]}\ \mathit{Switch}}{\dfrac{\vdash [B, [A, [\,]]]}{\vdash [B, A]}\ \mathit{Equivalence}}\ \mathit{Cut}$$

$\square$

...

3.1 Cut elimination

A fundamental feature of every logical system is the cut-elimination property, which can also be proved for MW. In the proof that we are going to present we will show not only the eliminability of the Cut rule, but also the eliminability of other rules. In particular we need to present a restricted calculus having a minimal set of rules. To motivate this restricted calculus a definition is necessary:

Definition 2 (Duality of rules). *Given of a rule S*

$$\frac{\vdash A}{\vdash B}\ S$$

the dual of S is the rule

$$\frac{\vdash \overline{B}}{\vdash \overline{A}}\ dS$$

For example the rule Axiom is dual to the rule Cut and the rules Equivalence, Switch, SwitchR are dual to themselves. It is also easy to observe that from any rule S it is possible to derive its dual, using the Axiom, the Switch, and the Cut rules. In the following we are going to prove that for any pair of dual rules one of them can be eliminated. Some special care has to be devoted to the Equivalence rule. The Equivalence rule can be seen as a compact way of expressing a set of rules: a rule saying that par is associative, another saying that par is commutative etc. Considering this expanded set of rules one can observe that it contains pairs of dual rules. For example the rule stating associativity of par is dual to the rule stating associativity of times, the rule for the introduction of the times unit is dual to the rule for the elimination of the par unit, and so on.

To admit just a single instance for each pair of dual rules, we define a restricted version of the Equivalence rule. This restricted version of the Equivalence considers a new relation $\leadsto$ on structures. The relation $\leadsto$ is strictly contained in $\sim$.

Definition 3. *The relation on structures $\leadsto$ is defined as follows:*

$$\begin{array}{rcll}
[[A,B],C] & \leadsto & [A,[B,C]] & \\
[A,[B,C]] & \leadsto & [[A,B],C] & \textit{par associative} \\
[A,[\,]] & \leadsto & A & \\
[[\,],A] & \leadsto & A & \textit{par unit} \\
(A,(\,)) & \leadsto & A & \\
((\,),A) & \leadsto & A & \textit{times unit}
\end{array}$$

The Restricted Equivalence rules is:

$$\frac{\vdash SA}{\vdash SB}\ \textit{if } B \leadsto A \qquad\qquad \textit{Restricted Equivalence}$$

Moreover it is useful to consider a restricted version of the Axiom rule. In fact it is possible to reduce the Axiom rule to its atomic version. We call AxiomA the Axiom rule restricted to atomic formulae.

$$\frac{\vdash S(\ )}{\vdash S[A, \overline{A}\,]} \quad \text{with } A = a \text{ or } A = \overline{a} \qquad\qquad AxiomA$$

Definition 4. *The restricted system on non-commutative structures, denoted by* $\mathsf{MW_r}$*, is formed by the rules: Empty, Restricted Equivalence, Switch, SwitchR, AxiomA.*

We write $\vdash_{\mathsf{r}} A$ to indicate that the structure A can be proved in the system $\mathsf{MW_r}$. We aim to prove that system $\mathsf{MW_r}$ is equivalent to the MW. In particular we will prove that all the rules in MW are admissible in $\mathsf{MW_r}$. The proof proceeds by several steps, each step proving the admissibility of one missing rule.

Lemma 1. *The Axiom rule is derivable in* $\mathsf{MW_r}$*, that is, for any structure* A*, and context* S*, from* $S(\ )$*, using the* $\mathsf{MW_r}$ *rules it is possible to derive* $S[A, \overline{A}\,]$*.*

Proof. By induction on the structure A. If A is a unit, then the thesis follows from the Restricted Equivalence rule. In the case where A is an atom, the thesis follows from the AxiomA rule. In the case where A is in the form (A', A'') we have the following chain of implications:
$\vdash_{\mathsf{r}} S(\ ) \Rightarrow$ (by inductive hypothesis)
$\vdash_{\mathsf{r}} S[A', \overline{A'}\,] \Rightarrow$ (by Restricted Equivalence rule)
$\vdash_{\mathsf{r}} S[(A', (\ )), \overline{A'}\,] \Rightarrow$ (by inductive hypothesis)
$\vdash_{\mathsf{r}} S[(A', [A'', \overline{A''}\,]), \overline{A'}\,] \Rightarrow$ (by SwitchR rule)
$\vdash_{\mathsf{r}} S[[(A', A''), \overline{A''}\,], \overline{A'}\,] \Rightarrow$ (by Restricted Equivalence rule)
$\vdash_{\mathsf{r}} S[(A', A''), [\overline{A''}, \overline{A'}\,]]$.

The case where A is in the form $[A', A'']$, is perfectly equivalent (mirror image) to the previous one. ☐

Next we prove admissibility of the Equivalence rule. A preliminary definition and a lemma are here necessary. We start by defining new classes of contexts.

Definition 5. *A* left par-context *is defined by the following grammar:*

$$T_l ::= \circ \mid [A, T_l\,]$$

symmetrically, a right par-context *is defined by the grammar:*

$$T_r ::= \circ \mid [T_r, A\,]$$

Lemma 2. *For any structures* A, B, C*, context* S*, left context* T_l *and right context* T_l*:*

(i) *if* $\vdash_{\mathsf{r}} S(T_l(A, B), C)$ *then* $\vdash_{\mathsf{r}} S(T_l(A, (B, C)))$*, and symmetrically if* $\vdash_{\mathsf{r}} S(A, T_r(B, C))$ *then* $\vdash_{\mathsf{r}} S(T_r((A, B), C))$*,*

(ii) *if* $\vdash_r S(A, T_r(\,))$ *then* $\vdash_r S(T_r A)$*, and symmetrically, if* $\vdash_r S(T_l(\,), A)$ *then* $\vdash_r S(T_l A)$*,*

(iii) *if* $\vdash_r S[[\,], A]$ *then* $\vdash_r SA$ *and symmetrically, if* $\vdash_r S[A, [\,]]$ *then* $\vdash_r SA$

Proof. All three points are proved by structural induction of the proof Δ of the judgement in the premise. There are several cases to consider, the simplest ones are the cases where the last rule in Δ, let us call it X, works inside one of the contexts S, T_l, T_r, or inside one of the structures A, B, C: in all these cases the thesis simply derives by inductive hypothesis and by an application of X.

The remaining cases can be dealt with as follows:
(i) the applications of the rule X that are not included in the easy cases above are the ones where X generates one of the structures A, B, C, or where X is a Switch rules bringing part of the context inside the structure. The structure A is generated by X if and only if A is the times unit and $\vdash_r S(T_l((\,), B), C)$ derives from $\vdash_r S(T_l B, C)$, in this case, from $\vdash_r S(T_l B, C)$, by several applications of the Switch rule we can derive $\vdash_r S(T_l(B, C))$, from which, by an application of the Reduced Equivalence rule we have the thesis, i.e. $\vdash_r S(T_l((\,), (B, C)))$. The cases where the structures B and C are created by X can be dealt with in a similar manner.

The Switch rule can mix the structures in $\vdash_r S(T_l(A, B), C)$ in several ways. A possible case is the one where T_l is in the form $T_l'[D, \circ]$ and $\vdash_r S(T_l'[D, (A, B)], C)$ derives, by a Switch rule, from $\vdash_r S(T_l'([D, A], B), C)$: in this case by inductive hypothesis we have $\vdash_r S(T_l'([D, A], (B, C))$, from which by an application of the Switch rule we have $\vdash_r S(T_l'[D, (A, (B, C))]$ which is exactly the thesis. Another possible case is the one where S is in the form $S'[\circ, D]$ and $S'[(T_l(A, B), C), D]$, derives, by a SwitchR rule, from $S'(T_l(A, B), [C, D])$: in this case by inductive hypothesis we have: $S'(T_l(A, (B, [C, D])))$, from which by applying twice the SwitchR rule we have $S'(T_l[(A, (B, C)), D])$, from which by several applications of the Reduced Equivalence rule we have $S'[T_l(A, (B, C)), D]$, ie the thesis.

It remains to consider the case where S is in the form $S'[D, \circ]$ and $S'[D, (T_l(A, B), C)]$ derives, by a SwitchR rule, from $S'([D, T_l(A, B), C])$, ... □

Notice that as a special case of point (i) and (ii) of the above lemma we have that

(i) if $\vdash_r S((A, B), C)$ then $\vdash_r S(A, (B, C))$, and symmetrically if $\vdash_r S(A, (B, C))$ then $\vdash_r S((A, B), C)$,

(ii) if $\vdash_r S(A, (\,))$ then $\vdash_r SA$, and symmetrically, if $\vdash_r S((\,), A)$ then $\vdash_r SA$,

it follows:

Proposition 2. *The Equivalence rule is admissible in* MW_r*.*

The following lemma implies the admissibility of atomic cuts.

Lemma 3. *For any atom a, context S, left context T_l and right context T_l, if* $\vdash_r S(T_l a, T_r \overline{a})$ *(or* $\vdash_r S(T_l \overline{a}, T_r a)$*) then* $\vdash_r S[T_l[\,], T_r[\,]]$.

Proof. Similarly to the previous lemma the proof proceeds by structural induction on the proof Δ of $\vdash_{\mathsf{r}} S(T_l a, T_r \overline{a})$. Also for this lemma the cases can be split in two groups. The simple cases are the ones where the last rule X in Δ works internally to one of the contexts S, T_l, T_r: in all these cases the thesis derives by inductive hypothesis and by an application of the rule X. The other cases are the ones where the rule X modifies more than one context, or eliminates one of the atoms a, $\overline{a}$. This can only happen if X is a Switch or an AxiomA rule, we now consider these two possible cases.

A Switch rule can modify two contexts, for example we can derive

$$\vdash_{\mathsf{r}} S'[A, (T_l a, T_r \overline{a})]$$

from

$$\vdash_{\mathsf{r}} S'([A, T_l a], T_r \overline{a})$$

but in this case by induction hypothesis we have,

$$\vdash_{\mathsf{r}} S'[[A, T_l[\,]], T_r[\,]]$$

and by an application of the Restricted Equivalence rule (associativity of par) we have the thesis.

By the AxiomA rule we can introduces one of the atoms, i.e. we can derive

$$\vdash_{\mathsf{r}} S(T_l a, T'_r[a, \overline{a}])$$

from

$$\vdash_{\mathsf{r}} S'(T_l a, T'_r(\,))$$

In this case, from $\vdash_{\mathsf{r}} S'(T_l a, T'_r(\,))$, by several applications of the Equivalence rules, which we proved admissible, we obtain $\vdash_{\mathsf{r}} S'([T_l[\,], a], T'_r(\,))$, from which by the Switch rule we have $\vdash_{\mathsf{r}} S'[T_l[\,], (a, T'_r(\,))]$, from which by Lemma 2.(ii), we have $\vdash_{\mathsf{r}} S'[T_l[\,], T'_r a]$, and by the Restricted Equivalence rule we derive the thesis, $\vdash_{\mathsf{r}} S'[T_l[\,], T'_r[a, [\,]]]$. □

Lemma 4. *The Cut rule is admissible in* $\mathsf{MW_r}$, *that is, for any structure* A, *context* S, *if* $\vdash_{\mathsf{r}} S(A, \overline{A})$ *then* $\vdash_{\mathsf{r}} S[\,]$.

Proof. By structural induction on the structure A. If A is a unit then the thesis follows from the admissibility of the Equivalence rule.

In the case where A is an atom, the thesis follows from the previous lemma and from the admissibility of the Equivalence rule.

If A is in the form $[A', A'']$ then by inductive hypothesis, by the Switch rule and by the admissibility of the Equivalence rule we can derive, one from the other, the following judgements,

$\vdash_{\mathsf{r}} S([A', A''], (\overline{A''}, \overline{A'}))$
$\vdash_{\mathsf{r}} S(([A', A''], \overline{A''}), \overline{A'})$
$\vdash_{\mathsf{r}} S([A', (A'', \overline{A''})], \overline{A'})$
$\vdash_{\mathsf{r}} S([A', [\,]], \overline{A'})$

$\vdash_{\mathsf{r}} S(\,A', \overline{A'}\,)$
$\vdash_{\mathsf{r}} S[\,]$

The case where A is in the form $(\,A', A''\,)$ is equivalent to the previous one. $\square$

Having proved that all the rules in MW are admissible in $\mathsf{MW_r}$ we can finally state:

Proposition 3. *For every structure A if $\vdash A$ the $\vdash_{\mathsf{r}} A$.*

That is, the restricted system $\mathsf{MW_r}$ is as powerful as the complete one MW and from this we have:

Theorem 2. *The system* MW *satisfies the cut-elimination property.*

References

[Gb01] A. Guglielmi and L. Straßburger. Non-commutativity and MELL in the calculus of structures. In L. Fribourg, editor, *CSL 2001*, volume 2142 of *Lecture Notes in Computer Science*, pages 54–68. Springer-Verlag, 2001.

[Gir87] J.-Y. Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

[Gug02] A. Guglielmi. A system of interaction and structure. Technical Report WV-02-10, Dresden University of Technology, 2002. http://pikas.inf.tu-dresden.de/ guglielm/paper.html.

[Yet90] D. N. Yetter. Quantales and (non-commutative) linear logic. *J. Symbolic Logic*, 55(1), 1990.