A note on the m-step Fibonacci numbers

Paolo Serafini, University of Udine

1 Introduction

By solving a particular counting problem related to the number of occurrences of strings in words, we have derived an analytic expression for this number. The generating function of this sequence is very close to the generating function of the m-step Fibonacci numbers. Putting together the two results it turns out that it is possible to express all m-step Fibonacci numbers via a family of sequences. In particular the n-th m-step Fibonacci number can be computed by a sum of $(n-1)/(m+1)$ terms with alternating signs from the family of sequences. The only difference for different m is that these sequences enter the sum shifted to the right by $(m+1)$ places. We have also investigated the possibility of inverting the formulas, i.e., of expressing the sequences of this family via sums of m-step Fibonacci numbers.

2 Exact occurrences of strings in words

Consider the following problem: let w be a word of m letters over an alphabet of k letters. Suppose that no final substring of w is also an initial string of w. Use the sieve method to count the words of n letters, over that alphabet of k letters, that do not contain the substring w.

This problem is listed as Exercise 10 page 159 in [2]. By exploiting the technique explained in Section 4.2 of [2] it is not difficult to find out the following expression

$$g_{k,m,t}(n) = \sum_{r \le \lfloor n/m \rfloor} (-1)^{r-t}\, k^{n-r\,m} \binom{r+n-m\,r}{r} \binom{r}{t}$$

for the number of words of length n that contain exact t occurrences of w. By considering k, m and t fixed, the generating function $G_{k,m,t}(x)$ of the sequence $g_{k,m,t}(n)$ can be easily computed as

$$G_{k,m,t}(x) = \sum_n g_{k,m,t}(n)\, x^n = \frac{x^{m\,t}}{(1-k\,x+x^m)^{t+1}}\ .$$

We are interested in the case $k=2$ (a binary alphabet) and $t=0$ (no occurrences) for which (by omitting from now on the reference to k and t in the indices)

$$g_m(n) = \sum_{r \le \lfloor n/m \rfloor} (-1)^r\, 2^{n-r\,m} \binom{r+n-m\,r}{r}, \qquad G_m(x) = \frac{1}{1-2\,x+x^m}\ .$$

Since $x=1$ is a zero of the denominator of $G_m(x)$ we may factor the denominator as

$$G_m(x) = \frac{1}{(1-x)\,(1-x-x^2-\ldots-x^{m-1})}\ .$$

Let

$$\hat{F}_m(x) = \frac{1}{1 - x - x^2 - \ldots - x^{m-1}} = \sum_n \hat{f}_m(n)\, x^n \;.$$

Then by standard techniques of generating functions

$$\hat{f}_m(0) = g_m(0) = 1, \qquad \hat{f}_m(n) = g_m(n) - g_m(n-1), \quad n > 0 \;.$$

Therefore

$$\hat{f}_m(n) = \sum_{r \le \lfloor n/m \rfloor} (-1)^r\, 2^{n-r\,m} \binom{r + n - m\,r}{r} - \sum_{r \le \lfloor (n-1)/m \rfloor} (-1)^r\, 2^{n-1-r\,m} \binom{r + n - 1 - m\,r}{r} \;.$$

The upper limits in the two sums are different when $r = n/m$. If we substitute this value in the binomial coefficient in the second sum we obtain

$$\binom{r + n - 1 - n}{r} = \binom{r-1}{r}$$

which is zero for $r > 0$. The case $r = 0$ and $r = n/m$ occurs when $n = 0$. Hence we may write for $n > 0$

$$\hat{f}_m(n) = \sum_{r \le \lfloor n/m \rfloor} (-1)^r \left(2^{n-r\,m} \binom{r + n - m\,r}{r} - 2^{n-1-r\,m} \binom{r + n - 1 - m\,r}{r} \right) \;. \qquad (1)$$

It is useful to define

$$h_r(k) := \frac{2^{k-1}}{r} \binom{k + r - 1}{r - 1} (k + 2\,r) = 2^{k-1} \left(\binom{k+r}{r} + \binom{k + r - 1}{r - 1} \right)$$

so that for $n > 0$

$$\hat{f}_m(n) = \sum_{r \le \lfloor n/m \rfloor} (-1)^r\, h_r(n - m\,r) \;. \qquad (2)$$

Clearly $h_0(k) = 2^{k-1}$. It is however convenient to redefine $h_0(0) := 1$ so that the formula (2) is valid also for $n = 0$. Note that $h_1(k)$ is the sequence A001792 in [1], $h_2(k)$ is the sequence A001793, $h_3(k)$ is the sequence A001794, $h_4(k)$ is the sequence A006974, etc.

The family of sequences

$$\Big\{ \{h_r(k)\}_{k \ge 0} : r \ge 0 \Big\}$$

should deserve a special name because, as we shall see in the next section, they fully define the m-step Fibonacci numbers.

3 m-step Fibonacci numbers

The m-step Fibonacci numbers $f_m(n)$ are defined as

$$f_m(n) = 0, \quad n \le 0, \qquad f_m(1) = 1, \qquad f_m(n) = \sum_{k=1}^{m} f_m(n-k), \quad n > 1 \;. \qquad (3)$$

For $m = 2$ they are just the usual Fibonacci numbers. For $m > 2$ they are also called with the special names Tribonacci ($m = 3$, sequence A058265 in [1]), Tetranacci ($m = 4$, A000078), Pentanacci ($m = 5$, A001591), etc. Their generating function is

$$F_m(x) = \frac{x}{1 - x - x^2 - \ldots - x^m} = x\,\hat{F}_{m+1}(x)\ .$$

Hence $f_m(n) = \hat{f}_{m+1}(n-1)$ and we have a direct formula for the m-step Fibonacci numbers provided by (1). The interesting fact is that all numbers can be computed by the same list of sequences, namely $h_r(k)$, as apparent from (2). Indeed

$$f_m(n) = \sum_{r \le \lfloor (n-1)/(m+1) \rfloor} (-1)^r\, h_r(n - 1 - (m+1)\, r), \qquad n > 0\ . \tag{4}$$

The only difference for different m is that the list $h_r(k)$ enters the computation of $f_m(n)$ displaced by $(m+1)$ entries with respect to $h_{r-1}(k)$. The situation is displayed for $m = 2$, $m = 3$, $m = 4$ and $m = 5$ in Table 1. The sequences are shown up to $n = 18$. For instance the Fibonacci numbers can be computed by taking the sequence $h_0(k)$, subtracting the sequence $h_1(k)$ shifted to the right three places, adding the sequence $h_2(k)$ shifted to the right six places, subtracting the sequence $h_3(k)$ shifted to the right nine places and so on, until there are only zeros. In general, for the m-step Fibonacci number, starting always with the sequence $h_0(k)$, one has to subtract the sequence $h_1(k)$ shifted to the right $(m+1)$ places, then to add the sequence $h_2(k)$ shifted to the right $2\,(m+1)$ places, and so on.

Note that by shifting the $h_r(k)$ sequences by two places we get the sequence of all ones, and by shifting the sequences by one entry we get the sequence of all zeros (except the first term), as from the definition (3).

The formula (4) may be compared with the known formula

$$f_m(n) = \sum_{k_1 \le k_2 \le \ldots \le k_{m-1}} \binom{n - k_1}{k_1 - k_2} \binom{k_1 - k_2}{k_2 - k_3} \cdots \binom{k_{m-3} - k_{m-2}}{k_{m-2} - k_{m-1}} \binom{k_{m-2} - k_{m-1}}{k_{m-1}}$$

where a number of terms exponentially growing with m has to be computed.

4 Inverse formulas

The formula (4) shows that the m-step Fibonacci numbers can be expressed as particular sums of the family $h_r(k)$. We may wonder whether it is possible to express each sequence $h_r(k)$ through particular sums of the m-step Fibonacci numbers. This is indeed possible, although the expressions we have found involve an exponentially increasing number of terms.

For $n - 1 < m + 1$ we have

$$f_m(n) = h_0(n-1)$$

which implies, by choosing $n = m + 1$,

$$h_0(m) = f_m(m+1)\ .$$

For $m + 1 \le n - 1 < 2\,(m+1)$ we have

$$f_m(n) = h_0(n-1) - h_1(n - 1 - m - 1) = f_{n-1}(n) - h_1(n - 1 - m - 1)$$

so that

$$h_1(n-1-m-1) = f_{n-1}(n) - f_m(n)$$

which implies, by choosing $n = 2\,(m+1)$,

$$h_1(m) = -f_m(2\,(m+1)) + f_{2\,m+1}(2\,(m+1))\ .$$

For $2\,(m+1) \le n-1 < 3\,(m+1)$ we have

$$f_m(n) = h_0(n-1) - h_1(n-1-m-1) + h_2(n-1-2\,m-2) =$$

$$f_{n-1}(n) - f_{2\,(n-m-1)-1}(2\,(n-m-1)) + f_{(n-m-2)}(2\,(n-m-1)) + h_2(n-1-2\,m-2)$$

so that

$$h_2(n-1-2\,m-2) = f_m(n) - f_{n-1}(n) + f_{2\,(n-m-1)-1}(2\,(n-m-1)) - f_{(n-m-2)}(2\,(n-m-1))$$

which implies, by choosing $n = 3\,(m+1)$,

$$h_2(m) = f_m(3\,(m+1)) - f_{3\,m+2}(3\,(m+1)) + f_{4\,m+3}(4\,(m+1)) - f_{2\,m+1}(4\,(m+1))\ .$$

For $3\,(m+1) \le n-1 < 4\,(m+1)$ we have

$$f_m(n) = h_0(n-1) - h_1(n-1-(m+1)) + h_2(n-1-2\,(m+1)) - h_3(n-1-3\,(m+1))$$

which, by choosing $n = 4\,(m+1)$, yields

$$\begin{aligned} h_3(m) = & -f_m(4\,(m+1)) + f_{4\,m+3}(4\,(m+1)) \\ & + f_{2\,m+1}(6\,(m+1)) + f_{3\,m+2}(6\,(m+1)) - 2\,f_{6\,m+5}(6\,(m+1)) \\ & - f_{4\,m+3}(8\,(m+1)) + f_{8\,m+7}(8\,(m+1))\ . \end{aligned}$$

By using the same tools we get

$$\begin{aligned} h_4(m) = & +f_m(5\,(m+1)) - f_{5m+4}(5\,(m+1)) \\ & - f_{2m+1}(8\,(m+1)) - f_{4m+3}(8\,(m+1)) + 2\,f_{8m+7}(8\,(m+1)) \\ & - f_{3m+2}(9\,(m+1)) + f_{9m+8}(9\,(m+1)) \\ & + f_{4m+3}(12\,(m+1)) + 2\,f_{6m+5}(12\,(m+1)) - 3\,f_{12m+11}(12\,(m+1)) \\ & - f_{8m+7}(16\,(m+1)) + f_{16m+15}(16\,(m+1))\ . \end{aligned}$$

All terms are of the form $f_{p\,(m+1)-1}(q\,(m+1))$ as can be seen inductively. Assume that $h_r(m)$, $r < k$, can be expressed as

$$h_r(m) = \sum_{p,q} \alpha^r_{p,q}\, f_{p\,(m+1)-1}(q\,(m+1))\ . \tag{5}$$

For $k\,(m+1) \le n-1 < (k+1)\,(m+1)$ we have

$$f_m(n) = \sum_{r \le k} (-1)^r h_r(n-1-r\,(m+1))$$

and by choosing $n = (k+1)\,(m+1)$ we have

$$f_m((k+1)\,(m+1)) = (-1)^k\, h_k(m) + \sum_{r<k} (-1)^r h_r((k+1-r)\,(m+1)-1)$$

i.e.,

$$h_k(m) = (-1)^k\, f_m((k+1)\,(m+1)) + \sum_{r<k} (-1)^{k+1-r} h_r((k+1-r)\,(m+1)-1)$$

which becomes, after substituting (5)

$$h_k(m) = (-1)^k\, f_m((k+1)\,(m+1)) + \sum_{r<k} (-1)^{k+1-r} \sum_{p,q} \alpha^r_{p,q}\, f_{p\,(k+1-r)\,(m+1)-1}(q\,(k+1-r)\,(m+1))\ .$$

Hence also $h_k(m)$ can be expressed as a linear combination of terms of the form $f_{p\,(m+1)-1}(q\,(m+1))$. Moreover, this expression shows how the coefficients α^k_{pq} can be computed from the coefficients α^r_{pq}, $r < k$. Indeed we must have

$$\alpha^k_{p',q'} = \sum_{p,q,r} [p' = p\,(k+1-r) \wedge q' = q\,(k+1-r)]\,(-1)^{k+1-r}\,\alpha^r_{p,q} \tag{6}$$

or equivalently

$$\alpha^k_{p',q'} = \sum_{p,q,r>1} [p' = p\,r \wedge q' = q\,r]\,(-1)^r\,\alpha^{k+1-r}_{p,q}\ . \tag{7}$$

Furthermore there is the coefficient

$$\alpha^k_{1,k+1} = (-1)^k\ .$$

In the following tables we see the values of α^k, $k = 0, \ldots, 4$:

$\alpha^0 =$

$p \setminus q$	1
1	1

$\alpha^1 =$

$p \setminus q$	2
1	-1
2	1

$\alpha^2 =$

$p \setminus q$	3	4
1	1	
2		-1
3	-1	
4		1

$\alpha^3 =$

$p \setminus q$	4	6	8
1	-1		
2		1	
3		1	
4	1		-1
6		-2	
8			1

$\alpha^4 =$

$p \setminus q$	5	8	9	12	16
1	1				
2		-1			
3			-1		
4		-1		1	
5	-1				
6				2	
8		2			-1
9			1		
12				-3	
16					1

It can be proven by induction that $\sum_p \alpha^k_{pq} = 0$, $k > 0$, and any q. According to (7) the entries in table α^4 come from those of table α^3 with row and column indices multiplied by 2, plus the entries in table α^2, inverted in sign, with row and column indices multiplied by 3, plus the entries in table α^1 with row and column indices multiplied by 4, plus the entry in table α^0, inverted in sign with row and column indices equal to 5 and plus the entry $(-1)^4$ in position $(1, 5)$.

+	1	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192	16384	32768	65536
−				1	3	8	20	48	112	256	576	1280	2816	6144	13312	28672	61440	131072
+							1	5	18	56	160	432	1120	2816	6912	16640	39424	92160
−										1	7	32	120	400	1232	3584	9984	26880
+													1	9	50	220	840	2912
−																1	11	72
	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987	1597	2584

Fibonacci numbers

+	1	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192	16384	32768	65536
−					1	3	8	20	48	112	256	576	1280	2816	6144	13312	28672	61440
+									1	5	18	56	160	432	1120	2816	6912	16640
−													1	7	32	120	400	1232
+																	1	9
	1	1	2	4	7	13	24	44	81	149	274	504	927	1705	3136	5768	10609	19513

Tribonacci numbers

+	1	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192	16384	32768	65536
−						1	3	8	20	48	112	256	576	1280	2816	6144	13312	28672
+											1	5	18	56	160	432	1120	2816
−																1	7	32
	1	1	2	4	8	15	29	56	108	208	401	773	1490	2872	5536	10671	20569	39648

Tetranacci numbers

+	1	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192	16384	32768	65536
−							1	3	8	20	48	112	256	576	1280	2816	6144	13312
+													1	5	18	56	160	432
	1	1	2	4	8	16	31	61	120	236	464	912	1793	3525	6930	13624	26784	52656

Pentanacci numbers

Table 1: How the sequences $h_r(k)$ enter the computation of $f_m(n)$

References

[1] N.J.A. Sloane: The On-Line Encyclopedia of Integer Sequences, visited on: December 20, 2013, `http://oeis.org`.

[2] H.S. Wilf, *Generatingfunctionology*, Academic Press, 1994.