The border of the giant component in very large random graphs

Paolo Serafini

University of Udine, Department of Mathematics and Computer Science
paolo.serafini@uniud.it

Abstract. In this paper we deal with random graphs specified by the degree probabilities. We investigate a special subset of the giant component, that consists of vertices which can be detached from the giant component by the removal of one edge. We call this subset the border of the giant component. The study of the border allows a better understanding of the dynamic properties of the graph and its structure. The investigation provides explicit expressions for the degree probabilities and the size probabilities in various components of the graph. A simulation on graphs with 20,000 vertices shows consistency with the theoretical values.

Keywords: random graphs, giant component, small component probabilities, Poisson graphs.

1 Introduction

Random graphs are useful models to describe social networks. See for instance [11] for a comprehensive survey. Usually the particular graphs under investigation are not totally random but some graph structure is predefined. A deep research has been devoted in the literature to random graphs with assigned degree probabilities. In this paper we deal with this type of graphs.

For these graphs a nice theory [10, 12, 3] has been built starting from the pioneering papers [5, 6] and [2]. This theory allows to characterize the so called small and giant components of the graph. This theory is based on the generating functions of the degree probabilities and cleverly exploits the recursive properties of generating functions together with the fact that small components are trees with high probability. By this theory we are able to statistically describe the small component sizes and predict under which conditions the so called giant component is present.

If we think that the graph changes dynamically by continuously adding and removing edges, it is interesting to understand how some small components become part of the giant component and, conversely, how some parts of the giant component become small components. Clearly these parts have to be loosely connected to the rest of the giant component.

This investigation has led to the definition of some special parts of the giant component which we have called the border and the linking set. A better understanding of the small components requires understanding also the border and the linking set. Loosely speaking the border is made up of those parts of the giant component which can be detached from the giant component by the simple removal of an edge. The border consists of trees (like the small components), which are connected to the main body of the giant component through the so called linking vertices.

In our opinion highlighting the border part of the giant component can shed new light on the dynamic behavior of the random graph, also with respect to the critical behavior near the transition point. In order to investigate the properties of the border, it is necessary to investigate also other parts of the graph, like the small components. Although facts related to small components are already well known, we have found necessary to report them for the sake of clarity.

The analytical results have been also tested with many simulations on large random graphs of various type showing a remarkable accordance of the empirical values with the theoretical ones. We have reported only two simulations in this paper, namely for a power-law graph and a Poisson graph.

The paper is organized as follows. In Section 2 we provide the necessary mathematical background. Then in the Sections 3, 4 and 5 we investigate the degree structure in the small components, in the border and in the linking set respectively. In Section 6 we study the small component sizes and in Section 7 the border component sizes. In Section 8 we give a different derivation of a fundamental fixed point equation which explicitly takes care of the different degree probabilities in the various parts of the graph. Given the size probabilities and the degree probabilities we verify in Section 9 that a particular random change in the graph does not change its statistical properties. Then in Section 10 the results are particularized to Poisson graphs. Finally in Section 12 we show the results of the simulations. Some conclusions are drawn in Section 13.

2 Background

Let $G = (V, E)$ be a random graph with assigned degree probabilities p_k, $k = 0, 1, 2, \ldots$, i.e., p_k is the probability that a randomly selected vertex has degree k. Let $d = \sum_{k \geq 0} k\, p_k$ be the average degree in the graph. We are interested in the asymptotic properties of the graph, i.e., when $|V|$ goes to infinity. Let us resume some basic and known facts. The graph consists of connected components. A *giant component* is a connected component whose size asymptotically goes to infinity and a *small component* is a connected component whose size asymptotically remains finite. The following quite standard assumptions are met asymptotically [11]. We do not attempt to justify them theoretically here. Simple experimental simulations show that they are typically met with very good approximation already for graphs with a few hundreds of vertices.

Assumption 1. *There is at most one giant component.* ∎

Let us say that a pair of vertices i and j are *strongly linked* if any cut separating i from j contains at least two edges. This relation is symmetric, transitive, as can be easily seen, and reflexive, if we consider a vertex to be strongly linked with itself. Hence it is an equivalence relation. Asymptotically we have:

Assumption 2. *There is at most one equivalence class that is not a singleton. If it exists, it is in the giant component.* ∎

As a consequence of this assumption all small components are trees. If the giant component is present we define as the *core* of the graph this equivalence class. The vertices in the giant component not in the core are defined as the *border* of the graph. By Assumption 2 the border consists of trees, called *border components*,

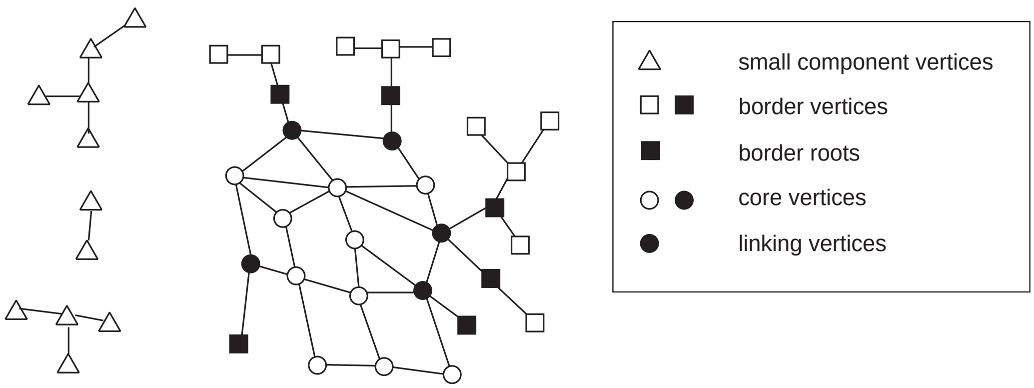


Fig. 1. Small components, border components and core of a graph

linked to the core by an edge that connects a vertex in the border to a vertex in the core. Let us call such edge a *linking edge*. The vertex of a linking edge in the border is the *root* of the border component which is therefore a rooted tree. Given an edge in the border let us call its *outbound* vertex the one more distant from the root. Let us call the vertex of a linking edge in the core a *linking vertex*. Note that a linking vertex can be linked to more than one border component. The set of linking vertices is called the *linking set*.

See in Figure 1 an example with three small components, six border components and the core. The vertices of the small components are drawn as triangles. The vertices of the border are drawn as squares with the border roots in black. The vertices in the core are drawn as circles with the linking vertices in black.

We define the following events:

- S: a randomly selected vertex belongs to a small component;
- B: a randomly selected vertex belongs to the border;
- L: a randomly selected vertex belongs to the linking set.

We also need to define the following event:

- R: given a vertex i, randomly selected among the vertices with degree at least one, and a vertex j, randomly selected among the vertices adjacent to vertex i, the vertex j belongs to a small component after removing the edge $\{i, j\}$.

Note that the event R can happen not only when the vertex i belongs to a small component but also when it belongs to the border or to the linking set.

The *excess degree* of a vertex of degree at least one is defined as its degree minus one. Let us also define the events:

- D_k: a randomly selected vertex has degree k;
- D'_k: a vertex randomly selected between the vertices of a randomly selected edge has degree k;
- D''_k: given a vertex i, randomly selected among the vertices with degree at least one, and a vertex j, randomly selected among the vertices adjacent to vertex i, the vertex j has degree k;
- $\tilde{D}''_k$: given a vertex i, randomly selected among the vertices with degree at least one, and a vertex j, randomly selected among the vertices adjacent to vertex i, the vertex j has excess degree k.

By definition we have $p_k = \Pr\{D_k\}$. We denote $q_k = \Pr\{\tilde{D}''_k\}$. Obviously $\Pr\{D''_k\} = \Pr\{\tilde{D}''_{k-1}\}$. It is easy to see that

$$\Pr\{D'_k\} = \frac{k\,p_k}{d}.$$

In general $\Pr\{D'_k\} \neq \Pr\{D''_k\}$, but it turns out that the two quantities are asymptotically equal. So we have

Assumption 3.

$$\Pr\{D''_k\} = \frac{k\,p_k}{d}$$

∎

Hence by this assumption

$$q_k = \frac{(k+1)\,p_{k+1}}{d}.$$

The generating functions of the probabilities p_k and q_k are respectively:

$$G_0(x) = \sum_{k \geq 0} p_k\,x^k, \qquad G_1(x) = \sum_{k \geq 0} q_k\,x^k = \sum_k \frac{(k+1)\,p_{k+1}}{d}\,x^k = \frac{G'_0(x)}{d}.$$

We first want to compute the following probabilities

$$u := \Pr\{R\}, \qquad \hat{v} := \Pr\{S\}, \qquad \bar{v} := \Pr\{B\}, \qquad \dot{v} = \Pr\{L\},$$

$$\hat{p}_k := \Pr\{D_k \,|\, S\}, \qquad \bar{p}_k := \Pr\{D_k \,|\, B\}, \qquad \dot{p}_k := \Pr\{D_k \,|\, L\}.$$

The value u can be computed by using the generating function $G_1(x)$. Given a vertex i the event R happens if and only if it happens for all other vertices incident to i. Hence, if vertex i has excess degree k, the probability of creating a small component is u^k. Averaging over all possible excess degrees yields

$$u = \sum_{k \geq 0} u^k\,q_k = G_1(u). \tag{1}$$

The fixed point equation (1) is the fundamental starting point of all subsequent computations [11]. However, some doubts can arise on the validity of (1) because the event R is not uniformly spread over all graph but can happen only on some particular parts of the graph, namely the small components, the border and the linking set. Hence one should condition the degree probability in turn to the events S, B and L. It turns out that this is unnecessary. We shall find again the same equation (1) in Section 8 by carrying out a more detailed investigation, which explicitly takes care of the various component probabilities.

Obviously $u = 1$ is always a root of $u = G_1(u)$. If there is another root $u < 1$ then the giant component is present. If there are other roots that are all greater than 1, then there are only small components. The case in which there are two roots at $u = 1$ is called *phase transition*. It is easy to derive $G'_1(1) = 1$ at the phase transition.

The quantities $\hat{v}$, $\bar{v}$ and $\dot{v}$ are the fraction of vertices belonging to the small components, to the border components and to the linking set respectively. The fraction of vertices in the core is clearly $1 - \hat{v} - \bar{v}$.

In order to compute $\hat{p}_k$, $\bar{p}_k$, and $\dot{p}_k$ we exploit Bayes' rule. So we have

$$\hat{p}_k = \frac{\Pr\{S \mid D_k\}\, p_k}{\Pr\{S\}}, \qquad \bar{p}_k = \frac{\Pr\{B \mid D_k\}\, p_k}{\Pr\{B\}}, \qquad \dot{p}_k = \frac{\Pr\{L \mid D_k\}\, p_k}{\Pr\{L\}}. \tag{2}$$

Let $\hat{d}$, $\bar{d}$ and $\dot{d}$ be the respective average degrees. Let $\hat{q}_k = (k+1)\,\hat{p}_{k+1}/\hat{d}$ be the excess degree probabilities in the small components.

3 Degree probabilities in the small components

The generating functions of the probabilities $\hat{p}_k$ and $\hat{q}_k$ are respectively:

$$\hat{G}_0(x) = \sum_k \hat{p}_k\, x^k, \qquad \hat{G}_1(x) = \sum_k \hat{q}_k\, x^k = \sum_k \frac{(k+1)\,\hat{p}_{k+1}}{\hat{d}}\, x^k = \frac{\hat{G}_0'(x)}{\hat{d}}.$$

Clearly $\Pr\{S \mid D_0\} = 1$ and consequently (recall (2)) $\hat{p}_0 = p_0/\hat{v}$. If $k > 0$, $\Pr\{S \mid D_k\}$ is the probability that *all* adjacent k vertices belong to a small component once we have removed the corresponding arcs. Hence

$$\hat{p}_k := \frac{\Pr\{R\}^k\, p_k}{\hat{v}} = \frac{u^k}{\hat{v}}\, p_k, \tag{3}$$

valid also for $k = 0$. Then we have

$$1 = \sum_{k \geq 0} \hat{p}_k = \frac{G_0(u)}{\hat{v}},$$

from which

$$\hat{v} = G_0(u). \tag{4}$$

The relation between $G_0(x)$ and $\hat{G}_0(x)$ is given by

$$\hat{G}_0(x) = \frac{G_0(u\,x)}{\hat{v}}, \qquad \hat{G}_0'(x) = \frac{u}{\hat{v}}\, G_0'(u\,x),$$

from which

$$\hat{d} = \hat{G}_0'(1) = \frac{u}{\hat{v}}\, G_0'(u) = \frac{u}{\hat{v}}\, G_1(u)\, d. \tag{5}$$

If we take into account the equation $u = G_1(u)$ we may write

$$\hat{d} = \frac{u^2}{\hat{v}}\, d, \qquad \hat{q}_k = \frac{(k+1)\,\hat{p}_{k+1}}{\hat{d}} = u^{k-1}\, q_k.$$

Since $\hat{v}\,\hat{d}/d$ is equal to the fraction of edges belonging to small components, this equation tells that this fraction decreases quadratically with u. Since $\hat{d} < d$ we have $u^2 < \hat{v}$ which in turn implies that $\hat{p}_k < p_k$ for $k \geq 2$. It is not difficult to find the following relations

$$\hat{G}_1(x) = \frac{G_1(u\,x)}{G_1(u)} = \hat{G}_1(x) = \frac{G_1(u\,x)}{u}, \tag{6}$$

where $u = G_1(u)$ has been used for the second equality. Hence we have the following 'reciprocal' equations on the small components:

$$\hat{G}_1(u^{-1}) = u^{-1}, \qquad \hat{G}_0(u^{-1}) = \hat{v}^{-1}. \tag{7}$$

4 Degree probabilities in the border

$\Pr\{B \mid D_k\}$ is the probability that *all but one* adjacent k vertices belong to a small component once we have removed the corresponding arcs. Hence

$$\bar{p}_k = \frac{k\, u^{k-1}\,(1-u)}{\bar{v}}\, p_k. \tag{8}$$

Then we have

$$1 = \sum_k \bar{p}_k = \frac{1-u}{\bar{v}}\, G_0'(u) = \frac{d\,(1-u)}{\bar{v}}\, G_1(u),$$

from which

$$\bar{v} = d\,(1-u)\,G_1(u), \tag{9}$$

so that

$$\bar{p}_k = \frac{k\, u^{k-1}}{G_1(u)\, d}\, p_k.$$

By using $u = G_1(u)$ we may also write

$$\bar{v} = (1-u)\, u\, d, \qquad \bar{p}_k = \frac{k\, u^{k-2}}{d}\, p_k.$$

We see that, as expected, $\lim_{u\to 1} \bar{v} = 0$, i.e., the border components vanish together with the giant component as u approaches unity. The generating functions of the probabilities $\bar{p}_k$ and $\bar{q}_k := (k+1)\,\bar{p}_{k+1}/\bar{d}$ are

$$\bar{G}_0(x) = \sum_k \bar{p}_k\, x^k, \qquad \bar{G}_1(x) = \sum_k \frac{(k+1)\,\bar{p}_{k+1}}{\bar{d}} x^k = \frac{\bar{G}_0'(x)}{\bar{d}}.$$

The relation between $G_0(x)$ and $\bar{G}_0(x)$ is given by

$$\bar{G}_0(x) = x\, \frac{G_1(u\,x)}{G_1(u)} = x\, \frac{G_1(u\,x)}{u}.$$

By comparing this expression with (6) we have $\bar{G}_0(x) = x\,\hat{G}_1(x)$, or equivalently

$$\bar{p}_k = \hat{q}_{k-1}. \tag{10}$$

Simple computations lead to

$$\bar{d} = 1 + \frac{u\, G_1'(u)}{G_1(u)} = 1 + G_1'(u) = 1 + \hat{G}_1'(1). \tag{11}$$

It is interesting to note that the $\bar{d}$ tends to the value 2 as the network is approaching the phase transition.

5 Degree probabilities in the linking set

$\Pr\{L \,|\, D_k\}$ is the probability that among the adjacent k vertices there exists a set of three vertices two of which are in the core and the remaining one is in the border. It does not matter whether the other adjacent vertices are in the core or in the border. Hence

$$\Pr\{L \,|\, D_k\} = 1 - u^k - k\, u^{k-1}\,(1-u) - (1-u)^k, \qquad k \geq 3, \tag{12}$$

where the three cases incompatible with the existence of the set are explicitly excluded. This expression is valid also for $k = 2$ when it is identically equal to 0. So we have

$$\dot p_0 = 0, \qquad \dot p_1 = 0, \qquad \dot p_k = \frac{1 - u^k - k\, u^{k-1}\,(1-u) - (1-u)^k}{\dot v}\, p_k, \qquad k \geq 2, \tag{13}$$

from which

$$1 = \sum_{k \geq 0} \dot p_k = \sum_{k \geq 2} \frac{1 - u^k - k\, u^{k-1}\,(1-u) - (1-u)^k}{\dot v}\, p_k,$$

i.e.

$$\dot v = \sum_{k \geq 2} (1 - u^k - k\, u^{k-1}\,(1-u) - (1-u)^k)\, p_k.$$

This expression can be rewritten as

$$\dot v = 1 - p_0 - p_1 - G_0(u) + p_0 + u\, p_1 - (1-u)\, G_0'(u) + (1-u)\, p_1 - G_0(1-u) + p_0 + (1-u)\, p_1,$$

i.e.,

$$\dot v = 1 - \hat v - \bar v + (1-u)\, p_1 - G_0(1-u) + p_0, \tag{14}$$

where the previous definitions and results have been taken into account. Note that $1 - \hat v - \bar v$ is the fraction of graph in the core. The fraction of vertices in the core that are not linking is therefore $G_0(1-u) - p_0 - (1-u)\, p_1$ as can be inferred also by reasoning like in the case of the small components. In this case after selecting a vertex, deleting one of its incident edges never creates a small component. This leads to a probability expression like $(1-u)^k\, p_k$ with generating function $G_0(1-u)$. Since zero and one degree vertices do not belong to the core we get indeed $G_0(1-u) - p_0 - (1-u)\, p_1$.

The generating functions are

$$\dot G_0(x) = \sum_k \dot p_k\, x^k = \sum_{k \geq 2} \frac{1 - u^k - k\, u^{k-1}\,(1-u) - (1-u)^k}{\dot v}\, p_k\, x^k =$$

$$\frac{G_0(x) - \hat v\, \hat G_0(x) + p_0 - \bar v\, \bar G_0(x) + p_1\,(1-u)\, x - G_0((1-u)\, x)}{\dot v},$$

from which

$$\dot d = \frac{d - \hat v\, \hat d - \bar v\, \bar d + p_1\,(1-u) - (1-u)\ G_0'(1-u)}{\dot v}. \tag{15}$$

Hence (4), (9) and (14) allow to express $\hat{v}$, $\bar{v}$ and $\dot{v}$ respectively as a function of u whereas (5), (11) and (15) allow to express $\hat{d}$, $\bar{d}$ and $\dot{d}$ respectively as a function of u.

We need also to know how many edges are linking a linking vertex to the border. Let us call the *linking degree* of a linking vertex the number of edges incident to border vertices. If a linking vertex has degree k, there can be $1 \leq h \leq k-2$ edges incident to the border, i.e., the linking degree is h, with probability

$$\frac{\binom{k}{h}\, u^h\,(1-u)^{k-h}}{1-u^k-k\,u^{k-1}\,(1-u)-(1-u)^k} = p_k\,\frac{\binom{k}{h}\, u^h\,(1-u)^{k-h}}{\dot{p}_k\,\dot{v}}, \tag{16}$$

where we have used also (13). As apparent from (16) we may have any combination of linking and non linking edges except the three cases that are not compatible with the constraint of at least one edge linking and at least two edges not linking as we have done in (12). The denominator in (16) reflects this condition. If the degree of the linking vertex is k, the average linking degree is

$$\sum_{h\geq 1}^{k-2} h\,p_k\,\frac{\binom{k}{h}\, u^h\,(1-u)^{k-h}}{\dot{p}_k\,\dot{v}} = \frac{p_k}{\dot{p}_k\,\dot{v}}\,u\,(k-k\,(k-1)\,u^{k-2}\,(1-u)-k\,u^{k-1}). \tag{17}$$

Therefore the average linking degree, which we denote as δ, is

$$\delta = \sum_{k\geq 3}\frac{p_k}{\dot{p}_k\,\dot{v}}\,u\,(k-k\,(k-1)\,u^{k-2}\,(1-u)-k\,u^{k-1})\,\dot{p}_k = \frac{u}{\dot{v}}\,(d-(1-u)\,G_0''(u)-G_0'(u)). \tag{18}$$

If we use $u = G_1(u)$ the expression for the average linking degree simplifies to

$$\delta = \frac{u\,(1-u)}{\dot{v}}\,(d-G_0''(u)).$$

We may also compute the probability of having linking degree $h \geq 1$ in general as follows

$$\frac{1}{\dot{v}}\sum_{k\geq h+2} p_k\binom{k}{h}\,u^h\,(1-u)^{k-h} = \frac{1}{\dot{v}}\,\frac{u^h}{h!}\sum_{k\geq h+2} p_k\,\frac{k!}{(k-h)!}(1-u)^{k-h} =$$

$$= \frac{u^h}{\dot{v}}\,(\frac{G_0^{(h)}(1-u)}{h!}-p_h-(h+1)\,(1-u)\,p_{h+1}), \tag{19}$$

where $G_0^{(h)}(x)$ is the hth derivative of $G_0(x)$.

6 Small component sizes

We define the events:

- $\hat{S}_k$: a vertex randomly selected among the vertices in the small components belongs to a component of size k;
- $\hat{R}_k$: given a vertex i, randomly selected among the vertices in the small components with degree at least one, and a vertex j, randomly selected among the vertices adjacent to vertex i, the vertex j belongs to a small component of size k after removing the edge (i,j);

– $\hat{T}_k$: a randomly selected small component has size k.

Let $\hat{s}_k = \Pr\{\hat{S}_k\}$, $\hat{r}_k = \Pr\{\hat{R}_k\}$ and $\hat{t}_k = \Pr\{\hat{T}_k\}$ with corresponding generating functions

$$\hat{H}_0(x) = \sum_k \hat{s}_k\, x^k, \qquad \hat{H}_1(x) = \sum_k \hat{r}_k\, x^k, \qquad \hat{K}(x) = \sum_k \hat{t}_k\, x^k.$$

Note that the average size $\hat{c} = \hat{K}'(1)$ of a small component is linked to the average degree $\hat{d}$ in the small components as

$$\hat{c} = \frac{2}{2-\hat{d}}\,,$$

due to the assumption that all small components are trees. It is known [11] that $\hat{H}_0(x)$ and $\hat{H}_1(x)$ obey the recursive equations

$$\hat{H}_0(x) = x\,\hat{G}_0(\hat{H}_1(x)), \qquad \hat{H}_1(x) = x\,\hat{G}_1(\hat{H}_1(x)), \tag{20}$$

and that $\hat{K}(x)$ is linked to $\hat{H}_0(x)$ and $\hat{H}_1(x)$ through

$$\hat{K}(x) = \hat{c}\,(\hat{H}_0(x) - \frac{\hat{d}}{2}\,\hat{H}_1^2(x)) = \frac{2}{2-\hat{d}}\,(\hat{H}_0(x) - \frac{\hat{d}}{2}\,\hat{H}_1^2(x)). \tag{21}$$

Moreover the following relation holds, as can be easily shown,

$$\hat{s}_k = \frac{k\,\hat{t}_k}{\hat{c}}.$$

We note that the equations that can be found in the literature [11] are not linked to the degree probabilities in the small components like in (20) but in the whole graph. There are functions $H_0(x)$ and $H_1(x)$ defined by the recursive equations

$$H_0(x) = x\,G_0(H_1(x)), \qquad H_1(x) = x\,G_1(H_1(x)),$$

with s_k and r_k the coefficients of the power series representation of $H_0(x)$ and $H_1(x)$ respectively.

By expressing the functions $\hat{G}_1(x)$ and $\hat{G}_0(x)$ in terms of $G_1(x)$ and $G_0(x)$ we obtain

$$\hat{H}_1(x) = x\,\hat{G}_1(\hat{H}_1(x)) = x\,\frac{G_1(u\,\hat{H}_1(x))}{u},$$

from which we get

$$H_1(x) = u\,\hat{H}_1(x). \tag{22}$$

We remark that $\hat{H}(1) = 1$ since there are no infinite size components among the small components. Hence $H_1(1) = u$. Furthermore

$$\hat{H}_0(x) = x\,\hat{G}_0(\hat{H}_1(x)) = x\,\frac{G_0(u\,\hat{H}_1(x))}{\hat{v}} = x\,\frac{G_0(H_1(x))}{\hat{v}},$$

from which

$$H_0(x) = \hat{v}\,\hat{H}_0(x). \tag{23}$$

Hence $H_0(1) = \hat{v}$. The average size of a small component obtained by randomly selecting a vertex among the adjacent vertices of a randomly selected vertex and removing the edge is

$$\hat{H}'_1(1) = \frac{1}{1 - \hat{G}'_1(1)} = \frac{1}{1 - G'_1(u)},$$

obtained by differentiating (23) and the average size of a small component chosen by selecting a random vertex in the small components is

$$\hat{H}'_0(1) = 1 + \frac{\hat{d}}{1 - \hat{G}'_1(1)}, \tag{24}$$

obtained again by differentiating (23). Therefore the average size of a small component selected by selecting a random edge is $2\,\hat{H}'_1(1)$.

7 Border component sizes

We may define similar generating functions for the border components. In this case we want to know how many vertices are detached from the border if we remove a random edge in the border. Hence we define:

- $\bar{S}_k$: a vertex randomly selected among the vertices in the border components belongs to a component of size k;
- $\bar{R}_k$: given a random vertex i in the border of degree at least two and a randomly selected vertex j among the outbound vertices adjacent to i, the vertex j belongs to a small component of size k after removing the edge (i, j).
- $\bar{T}_k$: a randomly selected border component has size k.

Let $\bar{s}_k = \Pr\{\bar{S}_k\}$, $\bar{r}_k = \Pr\{\bar{R}_k\}$ and $\bar{t}_k = \Pr\{\bar{T}_k\}$ with corresponding generating functions

$$\bar{H}_0(x) = \sum_k \bar{s}_k\, x^k, \qquad \bar{H}_1(x) = \sum_k \bar{r}_k\, x^k, \qquad \bar{K}(x) = \sum_k \bar{t}_k\, x^k.$$

Note that the average size $\bar{c} = \bar{K}'(1)$ of a border component is linked to the average degree $\bar{d}$ in the border components as

$$\bar{c} = \frac{1}{2 - \bar{d}}\,, \tag{25}$$

since each border component is a tree plus one edge linked to a linking vertex. As we have observed before, $\bar{d}$ tends to 2 as the transition phase is approached. The above equation tells that the average border component size becomes unbounded near the transition phase. Since the number of border vertices tends to vanish near the transition phase, this means that the core vanishes and the giant component becomes made up just of a few large border components, because most border components are detached from the core and contribute to the small components. We shall come back to this point later on investigating Poisson graphs.

The generating functions $\bar{H}_0(x)$, $\bar{H}_1(x)$ and $\bar{K}(x)$ do not obey recursive equations similar to (23) for the small components. We face a different situation in the border with respect to a small component. A border component is a rooted tree and choosing randomly and edge (i, j) in a rooted tree, with j outbound, is equivalent to choosing randomly the vertex j since there is a one-to-one correspondence between edges and vertices. This means that, choosing with uniform probability the edge (i,j), the probability that the outbound vertex j has degree k is simply $\bar{p}_k$. By Assumption 3 this probability is also asymptotically equal to the probability of finding the degree k by choosing a vertex i of degree at least two and a random outbound vertex j adjacent to i.

Hence, we have (by using also (10))

$$\bar{H}_1(x) = x \sum_{k\geq 1} \bar{H}_1^{k-1}(x)\, \bar{p}_k = x \sum_{k\geq 1} \bar{H}_1^{k-1}(x)\, \hat{q}_{k-1} = x\, \hat{G}_1(\bar{H}_1(x)), \tag{26}$$

from which we get the interesting fact

$$\bar{H}_1(x) = \hat{H}_1(x).$$

In other words the outbound part of a border component has the same statistical properties of the small components. We note that the generating function $\bar{H}_1(x)$ is defined irrespective of the starting vertex, which can therefore be also the root of the border component. Hence $\bar{r}_k$ is also the probability that a border component has size k, i.e., $\bar{r}_k = \bar{t}_k$, so that $\bar{K}(x) = \bar{H}_1(x)$. Since $\bar{s}_k = k\, \bar{t}_k/\bar{c}$, we derive

$$\bar{H}_0(x) = \frac{x}{\bar{c}}\, \bar{K}'(x) = \frac{x}{\bar{c}}\, \bar{H}_1'(x), \qquad \bar{H}_0'(x) = \frac{1}{\bar{c}}\, (\bar{H}_1'(x) + x\, \bar{H}_1''(x)).$$

By working out (26) we get

$$\bar{H}_1'(x) = \frac{\hat{G}_1(\bar{H}_1(x))}{1 - x\, \hat{G}_1'(\bar{H}_1(x))}$$

so that

$$\bar{c} = \bar{H}_1'(1) = \frac{1}{1 - \hat{G}_1'(1)}$$

that is consistent with the previous relations (11) and (25).

8 The equation $u = G_1(u)$ revisited

As stated before, u may be computed by the fixed point equation $u = G_1(u)$. Now we find out the same equation by using the previous results (without of course using any result coming from the same equation $u = G_1(u)$), that are based on the conditional probabilities of the event R with respect to the events S, B and L. We may reason as follows.

The number of border components is related to the number of linking vertices and their linking degrees. If δ is the average number of border components incident to a linking vertex, the number of border components is equal to the number of linking vertices times δ. Let n be the number of vertices in the graph. Then the number of linking vertices is

$$n\, \dot{v} = n\, (1 - \hat{v} - \bar{v} + (1-u)\, p_1 - G_0(1-u) + p_0)$$

and the number of border components is

$$\delta\, n\, \dot{v}.$$

Hence the total number of vertices in the border components is (by using (25))

$$\frac{\delta\, n\, \dot{v}}{2-\bar{d}},$$

that must be equal to

$$n\,\bar{v} = n\,(1-u)\,G_0'(u).$$

So we have the equation

$$\delta\,\dot{v} = (1-u)\,G_0'(u)\,(2-\bar{d}),$$

that, by applying (11) rewritten as

$$\bar{d} = 1 + \frac{u\,G_1'(u)}{G_1(u)} = 1 + \frac{u\,G_0''(u)}{G_0'(u)},$$

becomes

$$\delta\,\dot{v} = (1-u)\,(G_0'(u) - u\,G_0''(u)), \tag{27}$$

and, by applying the formula (18)

$$u\,(d - (1-u)\,G_0''(u) - G_0'(u)) = (1-u)\,(G_0'(u) - u\,G_0''(u)),$$

which turns out to be $u\,d = G_0'(u)$, i.e., $u = G_1(u)$.

9 Dynamic graph change and stationary probability

In this section we consider a change in the random graph obtained by taking two random edges, removing them and reconnecting the four vertices in one of the other two possible ways. This move corresponds to a Markov chain transition in the state space of all graphs with fixed vertex degrees [13]. See also Section 12. The important fact about this Markov chain is that its stationary probability is uniform over all graphs with fixed degrees. If we have sampled a graph with uniform probability, then the Markov chain transition must produce another graph with the same statistical properties, although clearly with different connected components.

We can check that the average number of vertices in the small components is invariant. We can carry out this analysis at least for the small components where it is quite clear the type of change involved in the graph. What happens for the core and the border is much more complex and we do not carry out this analysis.

Let us consider in detail all possible ways we may pick up the two edges and the changes in the component sizes resulting from the choice. Let ΔS be the average overall size change of the small components.

1) S–S: both edges belong to small components. We may assume that the probability of picking the two edges in the same small component is asymptotically zero given the infinite number of small components (as long as $\hat{v} > 0$). Hence the two edges belong to two different small components. The move destroys the small components and creates two new small components. See Fig. 2-(1) where the relevant edges are highlighted. Hence we have on the average $\Delta S = 0$.

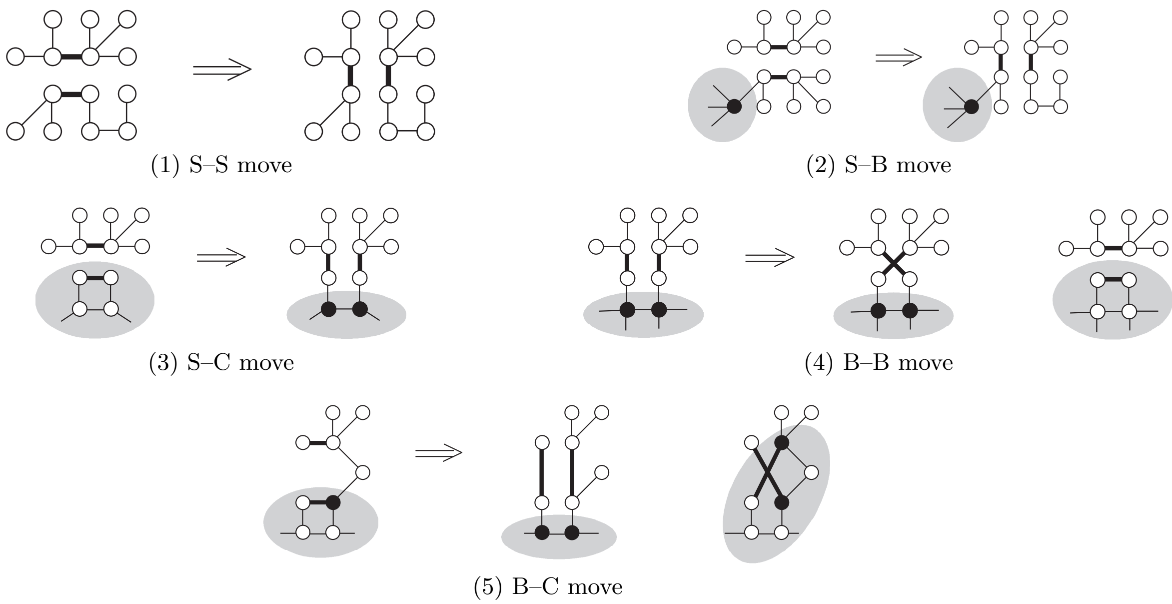


Fig. 2.

2) S–B: one edge is in a small component and the other one is in the border (including possibly the linking edge). The move destroys the small and the border components and creates a new small component and a new border component. See Fig. 2-(2). Here and in all other figures the linking vertices are in black and the core is highlighted with a gray background. We have on the average $\Delta S = -\hat{H}_1'(1) + \bar{H}_1'(1) = 0$.

3) S–C: one edge is in a small component and the other one is in the core. The move destroys the small component and creates two new border components. If the edge in the core is fully within the core the new border components are made up of just the vertices of the destroyed small component. In this case the core size is not changed. However, if the edge vertices in the core have degree two (like the one shown in Fig. 2-(3)) there may be extra vertices in the core that become part of the border. In this case the core size is decreased. We have on the average $\Delta S = -2\,\hat{H}_1'(1)$.

4) B–B: both edges are in the border (including possibly linking edges). Again we may assume that they do not belong to the same border component. The two border components are destroyed. According to the way we select the new edges, we may end up either with two new border components or with a small component and an increased core. See Fig. 2-(4). We have on the average $\Delta S = 0$, $\Delta S = 2\,\bar{H}_1'(1)$, in the first and in the second case respectively.

5) B–C: one edge is in the border (including linking edges) and the other one is in the core. If the edge in the core is fully within the core the border component is destroyed and new border components are formed, the core is increased. However if the edge vertices in the core have degree two it can happen that the core is decreased and there are two new border components. See Fig. 2-(5). In any case we have $\Delta S = 0$.

6) C–C: both edges are in the core. Clearly $\Delta S = 0$.

Note that the core decreases only in the special cases when edges are selected in the core with degree two. Hence we see that small components are formed from border components and these in turn are formed, not only from small components, but also from loosely connected parts of the core, namely, from vertices with degree two.

Now we have to compute the probabilities of these moves and the change in size due to these moves. An edge is selected in the small components with probability proportional to the fraction of edges in the small components, which is

$$\frac{\hat{v}\,\hat{d}}{d} = u^2.$$

An edge is selected in the border components with probability proportional to the fraction of edges in the border components plus the linking edges. The number of edges we have to count is exactly equal to the number of vertices in the border components. Hence the fraction of edges is

$$\frac{n\,\bar{v}}{n\,d/2} = \frac{2\,\bar{v}}{d} = 2\,u\,(1-u).$$

Consequently the fraction of edges in the core is

$$1 - u^2 - 2\,u\,(1-u) = (1-u)^2.$$

Hence the moves have the following probabilities

$$\Pr\{\text{S-S}\} = u^4, \quad \Pr\{\text{S-B}\} = 4\,u^3\,(1-u), \quad \Pr\{\text{S-C}\} = 2\,u^2\,(1-u)^2,$$

$$\Pr\{\text{B-B}\} = 4\,u^2\,(1-u)^2, \quad \Pr\{\text{B-C}\} = 4\,u\,(1-u)^3, \quad \Pr\{\text{C-C}\} = (1-u)^4.$$

Therefore we have

$$\Delta S = -2\,\hat{H}_1'(1)\,2\,u^2\,(1-u)^2 + \frac{1}{2}\,2\,\bar{H}_1'(1)\,4\,u^2\,(1-u)^2 = 0.$$

10 Analysis of Poisson graphs

In this section we particularize the previous general results to Poisson graphs. We recall that in a Poisson graph, for every pair of vertices, an arc is generated with probability $d/(n-1)$, where d is the average degree. As a consequence the degree probabilities and their generating functions are asymptotically

$$p_k = \frac{d^k}{k!}\,e^{-d}, \qquad G_0(x) = e^{d\,(x-1)} = G_1(x).$$

In a Poisson graph u is the solution of the equation $u = e^{d\,(u-1)}$, from which $d = \ln u/(u-1)$. If $d > 1$ there is a giant component and $u < 1$. For $d > 1$ the previous results particularize to

$$\hat{v} = G_0(u) = e^{d\,(u-1)} = u, \qquad \hat{d} = \frac{u\,G_0'(u)}{G_0(u)} = u\,d,$$

$$\hat{p}_k = \frac{u^k}{\hat{v}}\,p_k = \frac{u^k}{e^{d\,(u-1)}}\,\frac{d^k}{k!}\,e^{-d} = \frac{(u\,d)^k}{k!}\,e^{-u\,d} = \frac{\hat{d}^k}{k!}\,e^{-\hat{d}}.$$

Hence the small components are still Poisson, but with a smaller average degree, and clearly

$$\hat{G}_0(x) = \hat{G}_1(x) = e^{\hat{d}\,(x-1)}.$$

For the border we have

$$\bar{p}_k = \frac{k\,u^{k-1}}{G_0'(u)}\,p_k = \frac{k\,u^{k-1}}{d\,e^{d\,(u-1)}}\,e^{-d}\frac{d^k}{k!} = \frac{(u\,d)^{k-1}}{(k-1)!}e^{-u\,d} = \frac{\hat{d}^{k-1}}{(k-1)!}e^{-\hat{d}}.$$

It is interesting that the degrees in the border are distributed like in the small components, but with a shift of one degree. Clearly

$$\bar{d} = 1 + \hat{d} = 1 + u\,d.$$

Moreover

$$\bar{G}_0(x) = x\,e^{u\,d\,(x-1)} = x\,e^{\hat{d}\,(x-1)}, \qquad \bar{G}_1(x) = \frac{1 + x\,\hat{d}}{1+\hat{d}}\,e^{\hat{d}\,(x-1)}.$$

The fraction of graph in the border is

$$\bar{v} = (1-u)\,G_0'(u) = (1-u)\,d\,e^{d\,(u-1)} = (1-u)\,u\,d = -u\,\ln u,$$

The maximum of $\bar{v}$ is obtained at the value $u = e^{-1}$, i.e., at the average degree

$$d = \frac{1}{1-e^{-1}} = 1.58198,$$

for which $\bar{v} = e^{-1} = \hat{v} = 0.367879$. For the linking vertices we have

$$\dot{v} = 1 - e^{-u\,d} - (1 + d\,(1-u))\,(e^{d\,(u-1)} - e^{-d}) = (1-u)\,(1-u\,d)\,(1-e^{-u\,d}).$$

Also $\dot{v}$ exhibits a maximum. An analytic expression for the maximum is not immediately available. Numerically we find that the maximum of $\dot{v}$ is achieved at $u = 0.16062239$, i.e, $d = 2.17864$, for which $\dot{v} = 0.161112$. The average linking degree is

$$\delta = \frac{u}{\dot{v}}\,(d - (1-u)\,G_0''(u) - G_0'(u)) = \frac{u\,(d - (1-u)\,d^2\,e^{d(u-1)} - d\,e^{d(u-1)})}{(1-u)\,(1-u\,d)\,(1-e^{-u\,d})} = \frac{u\,d}{1-e^{-u\,d}}.$$

The limit of δ as d tends to ∞ (and consequently u tends to 0) is 1. This is consistent with the fact that the border component vanishes as the average degree increases, and so, if there are linking vertices, they have just one border component attached. In case of a Poisson graph the expression (19) can be easily worked out to produce the following values for the probabilities of having linking degree $h \geq 1$

$$\frac{u^h\,d^h}{h!}\,\frac{e^{-u\,d}}{1-e^{-u\,d}}.$$

11 A critical window for the border

It is interesting to analyze the behavior of the border components of a Poisson graph near the transition phase. We parametrize the solution of $u = G_1(u)$, as $u = 1 - \varepsilon$. Hence $G_1(1-\varepsilon) = e^{-d\,\varepsilon}$ and we have to solve $1 - \varepsilon = e^{-d\,\varepsilon}$, from which, by taking a first order approximation we may express d as a function of ε,

$$d = -\frac{\ln(1-\varepsilon)}{\varepsilon} = \sum_{k\geq 0} \frac{\varepsilon^k}{k+1} \approx 1 + \frac{\varepsilon}{2}.$$

Similarly we may express ε as a function of d as

$$(1-\varepsilon) \approx 1 - d\,\varepsilon + \frac{d^2\,\varepsilon^2}{2} \quad \Longrightarrow \quad \varepsilon \approx \frac{2\,(d-1)}{d^2} \approx 2\,(d-1) \quad \Longrightarrow \quad u \approx 1 - 2\,(d-1) = 3 - 2\,d.$$

Now the fraction of vertices in the border is

$$\bar{v} = u\,(1-u)\,d = \varepsilon\,(1-\varepsilon)\,d = \varepsilon\,(1-\varepsilon) \sum_{k\geq 0} \frac{\varepsilon^k}{k+1} \approx \varepsilon\,(1 - \frac{\varepsilon}{2}),$$

or, expressed as a function of d.

$$\bar{v} = \varepsilon\,(1-\varepsilon)\,d \approx 2\,(d-1)\,(1 - 2\,(d-1))\,d = 2\,(d-1)\,(3-2\,d)\,d$$

and the average border degree is

$$\bar{d} = 1 + u\,d \approx 2 - \frac{\varepsilon}{2},$$

with average border component size

$$\bar{c} = \frac{1}{2-\bar{d}} = \frac{2}{\varepsilon} = \frac{1}{d-1}. \tag{28}$$

Therefore, as in the general case, whilst the border vertices vanish, the average size of a border component becomes unbounded. The fraction of vertices in the core is

$$1 - \hat{v} - \bar{v} \approx 1 - (1-\varepsilon) - \varepsilon\,(1 - \frac{\varepsilon}{2}) = \frac{\varepsilon^2}{2},$$

Hence the core size decreases quadratically with ε, or, viewed from the opposite side, the core size has a sharp rise when the average degree increases after the transition point.

To better understand the critical behavior near the transition point, let us assume now that the number of vertices is a large finite number n. The previous relations are still valid with a good approximation. The number of border components, expressed as a function of ε, is therefore given by

$$\frac{n\,\bar{v}}{\bar{c}} = \frac{1}{2}\,n\,\varepsilon^2\,(1 - \frac{\varepsilon}{2}) \approx \frac{1}{2}\,n\,\varepsilon^2.$$

Note that this number is equal to the core size. This means that each vertex in the core is root of a border component. The birth of the giant component passes through this step when several trees are joined together via a small set of nodes that is about to become the core of the graph.

So we have that, as ε tends to zero, the number of border vertices is going to zero, the number of border components decreases quadratically as the core size, whilst the average size of a border component increases without bounds. However, since n is finite, $\bar{c}$ cannot grow unbounded. The formulas have been derived under Assumption 2. So when the formulas give an inconsistent result, this means that the very structure of a giant component with its border components is collapsing or, viewed from the opposite side, the structure of the giant component is still in formation. We may mark this moment by noting that the number of border components has to be greater than one, or, asymptotically, has to be greater than $O(1)$. Hence

$$\frac{1}{2}\, n\, \varepsilon^2 \geq O(1) \quad \Longrightarrow \quad \varepsilon \geq \frac{\lambda}{\sqrt{n}}$$

where λ is a $O(1)$ function with respect to n. In other words the interval

$$1 < d < 1 + \frac{\lambda}{\sqrt{n}}$$

is a critical window with respect to the border components. It is interesting to compare this critical time window with the one for the giant component. It is known that the critical time window for the giant component is [8, 14]

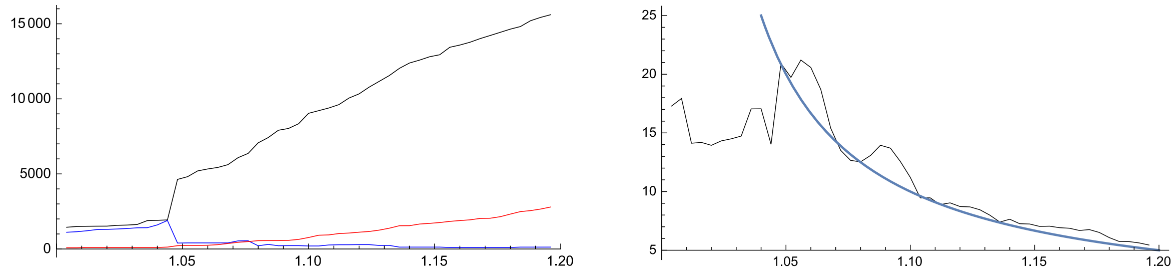
$$1 < d < 1 + \frac{\lambda}{\sqrt[3]{n}}$$

which is larger than the previous one. Hence when the giant component is born, it is already equipped with core and border components. In Fig. 3 we report the results of a simulation on a graph with $n = 50,000$ vertices. The arcs have been randomly added one by one using a Union-Find structure to track the components, until an average degree $d = 1.2$ has been reached. Then the arcs have been removed in the reverse order and the data relative to the border and the core have been computed on steps of 100 arcs. In Fig. 3 (a) we report, as a function of d, the size of the largest component (in black), the size of the second largest component (in blue) and the size of the core (in red). We may see that around $d = 1.045 \approx 1.65\, n^{-1/3}$ the giant component is formed. In Fig. 3 (b) we report the empirical average border size as a function of d (in black) plotted against the theoretical value given by (28) (in blue). We may see that the two functions agree with good approximation out of the critical window.

12 Simulations

We have carried out computational simulations to check the validity of the theoretical formulas found in the previous sections against experimental outcomes. We have carried out several simulations on different graph classes. They are all consistent with the theoretical results. In this paper we limit ourselves to report on the simulation carried out for a power law graph and a Poisson graph.

For the power law graph we have generated a graph with $n =$ 20,000 vertices. We recall that in a power-law graph the degree $k \geq 1$ is present with probability $p_k = k^{-\alpha}/\zeta(\alpha)$, with $\alpha > 1$ a given parameter and $\zeta(\alpha)$ the Riemann-zeta function. The average degree is $d = \zeta(1-\alpha)/\zeta(\alpha)$. In this simulation we have chosen



(a)
largest component (black), second largest component (blue) and core (red) sizes as a function of d.

(b)
average border size as a function of d: empirical (black), theoretical (blue)

Fig. 3.

$\alpha = 2.5$ which yields $d = 1.94737$. See [1] for a alternative definition where p_k is proportional to $k^{-\alpha}$ and $0 < \alpha < 1$ is allowed but the maximum degree is fixed.

For the Poisson graph we have generated a graph with $n = 50{,}000$ vertices and average degree $d = 1.3$. The graph has not been created by generating an arc for each pair of vertices with probability $p = d/(n-1)$ because this procedure is too much time consuming.

Both graphs have been generated by using the well known Markov chain model over the state space of all graphs with known and fixed degree for each vertex [13]. A transition from a graph to another graph is carried out by selecting two random edges (i, j) and (h, k). If the edges do not share a vertex, they are deleted and either the pair of edges (i, h), (j, k) or the pair (i, k), (j, h) is generated with probability 1/2. The probability of this transition is independent of the particular edges and has value $1/(|E|\,(|E| - 1))$. If the edges share a vertex the graph remains unchanged and in the Markov chain model this corresponds to a self-transition. The important fact is that the stationary probability of this Markov chain is uniform. Therefore, selecting the graph obtained after a sufficiently high number of transitions is equivalent to uniformly sampling a graph with given degrees.

A starting graph is generated as follows. Knowing the degree probabilities, the maximum degree value $\bar{k}$ is computed such that $n\, p_{\bar{k}} > 0.5$. We have found $\bar{k} = 61$ for the power law graph and $\bar{k} = 8$ for the Poisson graph. Then the fraction of vertices of degree k is given by rounding to the nearest integer $n\, p_k$, $k = 1, \dots, \bar{k}$. If there is no graph with the given degrees there is a slight adjustment to allow for a graph. In the simulation for the power law graph, the number of vertices with degree $1, \dots, 28$ is 14924, 2639, 956, 466, 267, 169, 115, 82, 61, 47, 37, 30, 24, 20, 17, 15, 13, 11, 9, 8, 7, 7, 6, 5, 5, 4, 4, 4; then there are 3 vertices for each of the degrees 29–32, 2 vertices for each of the degrees 33–39 and one vertex for each of the degrees 40–61. For the Poisson graph the number of vertices with degree $0, \dots, 8$ is 13627, 17715, 11515, 4990, 1622, 422, 91, 17, 3. The probabilities p_k are then recomputed on the basis of the actual fractions of vertices.

Then, from the starting graph 300,000 Markov chain transitions have been carried out. The resulting graph is the one on which the experimental values have been computed. All experimental values have been computed by simple counting operations. For instance, the value u has been computed, according to its definition, by summing the number of vertices in the small components, adding the value $(d_i - 1)/d_i$ for each vertex i in the border (with d_i its degree), and adding the value δ_i/d_i for each vertex in the linking set (with d_i its degree and δ_i its linking degree). The resulting sum is divided by the total number of vertices yielding the experimental value for u.

In Table 1 we report both the theoretical (in italics) and the experimental (in roman) data for the power law graph and in Table 2 we report the data for the Poisson graph. As can be seen, the experimental values are in accordance with the theoretical values.

13 Conclusions

In this paper a detailed investigation on some special components of the giant component of a random graph has been carried out. The random graphs we have considered are those specified by assigning the degree probabilities to the vertices. The special components are components that can be easily detached from the giant component and can provide a better understanding of the dynamic behavior of a graph when edges are added and removed. We have called these components the border of the giant component. The investigation has provided explicit expressions for the degree probabilities and the size probabilities in the various components of the graph, including the small components.

The general formulas have been particularized to Poisson graphs, where a critical window related to the birth of the border components can be identified. Then a simulation has been carried out on large power-law and Poisson graphs respectively to check the theoretical expressions against the experimental data. These simulations have shown a remarkable accordance between the theoretical and the experimental data.

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	u	$\hat{v}$	$\hat{d}$	$\bar{v}$	$\bar{d}$	$\dot{v}$	$\dot{d}$	δ
theor	*0.5403*	*0.4522*	*1.1414*	*0.4391*	*1.3322*	*0.0748*	*7.9012*	*3.9217*
exp	0.5416	0.4547	1.1397	0.4346	1.3284	0.0754	7.8633	3.8733
	$\hat{p}_0$	$\hat{p}_1$	$\hat{p}_2$	$\hat{p}_3$	$\hat{p}_4$	$\hat{p}_5$	$\hat{p}_6$	$\hat{p}_7$
theor	*0*	*0.8916*	*0.0852*	*0.0167*	*0.0044*	*0.0014*	*0.0005*	*0.0002*
exp	0	0.8926	0.0853	0.0168	0.0031	0.0009	0.0005	0.0004
	$\bar{p}_0$	$\bar{p}_1$	$\bar{p}_2$	$\bar{p}_3$	$\bar{p}_4$	$\bar{p}_5$	$\bar{p}_6$	$\bar{p}_7$
theor	*0*	*0.7812*	*0.1493*	*0.0438*	*0.0154*	*0.0060*	*0.0024*	*0.0010*
exp	0	0.7832	0.1488	0.0414	0.0161	0.0067	0.0017	0.0015
	$\dot{p}_2$	$\dot{p}_3$	$\dot{p}_4$	$\dot{p}_5$	$\dot{p}_6$	$\dot{p}_7$	$\dot{p}_8$	$\dot{p}_9$
theor	*0*	*0.2190*	*0.1808*	*0.1317*	*0.0948*	*0.0694*	*0.0516*	*0.0394*
exp	0	0.2216	0.1825	0.1307	0.0982	0.0650	0.0504	0.0398
	$\hat{s}_1$	$\hat{s}_2$	$\hat{s}_3$	$\hat{s}_4$	$\hat{s}_5$	$\hat{s}_6$	$\hat{s}_7$	$\hat{s}_8$
theor	*0*	*0.6965*	*0.1560*	*0.0628*	*0.0318*	*0.0181*	*0.0111*	*0.0071*
exp	0	0.7042	0.1478	0.0638	0.0280	0.0165	0.0115	0.0079
	$\hat{r}_1$	$\hat{r}_2$	$\hat{r}_3$	$\hat{r}_4$	$\hat{r}_5$	$\hat{r}_6$	$\hat{r}_7$	$\hat{r}_8$
theor	*0.7812*	*0.1166*	*0.0442*	*0.0219*	*0.0124*	*0.0076*	*0.0049*	*0.0032*
exp	0.7862	0.1119	0.0442	0.0195	0.0112	0.0078	0.0050	0.0033
	$\hat{t}_1$	$\hat{t}_2$	$\hat{t}_3$	$\hat{t}_4$	$\hat{t}_5$	$\hat{t}_6$	$\hat{t}_7$	$\hat{t}_8$
theor	*0*	*0.8096*	*0.1209*	*0.0365*	*0.0148*	*0.0070*	*0.0037*	*0.0021*
exp	0	0.8185	0.1145	0.0371	0.0130	0.0064	0.0038	0.0023
	$\bar{s}_1$	$\bar{s}_2$	$\bar{s}_3$	$\bar{s}_4$	$\bar{s}_5$	$\bar{s}_6$	$\bar{s}_7$	$\bar{s}_8$
theor	*0.5247*	*0.1566*	*0.0890*	*0.0589*	*0.0416*	*0.0305*	*0.0229*	*0.0174*
exp	0.5273	0.1576	0.0828	0.0575	0.0403	0.0276	0.0185	0.0212
	$\bar{r}_1$	$\bar{r}_2$	$\bar{r}_3$	$\bar{r}_4$	$\bar{r}_5$	$\bar{r}_6$	$\bar{r}_7$	$\bar{r}_8$
theor	*0.7812*	*0.1166*	*0.0442*	*0.0219*	*0.0124*	*0.0076*	*0.0049*	*0.0032*
exp	0.7890	0.1153	0.0363	0.0220	0.0126	0.0085	0.0054	0.0030
	$\bar{t}_1$	$\bar{t}_2$	$\bar{t}_3$	$\bar{t}_4$	$\bar{t}_5$	$\bar{t}_6$	$\bar{t}_7$	$\bar{t}_8$
theor	*0.7812*	*0.1166*	*0.0442*	*0.0219*	*0.0124*	*0.0076*	*0.0049*	*0.0032*
exp	0.7852	0.1174	0.0411	0.0214	0.0120	0.0069	0.0039	0.0039

Table 1. Theoretical and experimental values for a power-law graph with $\alpha = 2.5$ and n=20,000

	u	$\hat{v}$	$\hat{d}$	$\bar{v}$	$\bar{d}$	$\dot{v}$	$\dot{d}$	δ
theor	*0.5770*	*0.5770*	*0.7501*	*0.3173*	*1.7501*	*0.0558*	*3.6222*	*1.4215*
exp	0.5809	0.5791	0.7550	0.3185	1.7574	0.0539	3.6324	1.4345
	$\hat{p}_0$	$\hat{p}_1$	$\hat{p}_2$	$\hat{p}_3$	$\hat{p}_4$	$\hat{p}_5$	$\hat{p}_6$	$\hat{p}_7$
theor	*0.4723*	*0.3543*	*0.1329*	*0.0332*	*0.0062*	*0.0009*	*0.0001*	*0*
exp	0.4706	0.3554	0.1315	0.0347	0.0065	0.0012	0.0001	0
	$\bar{p}_0$	$\bar{p}_1$	$\bar{p}_2$	$\bar{p}_3$	$\bar{p}_4$	$\bar{p}_5$	$\bar{p}_6$	$\bar{p}_7$
theor	*0*	*0.4723*	*0.3543*	*0.1329*	*0.0332*	*0.0062*	*0.0009*	*0.0001*
exp	0	0.4662	0.3593	0.1340	0.0333	0.0058	0.0010	0.0001
	$\dot{p}_2$	$\dot{p}_3$	$\dot{p}_4$	$\dot{p}_5$	$\dot{p}_6$	$\dot{p}_7$	$\dot{p}_8$	$\dot{p}_9$
theor	*0*	*0.5542*	*0.3094*	*0.1040*	*0.0260*	*0.0053*	*0.0009*	*0.0001*
exp	0	0.5444	0.3167	0.1069	0.0260	0.0059	0	0
	$\hat{s}_1$	$\hat{s}_2$	$\hat{s}_3$	$\hat{s}_4$	$\hat{s}_5$	$\hat{s}_6$	$\hat{s}_7$	$\hat{s}_8$
theor	*0.4723*	*0.1673*	*0.0889*	*0.0560*	*0.0388*	*0.0285*	*0.0218*	*0.0172*
exp	0.4706	0.1636	0.0929	0.0571	0.0359	0.0247	0.0201	0.0185
	$\hat{r}_1$	$\hat{r}_2$	$\hat{r}_3$	$\hat{r}_4$	$\hat{r}_5$	$\hat{r}_6$	$\hat{r}_7$	$\hat{r}_8$
theor	*0.4723*	*0.1673*	*0.0889*	*0.0560*	*0.0388*	*0.0285*	*0.0218*	*0.0172*
exp	0.4668	0.1696	0.0886	0.0537	0.0358	0.0277	0.0245	0.0199
	$\hat{t}_1$	$\hat{t}_2$	$\hat{t}_3$	$\hat{t}_4$	$\hat{t}_5$	$\hat{t}_6$	$\hat{t}_7$	$\hat{t}_8$
theor	*0.7587*	*0.1344*	*0.0476*	*0.0225*	*0.0125*	*0.0076*	*0.0050*	*0.0035*
exp	0.7560	0.1314	0.0498	0.0229	0.0115	0.0066	0.0046	0.0037
	$\bar{s}_1$	$\bar{s}_2$	$\bar{s}_3$	$\bar{s}_4$	$\bar{s}_5$	$\bar{s}_6$	$\bar{s}_7$	$\bar{s}_8$
theor	*0.1146*	*0.0812*	*0.0647*	*0.0543*	*0.0470*	*0.0414*	*0.0370*	*0.0334*
exp	0.1125	0.0823	0.0669	0.0507	0.0553	0.0381	0.0343	0.0362
	$\bar{r}_1$	$\bar{r}_2$	$\bar{r}_3$	$\bar{r}_4$	$\bar{r}_5$	$\bar{r}_6$	$\bar{r}_7$	$\bar{r}_8$
theor	*0.4723*	*0.1673*	*0.0889*	*0.0560*	*0.0388*	*0.0285*	*0.0218*	*0.0172*
exp	0.4675	0.1670	0.0870	0.0562	0.0425	0.0284	0.0230	0.0173
	$\bar{t}_1$	$\bar{t}_2$	$\bar{t}_3$	$\bar{t}_4$	$\bar{t}_5$	$\bar{t}_6$	$\bar{t}_7$	$\bar{t}_8$
theor	*0.4723*	*0.1673*	*0.0889*	*0.0560*	*0.0388*	*0.0285*	*0.0218*	*0.0172*
exp	0.4636	0.1696	0.0919	0.0523	0.0456	0.0261	0.0202	0.0186

Table 2. Theoretical and experimental values for a Poisson graph with $d = 1.3$ and n=50,000