A note on uninominal electoral systems with two parties

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Abstract

A crucial question for uninominal electoral systems concerns the discrepancy between the vote percentages and the seat percentages across the state. Apart from unfair district subdivision due to gerrymandering it seems that uninominal electoral systems are inherently non proportional. We show that even if the two parties have the same chances to get any vote percentage, yet the vote percentage above 50% is amplified into a larger seat percentage above 50%. We compute the amplification factor for three different random variables. We compare these results with the concept of Efficiency gap.

Keywords: electoral systems; uninominal systems; first-pass-the-post; majority vote; gerrymandering.

1 Introduction

We consider a uninominal electoral system where a state is divided into several districts and in each district a seat is assigned to the party that gets more votes in that district. This system is also referred to as first-pass-the-post and is used in UK to elect the members of the House of Commons and in US to elect the members of the House of Representatives state by state. In this paper we restrict the attention to the case of only two parties competing for the seats.

It is well known that there can be a large difference between the percentages of votes and seats gained by one party all over the state. It may also very well happen that a minority of votes turns into a majority of seats. Very large discrepancies between the two percentages are usually attributed to an unfair district subdivision due to gerrymandering practice. This is certainly true. However, it is inherent in the very mechanism of uninominal systems that, even with a fair district subdivision and under partisan symmetry ([3, 5]), the percentage of votes above 50% is amplified into a larger percentage of seats above 50%.

In this note we show this fact by an elementary probabilistic analysis. If the votes are uniformly distributed the amplification factor is 1.5, but if the votes follow a gaussian distribution, a case closer to reality, the amplification factor can reach the value 4.

It has been objected by several authors that the final seat outcome should follow more closely the vote outcome across the state. For instance we may quote the proposal by Balinski [1, 2] that invokes leaving the uninominal system in favor of the so called ‘Fair Majority Voting’. On the opposite side one might argue that the amplification factor makes a winning party stronger thereby providing a more stable majority in the House. However, in this paper we do not deal with the political issue whether uninominal systems are fair or not. We just point out the existence of this discrepancy between votes and seats even under the best partisan symmetry.

The paper is organized as follows: in Section 2 we formally state the problem. Then in Section 3 we prove the existence of the amplifcation factor for three simple types of random variables if the two parties have the same strength. We confirm the analytical results by using simulation in Section 4. In Section 5 we compare the concept of Efficiency Gap [7] to the results of the paper. A hint for an extension to three parties is outlined in Section 6. Finally we draw some conclusions in Section 7.

2 Mathematical statement

A state is divided into n districts. Each district is assigned a seat, that is won by the candidate who gets more votes. We assume that there are only two parties A and B and two candidates associated to either party. Let $X_i \in [0,1]$ be a random variable representing the fractions of votes party A receives in district i and $Y_i \in \{0,1\}$ a random variable denoting the gained or lost seat by party A in district i, defined as

$$Y_i = \begin{cases} 0 & \text{if } X_i \le 0.5 \\ 1 & \text{if } X_i > 0.5 \end{cases}, \qquad i = 1, 2, \dots, n.$$

We assume that the variables $X_1, \dots, X_n$ are equally distributed with distribution function $F(x)$ and density function $f(x)$. Let $\bar{X}$ and $\bar{Y}$ be the expected values of X_i and Y_i respectively.

The expected value of the fraction of votes received by party A in the whole state is also $\bar{X}$. The expected value of the fraction of seats (with respect to the n seats at stake) received by party A is given also by $\bar{Y}$.

We define the random variables $X = \sum_i^n X_i/n$ and $Y = \sum_i^n Y_i/n$. Clearly their expected values are $\bar{X}$ and $\bar{Y}$ respectively. If all districts have the same total number of expressed votes, X is also the fraction of votes received by party A in the whole state (otherwise the overall fraction of votes is a weighted average). We assume that each district has the same number of total votes so that X is the fraction of votes received by party A in the whole state. The results of this paper depend on this assumption. If the votes of each district are almost the same, a situation that is considered desirable, the results hold with very good approximation. If they differ consistently, say a district is two or three times larger than another one, the amplification factor (see later) is smaller.

We say that the two parties have the same *strength* if they have the same probability of receiving a certain fraction x of votes, for any x, in each district. For this to be possible we must have a symmetric density function, i.e., $f(x) = f(1-x)$. In this paper we assume that the two parties have the same strength. Due to this assumption $\bar{X} = 0.5$ and $\bar{Y} = 0.5$.

We quote from [3]: ‘The [partisan] symmetry standard requires that the electoral system treat similarly-situated parties equally, so that each receives the same fraction of legislative seats for a particular vote percentage as the other party would receive if it had received the same percentage [of the vote]'. If we translate the term 'similarly-situated' into 'having the same probability of receiving a certain fraction of votes', we see that when two parties have the same strength partisan symmetry is satisfied.

We are interested in the relationship between $X - 0.5$ and $Y - 0.5$, i.e., how the excess percentage of votes, i.e., the fraction above 50%, in the whole state is reflected in the excess percentage of seats (the two figures can be both negative, but clearly in this case the excess percentage refers to the other party). More exactly, we address the following question: given $X = x$, what it is the most likely value of Y?

Ideally the best prediction is $E[Y|x]$, the expected value of Y conditioned to the outcome $X = x$. In general this is a cumbersome computation. For X_i uniformly distributed and $n = 2$ the computation is simple and one gets

$$E[Y|x] = \begin{cases} 0 & 0 \le x \le \frac{1}{4}, \\ 1 - \frac{1}{4\,x} & \frac{1}{4} \le x \le \frac{1}{2}, \\ \frac{1}{4\,(1-x)} & \frac{1}{2} \le x \le \frac{3}{4}, \\ 1 & \frac{3}{4} \le x \le 1. \end{cases}$$

This is not a linear function, although it can be fit, not too badly, by a linear function in the range $[0.25, 0.75]$. For $n = 3$ and X_i uniformly distributed an analytical expression for $E[Y|x]$ is very complicated. We can carry out a numerical computation and see that $E[Y|x]$ is almost linear in x in the range $[1/6, 5/6]$. A numerical computation for $n = 4$ and X_i uniformly distributed displays an almost perfect linearity in the same range.

A theorem in probability ([6] pag. 243, Theorem 9.2, or [4] Section 4.3, pag. 143) states that if $E[Y|x]$ is linear in x then

$$E[Y|x] = \bar{Y} + \frac{\sigma_{XY}}{\sigma_X^2}\,(x - \bar{X}) \tag{1}$$

where σ_X^2 is the variance of X, σ_{XY} is the covariance of the pair (X, Y). Due to the simplicity of the expression (1), we approximate the best prediction by assuming a linear model, at least in a restricted range around the mean $\bar{X}$, and define the best prediction as

$$\hat{Y}(x) = \bar{Y} + \frac{\sigma_{XY}}{\sigma_X^2}\,(x - \bar{X}) = \bar{Y} + \alpha\,(x - \bar{X}) \tag{2}$$

where α is here defined as the ratio between σ_{XY} and σ_X^2. Clearly the random variable $\hat{Y}(X)$ has the same expected value as Y and it can be easily proved that $\hat{Y}(X) - Y$ has minimum variance among all linear estimates. Since

$$\alpha = \frac{\hat{Y}(x) - 0.5}{x - 0.5}$$

we refer to α as an *amplification factor*, because the excess percentage of votes is 'amplified' into a larger excess percentage of seats.

With only one district there is an obvious amplification in passing from $X - 0.5$ to $Y - 0.5$ (and the 'prediction' is actually deterministic). However,

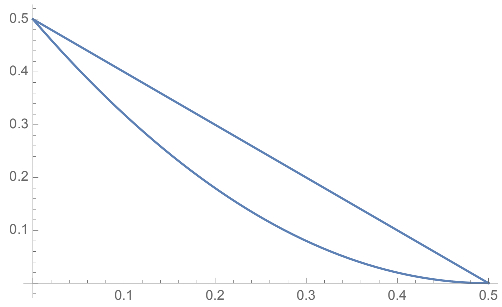


Figure 1: The ratio α

with many districts and flat random variables X_i, we may wonder whether gains and losses in the various districts can be compensated so that the overall excess percentage of seats is almost the same as the overall excess percentage of votes, i.e., the amplification factor is almost one. If the two parties have the same strength, i.e., partisan symmetry is satisfied, the answer is negative, as we shall see in the next section. The excess percentage of seats is always larger than the excess percentage of votes by a factor of 1.5 or more.

3 Majority votes vs majority seats

In order to compute α we have to compute the covariance matrix of (X, Y). Since X and Y are averages of independent random variables equally distributed, the covariance matrix of (X, Y) is the covariance matrix of (X_i, Y_i) divided by n. So we have

$$\sigma_X^2 = \frac{1}{n}\int_0^1 f(x)\,(x-\frac{1}{2})^2\,dx = \frac{2}{n}\int_0^{0.5} f(x)\,(x-\frac{1}{2})^2\,dx, \qquad \sigma_Y^2 = \frac{1}{4\,n},$$

$$\sigma_{XY} = \frac{1}{n}\int_0^{0.5} f(x)\,(x-\frac{1}{2})\,(0-\frac{1}{2})\,dx + \frac{1}{n}\int_{0.5}^1 f(x)\,(x-\frac{1}{2})\,(1-\frac{1}{2})\,dx = -\frac{1}{n}\int_0^{0.5} f(x)\,(x-\frac{1}{2})\,dx.$$

Hence

$$\alpha = \frac{-\int_0^{0.5} f(x)\,(x-0.5)\,dx}{2\int_0^{0.5} f(x)\,(x-0.5)^2\,dx}.$$

In Fig. 1 we see the plots of the functions $-(x-0.5)$ and $2\,(x-0.5)^2$. For a constant function $f(x)$, α is the ratio between the area under the segment and the area under the parabola. For non constant $f(x)$ the two areas are weighted by $f(x)$. It is clear that α is always larger than 1, unless $f(x)$ is concentrated on 0, 0.5 and 1. The value of α is larger for functions $f(x)$ that are larger near 0.5. It is interesting to compute α in a few particular cases:

1) X_i is a flat unbiased random variable, i.e., $f(x) = 1$,

2) $f(x)$ has a triangular shape. It is linearly raising between 0 and 0.5 and decreasing with the same slope between 0.5 and 1, i.e.,

$$f(x) = \begin{cases} 1 - a + 4\,a\,x & 0 \le x \le 0.5 \\ 1 + 3\,a - 4\,a\,x & 0.5 \le x \le 1 \end{cases} \tag{3}$$

where $0 \le a \le 1$ (note that case 1 falls into this more general case if $a = 0$),

3) X_i is a normal random variable without tails, i.e.,

$$f(x) = K \frac{1}{\sqrt{2\,\pi\,\sigma^2}}\, e^{-(x-0.5)^2/(2\,\sigma^2)}, \qquad 0 \le x \le 1, \tag{4}$$

where K is chosen so that the integral of $f(x)$ over the interval $[0,1]$ is indeed 1. For small values of σ, K is slightly larger than 1.

Case 1: It is immediate to compute $\sigma_X^2 = 1/(12\,n)$ and $\sigma_{XY} = 1/(8\,n)$ so that

$$\alpha = \frac{3}{2}.$$

Case 2: We get

$$\sigma_X^2 = \frac{2-a}{24\,n}, \qquad \sigma_{XY} = \frac{3-a}{24\,n},$$

so that

$$\alpha = \frac{3-a}{2-a}.$$

We see that for $0 \le a \le 1$ we have $3/2 \le \alpha \le 2$. Note that the sharper the function $f(x)$ is the larger the amplification factor is.

Case 3: Note that $n\,\sigma_X^2 < \sigma^2$ because the tails of the distribution outside the interval $[0,1]$ are cut. For values $\sigma \le 0.1$, $n\,\sigma_X^2$ differs from σ^2 less than 10^{-5}. The computation of σ_{XY} gives

$$\sigma_{XY} = K\,\sigma\,\frac{1-e^{-1/(8\sigma^2)}}{n\,\sqrt{2\,\pi}},$$

so that we have

$$\alpha = \frac{\sigma_{XY}}{\sigma_X^2} > \frac{n\,\sigma_{XY}}{\sigma^2} = K\,\frac{1-e^{-1/(8\sigma^2)}}{\sigma\,\sqrt{2\,\pi}}.$$

For small values of σ, we may approximate

$$\alpha \approx \frac{1}{\sigma\,\sqrt{2\,\pi}}.$$

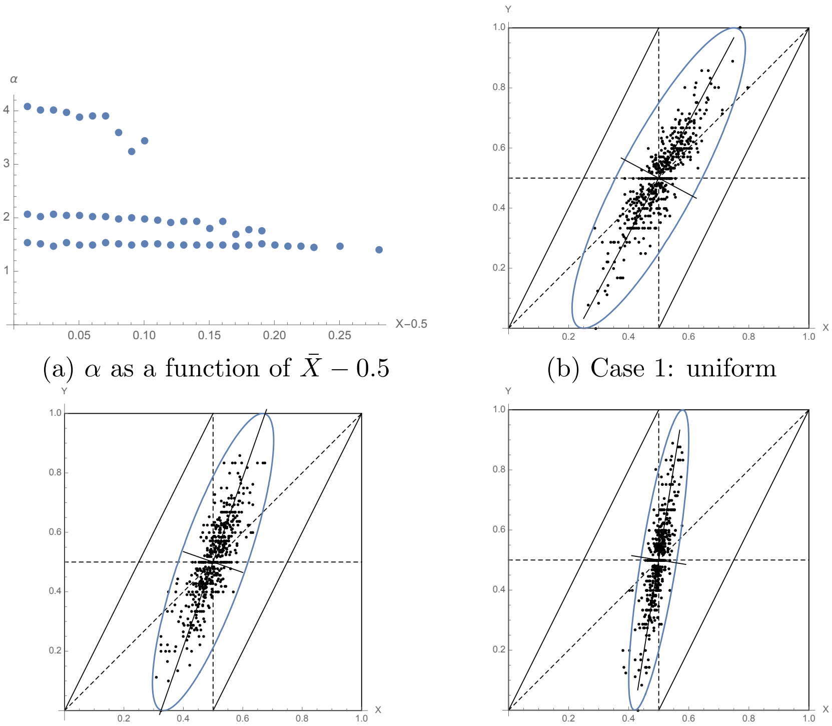
As apparent from the expression the amplification factor can be large. For $\sigma = 0.1$ (a value that can closely fit a real situation) α is almost equal to 4.

This is an interesting conclusion if we know that the two parties not only have the same strength but they are also very close to 50% of votes in all districts. This circumstance blows up the amplification factor to large values.

4 Simulation

The previous analytical findings can be substantiated by simulation. In particular we consider the same three cases examined in Section 3: 1) $f(x) = 1$; 2) $f(x)$ as in (3) with $a = 1$; 3) $f(x)$ as in (4) with $\sigma = 0.1$ (actually we have generated an approximate gaussian variable as the sum of twelve uniform random variables).

We show the results of the simulation in two different ways. First we have fixed the number of districts to 20 and we have performed 100,000 runs for each



(a) α as a function of $\bar{X} - 0.5$

(b) Case 1: uniform

(c) Case 2: triangular shape

(d) Case 3: gaussian

Figure 2: Simulation results

one of the three cases. For each run the X_i, $i = 1, \ldots, 20$, random variables were generated, the Y_i were derived from the X_i variables and the pair (X, Y) was computed. Then, among the 100,000 generated pairs (X, Y), we have selected those pairs such that $0.01\, k \leq X - 0.5 \leq 0.01\,(k + 1)$, for $k = 0, \ldots, 25$. This means that we have selected those cases such that $X - 0.5$ is almost equal to $k\%$. For each value of k we have computed the averages $(\bar{X}, \bar{Y})$ of the selected pairs (X, Y) and then we have computed the ratio $\alpha = (\bar{Y} - 0.5)/(\bar{X} - 0.5)$. In Fig. 2 (a) we show the values of α for each k (for which there exist selected pairs) and for the three cases. We see a remarkable fitting of the simulated data with the analytical values 1.5, 2 and 4, except for the gaussian variables with $X - 0.5 > 0.07$ for which there are too few values to have a significative simulation and also the values can be outside the linear range.

The second type of simulation shows the distribution of all generated pairs (X, Y). In this case we have generated only 500 runs in order to be able to see the points in the figure. For each run the number of districts was uniformly randomly chosen between 5 and 30 and then the X_i variable was generated for each district. The Y_i variables were derived from the X_i variables. For each run the point (X, Y) is shown in Fig. 2 (b), (c) and (d), according to the case. Besides we have shown the ellipsis $(x, y)^T V^{-1}\,(x, y) = C$, (with V the covariance matrix of (X, Y)) and the eigenvectors of V. Also this simulation confirms the analytical findings.

5 Efficiency gap

Recently, in order to measure how fair the district subdivision is, the concept of Efficiency gap has been introduced [7]. Let us briefly summarize the basic facts about the Efficiency gap.

The votes above 50% in a particular district are considered 'wasted' for the winner party because they are not necessary to win the seat. Also the votes of the looser party are wasted because they don't contribute to win the seat. The concept of Efficiency gap takes into account the wasted votes in all districts by both parties. Let W_A and W_B be these values. The Efficiency gap is defined as $|W_A - W_B|$ divided by the number of all votes in all districts. A subdivision is declared fair if the Efficiency gap is zero, a circumstance that should correspond to equal treatment of the two parties [7].

If we assume that the same number of votes is casted in each district it is not difficult to show that zero Efficiency gap corresponds to

$$Y - \frac{1}{2} = 2\,(X - \frac{1}{2}).$$

This means that, according to the concept of Efficiency gap, a fair district subdivision implies an amplifying factor of two of the excess percentage of seats with respect to the excess percentage of votes.

6 Extension to three parties

We may extend the previous analysis to the case of three parties competing for the seats. The analysis is slightly more complex and we just provide details for the case of uniform random variables. The fraction of votes received by the three parties in districts i are three random variables X_i^1, X_i^2, X_i^3, taking values on the simplex $\left\{(x^1, x^2, x^3) \geq 0 : \sum_i x^i = 1\right\}$ with a joint density function $\phi(x^1, x^2, x^3)$. We say that the parties have the same strength if ϕ is symmetric. Let $X_i = X_i^1$ and

$$Y_i = \begin{cases} 1 & \text{if } X_i^1 > X_i^2 \text{ and } X_i^1 > X_i^3 \\ 0 & \text{otherwise.} \end{cases}$$

Let $X = \sum_i X_i/n$ and $Y = \sum_i Y_i/n$. Let $F(x) = \Pr\{X \leq x\}$. If ϕ is uniform, then $F(x) = 2\,x - x^2$, and the density function is $f(x) = 2 - 2\,x$. Clearly $\bar{X} = \bar{Y} = 1/3$. Then

$$\sigma_X^2 = \frac{1}{n} \int_0^1 2\,(1-x)\,(x - \frac{1}{3})^2\,dx = \frac{1}{18\,n}.$$

To compute σ_{XY} note that $X_i > 0.5$ implies $Y_i = 1$ and $X_i < 1/3$ implies $Y_i = 0$. If $1/3 \leq X_i \leq 0.5$

$$\Pr\{Y_i = 1 \,|\, X_i = x\} = \frac{3\,x - 1}{1 - x},$$

$$\Pr\{X_i = x, Y_i = 1\} = \Pr\{Y_i = 1 \,|\, X_i = x\}\ \Pr\{X_i = x\} = 2\,(3\,x - 1),$$

$$\Pr\{Y_i = 0 \,|\, X_i = x\} = 1 - \frac{3\,x - 1}{1 - x} = 2\,\frac{1 - 2\,x}{1 - x},$$

$$\Pr\{X_i = x, Y_i = 0\} = \Pr\{Y_i = 0 \,|\, X_i = x\} \,\Pr\{X_i = x\} = 4\,(1 - 2\,x).$$

From these expressions we may easily compute $\sigma_{XY} = 5/(54\,n)$ and

$$\frac{\sigma_{XY}}{\sigma_X^2} = \frac{5}{3},$$

that shows an amplifying factor larger than the value 3/2 obtained for two parties. Note that for three parties the amplifying factor is respect to the ratio $(\hat{Y}(x) - 1/3)/(x - 1/3)$. We also get $\sigma_Y^2 = 2/(9\,n)$.

7 Conclusions

In this paper we have carried out a probabilistic analysis of a uninominal electoral system with two parties to understand how the vote percentage is turned into a seat percentage when the two parties have the same probability of obtaining any vote percentage. We have shown that the percentage of votes above 50% turns into a larger percentage of seats above 50%. In particular for a uniform vote distribution the amplification is 3/2, whereas for sharper density functions the amplification factor is larger. Therefore a uninominal system seems to be inherently non proportional, even if the district subdivision is fair and gerrymandering practice is banned.

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