# Games Characterizing Levy-Longo Trees

C.-H. L. Ong [a] P. Di Gianantonio [b]

[a] *Oxford University Computing Laboratory, Wolfson Building, Parks Road, Oxford OX1 3QD, United Kingdom.*

[b] *Dipartimento di Matematica e Informatica, Università di Udine, via delle Scienze, 206, 33100 Udine, Italy.*

**Abstract**

We present a simple *strongly universal* innocent game model for Levy-Longo trees i.e. every point in the model is the denotation of a unique Levy-Longo tree. The observational quotient of the model then gives a universal, and hence fully abstract, model of the pure Lazy Lambda Calculus.

*Key words:* Game Semantics, Lambda Calculus, Levy-Longo Trees, Universality, Full Abstraction.

**Contents**



*Preprint submitted to Elsevier Science* *23 January 2005*

## 1 Introduction

This paper presents a *strongly universal* innocent game model for Levy-Longo trees [Lev75,Lon83] (i.e. every point in the model is the denotation of a unique Levy-Longo tree). We consider arenas in the sense of [HO00,McC98] in which questions may justify either questions or answers, but answers may only justify questions; and we say that an answer (respectively question) is *pending* in a justified sequence if no question (respectively answer) is explicitly justified by it. Plays are justified sequences that satisfy the standard conditions of Visibility and Well-Bracketing, and a new condition, which is a kind of *dual* of Well-Bracketing, called

> *Persistence*: Every question is explicitly justified by the *last* pending answer, provided a pending answer exists at that point; otherwise it is explicitly justified by a question.

We then consider *conditionally copycat* strategies, which are *innocent* strategies (in the sense of [HO00]) that behave in a *copycat* fashion as soon as an O-answer is followed by a P-answer. Together with a condition called *Relevance*, we prove that the recursive such strategies give a *strongly universal* model of Levy-Longo trees i.e. every strategy is the denotation of a unique Levy-Longo tree. To our knowledge, this is the first universal model of Levy-Longo trees. The observational quotient of the model then gives a universal and fully abstract model of the pure Lazy Lambda Calculus [Plo75,AO93].

### *1.1 Related work*

Universal (game) models for the Lazy Lambda Calculus with convergence test were first presented in [AM95] and [McC96]. The model studied in the former is in the AJM style [AJM00], while that in the latter, by McCusker, is based on an innocent-strategy [HO00] universal model for call-by-name FPC, and is obtained via a universal and fully abstract translation from the Lazy

Lambda Calculus into call-by-name FPC. The present paper considers the *pure* (i.e. without any constant) Lazy Lambda Calculus. Our model builds on what is essentially McCusker's model by adding three constraints: Persistence, which is a constraint on plays, and Conditional Copycat and Relevance, which are constraints on strategies. Indeed ours is a submodel of McCusker's (see Remark 12).

The first fully abstract (game) model of the the pure Lazy Lambda Calculus was constructed by the second author in [Gia01]. The strategies therein are history-free and satisfy a *monotonicity* condition. The model is not universal (there are finite monotone strategies that are not denotable). However we believe it is possible to achieve universality by introducing a condition similar to Relevance. In [KNO02,KNO99] game models based on *effectively almost-everywhere copycat* (or EAC) strategies are constructed which are strongly universal for Nakajima trees and Böhm trees respectively. Several local structure results for AJM-style game models can be found in [GFH99].

## 2 Arenas, legal positions and nested levels

We begin this section by introducing a formal setting for playing games called *arenas*. *Legal positions* are then introduced as justified sequences (which are sequences of moves with pointers) that satisfy three conditions, namely, Visibility, Well-Bracketing and Persistence. The second part of the section is about *nested levels* of sequences of questions and answers, a notion useful for several technical proofs in the sequel.

### *2.1 Arenas and legal positions*

An ***arena*** is a triple $A = \langle M_A, \lambda_A, \vdash_A \rangle$ where $M_A$ is a set of moves; $\lambda_A : M_A \longrightarrow \{ \text{PQ}, \text{PA}, \text{OQ}, \text{OA} \}$ is a labelling function that indicates whether a given move is a P-move or an O-move, and whether it is a question (Q) or an answer (A); and $\vdash_A \subseteq (M_A + \{ * \}) \times M_A$ (where $*$ is a dummy move), called *justification relation* (we read $m_1 \vdash_A m_2$ as "$m_1$ justifies $m_2$"), satisfies the following axioms: for $m, m', m_i$ ranging over $M_A$

(1) For each $m \in M_A$, there is a unique $m^- \in M_A + \{ * \}$ such that $m^- \vdash_A m$; in case $* \vdash_A m$, we call $m$ *initial*.
(2) Every initial move is an O-question.
(3) If $m \vdash_A m'$ then $m$ and $m'$ are moves by different players.
(4) If $m \vdash_A m'$ and $m$ is an answer then $m'$ is a question ("Answers may only justify questions.").

It is useful to think of the justification relation $\vdash_A$ (restricted to $M_A \times M_A$) as defining the edge-set of a vertex-labelled directed graph whose vertex-set is $M_A$. We shall refer to the graph as the ***arena graph*** of $A$.

We use square and round parentheses in bold type as meta-variables for moves as follows:

| O-question | P-answer | P-question | O-answer |
|---|---|---|---|
| **[** | **]** | **(** | **)** |

We write $M_A^{\mathrm{Init}}$ for the set of initial moves of $A$, and write $\overline{(-)}$ for the function that inverts the P/O-designation of a move, so that e.g. $\overline{\mathrm{PQ}} = \mathrm{OQ}$ and $\overline{\mathrm{OA}} =$ PA etc.

The simplest arena is the empty arena $\mathbf{1} = \langle \varnothing, \varnothing, \varnothing \rangle$. Let $A$ and $B$ be arenas. The ***product arena*** $A \times B$ is just the disjoint union of the arena graphs of $A$ and $B$. Formally we have

$$\begin{array}{rcl} M_{A\times B} & = & M_A + M_B \\ \lambda_{A\times B} & = & [\lambda_A, \lambda_B] \\ * \vdash_{A\times B} m & \Longleftrightarrow & * \vdash_A m \;\vee\; * \vdash_B m \\ m \vdash_{A\times B} n & \Longleftrightarrow & m \vdash_A n \;\vee\; m \vdash_B n. \end{array}$$

Given an arena $A$, we write $\overline{A}$ for the graph that is obtained from the arena graph of $A$ by inverting the P/O-label at each vertex. As an operation on arena graphs, the ***function space arena*** $A \Rightarrow B$ is obtained from the arena graph of $B$ by grafting a copy of $\overline{A}$ just under each initial move $b$ of $B$ (so that each tree of $\overline{A}$ is a subtree of $b$). Formally we have

$$\begin{array}{rcl} M_{A\Rightarrow B} & = & M_A \times M_B^{\mathrm{Init}} + M_B \\ \lambda_{A\Rightarrow B} & = & [\overline{\pi_1 \,;\, \lambda_A}, \lambda_B] \end{array}$$

and $\vdash_{A\Rightarrow B}$ is defined by:

$$\begin{aligned} * \vdash_{A\Rightarrow B} b &\iff * \vdash_B b \\ b \vdash_{A\Rightarrow B} (a, b') &\iff b = b' \ \wedge \ * \vdash_A a \\ (a, b) \vdash_{A\Rightarrow B} (a', b') &\iff b = b' \ \wedge \ a \vdash_A a' \\ b \vdash_{A\Rightarrow B} b' &\iff b \vdash_B b'. \end{aligned}$$

Note that we shall refer to a move of the form $(a, b) \in M_{A\Rightarrow B}$ simply as a copy of $a$.

The ***lifted arena*** $A_\perp$ is obtained from $A$ by adding two moves, namely, $q$, which is the new initial move, and $a$, which is a P-answer, such that $q$ justifies $a$ which justifies each initial move of $A$, and moves from $A$ inherit the relation $\vdash_A$.

A ***justified sequence*** over an arena $A$ is a finite sequence of alternating moves such that, except the first move which is initial, every move $m$ has a *justification pointer* (or simply *pointer*) to some earlier move $m^-$ satisfying $m^- \vdash_A m$; we say that $m$ is *explicitly justified* by $m^-$. A question (respectively answer) in a justified sequence $s$ is said to be ***pending*** just in case no answer (respectively question) in $s$ is explicitly justified by it. This extends the standard meaning of "pending questions" to "pending answers". Recall the definition of the ***P-view*** $\ulcorner s \urcorner$ of a justified sequence $s$:

$$\begin{aligned} \ulcorner \epsilon \urcorner &= \epsilon \\ \ulcorner s\, m \urcorner &= \ulcorner s \urcorner m && \text{if } m \text{ is a P-move} \\ \ulcorner s\, m \urcorner &= m && \text{if } m \text{ is initial} \\ \ulcorner s\, m_0\, u\, m \urcorner &= \ulcorner s \urcorner m_0\, m && \text{if the O-move } m \text{ is explicitly justified by } m_0 \end{aligned}$$

In $\ulcorner s\, m_0\, u\, m \urcorner$ the pointer from $m$ to $m_0$ is retained, similarly for the pointer from $m$ in $\ulcorner s\, m \urcorner$ in case $m$ is a P-move. The definition of the ***O-view*** $\llcorner s \lrcorner$ of a justified sequence $s$ is obtained from the above definition of P-view by swapping P and O.

**Definition 1** A justified sequence $s$ over $A$ is said to be a ***legal position*** (or ***play***) just in case it satisfies:

(1) *Visibility*: Every P-move (respectively non-initial O-move) is explicitly justified by some move that appears in the P-view (respectively O-view)

at that point.

(2) *Well-Bracketing*: Every answer is explicitly justified by the last pending question at that point.

(3) ***Persistence***: Every question is explicitly justified by the last pending answer, provided there is one such at that point, otherwise it is explicitly justified by a question.

For example in the following justified sequence

**[ ( [ ] ) ] [ (**

Persistence requires that the last move "**(**" be explicitly justified by "**)**". For another example, take the following justified sequence that satisfies Persistence:

**[ ( ) ( [ (**

The last "**(**" must be explicitly justified by one of the two "**[**"; it may not be explicitly justified by "**)**".

**Remark 2** (i) Except for Persistence, all that we have introduced so far are standard notions of the *innocent* approach to Game Semantics in the sense of [HO00]. Note that there can be at most one pending O-answer (respectively P-answer) in a P-view (respectively O-view). It is an immediate consequence of Well-Bracketing that no question may be answered more than once in a legal position.

(ii) It is a consequence of the definition that in an odd-length (respectively even-length) legal position, the last pending question (if any) is an O-question (respectively P-question), and the last pending answer (if any) is an O-answer (respectively P-answer).

(iii) As a consequence of Persistence, if a question in a legal position is explicitly justified by an answer, the answer must be pending at that point.

Persistence may be regarded as a *dual* of Well-Bracketing: it is to questions what Well-Bracketing is to answers. The effect of Persistence is that, whenever there is a pending O-answer, a strategy is restricted in which question it can ask, or equivalently over which argument it can interrogate, *at that point* (of course it may decide to answer an O-question instead). An apparently similar restriction on the behaviour of strategies is captured by the *rigidity* condition introduced by Danos and Harmer [DH01], namely, for any legal position of a rigid strategy, the pointer from a question is to some move that appears in the *R-view* of the play at that point. However since Persistence is a constraint on *plays* consisting of answers that may justify questions, whereas rigidity is a condition on *strategies* over arenas whose answers do *not* justify any move, it is not immediately obvious how the two notions are related.

### *2.2 Nested levels*

Take any set $M$ that is equipped with a function $\lambda : M \longrightarrow \{\,\mathrm{Q}, \mathrm{A}\,\}$ which labels elements as either questions or answers. Let $s$ be a finite sequence of elements from $M$ – call $s$ a ***dialogue***. The nested level of a dialogue is closely related to the number of pending questions at that point. Formally, set $\#_{\mathrm{qn}}(s)$ and $\#_{\mathrm{ans}}(s)$ respectively to be the number of questions and the number of answers in $s$; following [Gia01], we define the ***nested level*** at $s\,m$ (or simply the ***level*** of $m$ whenever $s$ is understood) to be

$$\mathsf{NL}(s\,m) \;=\; \begin{cases} \delta - 1 & \text{if } m \text{ is a question} \\ \delta & \text{if } m \text{ is an answer} \end{cases}$$

where $\delta = \#_{\mathrm{qn}}(s\,m) - \#_{\mathrm{ans}}(s\,m)$; we define $\mathsf{NL}(\epsilon) = 0$. Take, for example, the dialogue

$$\mathbf{[\,(\,)\,(\,[\,]\,[\,(\,)\,]\,)\,(\,)\,]\,[\,(}\cdot$$

We present the same sequence by displaying the elements at their respective levels as follows:

| Nested Level | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | | | | | | | | **(** | **)** | | | | | | |
| 2 | | | | | **[** | **]** | **[** | | | **]** | | | | | |
| 1 | | **(** | **)** | **(** | | | | | | | **)** | **(** | **)** | | **(** |
| 0 | **[** | | | | | | | | | | | | | **] [** | |

For $l \geq 0$, we write $s \upharpoonright l$ to mean the subsequence of $s$ consisting of moves at level $l$.

We state some basic properties of nested levels of dialogues.

**Lemma 3** *In the following, we let $s$ range over dialogues.*

*(i) For any $s = u\,m\,m'$, if $m$ and $m'$ are at different levels $l$ and $l'$ respectively,*

*then $m$ and $m'$ are either both questions (in which case $l' = l+1$) or both answers (in which case $l' = l-1$). As a corollary we have:*

*(i') If $a$ and $b$ in a dialogue are at levels $l_1$ and $l_2$ respectively, then for any $l_1 \leq l \leq l_2$, there is some move between $a$ and $b$ (inclusive) at level $l$.*

*(ii) For any $l \geq 0$, if $l < \mathsf{NL}(s)$ (respectively $l > \mathsf{NL}(s)$) then the last move in $s$ at level $l$, if it exists, is a question (respectively answer).*

*(iii) Suppose $s$ begins with a question. For each $l$, if $s \upharpoonright l$ is non-empty, the first element is a question, thereafter the elements alternate strictly between answers and questions.*

**PROOF.** (i): By a straightforward question-answer case analysis of $m$ and $m'$.

(ii): Take an $m$ which is the last in $s$ at level $l < \mathsf{NL}(s)$. The element $m'$ (say) after $m$ in $s$ is at a level not equal to $l$, which must be $l+1$; for if it were $l-1$, by (i'), there must be some move after $m'$ at level $l$, which contradicts the assumption that $m$ is the last such. The required result then follows from (i). The other case is symmetrical.

(iii): We prove by induction on the length $|s|$ of $s$. The base case of $|s| = 0$ is trivial. For the inductive case, take $sm$ such that $\mathsf{NL}(s) = l$ and $\mathsf{NL}(sm) = l+1$; by (i) above, $m$ is a question. Suppose $s \upharpoonright (l+1)$ is non-empty, the last move $m'$ (say) by (ii) must be an answer. We leave the other cases of $\mathsf{NL}(sm) = l$ and $l-1$ to the reader as an easy exercise. $\square$

We shall see shortly that the notion of nested level is useful for proving the compositionality of strategies. Note that Lemma 3 holds for dialogues in general – there is no assumption of justification relation or pointers, nor of the distinction between P and O.

Before we conclude the section, we prove another result about nested levels. Unlike the first, this result concerns dialogues that are equipped with justification pointers. First we introduce anonymous arenas which are arenas except that the moves are not designated as either P-moves or O-moves. Formally an ***anonymous arena*** is a structure $\langle M, \lambda, \vdash \rangle$ such that $M$ is a set, $\lambda : M \longrightarrow \{ Q, A \}$ is a map that labels each element of $M$ as either a question (Q) or an answer (A), and $\vdash \subseteq (M + \{ * \}) \times M$ is a relation that satisfies axioms (1) and (4) of justification relations, and (2'): Every initial move is a question. Note that an anonymous arena is just an arena graph except that its vertices are labelled by either Q or A.

A ***dialogue with pointers*** over an anonymous arena $\langle M, \lambda, \vdash \rangle$ is a finite sequence of elements of $M$ in which each element $m$, except the first which is initial, is equipped with a pointer to some earlier element $m^-$ in the sequence such that $m^- \vdash m$. The prime examples of dialogues with pointers are legal positions and interaction sequences (which we shall introduce in the following section). Note that it is clear what it means for a question (or an answer) in a dialogue with pointers to be pending; note also that as conditions for dialogues with pointers, Well-Bracketing and Persistence are well-defined.

**Lemma 4** *Let $s\,m$ be a dialogue with pointers over an anonymous arena $\langle M, \lambda, \vdash \rangle$. Suppose $s\,m$ satisfies Well-Bracketing and Persistence.*

*(i) The pending questions in $s\,m$ are the last moves in $s$ at a level $l < \mathsf{NL}(s\,m)$, together with $m$ if $m$ is a question. Symmetrically the pending answers in $sm$ are the last moves in $s$ at a level $l > \mathsf{NL}(s\,m)$ together with $m$ if $m$ is an answer.*
*(ii) For any $l > 0$, if the segment $a\,b$ appears in $s\,m \restriction l$ then $b$ is explicitly justified by $a$.*

**PROOF.** We prove both parts by induction on the length $|s|$ of $s$. The base case of $|s| = 0$ is trivial. For the inductive case, we reason by cases. Take $s = u\,q$ and suppose $q$ and $m$ are question moves. By the induction hypothesis and by Lemma 3(i'), the last pending answer in $u\,q$, if it exists, is the last move $a$ in $u\,q$ at level $\mathsf{NL}(u\,q) + 1$. Since by Persistence $m$ is explicitly justified by $a$ and since $\mathsf{NL}(u\,q\,m) = \mathsf{NL}(u\,q) + 1$, it follows that (i) and (ii) hold. All the remaining cases, i.e. when one or both the moves $m$ and $q$ are answer moves, can be proved in a similar, or simpler, way. $\square$

## 3 Conditionally copycat strategies and relevance

This section introduces a Cartesian closed category $\mathbb{L}$ whose objects are arenas and whose maps are innocent strategies that satisfy two new conditions: Conditionally Copycat and Relevance.

### *3.1 Innocence and conditionally copycat*

Recall that a *P-strategy* (or simply *strategy*) $\sigma$ for a game $A$ is defined to be a non-empty, prefix-closed set of legal positions of $A$ satisfying:

(1) For any even-length $s \in \sigma$, if $sm$ is a legal position then $sm \in \sigma$.
(2) (*Determinacy*). For any odd-length $s$, if $sm$ and $sm'$ are in $\sigma$ then $m = m'$.

A strategy is said to be ***innocent*** [HO00] if whenever even-length $s\,m \in \sigma$ then for any odd-length $s' \in \sigma$ such that $\ulcorner s \urcorner = \ulcorner s' \urcorner$, we have $s'm \in \sigma$. That is to say, $\sigma$ is completely determined by a partial function $f$ (say), which maps P-views $p$ to *justified P-moves* (i.e. $f(p)$ is a P-move together with a pointer to some move in $p$). We write $f_\sigma$ for the minimal such function that defines $\sigma$. We say that an innocent strategy $\sigma$ is *compact* just in case $f_\sigma$ is a finite function (or equivalently $\sigma$ contains only finitely many even-length P-views).

**Definition 5** We say that an innocent strategy $\sigma$ is ***conditionally copycat*** (or simply ***CC***) if for any odd-length P-view $p \in \sigma$ in which there is an O-answer which is immediately followed by a P-answer (i.e. $p$ has the shape "$\cdots \boldsymbol{)\,]} \cdots$"), then $p\,m \in \sigma$ for some P-move $m$ which is explicitly justified by the penultimate O-move in $p$.

CC strategies can be characterized as follows.

**Lemma 6 (CC)** *An innocent strategy $\sigma$ is CC if and only if for every even-length P-view $p$ in $\sigma$ that has the shape $u\,\boldsymbol{)}_0\,\boldsymbol{]}_0\,v$*

*(1) for any O-move $m$, if $pm \in \sigma$ then $pmm' \in \sigma$ for some P-move $m'$, and*
*(2) the sequence $\boldsymbol{)}_0\,\boldsymbol{]}_0\,v$ is a **copycat block** of moves, i.e. it has the form*

$$a_0\,b_0\,a_1\,b_1\,\cdots\,a_n\,b_n$$

*and*
*(a) for each $i \le n$, the P-move $b_i$ is a question iff the preceding O-move $a_i$ is a question*
*(b) for each $i < n$, $b_i$ explicitly justifies $a_{i+1}$ (and does so uniquely), and each $a_i$ explicitly justifies $b_{i+1}$ (and does so uniquely).*
*In other words $\boldsymbol{)}_0\,\boldsymbol{]}_0\,v$ is an interleaving of two sequences $v_1$ and $v_2$, such that in each $v_i$, each element (except the first) is explicitly justified by the preceding element in the other sequence.*

**PROOF.** The $\Leftarrow$-direction is straightforward. We prove the other direction. We omit the proof of (1) as it is obvious. Since $p$ is a P-view, $a_{n+1}$ must be justified by $b_n$. By CC, $b_{n+1}$ must be justified by $a_n$. It follows that (2b) holds.

We prove (2a) by induction on the length $|v|$ of $v$. The base case of $|v| = 0$ is obvious. Now take any even-length P-view $p$ where the corresponding $v$ has length $2n + 2$. We shall consider all possible cases.

If $a_n$ is an answer, by the induction hypothesis so is $b_n$, and since an answer can only justify a question, $a_{n+1}$ must be an O-question; and $b_{n+1}$ must be a P-question because of Well-Bracketing.

If $a_n$ is a question, by the induction hypothesis so is $b_n$, now if $a_{n+1}$ is a question then so is $b_{n+1}$ because of Well-Bracketing; on the other hand, if $a_{n+1}$ is an answer then so is $b_{n+1}$ because of Persistence.

The three cases considered are as follows:

(1) $u\,\mathbf{)}_0\,\mathbf{]}_0\,a_1\,b_1\,\cdots\,\mathbf{)}_n\,\mathbf{]}_n\,\mathbf{[}_{n+1}\,\mathbf{(}_{n+1}$
(2) $u\,\mathbf{)}_0\,\mathbf{]}_0\,a_1\,b_1\,\cdots\,\mathbf{[}_n\,\mathbf{(}_n\,\mathbf{[}_{n+1}\,\mathbf{(}_{n+1}$
(3) $u\,\mathbf{)}_0\,\mathbf{]}_0\,a_1\,b_1\,\cdots\,\mathbf{[}_n\,\mathbf{(}_n\,\mathbf{)}_{n+1}\,\mathbf{]}_{n+1}$

$\square$

### *3.2 Composition of strategies*

For arenas $A_1, A_2$ and $A_3$, a ***local sequence*** over $(A_1, A_2, A_3)$ is a sequence $u$ of elements from the set $M_{A_1} + M_{A_2} + M_{A_3}$ such that every element $m$ in $u$ other than the first (which must be initial in $A_3$) has a pointer to some earlier element $m^-$ satisfying:

(1) for $i = 1, 2$, if $m$ is initial in $A_i$ then $m^-$ is initial in $A_{i+1}$
(2) if $m$ is non-initial in $A_i$, then $m^-$ is in $A_i$ and $m^- \vdash_{A_i} m$

further $u$ satisfies *locality*: If $m'$ and $m''$ occur consecutively in $s$ such that $m' \in M_{A_i}$ and $m'' \in M_{A_j}$ then $|i - j| \leq 1$. We write $\mathcal{L}(A_1, A_2, A_3)$ for the set of *local sequences* over $(A_1, A_2, A_3)$.

Now suppose $\sigma$ and $\tau$ are strategies over arenas $A \Rightarrow B$ and $B \Rightarrow C$ respectively. The set of ***interaction sequences*** arising from $\sigma$ and $\tau$, written $\mathbf{ISeq}(\sigma, \tau)$, consists of local sequences $u \in \mathcal{L}(A, B, C)$ such that

(i) $u \upharpoonright (A, B, b) \in \sigma$, for each occurrence $b$ of an initial $B$-move in $u$
(ii) $u \upharpoonright (B, C) \in \tau$

where $u \upharpoonright (A, B, b)$, called the $(A, B, b)$-*component* of $u$, is the subsequence of $u$ consisting of moves from the arena $A \Rightarrow B$ that are hereditarily justified by the occurrence $b$ (note that the subsequence inherits the pointers associated with the moves); similarly $u \upharpoonright (B, C)$, called the $(B, C)$-*component* of $u$, is the subsequence of $u$ consisting of moves from the arena $B \Rightarrow C$. We can now define the composite strategy $\sigma\,\mathbf{;}\,\tau$ over $A \Rightarrow C$:

$$\sigma\,\mathbf{;}\,\tau \;=\; \{\, u \upharpoonright (A, C) : u \in \mathbf{ISeq}(\sigma, \tau) \,\}.$$

In $u \upharpoonright (A, C)$ the pointer of every initial $A$-move is to the unique initial $C$-move.

It is straightforward to verify that an interaction sequence $u \in \mathbf{ISeq}(\sigma, \tau)$ is a dialogue with pointers over the anonymous arena $\langle M, \lambda, \vdash \rangle$ where $M = (M_A \times M_B^{\mathrm{Init}} + M_B) \times M_C^{\mathrm{Init}} + M_C$, the question-answer labelling $\lambda : M \longrightarrow \{ Q, A \}$ is inherited from arenas $A, B$ and $C$, and $\vdash \subseteq (M + \{ * \}) \times M$ is defined as:

$$\begin{aligned}
* \vdash c &\iff * \vdash_C c \\
c \vdash (b, c') &\iff c = c' \wedge * \vdash_B b \\
(b, c) \vdash ((a, b'), c') &\iff c = c' \wedge b = b' \wedge * \vdash_A a \\
c \vdash c' &\iff c \vdash_C c' \\
(b, c) \vdash (b', c') &\iff c = c' \wedge b \vdash_B b' \\
((a, b), c) \vdash ((a', b'), c') &\iff c = c' \wedge b = b' \wedge a \vdash_A a'
\end{aligned}$$

(As is the case with moves of function space arenas, we shall refer to a move of the form $((a, b), c)$ (say) simply as a copy of $a$.) Thus the nested level of an interaction sequence is well-defined. We say that two moves in $u \in \mathbf{ISeq}(\sigma, \tau)$ are *from the same subarena* if both are from $A$, or both are from $B$, or both are from $C$.

**Lemma 7** *Let $\sigma$ and $\tau$ be as before. Take any $u \in \mathbf{ISeq}(\sigma, \tau)$.*

*(i) For any component $u \upharpoonright \theta$, and for any $m$ in $u \upharpoonright \theta$, we have $m$ is pending in $u \upharpoonright \theta$ iff $m$ is pending in $u$.*
*(ii) $u$ satisfies Persistence and Well-Bracketing.*

**PROOF.** (i) By definition of functions space arena, any pair of answer and question moves in $u$ such that one is explicitly justifying by the other are from the same subarena. From this the thesis follows immediately.

(ii) By induction on the length $|u|$ of $u$. The base case of $|u| = 0$ is trivial. For the inductive case, let $u = v\,m$. There are two subcases according to whether $m$ is a question or an answer. We shall just consider the former since the latter is similar. If there is no pending answer in $v$ then there is also no pending answer in the component $v \upharpoonright \theta$ to which $m$ belongs. Thus, by Persistence, $m$ is justified by a question, and so, Persistence is satisfied by $m$ in $u$. Otherwise if there are pending answers in $v$, let $a$ be the last such. If $a$ and $m$ both belong to the same *component* $\theta$, the thesis follows immediately from (i): $a$ is the last pending answer in $v \upharpoonright \theta$, and, by Persistence of justified sequences in $\sigma$ and $\tau$, $m$ is explicitly justified by $a$.

We now prove that it is impossible for $a$ and $m$ to belong to different components. Let $a'$ be the move following $a$ and let $m'$ be the move preceding $m$. Then $a'$ must be an answer (for if not $a$ would not be pending) and $m'$ must be a question (for if not $a$ would be equal to $m'$). By the induction hypothesis, we can apply Lemma 4(i) to $v$, and so, by Lemma 3(i'), $a$ and $m$ are at the same nested level $l$ (say) in $v\,m$; by Lemma 3(i), $a'$ and $m'$ are at the same level $l-1$. Again by the induction hypothesis, we can apply Lemma 4(ii) to $v$, and so, $m'$ is hereditarily justified by $a'$ through an alternating sequence of questions and answers; thus it follows that $a'$ and $m'$ are from the same subarena. Now suppose, for a contradiction, $a$ and $m$ are in different components. Then it follows that one component is $(B, C)$ and the other is $(A, B, b)$ for some occurrence $b$ of an initial $B$-move, and so, $a'$ and $m'$ must be $B$-moves. Suppose $a$ is an $A$-move (say) and $m$ a $C$-move. By the Switching Convention [1], we have $a'$ is a P-move in $A \Rightarrow B$. Since $a'$ and $m'$ are at the same level, it follows, from the induction hypothesis and axiom (3) of arena, that the question $m'$ is a P-move from $B$ in $B \Rightarrow C$; but the following move $m$ in $B \Rightarrow C$ switches to $C$, contradicting the Switching Convention. $\square$

*Notation.* We write $s_{\leqslant m}$ for the prefix of $s$ that terminates at $m$; and write $s_{<m}$ for the prefix of $s$ that terminates at the move just before $m$.

We are now in a position to prove that the composition of (innocent) strategies is well-defined and preserves Conditionally Copycat (CC).

**Lemma 8** *Suppose $\sigma$ and $\tau$ are strategies over arenas $A \Rightarrow B$ and $B \Rightarrow C$ respectively.*

*(i) The composite $\sigma\,;\tau$ is a well-defined strategy over $A \Rightarrow C$.*
*(ii) If $\sigma$ and $\tau$ are CC innocent, so is $\sigma\,;\tau$.*

**PROOF.** (i): We need to show that the elements in $\sigma\,;\tau$, which are justified sequences over $A \Rightarrow C$ (see e.g. [HO00] for a proof), satisfy the three axioms of legal positions. The argument for the first can be found in [HO00]. For Persistence, take a justified sequence $s\,q = u\,q \upharpoonright (A, C)$ from $\sigma\,;\tau$ where $q$ is a question. If there is a pending answer in $s$, it follows from Lemma 7(i) that there is also a pending answer in $u$. Since $u\,q$ satisfies Persistence (thanks to Lemma 7(ii)), $q$ is explicitly justified by the last pending answer $a$ (say) in $u$, and by Lemma 7(i) $a$ is also the last pending answer in $s$. Moreover, suppose $q$ is justified in $s$ by an answer $a$; we need to prove that $a$ is pending in $s$. Now

[1] *Switching Convention*: if $m_1\,m_2$ are consecutive moves in a legal position of $A \Rightarrow B$ such that one is an $A$-move and the other a $B$-move, then $m_2$ is a P-move. I.e. only P is allowed to switch games.

(by the definition of function space arena) it follows that $q$ is justified in $u$ by the same answer $a$. Since $u$ satisfies Persistence, $a$ is pending in $u$, and so, by Lemma 7(i), $a$ is also pending in $s$.

(ii): Suppose $\sigma$ and $\tau$ are CC innocent strategies. We show that the composite is CC. (For a proof that the composite is innocent, see e.g. [HO00].) By the characterization of CC in Lemma 6, it suffices to prove that even-length P-views in $\sigma\,;\tau$ of the form $p_0\,\mathbf{)}_0\,\mathbf{]}_0\,v$, where $|v| \geq 1$, satisfy the condition given in the Lemma. We shall only give the proof for the case of $|v| = 2$ (since the inductive case is a tedious repetition of the same argument):

> If the odd-length P-view $p_0\,\mathbf{)}_0\,\mathbf{]}_0\,\mathbf{[}_1 \in \sigma\,;\tau$, then $p_0\,\mathbf{)}_0\,\mathbf{]}_0\,\mathbf{[}_1\,\mathbf{(}_1 \in \sigma\,;\tau$, for some $\mathbf{(}_1$ which is explicitly justified by $\mathbf{)}_0$

Let $u \in \mathbf{ISeq}(\sigma,\tau)$ be the least $u$ such that $p_0\,\mathbf{)}_0\,\mathbf{]}_0 = u \upharpoonright (A, C)$ so that the last move of $u$ is $\mathbf{]}_0$. W.l.o.g. suppose $\mathbf{)}_0$ is a $C$-move. There are two cases: either $\mathbf{]}_0$ is a $C$-move or it is an $A$-move. We shall consider the latter, since it is harder. Suppose $\mathbf{)}_0$ and $\mathbf{]}_0$ are at levels $l'$ and $l$ in $u$ respectively. By Lemma 3(ii), we have $l' > l$. Set $l_0 = l' - l$ which is even. By Lemma 3(i'), for each $1 \leq k < l_0$, there is a move, which must be a $B$-move, occurring between $\mathbf{)}_0$ and $\mathbf{]}_0$ in $u$ at level $l + k$, and by Lemma 3(ii) the last such at that level is an answer. Suppose $u = u_0\,\mathbf{)}_0\,b_1\,\cdots\,b_L\,\mathbf{]}_0$.

**Claim**. The block of $B$-moves $b_1\,\cdots\,b_L$ between $\mathbf{)}_0$ and $\mathbf{]}_0$ consists of one move (which must be an answer) per level, starting from $l + l_0 - 1$ and going down to $l + 1$. I.e. $L = l_0 - 1$, and for each $1 \leq k \leq l_0 - 1$, $b_k$ is an answer at level $l + l_0 - k$.

We prove the claim by contradiction. Suppose $b_1, b_2 \cdots, b_k$ respectively are answers at levels $l + l_0 - 1, \cdots, l + l_0 - k$ but $b_{k+1}$ is a question, which must be at the same level as $b_k$. Suppose $b_{k-1}$ and $b_k$ are both from the component (say) $(A, B, b)$ for some occurrence $b$ of an initial $B$-move, $b_{k-1}$ is an O-answer and $b_k$ is a P-answer in $u \upharpoonright (A, B, b)$. By Lemma 4(ii), $b_{k+1}$ is an O-question explicitly justified by $b_k$; since $\sigma$ is assumed to be CC, $b_{k+2}$ is a P-question explicitly justified by $b_{k-1}$ at level $l + l_0 - k - 1$. Now continuing in this fashion, and by appealing to the assumption that $\sigma$ and $\tau$ are CC, we have $u_0\,\mathbf{]}_0\,b_1 \cdots b_k\,b_{k+1} \cdots b_{2k}\,c \in \mathbf{ISeq}(\sigma,\tau)$, where for each $1 \leq i \leq k$, the $B$-question $b_{2k-i+1}$ is explicitly justified by the answer $b_i$, and $c$ is a $C$-*question* explicitly justified by $\mathbf{)}_0$, which is a contradiction.

By assumption, $\mathbf{[}_1$ is explicitly justified by $\mathbf{]}_0$, and so, it is an $A$-move. It suffices to prove that $v = u\mathbf{[}_1\,d_1 \cdots d_{l_0-1}\,\mathbf{(}_1 \in \mathbf{ISeq}(\sigma,\tau)$, where $\mathbf{(}_1$ is a $C$-move which is at the same level as, and so is explicitly justified by, $\mathbf{)}_0$, and for each

$1 \leq i \leq l_0 - 1$, $d_i$ is a $B$-question at level $l+i$ which is explicitly justified by the $B$-answer $b_{l_0 - i}$. We leave this as a straightforward exercise for the reader. $\square$

### *3.3 A notion of relevance*

We consider a notion of Relevance whereby P is not allowed to respond to an O-question by engaging O indefinitely in a dialogue at one level higher, nor is P allowed to "give up"; instead he must answer the O-question eventually.

**Definition 9** We say that a CC strategy $\sigma$ is ***relevant*** if

(1) for each P-view $p \boldsymbol{)} \in \sigma$, there is a P-move $m$ such that $p \boldsymbol{)}\, m \in \sigma$
(2) there is no infinite sequence $p \boldsymbol{(}_0 \boldsymbol{)}_0 \boldsymbol{(}_1 \boldsymbol{)}_1 \cdots$ such that for every $n$

$$p \boldsymbol{(}_0 \boldsymbol{)}_0 \boldsymbol{(}_1 \boldsymbol{)}_1 \cdots \boldsymbol{(}_n \boldsymbol{)}_n \ \in \ \sigma.$$

**Theorem 10** *If $\sigma$ and $\tau$ are relevant CC strategies over arenas $A \Rightarrow B$ and $B \Rightarrow C$ respectively then the composite $\sigma \, ; \tau$ is also a relevant CC strategy.*

**PROOF.** Thanks to Lemma 8, it remains to prove that the composite $\sigma \, ; \tau$ satisfies conditions (1) and (2) of Relevance.

(1): Take a P-view $p \boldsymbol{)} \in \sigma \, ; \tau$. Let $u \in \mathbf{ISeq}(\sigma, \tau)$ be the least sequence such that $u \boldsymbol{)} \upharpoonright (A, C) = p \boldsymbol{)}$. W.l.o.g. suppose $\boldsymbol{)}$ is an O-move in the component $(B, C)$ (it follows that $\boldsymbol{)}$ is a $C$-move). Since $\tau$ is relevant and $\ulcorner u \upharpoonright (B, C) \urcorner \boldsymbol{)} \in \tau$, there is a P-move $m$ such that $\ulcorner u \upharpoonright (B, C) \urcorner \boldsymbol{)}\, m \in \tau$. There are two cases: $m$ is either a $C$-move or a $B$-move. If the former, $p \boldsymbol{)}\, m \in \sigma \, ; \tau$ (see e.g. the analysis of the composition of innocent strategies in [HO00]) and we are done. If the latter, let $n$ be the largest such that $u \boldsymbol{)}\, a_1 \, a_2 \, \cdots \, a_n \in \mathbf{ISeq}(\sigma, \tau)$ and each $a_i$ is an answer. Note that $n$ is a well-defined number (since the level of $u \boldsymbol{)}\, a_1 \, a_2 \, \cdots \, a_i$ decreases as $i$ increases, provided each $a_i$ is an answer) and at least one by assumption. Now if some $a_i$ is a move in $A \Rightarrow C$ (and let $i$ be the least such), we have $p \boldsymbol{)}\, a_i \in A \Rightarrow C$ are we are done. If not, by considering the P-view $\ulcorner u \boldsymbol{)}\, a_1 \, a_2 \, \cdots \, a_n \upharpoonright (B, C) \urcorner$ in $\tau$ (w.l.o.g. assuming that $a_n$ is an O-move in the component $(B, C)$) which is relevant, we must have $u \boldsymbol{)}\, a_1 \, a_2 \, \cdots \, a_n \, q_{n+1} \in \mathbf{ISeq}(\sigma, \tau)$, for some *question* $q_{n+1}$ which is explicitly justified by $a_n$ (by Persistence), and so, $q_{n+1}$ is a $B$-move. Now since both $\sigma$ and $\tau$ are CC, it follows inductively that there are $B$-questions $q_{n+2}, \cdots, q_{2n}$ such that $u \boldsymbol{)}\, a_1 \, a_2 \, \cdots \, a_n \, q_{n+1} \, \cdots \, q_{2n} \in \mathbf{ISeq}(\sigma, \tau)$ and for each $i \geq 1$, the question $q_{n+i}$ is explicitly justified by the answer $a_{n-i+1}$. Thus it follows that $u \boldsymbol{)}\, a_1 \, a_2 \, \cdots \, a_n \, q_{n+1} \, \cdots \, q_{2n} \, q \in \mathbf{ISeq}(\sigma, \tau)$ for some question $q$ which is explicitly justified by $\boldsymbol{)}$, and so, $q$ is a $C$-move and we are done.

(2): Suppose, for a contradiction, there is an infinite sequence $p \,\boldsymbol{(}_0 \boldsymbol{)}_0 \boldsymbol{(}_1 \boldsymbol{)}_1 \cdots$ such that for every $n$, $p \,\boldsymbol{(}_0 \boldsymbol{)}_0 \boldsymbol{(}_1 \boldsymbol{)}_1 \cdots \boldsymbol{(}_n \boldsymbol{)}_n \;\in\; \sigma \,\boldsymbol{;}\, \tau$. W.l.o.g. suppose $\boldsymbol{(}_0$ is an $A$-move in the component $(A, B, b)$ for some occurrence $b$ of an initial $B$-move; it then follows that $\boldsymbol{)}_0, \boldsymbol{(}_1, \boldsymbol{)}_1, \cdots$ are all $A$-moves (since they are all hereritarily justified by $\boldsymbol{(}_0$). By definition of composition, there is an infinite sequence $u \,\boldsymbol{(}_0 \, u_0 \,\boldsymbol{)}_0 \boldsymbol{(}_1 \, u_1 \,\boldsymbol{)}_1 \cdots$ such that for each $n$, $u \,\boldsymbol{(}_0 \, u_0 \,\boldsymbol{)}_0 \boldsymbol{(}_1 \, u_1 \,\boldsymbol{)}_1 \cdots \boldsymbol{(}_n \, u_n \,\boldsymbol{)}_n \in \mathbf{ISeq}(\sigma, \tau)$. Thus, by first projecting to $(A, B, b)$ and then taking the P-view, we have

$$\ulcorner u \urcorner \boldsymbol{(}_0 \boldsymbol{)}_0 \boldsymbol{(}_1 \boldsymbol{)}_1 \cdots \boldsymbol{(}_n \boldsymbol{)}_n \;\in\; \sigma$$

for each $n$, which contradicts the assumption that $\sigma$ is relevant. $\square$

### *3.4 The category $\mathbb{L}$*

We define a category called $\mathbb{L}$ whose objects are arenas and whose maps $A \longrightarrow B$ are relevant CC strategies of the arena $A \Rightarrow B$. It is completely straightforward to verify that $\mathbb{L}$ is Cartesian closed (see e.g. [HO00] for a similar proof): the terminal object is the empty arena; for any arenas $A$ and $B$, their Cartesian product is given by $A \times B$, and the function space arena is $A \Rightarrow B$. However lifting $(-)_\perp$ is *not* functorial (see Remark 11). We write $\mathbb{L}_{\mathrm{rec}}$ for the subcategory whose objects are arenas but whose maps are *recursive* (in the sense of [HO00, §5.6]) relevant CC strategies.

## 4 Universality and full abstraction

We introduce an arena $\mathcal{D}$, which is the initial solution of the recursive equation $D = [D \Rightarrow D]_\perp$, and interpret (closed) $\lambda$-terms as relevant CC strategies over it. By an analysis of the structure of P-views over $\mathcal{D}$, we obtain the main result of the paper: Every recursive relevant CC strategy over $\mathcal{D}$ is the denotation of a closed $\lambda$-term; further two terms have the same denotation iff they have the same Levy-Longo tree.

### *4.1 The model*

Following [McC98], for arenas $A$ and $B$, we define the *subarena relation* $A \trianglelefteq B$ by

(1) $M_A \subseteq M_B$

(2) $\lambda_A = \lambda_B \restriction M_A$
(3) $\vdash_A = \vdash_B \cap (M_A + \{\,*\,\}) \times M_A$

Equipped with the ordering $\trianglelefteq$, the collection of arenas is a (large) dcpo $\mathbb{A}$, with least element the empty arena $\mathbf{1}$, and directed suprema given by taking component-wise union. Take any operation $F$ on arenas. If $F$ is monotone and continuous with respect to $\trianglelefteq$, there is an arena solving the recursive equation $D = F(D)$ by taking $D = \bigsqcup_{n\geq 0} F^n(\mathbf{1})$.

Let $F$ be the arena operation $A \mapsto [A \Rightarrow A]_\perp$. It is straightforward to verify that $F$ is monotone and continuous. We define the arena $\mathcal{D}$ as the initial solution of the recursive equation $D = F(D)$ in the category $\mathbb{A}$. The arena graph of $\mathcal{D}$ (see Figure 1) is a finitely-branching tree that satisfies the following:

(1) Every question justifies a unique answer, and at most one question.
(2) Every answer justifies a unique question.

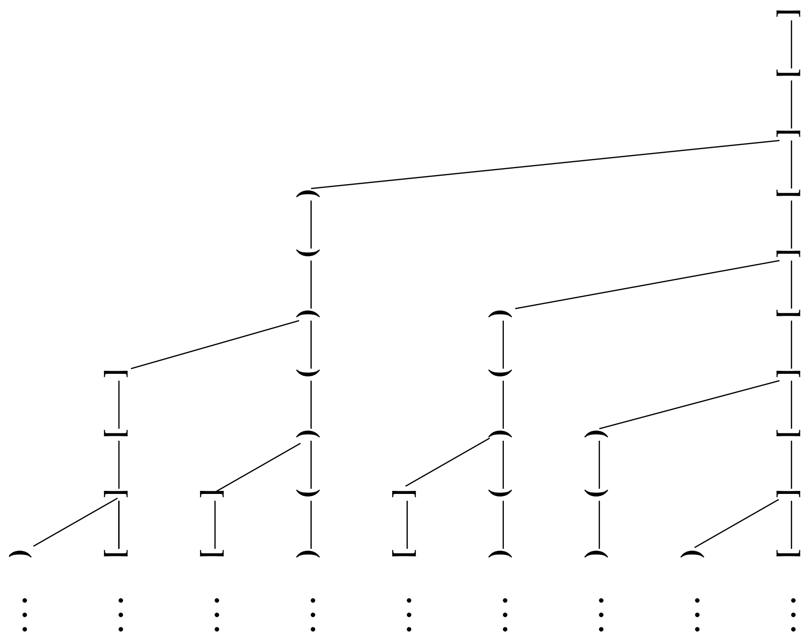


Fig. 1. The arena graph of $\mathcal{D}$

Let $\mathsf{app}$ be the "evaluation map" $[\mathcal{D}_1 \Rightarrow \mathcal{D}_2]_\perp \times \mathcal{D}_3 \longrightarrow \mathcal{D}_4$ (we label the four copies of $\mathcal{D}$) which is the following strategy: P responds to the opening move with the initial move of $[\mathcal{D} \Rightarrow \mathcal{D}]_\perp$, and responds to the answer justified by the latter with the answer justified by the opening move; and thereafter P plays copycat between $\mathcal{D}_2$ and $\mathcal{D}_4$, and between $\mathcal{D}_1$ and $\mathcal{D}_3$. We write $\overline{f}$ for the transpose in the following bijection between $\mathbb{L}$-maps:

$$\frac{f : C \times \mathcal{D} \longrightarrow \mathcal{D}}{\overline{f} : C \longrightarrow_t [\mathcal{D} \Rightarrow \mathcal{D}]_\perp}$$

which is natural in $C$, where $\tau : C \longrightarrow_t B$ denotes a *convergent* strategy in the sense that $\tau$ responds to the opening move immediately with an answer. As $\mathsf{app}$ is the inverse transpose of the identity map on $[\mathcal{D} \Rightarrow \mathcal{D}]_\perp$, we have

$$\langle\, \mathrm{id}_C, a \,\rangle \,;\, f = \langle\, \overline{f}, a \,\rangle \,;\, \mathsf{app} \;:\; C \longrightarrow \mathcal{D} \tag{1}$$

for any $a : C \longrightarrow \mathcal{D}$ in $\mathbb{L}$. We define the $\mathbb{L}$-map $[\![\, \Gamma \vdash s \,]\!] : \mathcal{D}^n \longrightarrow \mathcal{D}$, where $\Gamma = \{\, x_1, \cdots, x_n \,\}$ is a finite set of variables including the free variables of $s$, by recursion over $s$ as follows:

$$\begin{array}{rcl} [\![\, \Gamma \vdash x_i \,]\!] & = & \pi_i : \mathcal{D}^n \longrightarrow \mathcal{D} \\ [\![\, \Gamma \vdash st \,]\!] & = & \langle\, [\![\, \Gamma \vdash s \,]\!], [\![\, \Gamma \vdash t \,]\!] \,\rangle \,;\, \mathsf{app} \\ [\![\, \Gamma \vdash \boldsymbol{\lambda} x.s \,]\!] & = & \overline{[\![\, \Gamma, x \vdash s \,]\!]} \end{array}$$

where $\pi_i$ is the standard projection map, and $\langle\, -, - \,\rangle$ is pairing. Standardly (see e.g. [AO93]) this gives a model of the (Lazy) $\lambda$-calculus.

**Remark 11** (i) There is no way lifting can be functorial in a category of arenas and conditionally copycat strategies. Take a CC strategy $\sigma : A \longrightarrow B$. Since $\mathrm{id}_\perp = \mathrm{id} : A_\perp \longrightarrow B_\perp$, $\sigma_\perp$ is forced to respond to the initial move $q_B$ in $B_\perp$ with the initial move $q_A$ in $A_\perp$, and to respond to the P-view $q_B q_A a_A$ with the move $a_B$. Now almost all P-views in $\sigma_\perp$ contain an O-answer $a_A$ immediately followed by a P-answer $a_B$, and so, by Lemma 6, $\sigma_\perp$ is almost always constrained to play copycat, whereas $\sigma$ may not be restricted in the same way. (It is easy to construct concrete instances of $\sigma$ and $\sigma_\perp$.)

(ii) Functoriality of lifting is not necessary for the construction of our model. The domain equation $D = [D \Rightarrow D]_\perp$ is solved in the auxiliary category $\mathbb{A}$, and lifting *is* functorial in this category. All we need are two (relevant, CC) strategies, $\mathsf{up}_D : D \longrightarrow D_\perp$ and $\mathsf{dn}_D : D_\perp \longrightarrow D$, such that $\mathsf{dn}_D \circ \mathsf{up}_D = \mathrm{id}_D$, which are easily constructible for any arena $D$.

(iii) Indeed functoriality of lifting is *inconsistent* with our model being fully abstract. A feature of our model is that there are “few” denotable strategies that are compact-innocent; indeed the innocent strategy denoted by a closed term is compact if and only if the term is unsolvable of a finite order (Lemma 16). Now we know from [AO93, Lemma 9.2.8] that projections on the finite approximations $\mathcal{D}_n$ of the fully abstract model $\mathcal{D}$ of the Lazy Lambda Calculus are not $\lambda$-definable. If *all* the domain constructions involved in the domain equation $D = [D \Rightarrow D]_\perp$ were functorial, these projections would be maps that are definable *categorically*, which would imply that our model is not fully abstract.

**Remark 12** $\mathcal{D}$ is a *submodel* of McCusker's game model for the lazy $\lambda$-calculus, $\mathcal{D}_M$, as constructed in [McC96].

Here we sketch a proof in stages (and explain what we mean by submodel) as follows:

(1) First we prove that the two models are defined over the same arena; an important difference is that $\mathcal{D}_M$ contains plays in violation of Persistence.
(2) We then define an embedding $e$ of strategies in $\mathcal{D}$ to strategies in $\mathcal{D}_M$, and
(3) prove that the embedding $e$ preserves application.

By analogy with McCusker's category of games and innocent strategies, we define an $\mathbb{L}$-game to be a pair consisting of an object of $\mathbb{L}$ (i.e. an arena) and the set of legal positions over it. We can now define a relation $\blacktriangleleft$ between $\mathbb{L}$-games and the innocent games in the sense of [McC96]. We say that $A \blacktriangleleft A'$ just in case $A$ and $A'$ are defined over the same arena, $A'$ contains all the plays in $A$, and $A'$ does not contain any play in which there is (an occurrence of) an answer that justifies more that one question. It is straightforward to show that the relation $\blacktriangleleft$ is preserved by lifting and the functions space construction. As both models are appropriate limit constructions, it follows that $\mathcal{D} \blacktriangleleft \mathcal{D}_M$.

It is straightforward to check that Persistence is satisfied by every O-move that occurs in a P-view. Since an innocent strategy is completely determined by the set of P-views it contains, for any pair of games $A$ and $A'$ such that $A \blacktriangleleft A'$, given a strategy $\sigma$ over $A$, we define its embedding $e_{A,A'}(\sigma)$ to be the strategy over $A'$ given by the set of P-view in $\sigma$. (To save writing, we shall omit the subscripts in $e_{A,A'}$ in the following.)

The composition of strategies depends on the set of plays in the strategies. Now the strategy $e(\sigma)$ may contain plays that violate Persistence. We need to prove that $e$ preserves the composition of strategies. That is to say, we need to prove that for every three pairs of games $A, A', B, B', C$ and $C'$ such that $A \blacktriangleleft A'$, $B \blacktriangleleft B'$ and $C \blacktriangleleft C'$, and for every pair of strategies $\sigma : A \Rightarrow B$ and $\tau : B \Rightarrow C$ in $\mathbb{L}$, the equality $e(\sigma); e(\tau) = e(\sigma; \tau)$ holds. We shall establish $e(\sigma); e(\tau) \subseteq e(\sigma; \tau)$ by regarding strategies as sets of P-views; the opposite inclusion is omitted as it is straightforward. Take a P-view $p$ in $e(\sigma); e(\tau)$; there exists an interaction sequence $u \in \mathbf{ISeq}(e(\sigma), e(\tau))$ such that $p = u \upharpoonright (A', C')$. We claim that all moves in $u \upharpoonright (B', C')$ and $u \upharpoonright (A', B', b)$ (for each $b$) satisfy Persistence:

- P-moves in $u \upharpoonright (A', B', b)$ and $u \upharpoonright (B', C')$ satisfy Persistence since each such move is determined by either the strategy $\sigma$ or $\tau$.
- O-moves of $A'$ in $u \upharpoonright (A', B', b)$ and of $C'$ in $u \upharpoonright (B', C')$ satisfy Persistence since $p = u \upharpoonright (A', C')$ is a P-view.
- O-moves of $B'$ in $u \upharpoonright (A', B', b)$ and in $u \upharpoonright (B', C')$ satisfy Persistence since

these are P-moves when viewed in the other projection.

It follows that $u \upharpoonright (B', C') \in \tau$ and $u \upharpoonright (A', B', b) \in \sigma$ and so $p \in \sigma; \tau$, and hence, we have $p \in e(\sigma; \tau)$ as desired.

Thus we have an embedding $e_{\mathcal{D},\mathcal{D}_M}$ from $\mathcal{D}$ to $\mathcal{D}_M$ that preserves application i.e. $\mathcal{D}$ is a submodel of $\mathcal{D}_M$. $\square$

**Lemma 13 (Adequacy)** *For any closed term $s$, we have $[\![\, s \,]\!] = \perp$, the strategy that has no response to the opening move, if and only if $s$ is strongly unsolvable (i.e. $s$ is not $\beta$-convertible to a $\lambda$-abstraction).*

**PROOF.** By adapting a standard method in [Bar84] and as a corollary of an approximation theorem. $\square$

For any $\lambda$-term $s$, if the set $\{\, i \geq 0 : \exists t. \boldsymbol{\lambda\beta} \vdash s = \boldsymbol{\lambda} x_1 \cdots x_i.t \,\}$ has no supremum in $\mathbb{N}$, we say that $s$ has *order infinity*; otherwise if the supremum is $n$, we say that $s$ has *order $n$*. A term that has order infinity is unsolvable (e.g. $\mathbf{yk}$, for any fixpoint combinator $\mathbf{y}$).

### *4.2 Structure of P-views*

We aim to describe P-views of $\mathcal{D}$ in terms of blocks (of moves) of two kinds, called $\alpha$ and $\beta$ respectively.

For $n \geq 0$, an $\alpha_n$***-block*** is an alternating sequence of O-questions and P-answers of length $2n+1$, beginning with an O-question, such that each element except the first is explicitly justified by the preceding element, as follows:

$$\mathbf{[}_0\, \mathbf{]}\, \mathbf{[}_1\, \mathbf{]} \cdots \mathbf{[}_{n-1}\, \mathbf{]}\, \mathbf{[}_n$$

We call $\mathbf{[}_i$ the $i$-th question of the block.

For $m \geq 0, i \geq 0$ and $j \geq 1$, a $\beta_m^{(i,j)}$***-block*** is an alternating sequence of P-questions and O-answers of length $2m+1$, beginning with a P-question, such that each element except the first is explicitly justified by the preceding element, as follows:

$$\mathbf{(}_0\, \mathbf{)}\, \mathbf{(}_1\, \mathbf{)} \cdots \mathbf{(}_{m-1}\, \mathbf{)}\, \mathbf{(}_m$$

We call $\mathbf{(}_i$ the $i$-th question of the block. The superscript $(i,j)$ in $\beta_m^{(i,j)}$ encodes the target of the justification pointer of $\mathbf{(}_0$ relative to the P-view of which the $\beta_m^{(i,j)}$-block is a part, about which more anon. A $\overline{\beta}_m^{(i,j)}$-block is just a $\beta_m^{(i,j)}$-block followed by a $\mathbf{)}$, which is explicitly justified by the last question $\mathbf{(}_m$. An $\alpha$-block is just an $\alpha_n$-block, for some $n$; similarly for a $\beta$-block.

Suppose we have a P-view of the form

$$p \quad = \quad A_1 \; B_1 \; A_2 \; B_2 \; \cdots \; A_k \; B_k \; \cdots$$

where each $A_k$ is an $\alpha_{n_k}$-block and each $B_k$ is a $\beta_{l_k}^{(i_k,j_k)}$-block. The superscript $(i_k, j_k)$ encodes the fact that the 0-th question of the block $B_k$ is explicitly justified by the $j_k$-th question of the block $A_{k-i_k}$. Thus we have the following constraints: for each $k \geq 1$

$$0 \leq i_k < k \qquad \wedge \qquad 1 \leq j_k \leq n_{k-i_k} \tag{2}$$

The lower bound of $j_k$ is 1 rather than 0 because, by definition of $\mathcal{D}$ (see Figure 1), the only move that the 0-th question of any $\alpha$-block can justify is an answer. Note that since $p$ is a P-view by assumption, for each $k \geq 2$, the 0-th question of the $\alpha$-block $A_k$ is explicitly justified by the last question of the preceding $\beta$-block.

**Remark 14** It is straightforward to see that given any finite alternating sequence $\gamma$ of $\alpha$- and $\beta$-blocks

$$\gamma \; = \; \alpha_{n_1} \; \beta_{l_1}^{(i_1,j_1)} \; \cdots \; \alpha_{n_k} \; \beta_{l_k}^{(i_k,j_k)} \; \cdots$$

subject to the constraints (2), there is exactly one P-view $p$ of $\mathcal{D}$ that has the shape $\gamma$. Therefore there is no harm in referring to the P-view $p$ simply as $\gamma$, and we shall do so in the following.

**Lemma 15 (P-view Characterization)** *Suppose, for some $m \geq 0$, the even-length P-view*

$$W \quad = \quad \alpha_{n_1} \; \beta_{l_1} \; \cdots \; \alpha_{n_m} \; \beta_{l_m}$$

*is in a relevant CC strategy $\sigma$ over $\mathcal{D}$. Then exactly one of the following holds:*

*(1) For each $j \geq 0$, $W \, \alpha_j \in \mathrm{dom}(f_\sigma)$.*
*(2) There is some $n \geq 0$ such that $W \, \alpha_n \in \sigma \setminus \mathrm{dom}(f_\sigma)$.*

(3) *There are some* $n_{m+1} \geq 0$, *some* $0 \leq i \leq m$ *and some* $1 \leq j \leq n_{m+1-i}$ *such that* $f_\sigma : W\,\alpha_{n_{m+1}} \;\mapsto\; \boldsymbol{(}^{(i,j)}$; *further by Relevance, for some* $l \geq 0$, *we have*

$$f_\sigma \;:\; W\,\alpha_{n_{m+1}}\,\overline{\beta}_l^{(i,j)} \;\mapsto\; \boldsymbol{]}.$$

*Moreover by CC we have* $W\,\alpha_{n_{m+1}}\,\overline{\beta}_l^{(i,j)}\,\boldsymbol{]}\,C \in \mathrm{dom}(f_\sigma)$, *for each (odd-length) copycat block* $C$, *as defined in Lemma 6.*

**PROOF.** Suppose for some $m \geq 0$, the even-length $W \in \sigma$. Then $W\,\boldsymbol{[}$, where $\boldsymbol{[}$ is explicitly justified by the last P-question, is a P-view in $\sigma$. Clearly if neither (1) nor (2) above holds, then there is some $n_{m+1} \geq 0$ such that $f_\sigma$ maps $W\,\alpha_{n_{m+1}}$ to a P-question which (because the current P-view has no pending O-answer) is explicitly justified by an O-question that is currently P-visible i.e. by one of the O-questions (except the 0-th) in one of the $m+1$ preceding $\alpha$-blocks. Formally we have $f_\sigma : W\,\alpha_{n_{m+1}} \;\mapsto\; \boldsymbol{(}^{(i,j)}$ where $0 \leq i \leq m$ and $1 \leq j \leq n_{m+1-i}$ as required. The rest of (3) above follows immediately from Relevance and by Lemma 6 respectively. □

**Lemma 16** *The denotation of any closed term in* $\mathcal{D}$ *is a compact innocent strategy if and only if it is unsolvable of a finite order.*

**PROOF.** It suffices to prove that

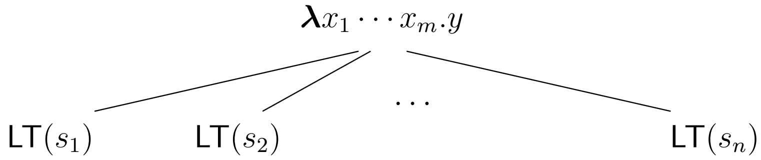
(1) unsolvables of infinite order
(2) solvable terms

are denoted by non-compact strategies. For the first, note that every finite-length legal position consisting of alternating questions and answers, such that all of which are at level 0, is in the denotation of any unsolvable of order infinity. For any solvable term $s$ which has head normal form $\boldsymbol{\lambda} x_1 \cdots x_n.x_i t_1 \cdots t_m$ (say) where $n \geq 1$ and $m \geq 0$, we observe that the P-view $\alpha_n\;\beta_m^{(0,i)}\;\boldsymbol{)\,]}$ is in $[\![\, s \,]\!]$. The denotation is a CC strategy, by Lemma 6; hence it is not compact. □

### *4.3 Levy-Longo trees*

We give an informal definition of $\mathsf{LT}(s)$, the ***Levy-Longo tree*** [Lev75,Lon83] of a $\lambda$-term $s$.

- Suppose $s$ is unsolvable: If $s$ has order infinity then $\mathsf{LT}(s)$ is the singleton tree $\top$; if $s$ has order $n \geq 0$ then $\mathsf{LT}(s)$ is the singleton tree $\bot_n$.




- Suppose $s =_\beta \boldsymbol{\lambda} x_1 \cdots x_m.y s_1 \cdots s_n$ where $m, n \geq 0$. Then $\mathsf{LT}(s)$ is the tree:

It is useful to fix a *variable-free representation* of Levy-Longo trees. We write $\mathbb{N} = \{0, 1, 2, \cdots\}$ and $\mathbb{N}_+ = \{1, 2, 3, \cdots\}$. A ***Levy-Longo pre-tree*** is a partial function $T$ from the set $(\mathbb{N}_+)^*$ of *occurrences* to the following set of *labels*

$$\mathbb{N} \times (\mathbb{N} \times \mathbb{N}_+) \times \mathbb{N} \quad \cup \quad \{\bot_i : i \geq 0\} \quad \cup \quad \{\top\}$$

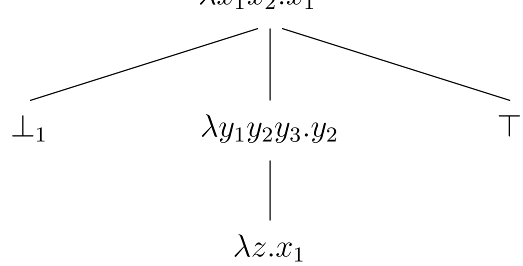
such that

(1) $\mathrm{dom}(T)$ is prefix-closed.
(2) Every occurrence that is labelled by any of $\bot_i$ and $\top$ is maximal in $\mathrm{dom}(T)$.
(3) If $T(l_1 \cdots l_m) = \langle n, (i, j), b \rangle$ then:
   (a) $l_1 \cdots l_m l \in \mathrm{dom}(T) \iff 1 \leq l \leq b$, and
   (b) $0 \leq i \leq m + 1$, and
   (c) If $i \leq m$ then $T(l_1 \cdots l_{m-i})$ is a triple, the first component of which is at least $j$.

(The case of $i = m+1$ corresponds to the head variable at $l_1 \cdots l_m$ being a free variable.) We say that the pre-tree is *closed* if $T(l_1 \cdots l_m) = \langle n, (i, j), b \rangle \Longrightarrow i \leq m$. A ***Levy-Longo tree*** is the Levy-Longo pre-tree given by $\mathsf{LT}(s)$ for some $\lambda$-term $s$. In the following, we shall only consider closed pre-trees and trees.

To illustrate the variable-free representation, consider the following (running) example.

**Example 17** Set $s = \boldsymbol{\lambda} x_1 x_2.x_1 \bot_1 (\boldsymbol{\lambda} y_1 y_2 y_3.y_2 (\boldsymbol{\lambda} z.x_1)) \top$. The Levy-Longo tree $\mathsf{LT}(s)$, as shown in the figure below



is the partial function:

$$\begin{cases} \epsilon \mapsto \langle\, 2, (0,1), 3 \,\rangle \\ 1 \mapsto \bot_1 \\ 2 \mapsto \langle\, 3, (0,2), 1 \,\rangle \\ 3 \mapsto \top \\ 21 \mapsto \langle\, 1, (2,1), 0 \,\rangle \end{cases}$$

Take $\mathsf{LT}(s) : 21 \mapsto \langle\, 1, (2,1), 0 \,\rangle$ which encodes the label $\boldsymbol{\lambda} z.x_1$ of the tree at occurrence 21: the first component is the *nested depth* of the $\lambda$-abstraction: in this case it is a 1-deep $\lambda$-abstraction (i.e. of order one); the second component $(i, j)$ says that the head variable ($x_1$ in this case) is a copy of the $j$-th (in this case, first) variable bound at the occurrence $i$ (in this case, two) levels up; and the third component is the *branching factor* at the occurrence, which is 0 in this case i.e. the occurrence 21 has 0 children.

Thanks to Lemma 15, we can now explain the correspondence between relevant CC strategies over $\mathcal{D}$ and closed Levy-Longo pre-trees; we shall write the pre-tree corresponding to the strategy $\sigma$ as $T_\sigma$. Using the notation of Lemma 15, the action of the strategy $\sigma$ on a P-view $p \in \sigma$ of the shape $\alpha_{n_1}\ \beta_{l_1}^{(i_1,j_1)}\ \cdots\ \alpha_{n_m}\ \beta_{l_m}^{(i_m,j_m)}\ \boldsymbol{[}$ determines precisely the label of $T_\sigma$ at the occurrence $l_1 \cdots l_m$. Corresponding to each of the three cases in the Lemma 15, the label defined at the occurrence is as follows:

(1) $\top$
(2) $\bot_n$ where $n \geq 0$
(3) $\langle\, n, (i,j), b \,\rangle$

It is easy to see the occurrence in question is maximal in $\mathrm{dom}(T_\sigma)$ in cases 1 and 2. In case 3, i.e., $T_\sigma(l_1 \cdots l_m) = \langle\, n, (i,j), b \,\rangle$, from the P-view $p$, we can work out the label of $T_\sigma$ at each prefix $l_1 \cdots l_k$ (where $k \leq m$) of the corresponding occurrence, which is $\langle\, n_{k+1}, (i_{k+1}, j_{k+1}), b_{k+1} \,\rangle$, as determined by

$$f_\sigma : \alpha_{n_1}\ \beta_{l_1}^{(i_1,j_1)}\ \cdots\ \alpha_{n_k}\ \beta_{l_k}^{(i_k,j_k)}\ \alpha_{n_{k+1}}\ \overline{\beta}_{b_{k+1}}^{(i_{k+1},j_{k+1})}\ \mapsto\ \boldsymbol{]}$$

we set $\langle\, n_{m+1}, (i_{m+1}, j_{m+1}), b_{m+1} \,\rangle = \langle\, n, (i,j), b \,\rangle$. Note that $b_{k+1}$ is well-defined because of Relevance. Thus the domain of $T_\sigma$ is prefix-closed. Take any $k \leq m$. For each $1 \leq l \leq b_{k+1}$, we have the odd-length P-view

$$\alpha_{n_1}\ \beta_{l_1}^{(i_1,j_1)}\ \cdots\ \alpha_{n_k}\ \beta_{l_k}^{(i_k,j_k)}\ \alpha_{n_{k+1}}\ \beta_{l}^{(i_{k+1},j_{k+1})}\ \boldsymbol{[}\quad \in \quad \sigma$$

and so, we have $l_1 \cdots l_k l \in \mathrm{dom}(T_\sigma) \iff 1 \le l \le b_{k+1}$. Finally, we must have $j_{k+1} \le n_{k-i_{k+1}}$, as the pointer of the 0-th (P-)question of the $\beta$-block $\beta_l^{(i_{k+1}, j_{k+1})}$ is to the $j_{k+1}$-th question of the $\alpha$-block $\alpha_{n_{k-i_{k+1}}}$.

To summarize, we have shown:

**Lemma 18 (Correspondence)** *There is a one-to-one correspondence between relevant CC strategies over $\mathcal{D}$ and closed Levy-Longo pre-trees.* $\Box$

**Example 19** Take the term $s = \boldsymbol{\lambda} x_1 x_2 . x_1 \perp_1 (\boldsymbol{\lambda} y_1 y_2 y_3 . y_2 (\boldsymbol{\lambda} z . x_1)) \top$ in the preceding example. In the following table, we illustrate the exact correspondence between the relevant CC strategy $[\![ s ]\!]$ denoted by $s$ on the one hand, and the Levy-Longo tree $\mathsf{LT}(s)$ of the term on the other.

| P-views in $[\![ s ]\!]$ | | occurrences | labels of $\mathsf{LT}(s)$ |
|---|---|---|---|
| $\boxed{\alpha_2\,\overline{\beta}_3^{(0,1)}}$ | $\mapsto \mathbf{]}$ | $\epsilon$ | $\langle 2, (0,1), 3 \rangle$ |
| $\alpha_2\,\beta_{\mathbf{1}}^{(0,1)}\,\alpha_1$ | $\in \sigma \setminus \mathrm{dom}(f_\sigma)$ | 1 | $\perp_1$ |
| $\alpha_2\,\beta_{\mathbf{2}}^{(0,1)}\,\boxed{\alpha_3\,\overline{\beta}_1^{(0,2)}}$ | $\mapsto \mathbf{]}$ | 2 | $\langle 3, (0,2), 1 \rangle$ |
| $\alpha_2\,\beta_{\mathbf{3}}^{(0,1)}\,\alpha_n$ | $\mapsto \mathbf{]}$ for $n \ge 0$ | 3 | $\top$ |
| $\alpha_2\,\beta_{\mathbf{2}}^{(0,1)}\,\alpha_3\,\beta_{\mathbf{1}}^{(0,2)}\,\boxed{\alpha_1\,\overline{\beta}_0^{(2,1)}}$ | $\mapsto \mathbf{]}$ | 21 | $\langle 1, (2,1), 0 \rangle$ |

For each P-view shown above, note that the subscripts in bold give the corresponding occurrence in the Levy-Longo tree, and the label at that occurrence is specified by the (subscripts and the superscript in the) block that is framed. The first, third and fifth P-views define the "boundary" beyond which the copycat response sets in.

Using an argument similar to the proof of [Bar84, Thm 10.1.23], we can show that every *recursive* closed Levy-Longo pre-tree $T$ is the Levy-Longo tree of some closed $\lambda$-term. Thus we have:

**Theorem 20 (Universality)** *(i) The denotation of a closed $\lambda$-term $s$ is a recursive, relevant, CC strategy which corresponds to $\mathsf{LT}(s)$ in the sense of Lemma 18.*

*(ii) Every recursive, relevant, CC strategy over $\mathcal{D}$ is the denotation of a closed $\lambda$-term. I.e. for every $\sigma \in \mathbb{L}_{\mathrm{rec}}(\mathbf{1}, \mathcal{D})$ there is some $s \in \Lambda^o$ such that $[\![ s ]\!] = \sigma$.* $\Box$

As a consequence, two closed $\lambda$-terms have the same denotation in $\mathcal{D}$ iff they have the same Levy-Longo tree.

From the Universality Theorem, it is a small step to show that the model is fully abstract for the Lazy Lambda Calculus. *Programs* of the (pure) Lazy Lambda Calculus [Plo75,AO93] are closed λ-terms, and *values* are closed abstractions (ranged over by $v, v'$). The evaluation relation $\Downarrow$ is a binary relation over closed λ-terms, defined by induction over the rules:

$$\frac{}{\boldsymbol{\lambda} x.p \Downarrow \boldsymbol{\lambda} x.p} \qquad \frac{s \Downarrow \boldsymbol{\lambda} x.p \quad p[t/x] \Downarrow v}{st \Downarrow v}$$

We write $s\Downarrow$ for the predicate $\exists v \,.\, s \Downarrow v$. We define *observational preorder* $\precapprox$ as follows: for λ-terms $s$ and $t$, $s \precapprox t$ if and only if for any context $C[X]$ such that both $C[s]$ and $C[t]$ are programs, if $C[s]\Downarrow$ then $C[t]\Downarrow$. We write $s \approx t$ for $s \precapprox t$ and $t \precapprox s$.

**Remark 21** Equivalently we can define $s \precapprox t$ by either of the following:

(1) *Closure by abstraction*: For every closed context $C[X]$, if $C[\boldsymbol{\lambda} y_1 \cdots y_m.s]\Downarrow$ then $C[\boldsymbol{\lambda} y_1 \cdots y_m.t]\Downarrow$, where $\{ y_1, \cdots, y_m \}$ is set of variables that occur free in either $s$ or $t$.
(2) *Closure by substitution*: For every closing substitution $\theta$ of $s$ and $t$ (i.e. $s_\theta$ and $t_\theta$ are closed), for every closed context $C[X]$, if $C[s_\theta]\Downarrow$ then $C[t_\theta]\Downarrow$.

Note that $\precapprox$ is a rich theory; e.g. we have

$$\boldsymbol{\lambda} x.x\,(x\bot_1\bot)\,\bot_1 \;\approx\; \boldsymbol{\lambda} x.x\,(\boldsymbol{\lambda} y.x\bot_1\bot y)\,\bot_1$$

where $\bot$ is any unsolvable term of order 0 such as $(\boldsymbol{\lambda} x.xx)(\boldsymbol{\lambda} x.xx)$, and $\bot_1$ is any unsolvable term of order 1 (see [AO93, p. 226] for a proof).

Set $\mathbf{2}$ to be $\mathbf{1}_\bot$. We write $\top : \mathbf{1} \longrightarrow \mathbf{2}$ for the convergent strategy (i.e. P responds to the opening question with the only answer). For any $\sigma, \tau : A \longrightarrow B$ in $\mathbb{L}_{\mathrm{rec}}$, we define $\sigma \lesssim \tau$ to mean for every $f : \mathbf{1} \longrightarrow A$ and every $g : B \longrightarrow \mathbf{2}$ in $\mathbb{L}_{\mathrm{rec}}$, if $f \,;\, \sigma \,;\, g = \top$ then $f \,;\, \tau \,;\, g = \top$. (This can be seen as the preorder generated by a notion of observables in the sense of [HO00].) For any $\mathbb{L}_{\mathrm{rec}}$-map $\rho : \mathbf{1} \longrightarrow \mathcal{D}$, we write $\rho\Downarrow$ to mean that $\rho$ is convergent.

**Lemma 22** *For $\sigma, \tau : \mathcal{D}^m \longrightarrow \mathcal{D}$ in $\mathbb{L}_{\mathrm{rec}}$, we have $\sigma \lesssim \tau$ if and only if for every $f : \mathbf{1} \longrightarrow \mathcal{D}^m$ and every $g : \mathcal{D} \longrightarrow \mathcal{D}$ in $\mathbb{L}_{\mathrm{rec}}$, if $f \,;\, \sigma \,;\, g\Downarrow$ then $f \,;\, \tau \,;\, g\Downarrow$.*

**PROOF.** For "⇒", we use the retraction map $\mathcal{D} \longrightarrow \mathbf{2}$; and for "⇐", we note that every $\mathbb{L}_{\mathrm{rec}}$-map $\mathcal{D} \longrightarrow \mathbf{2}$ extends by Conditionally Copycat to a map $\mathcal{D} \longrightarrow \mathcal{D}$. □

Take λ-terms $s$ and $t$ such that $\Gamma = \{ y_1, \cdots, y_m \}$ is the set of variables that occur free in either $s$ or $t$. Using (2) of Remark 21 and by Lemma 13, we have $s \precapprox t$ iff for every closing substitution $\theta$, and for every closed context $C[X]$, if $[\![ \; \vdash C[s_\theta] ]\!]\Downarrow$ then $[\![ \; \vdash C[t_\theta] ]\!]\Downarrow$. Now

$$[\![ \; \vdash C[s_\theta] ]\!] \; = \; \langle \overline{[\![ \; \vdash \theta(y_i) ]\!]} \rangle \, ; [\![ \Gamma \vdash s ]\!] \, ; [\![ y \vdash C[y] ]\!]$$

Hence, by the Universality Theorem, we have $s \precapprox t$ iff for every $f : \mathbf{1} \longrightarrow \mathcal{D}^m$ and for every $g : \mathcal{D} \longrightarrow \mathcal{D}$ in $\mathbb{L}_{\mathrm{rec}}$, if $f \, ; [\![ \Gamma \vdash s ]\!] \, ; g\Downarrow$ then $f \, ; [\![ \Gamma \vdash t ]\!] \, ; g\Downarrow$, which, by Lemma 22, is equivalent to $[\![ \Gamma \vdash s ]\!] \lesssim [\![ \Gamma \vdash t ]\!]$. To summarize, we have proved:

**Theorem 23 (Full Abstraction)** *For any λ-terms $s$ and $t$ such that $\Gamma$ is the set of variables that occur free in either $s$ or $t$, we have*

$$s \precapprox t \iff [\![ \Gamma \vdash s ]\!] \lesssim [\![ \Gamma \vdash t ]\!].$$

□

*Acknowledgments*

The work reported here was partially funded by the EU (Training and Mobility for Researchers Programme *Linear Logic in Computer Science*) and by Merton College, Oxford, under the Visiting Scholar Scheme. The authors are grateful to them for their support.

## References

[AJM00] S. Abramsky, R. Jagadeesan, and P. Malacaria.
Full abstraction for PCF.
*Information and Computation*, 163, 2000.

[AM95] S. Abramsky and G. McCusker.
Games and full abstraction for the Lazy Lambda Calculus.
In *Proc. 10th IEEE Symp. Logic in Computer Science*, pages 234–243. The Computer Society, 1995.

[AO93] S. Abramsky and C.-H. L. Ong.
Full abstraction in the Lazy Lambda Calculus.
*Information and Computation*, 105:159–267, 1993.

[Bar84] H. Barendregt.
*The Lambda Calculus.*
North-Holland, revised edition, 1984.

[DH01] V. Danos and R. Harmer.
The anatomy of innocence.
In *Proceedings, Tenth Annual Conference of the European Association for Computer Science Logic*, pages 188–202. Springer Verlag, 2001.
LNCS Vol. 2142.

[GFH99] P. Di Gianantonio, G. Franco, and F. Honsell.
Game semantics for the untyped $\lambda\beta\eta$-calculus.
In *Proceedings of the International Conference on Typed Lambda Calculus and Applications*, pages 114–128. Springer-Verlag, 1999.
LNCS Vol. 1591.

[Gia01] P. Di Gianantonio.
Game semantics for the Pure Lazy Lambda Calculus.
In *Proc. 5th Int. Conf. Typed Lambda Calculi and Applications, Krakow, Poland, May 2001*, pages 106–120. Springer-Verlag, 2001.

[HO00] J. M. E. Hyland and C.-H. L. Ong.
On Full Abstraction for PCF: I. Models, observables and the full abstraction problem, II. Dialogue games and innocent strategies, III. A fully abstract and universal game model.
*Information and Computation*, 163:285–408, 2000.

[KNO99] A. D. Ker, H. Nickau, and C.-H. L. Ong.
A universal innocent game model for the Böhm tree lambda theory.
In *Computer Science Logic: Proceedings of the 8th Annual Conference on the EACSL Madrid, Spain, September 1999*, pages 405 – 419.
Springer-Verlag, 1999.
LNCS Volume 1683.

[KNO02] A. D. Ker, H. Nickau, and C.-H. L. Ong.
Innocent game models of untyped $\lambda$-calculus.
*Theoretical Computer Science*, 272:247–292, 2002.

[Lev75] J.-J. Levy.
An algebraic interpretation of equality in some models of the lambda calculus.
In C. Böhm, editor, *Lambda Calculus and Computer Science Theory*, pages 147–165. Springer-Verlag, 1975.
*LNCS* No. 37.

[Lon83] G. Longo.
Set-theoretical models of lambda calculus: Theories, expansions and isomorphisms.
*Annals of Pure and Applied Logic*, 24:153–188, 1983.

[McC96] G. McCusker.
Full abstraction by translation.
In *Advances in Theory and Formal Methods of Computing*. IC Press, 1996.

[McC98] G. McCusker.
*Games for recursive types.*
BCS Distinguished Dissertation. Cambridge University Press, 1998.

[Plo75] G. D. Plotkin.
Call-by-name, call-by-value and the lambda calculus.
*Theoretical Computer Science*, 1:125–159, 1975.