Error Minimization Methods in Biproportional Apportionment

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Abstract

One of the most active research lines in the area of electoral systems to date deals with the Biproportional Apportionment Problem, which arises in those proportional systems where seats must be allocated to parties within territorial constituencies. A matrix of the vote counts of the parties within the constituencies is given, and one has to convert the vote matrix into an integer matrix of seats "as proportional as possible" to it - subject to the constraints that each district be granted its pre-specified number of seats, each party be allotted the total number of seats it is entitled to on the basis of its national vote count, and a zero-vote zero-seat condition be satisfied. The matrix of seats must simultaneously meet the integrality and the proportionality requirement and this not infrequently gives rise to self-contradictory procedures in the electoral laws of some countries. Here we discuss a class of methods for Biproportional Apportionment characterized by an "error minimization" approach. If the integrality requirement is relaxed, fractional seat allocations (target shares) can be obtained so as to achieve proportionality at least in theory. In order to restore integrality, one then looks for integral apportionments that are as close as possible to the ideal ones in a suitable metric. This leads to the formulation of constrained optimization problems called "best approximation problems" which are solvable in polynomial time through the use of network flow techniques. These error minimization methods can be viewed as an alternative to the classical axiomatic approach introduced by Balinski and Demange in 1989. We provide an empirical comparison between these two approaches against a real example from the Italian Elections and a theoretical discussion about the axioms that are not necessarily satisfied by the error minimization methods.

Keywords: *biproportional apportionment, error minimization, metric spaces, network flows.*

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1 Introduction

The *Biproportional Apportionment Problem* (BAP) arises in those countries that adopt a proportional electoral system in which the territory is subdivided into a set of constituencies (or regions). Given a matrix of votes of the parties (columns) within the regions (rows), the problem consists in computing the number of seats that each party is entitled to in each constituency on the basis of the number of votes that it has received in that constituency. The main difficulty in solving this problem relies on the simultaneous requirements of double proportionality and integrality by the matrix of seats. In order to obtain a correct biproportional seat distribution the problem must be rigorously formulated and solved through the use of suitable mathematical models. Mathematical problems with the same structure as BAP frequently arise in the literature also in other application contexts. For example, this happens in Statistics for the problem of approximating a table of noninteger counts by an integer one, so as to match the prescribed row- and column-sums [9, 10, 2, 3, 16]. For this reason, BAP was widely studied both in the mathematical and in the statistical literature.

A milestone theoretical setting was given by Balinski and Demange in 1989 [5, 6], but also several other authors provided different mathematical models and procedures to find correct biproportional seat allocations (see, [11, 12, 25, 28, 31], the collection of papers in [30] and the recent survey on network flow techniques for electoral problems [27]). In all these papers both models and solution procedures for BAP are provided. However, in spite of the abundance of contributions and results, it is quite difficult to find some references to them in an actual electoral law. An exception occurred when the Zurich Canton in the 2006 elections actually applied the *Discrete Alternating Scaling* procedure introduced by Pukelsheim [26] through the use of the public domain software BAZI. The Cantons of Schaffhausen and Aargau followed. Actually, BAP arises in the electoral procedures of several countries (e.g., Italy, Mexico, Switzerland, Faroe Islands), but sometimes the problem is not completely understood, with the consequence that the institutional rules to solve it may turn out to be wrong and lead to inconsistent allocations of seats that are not coherent with the voters' will. For instance, the Italian electoral law for the national parliament (Law n. 270/2005), as well as the great majority of the (recently reformed) laws for the Italian regional elections, are affected by errors [21, 22, 23], and the problem has not been yet faced by the political institutions.

Balinski and Demange base their analysis on an axiomatic characterization of proportionality between integral matrices. They provide a method to find a biproportional allocation satisfying a specified set of proportionality axioms, namely, i) *Exactness*, ii) *Relevance*, iii) *Uniformity*, iv) *Monotonicity*, v) *Homogeneity*, and vi) *Completeness*, which correspond to reasonable properties that an apportionment should satisfy (see Section 11 for details). Following this axiomatic approach, they propose a divisor based method to find biproportional apportionments. First a fractional matrix satisfying axioms i)-v) (and called *fair share* matrix) is computed. Then, starting from the fair share matrix, a primal-dual iterative procedure is applied to obtain a rounding of it satisfying i)-vi). In the Balinski and Demange approach the fair share matrix represents the "ideal" biproportional seat assignment in the above axiomatic sense, but, since fractional seats are not allowed, it is necessary to suitably round it in order to keep satisfied the row- and column-sums, maintaining its proportionality feature.

Although the main biproportional allocation procedures known in the literature are based on an axiomatic approach and follow the rules of divisor methods, a different approach for BAP can be also considered. A different class of BAP methods was recently developed in the literature following an error minimization approach [25, 28]. According to these methods, a fractional matrix is taken as a target (*target quotas* or *ideal shares*) and the biproportional seat apportionment is obtained through the solution of a constrained optimization problem (called *best approximation problem*) where the objective corresponds to a suitable error measure between the solution and the matrix of target quotas.

In this paper we review the class of error minimization methods for BAP. Some methods are already known in the literature, but some others are considered here for the first time for the solution of BAP. Actually, new linear and quadratic models are suggested to formulate BAP as a best approximation problem and it is shown that already known optimization techniques can be exploited to solve it efficiently.

As a result, our paper provides a detailed account of a new methodology for the solution of BAP which can be seen as an alternative to the classical axiomatic approach of Balinski and Demange. It must be pointed out here that the nature of the two approaches is completely different and, in principle, there is no reason to prefer one or the other. We believe that, once the characteristics of each method are clear, it has to be left to the subjective task of the legislators the choice of the method that 'best' fits their country.

In Section 2 a formal description of the BAP is given. In Section 3 the general error minimization approach for BAP is introduced and an account is given of the different metrics that can be adopted to measure the error objective function. We start in Section 4 with the description of the *Controlled*

Rounding procedure introduced by Cox and Ernst in 1982 [9]. They seem to be the first authors who suggest an error minimization approach, even if the actual application of their model to BAP was pointed out by Gassner later in 1988 [13]. In [9] the error is given by the L_p-norm computed w.r.t. the ideal shares, but the rounding of the entries of the target quotas are constrained to match the up or down nearest integer. This model corresponds to a polynomially solvable capacitated transportation problem, but, as we will see, the solution method does not minimize in general the L_p-norm, because there exist instances where the optimal apportionment is obtained outside the imposed rounding interval. Focusing on the L_1- and L_2-norm cases (Sections 5 and 7), we point out a common feature of the corresponding error minimization models showing that they are solvable in polynomial time via the solution of either a single linear minimum cost flow problem, or a sequence of them. In Section 8 we describe the L_∞–norm model proposed by Serafini and Simeone [28] which can be solved in strongly polynomial time via the solution of a sequence of maximum flow problems. In Section 10 we exhibit the matrices of seats obtained through the different best approximation models for the most recent Italian Elections of the Chamber of Deputies (2008) and compare them with each other, with the fair share matrix, and with the optimal allocation provided by the method of Balinski and Demange. In Section 11 we recall the proportionality axioms of Balinski and Demange [5, 6] and analyze the best approximation methods under this viewpoint. Finally, in Section 12 we draw some conclusions summarizing the main results of the paper.

2 The Biproportional Seat Apportionment Problem

Let H be the *house size* of a parliament, i.e., the total number of seats to be assigned and suppose that they are already apportioned among m electoral constituencies proportionally to their population counts, so that r_i seats are allocated to constituency $i \in M = \{1, \dots, m\}$. For the Italian case the rule to allocate the total number of seats in each constituency is stated in the Constitution. Let V be a $m \times n$ matrix such that $v_{ij}, i \in M, j \in N$ is the number of votes for party j in region i.

After the voting, the H seats are also apportioned among the n contending parties proportionally to the number of votes each party has received. Let c_j be the corresponding total number of seats received by party $j \in N = \{1, \dots, n\}$ on the basis of its total (national) number of votes $\sum_{i \in M} v_{ij}$. Clearly, one has $\sum_{i \in M} r_i = \sum_{j \in N} c_j = H$. Thus, apportioning the H seats among the constituencies on one hand and among the parties on the other one, produces two distributions of seats that we call *super-apportionments*.

When formulating the problem the r_i and the c_j are assumed to be known input data so that an instance of BAP is given by the triplet (V, r, c). The problem is to find a final allocation of the seats that is consistent with both the super-apportionments, and such that the number of seats that a party obtains in each constituency is as proportional as possible w.r.t. the number of votes received by the same party in that constituency.

The above requirements can be formally stated as finding a $m \times n$ integer matrix of seats X such that:

(1) each row-sum (*constituency-sum*) is equal to the total number of seats of the corresponding constituency;

(2) each column-sum (*party-sum*) is equal to the total number of seats of the corresponding party;

(3) a party cannot obtain seats in a constituency where it received no votes;

(4) the number of seats assigned to each party in each constituency is as *proportional as possible* to the corresponding number of votes.

In BAP conditions (1)–(3) are constraints, while (4) generally corresponds to the objective. In particular, condition (3) is referred to as "zero vote-zero seat" condition and it means that a party is not awarded any seat in a constituency in which it does not receive any vote. This is a relevant condition especially in those countries - like Italy - were local (regional) political parties exist that present their lists (and get votes) only in some regions of the country.

From a mathematical viewpoint BAP is not a trivial problem. A variety of models and mathematical formulations have been suggested in the literature: all provide integer solutions that satisfy conditions (1)–(3), while the objective (4) is formulated in different ways.

The Balinski and Demange's algorithm for BAP first computes the fair share matrix that satisfies the following *Rounding Property*: one can always obtain an apportionment by rounding either up or down the entries of the fair share matrix. Starting from the fair shares, the procedure follows a "scale and round" approach to find a rounding of it. In order to do this, suitable scaling factors are computed through a primal-dual iterative procedure known as the "Tie and Transfer" (TT) algorithm [6].

In this approach the fair share matrix is regarded as an ideal proportional matrix, but the integrality of the seat matrix requires the rounding procedure. Maintaining the idea that the fair share is a good "target matrix", an alternative error minimization approach can be followed for BAP, leading to the formulation of a best approximation problem. The rounding procedure can be actually replaced by the solution of an integer optimization model in which conditions (1)–(3) correspond to constraints and one searches for an integer matrix X as close as possible to the target matrix w.r.t. a suitable distance measure.

It must be noted that, in each country, the ideal proportional matrix can be defined in different ways according to the particular electoral law. Some countries, like Italy and Belgium, adopt *regional quotas* which for each region i and party j are defined as

$$r_i \frac{v_{ij}}{\sum_{h \in N} v_{ih}}. \tag{1}$$

Regional quotas correspond to proportionality within constituencies (i.e., row-wise quotas) and they are also called *natural quotas*.

Whatever the selected target matrix is, we will illustrate in the next section some best approximation models for BAP and we will show how all of them can be equivalently formulated as network flow problems.

3 Optimal biproportional apportionment via best approximation

Let $Z = \{(i,j) : v_{ij} = 0\}$ be the set of the structural zeros of V. Let x_{ij} be the seats to be allocated to party j in region i. Conditions (1)-(3) can be formalized through the following set of linear constraints

$$\begin{array}{lll} \sum_{j \in N} x_{ij} = r_i, & i \in M & \text{(constituency-sum)} \\ \sum_{i \in M} x_{ij} = c_j, & j \in N & \text{(party-sum)} \\ x_{ij} = 0, & (i,j) \in Z & \text{(zero vote-zero seat).} \end{array} \tag{2}$$

For $x_{ij} \geq 0$, $i \in M$, $j \in N$, the system (2) is easily recognizable as the set of constraints of an uncapacitated transportation problem with forbidden routes defined over a bipartite graph $G = (M, N; E)$, where the two sets of nodes M and N correspond to the regions and the parties, respectively. The arc set E is the set of all those pairs (i,j), $i \in M$, $j \in N$, such that $v_{ij} > 0$. From each node $i \in M$ there is an outgoing flow equal to r_i and into each node $j \in N$ there is an incoming flow equal to c_j.

A best approximation problem can be formulated adding a proper objective function to the above constraints. In the following we focus on models in which the L_1- L_2- and L_∞-error measures are considered for BAP and discuss how they can be solved in polynomial time through the application of network flow techniques. Before going into details of these models, we recall the "Controlled Rounding Problem" by Cox and Ernst [9] which can be referred to as the first error minimization method for a matrix problem of the same structure of BAP.

4 The Controlled Rounding Problem

The *Controlled Rounding Problem* (CRP) was introduced by Cox and Ernst in 1982 [9]. If Q is a $m \times n$ real matrix and X is a $m \times n$ integer matrix, X is a *rounding* of Q if either $x_{ij} = \lfloor q_{ij} \rfloor$ or $x_{ij} = \lceil q_{ij} \rceil$, $i \in M$, $j \in N$.

For a given input of the BAP and for a given nonnegative $m \times n$ real matrix Q of target quotas, the controlled rounding problem asks for an apportionment X^* that is a rounding of Q and minimizes a prescribed L_p-norm $\|X - Q\|_p$.

Following [9], it is convenient – and natural – to model CRP via the introduction of binary variables y_{ij}, and to write the desired apportionment X under the form $x_{ij} = \lfloor q_{ij} \rfloor + y_{ij}$, $i \in M$, $j \in N$.

In this way, one can write CRP as

$$\begin{aligned} \min \quad & \sum_{i \in M} \sum_{j \in N} |\lfloor q_{ij} \rfloor + y_{ij} - q_{ij}|^p & \\ & \sum_{j \in N} (\lfloor q_{ij} \rfloor + y_{ij}) = r_i & i \in M \\ & \sum_{i \in M} (\lfloor q_{ij} \rfloor + y_{ij}) = c_j & j \in N \\ & y_{ij} = 0 & (i,j) \in \bar{Z} \\ & y_{ij} \in \{0,1\} & i \in M, j \in N, (i,j) \notin \bar{Z}, \end{aligned} \tag{3}$$

where the set $\bar{Z}$ is the set Z (entries with zero votes) plus the entries (if any) with integral q_{ij}, if one wants the seats to be a correct rounding of the quotas. In view of the well-known fact that any real-valued function $f(z)$ of a single binary variable z can be written as a linear function, $f(z) = f(0)\,(1-z) + f(1)\,z$, the problem (3) can be equivalently written as

$$\begin{aligned} \min \quad & \sum_{i \in M} \sum_{j \in N} d_{ij}\, y_{ij} & \\ & \sum_{j \in N} y_{ij} = \bar{r}_i & i \in M \\ & \sum_{i \in M} y_{ij} = \bar{c}_j & j \in N \\ & y_{ij} = 0 & (i,j) \in \bar{Z} \\ & 0 \le y_{ij} \le 1 & i \in M, j \in N, (i,j) \notin \bar{Z}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} d_{ij} &= (1 - <q_{ij}>)^p - <q_{ij}>^p \\ \bar{r}_i &= r_i - \sum_{j \in N} \lfloor q_{ij} \rfloor \\ \bar{c}_j &= c_j - \sum_{i \in M} \lfloor q_{ij} \rfloor\,, \end{aligned}$$

with $<q_{ij}> = q_{ij} - \lfloor q_{ij} \rfloor$, i.e., the fractional part of q_{ij}. The objective functions in (3) and (4) differ by the constant term $\sum_{i \in M, j \in N} <q_{ij}>^p$. Notice that the constraints $y_{ij} \in \{0,1\}$ in (3) have been relaxed into the bounds $0 \leq y_{ij} \leq 1$ in (4). Nonetheless, (3) and (4) remain equivalent since the coefficient matrix of (4) is totally unimodular. Notice also that (4) is a capacitated linear transportation problem, which can be solved in $O(|E| \log(m+n) (|E| + (m+n) \log(m+n)))$ time, where $|E|$ is the number of nonnull entries in the vote matrix (see [1], Chap.10).

For the L_∞-norm a more complex linearization is suggested involving the definition of new variables (for details, see [9]).

As noticed by Gassner [13], the above model can be applied for the solution of BAP if one starts from the fair share matrix as the ideal apportionment. Here it is taken for granted that the seats are obtained only by rounding up or down the fair shares so that in the model by Cox and Ernst this condition can be set as a constraint.

However, it is important to remark that limiting the seat values to either rounding down or up the fair share quotas, as in the Cox and Ernst approach, introduces a constraint which may cut off the true solution minimizing either the L_1-norm or the L_2-norm over all possible apportionments. To show this point, we report an example introduced in [27].

Example 1

Consider the $(n+2) \times (n+2)$ matrix Q of fair share quotas

$$Q = \begin{pmatrix} \frac{n-1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} \\ \frac{1}{n} & \frac{n-1}{n} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n} & 0 & \cdots & \frac{n-1}{n} & 0 \\ \frac{1}{n} & 0 & \cdots & 0 & \frac{n-1}{n} \end{pmatrix}, \qquad r = (\,2 \quad 1 \quad \cdots \quad 1\,), \qquad c = (\,2 \quad 1 \quad \cdots \quad 1\,).$$

There are essentially three apportionments up to permutation of the indices $\{2, \ldots, n+2\}$, namely

$$X^1 = \begin{pmatrix} 2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad X^2 = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad X^3 = \begin{pmatrix} 0 & 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1, \end{pmatrix}$$

for which we have

$$\|X^1 - Q\|_1 = 4 + \frac{4}{n}, \qquad \|X^2 - Q\|_1 = 6 - \frac{2}{n}, \qquad \|X^3 - Q\|_1 = 10 - \frac{10}{n},$$

and

$$\|X^1 - Q\|_2^2 = 1 + \frac{5}{n} + \frac{4}{n^2}, \qquad \|X^2 - Q\|_2^2 = 3 - \frac{3}{n} + \frac{4}{n^2}, \qquad \|X^3 - Q\|_2^2 = 7 - \frac{11}{n} + \frac{4}{n^2}.$$

Hence for $n > 4$ the optimal apportionment (both for the L_1- and the L_2-norm) is X^1, with x^1_{11} outside the range $\{0, 1\}$. If we restrict the apportionments to $\{0, 1\}$, then the optimal apportionment is X^2. ∎

We may therefore wonder whether it is possible to solve efficiently the L_1 and L_2 minimization without the restriction of finding seats within $\{\lfloor q_{ij} \rfloor, \lceil q_{ij} \rceil\}$. The answer is affirmative thanks to the properties of network flows. In the following Section 5 and Section 7 we formalize this approach by introducing new

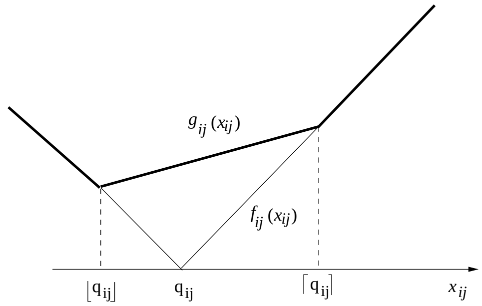

Figure 1: Functions $f_{ij}(x_{ij})$ and $g_{ij}(x_{ij})$ are both convex piecewise linear and coincide at all integral points.

linear and quadratic models for BAP. If, on the one hand, we show that these models can be efficiently solved by using optimization techniques already known in the literature, on the other hand, we point out that they are applied for the solution of BAP for the first time here, thus providing new ways of formalizing BAP as a best approximation problem.

5 A min-cost flow model for finding apportionments with minimum L_1-error

In this section we discuss a model for BAP in which the L_1-error must be minimized but under no restrictions on the possible rounding of the elements q_{ij}. Such model is, in fact, a minimum cost flow one and it can be efficiently solved to find a minimum error BAP for a given arbitrary target matrix Q.

To this purpose, consider the L_1-error computed w.r.t. Q

$$\sum_{i \in M} \sum_{j \in N} |x_{ij} - q_{ij}|. \tag{5}$$

Starting from G, a new bipartite graph $\mathcal{G} = (M, N; \mathcal{E})$ can be defined in which the set of vertices is the same as in G, and each arc $(i, j) \in E$ is replaced by three parallel arcs in $\mathcal{E}$. All such arcs have lower capacity equal to 0; the first arc has upper capacity $\lfloor q_{ij} \rfloor$ and cost -1; the second one has upper capacity 1 and cost $1 - 2 <q_{ij}>$; the third arc has infinite upper capacity and cost 1.

We can consider the number of seats x_{ij} to be assigned to party j in constituency i as the sum of three separate flows from i to j, namely y_{ij}, z_{ij} and t_{ij}, in the above network $\mathcal{G}$, i.e.,

$$x_{ij} = y_{ij} + z_{ij} + t_{ij}, \tag{6}$$

with $0 \le y_{ij} \le \lfloor q_{ij} \rfloor$, $0 \le z_{ij} \le 1$, $t_{ij} \ge 0$.

In this way, the function $f_{ij}(x_{ij}) = |x_{ij} - q_{ij}|$, which is convex piecewise linear but has a breakpoint at the fractional value q_{ij}, is approximated by another convex piecewise linear function $g_{ij}(x_{ij})$ with integral breakpoints, and taking the same values as $f_{ij}(x_{ij})$ in all integral points x_{ij} (see Fig. 1):

$$g_{ij}(x_{ij}) = \begin{cases} q_{ij} - x_{ij} & \text{if } x_{ij} \le \lfloor q_{ij} \rfloor \\ (1 - 2 <q_{ij}>)\,(x_{ij} - \lfloor q_{ij} \rfloor) + <q_{ij}> & \text{if } \lfloor q_{ij} \rfloor \le x_{ij} \le \lceil q_{ij} \rceil \\ x_{ij} - q_{ij} & \text{if } x_{ij} \ge \lceil q_{ij} \rceil. \end{cases} \tag{7}$$

Since in BAP we are interested only in integral values of x, we may replace $f(x)$ with $g(x)$, with the consequence that $g(x)$ has breakpoints at integral values and network flow techniques can be applied to produce integral solutions.

Notice that, since the cost $1 - 2 <q_{ij}>$ always lies in the open interval $(-1, 1)$, the function $g_{ij}(x_{ij})$ is convex piecewise linear, implying that

$$
\begin{aligned}
& y_{ij} < \lfloor q_{ij} \rfloor && \Rightarrow && z_{ij} = t_{ij} = 0 \\
& 0 < z_{ij} < 1 && \Rightarrow && y_{ij} = \lfloor q_{ij} \rfloor ,\, t_{ij} = 0 \\
& t_{ij} > 0 && \Rightarrow && y_{ij} = \lfloor q_{ij} \rfloor ,\, z_{ij} = 1.
\end{aligned}
\tag{8}
$$

Then BAP can be formulated and solved by means of a standard minimum cost flow problem on $\mathcal{G}$ with at most $4\,m\,n$ variables and linear objective function:

$$\sum_{(hk)\in\mathcal{E}} c_{hk}\,\varphi_{hk},$$

where φ_{hk} denotes the generic flow variable on $\mathcal{G}$ and c_{hk} is the corresponding unit cost.

Let X^* be the solution of the above problem. Since all capacities are integral, in view of the Integrality Theorem for minimum linear cost flows, the apportionment X^* is integral as well. In addition, since $g(x)$ coincides with the L_1-error in all integral X's, it follows that X^* minimizes the L_1-error over all the apportionments. We remark here that the L_1-error method may lead to multiple optimal solutions. This aspect will be investigated in Section 6.

6 On the uniqueness of L_1 optimal apportionments

Given a vote matrix V, a *cycle* on V is induced by the corresponding cycle in the bipartite graph G (see, for example, [27]). Given a feasible apportionment X, let $f(X) = \sum_{i\in M}\sum_{j\in N} |x_{ij} - q_{ij}|$ be its *value*. An *adjacent solution* of X is a solution obtained by adding or subtracting a seat to X along a cycle. In the following, we investigate under which conditions there may exist adjacent optimal solutions for an optimal solution of a problem instance (V, r, c).

Given an apportionment X, let us call *excess pairs* those (i,j) such that $x_{ij} > q_{ij}$ and *defect pairs* those (i,j) such that $x_{ij} < q_{ij}$. Furthermore, define *strongly excess pairs* those (i,j) such that $x_{ij} > q_{ij}+1$ and *strongly defect pairs* those (i,j) such that $x_{ij} < q_{ij} - 1$. The other excess and defect pairs are called *weakly excess pairs* and *weakly defect pairs*, respectively. Let us denote by S^+ and S^- the strongly excess and defect pairs (i,j), respectively and by W^+ and W^- the weakly excess and defect pairs (i,j), respectively. Given a (necessarily even) cycle C of pairs, let us partition C into two sets by putting in each partition set the pairs in C that are at even distance among themselves and denote the two sets as C^+ and C^-, with an arbitrary choice. An adjacent solution of X is obtained by adding one seat to pairs in C^+ and decreasing one seat from pairs in C^-. Note that strongly excess or defect pairs remain excess or defect respectively in any adjacent solution, whereas weakly excess or defect pairs may become weakly defect or excess pairs, respectively.

The difference in value between X and an adjacent solution Y obtained from the cycle $C^+ \cup C^-$ is given by

$$
\begin{aligned}
f(Y) - f(X) = & \sum_{(ij)\in S^+\cap C^+} 1 + \sum_{(ij)\in W^+\cap C^+} 1 + \sum_{(ij)\in S^-\cap C^-} 1 + \sum_{(ij)\in W^-\cap C^-} 1 - \sum_{(ij)\in S^-\cap C^+} 1 - \sum_{(ij)\in S^+\cap C^-} 1 + \\
& \sum_{(ij)\in W^-\cap C^+} (x_{ij} + 1 - q_{ij}) - (q_{ij} - x_{ij}) + \sum_{(i,j)\in W^+\cap C^-} (q_{ij} - x_{ij} + 1) - (x_{ij} - q_{ij}) = \\
& |(S^+ \cup W^+) \cap C^+| + |(S^- \cup W^-) \cap C^-| - |S^- \cap C^+| - |S^+ \cap C^-| + \\
& \sum_{(i,j)\in W^-\cap C^+} (1 - 2 \ <q_{ij}>) + \sum_{(ij)\in W^+\cap C^-} (1 - 2 \ <q_{ij}>).
\end{aligned}
$$

We may assume that sums and/or subtractions of fractional parts of quotas are integral with "very low probability". Hence we very likely expect $f(Y) = f(X)$ only if $W^- \cap C^+ = \emptyset$ and $W^+ \cap C^- = \emptyset$. In this case $f(Y) = f(X)$ if and only if

$$|(S^+ \cup W^+) \cap C^+| + |(S^- \cup W^-) \cap C^-| = |S^- \cap C^+| + |S^+ \cap C^-|$$

In turn this condition can be fulfilled only if there are strongly excess and defect pairs. Consider the following example.

Example 2

$$V = \begin{pmatrix} 796 & 965 & 352 & 12 & 923 \\ 883 & 132 & 95 & 61 & 880 \\ 614 & 972 & 710 & 658 & 433 \end{pmatrix}, \qquad r = \begin{pmatrix} 29 \\ 20 \\ 16 \end{pmatrix}, \qquad c = \begin{pmatrix} 18 & 12 & 14 & 12 & 9 \end{pmatrix},$$

with regional quotas Q and an optimal seat assignment X

$$Q = \begin{pmatrix} 7.574 & 9.181 & 3.349 & 0.114 & 8.782 \\ 8.610 & 1.288 & 0.926 & 0.595 & 8.581 \\ 2.900 & 4.592 & 3.354 & 3.108 & 2.046 \end{pmatrix}, \qquad X = \begin{pmatrix} 7 & 7 & 8 & 7 & 0 \\ 8 & 1 & 2 & 1 & 8 \\ 3 & 4 & 4 & 4 & 1 \end{pmatrix}.$$

The optimal value of X is 29.3054. The sets S^+, S^-, W^+ and W^- are as follows (denoted as $S^+ \to ++$, $W^+ \to +$, $S^- \to$ - -, $W^- \to$ -)

$$\begin{pmatrix} - & -- & ++ & ++ & -- \\ - & - & ++ & + & - \\ + & - & + & + & -- \end{pmatrix},$$

and a possible cycle is $C^+ = \{(1,5),(2,4)\}$, $C^- = \{(1,4),(2,5)\}$ with adjacent solution and new sets

$$Y = \begin{pmatrix} 7 & 7 & 8 & 6 & 1 \\ 8 & 1 & 2 & 2 & 7 \\ 3 & 4 & 4 & 4 & 1 \end{pmatrix}, \qquad \begin{pmatrix} - & -- & ++ & ++ & -- \\ - & - & ++ & ++ & -- \\ + & - & + & + & -- \end{pmatrix}.$$

Through a sequence of equivalent adjacent solutions it is possible to obtain the following apportionment

$$Y' = \begin{pmatrix} 7 & 9 & 9 & 1 & 3 \\ 8 & 1 & 1 & 4 & 6 \\ 3 & 2 & 4 & 7 & 0 \end{pmatrix},$$

with the same optimal value 29.3054. ∎

The astonishing fact in Example 2 is that it is possible to vary in the cell $(1,4)$ the seats from 7 to 1, remaining always within optimal solutions!

This is clearly a highly undesirable feature of the L_1 norm minimization. In this example we have used regional quotas and, admittedly, the seats values r_i and c_j preassigned to the regions and to the parties are anomalous w.r.t. to the matrix of votes. However, from a theoretical point of view we have to take care of any possible situation. The behavior of the L_1 norm is much more comfortable if we use the fair share quotas, which for Example 2 are given by

$$Q = \begin{pmatrix} 7.625 & 8.638 & 7.886 & 0.609 & 4.241 \\ 8.946 & 1.250 & 2.251 & 3.277 & 4.276 \\ 1.428 & 2.113 & 3.862 & 8.114 & 0.483 \end{pmatrix}.$$

The optimal solution is

$$X = \begin{pmatrix} 8 & 9 & 8 & 0 & 4 \\ 9 & 1 & 2 & 4 & 4 \\ 1 & 2 & 4 & 8 & 1 \end{pmatrix}.$$

All pairs are weakly excess or weakly defect (in other words the solution is within the rounded quotas, like in the controlled rounding procedure) and therefore the solution is in general unique with high probability.

The conclusion is that the L_1 norm minimization is a viable method if the fair shares are used as quotas. Otherwise the solution may be not unique producing very anomalous seat assignments.

7 A min-cost flow model for finding apportionments with minimum L_2-error

The same technique used in Section 5 for the L_1-error might be applied to any convex objective function, by sampling the function at the integral points and building an equivalent (on the integral points) convex piecewise linear function. However – differently from the L_1-error case – for an arbitrary convex function the number of breakpoints might grow in a non polynomial way.

The L_2-error minimization model was already considered by Minoux in 1984 [18] when he studied a particular quadratic minimum cost flow problem. We show that BAP can be formulated exactly as the quadratic problem analyzed by Minoux, so that the same solution approach can be exploited to solve BAP, too.

Consider an arbitrary matrix of target shares Q. Following the approach already used in Section 5, and according to the objective functions suggested in [9], BAP can be formulated as a minimum cost flow problem w.r.t. any L_p-error. In particular, when the L_2-error must be minimized the problem becomes a minimum quadratic cost flow problem, that is, a special case of a non linear separable cost network flow problem with convex and continuously separable objective function [19].

As already noticed by Hochbaum in 2005 [15], a first polynomial algorithm for the quadratic minimum cost flow problem was introduced in [18] where the idea of the out-of-kilter method for linear minimum cost flow problems is extended to the case of quadratic separable cost functions. Other polynomial algorithms for the same problem were suggested in [1]. All these algorithms are polynomial but not strongly polynomial since their computational complexity depends on the magnitude of some parameters in the problem instance input, such as the maximum upper capacity of an arc, or the maximum supply of a node.

The method suggested by Minoux for the quadratic problem [18] essentially follows a methodology similar to the one used in Section 5, but, instead of discretizing the cost function once according to some desired accuracy, a scaling approach is used to repeatedly replace the original quadratic cost function by different (scaled) piecewise linear convex approximations - each with a different number of breakpoints - until an approximation sufficiently close to the (integral) optimum of the original problem is reached. Minoux also provides basic results guaranteeing that such an optimum is reached after a polynomial number of scalings. In the following, we show how the L_2-error minimization model for BAP can be formulated in the form of the model studied by Minoux and provide some details of the corresponding solution procedure.

The L_2-error model for BAP is formulated as the following quadratic cost transportation problem:

$$
\begin{aligned}
\min \quad & \sum_{i \in M} \sum_{j \in N} (x_{ij} - q_{ij})^2 \\
& \sum_{j \in N} x_{ij} = r_i && i \in M \\
& \sum_{i \in M} x_{ij} = c_j && j \in N \\
& \quad x_{ij} = 0 && (i,j) \in Z \\
& \quad x_{ij} \geq 0 \quad \text{and integral} && (i,j) \notin Z.
\end{aligned}
\tag{9}
$$

Recall the bipartite graph $G = (M, N; E)$ underlying the set of constraints in (9). It is well known that the above problem can be equivalently formulated as a minimum cost flow problem with a single source s and a single sink t on a suitable network $\mathcal{M}$ obtained by augmenting G and introducing proper lower and upper capacities for the outgoing arcs of the source and the incoming arcs of the sink. It can be also shown that any feasible solution of the resulting minimum cost flow problem can be obtained as a feasible solution of the corresponding *circulation problem* on a network $\mathcal{N} = (\mathcal{V}, \mathcal{A})$ obtained by adding in $\mathcal{M}$ the arc (t, s) with lower capacity equal to 0 (see [1]).

W.l.o.g., in our BAP problem we can introduce lower and upper capacities ℓ_{ij}, u_{ij} also for arcs from a constituency-node i to a party-node j as follows:

$$
\ell_{ij} := 0 \leq x_{ij} \leq \min\{r_i\,,\, c_j\} =: u_{ij}, \quad i \in M, j \in N,
$$

and an upper capacity on the arc (t, s) is set equal to $u_{ts} := H$.

After the above considerations, the L_2-error model for BAP (9) can be rewritten in the following form:

$$
\begin{aligned}
\min \quad & \sum_{(h,k) \in \mathcal{A}} (x_{hk} - q_{hk})^2 \\
& A\,x = 0 \\
& 0 \leq x_{hk} \leq u_{hk}, && (h,k) \in \mathcal{A} \\
& x_{hk} \quad \text{integral} && (h,k) \in \mathcal{A},
\end{aligned}
\tag{10}
$$

where A is the node-arc incidence matrix of $\mathcal{N}$. Problem (10) has the form of the quadratic separable min cost flow problem (QCFP) studied by Minoux in [18].

Since the objective function in (10) is separable, for each x_{ij}, the quadratic function $f_{ij}(x_{ij}) = (x_{ij} - q_{ij})^2$ can be approximated in the corresponding interval $[0, u_{ij}]$ by a piecewise linear function (see Fig. 2). Any such approximation of problem (10) can be solved in polynomial time by applying the out-of-kilter method generalized to the case of minimum cost network flow problems with a convex piecewise linear objective function (see, [8, 14]).

Minoux proves that problem (10) can be solved in polynomial time by the solution of a finite sequence of "scaled" problems each corresponding to a different piecewise linear approximation of the quadratic cost function of (10), and that, after a polynomial number of scalings, the optimum of the scaled problem coincides with the optimum of (10). For a given integer p, consider the *p-th order approximation* of QCFP - that we denote by QCFP(p) - which scales the original problem by a factor $\sigma = 2^p$ so that the scaled problem has a piecewise linear objective function taking the same value of $f_{ij}(x_{ij})$ in all the x_{ij} that are integer multiples of σ. The sequence of scaled problems is given by QCFP(p), $p := \bar{p}, \bar{p} - 1, \ldots, \rho + 1, \rho$, with $\bar{p} > \rho$, and where $\bar{p} = \lceil \log_2 u \rceil$, with $u = \max_{ij} u_{ij}$, and ρ is an integer (non necessarily positive). The idea is to refine at each step of the sequence the piecewise linear approximations of the cost function

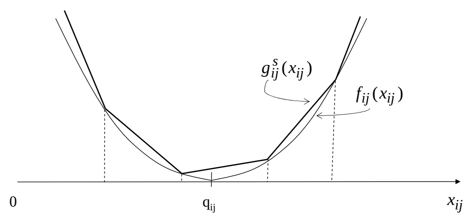

Figure 2: Piecewise linear approximation of function $(x_{ij} - q_{ij})^2$.

of the original problem until a point sufficiently close to the optimal one is obtained and it is shown that this can be done through a suitable choice of (a small) ρ. In addition, for every p, using as a starting solution at step $p-1$ the optimal flow obtained at step p, the optimal solution of QCFP(p) can be obtained in polynomial time by the out-of-kilter algorithm. This leads to an overall time complexity of $O(\mu\,\nu^2\,\log u)$, where μ and ν are the number of arcs and vertices of $\mathcal{N}$, respectively.

8 Error minimization apportionments with minimum L_∞-error

When the L_∞-error is considered, BAP leads to a nonlinear minimum cost flow problem on the bipartite network $G = (M, N; E)$, with the objective function representing the *maximum absolute error* [24, 28]:

$$
\begin{array}{lll}
\min & \displaystyle\max_{i\in M,\, j\in N} |x_{ij} - q_{ij}| & \\
& \displaystyle\sum_{j\in N} x_{ij} = r_i & i \in M \\
& \displaystyle\sum_{i\in M} x_{ij} = c_j & j \in N \\
& x_{ij} = 0 & (i,j) \in Z \\
& x_{ij} \geq 0 \quad \text{and integral} & (i,j) \notin Z.
\end{array}
$$

The above model can be reformulated as follows:

$$
\begin{array}{lll}
\min & \tau & \\
& \displaystyle\sum_{j\in N} x_{ij} = r_i & i \in M \\
& \displaystyle\sum_{i\in M} x_{ij} = c_j & j \in N \\
& \lceil q_{ij} - \tau \rceil^+ \leq x_{ij} \leq \lfloor q_{ij} + \tau \rfloor & (i,j) \notin Z \\
& x_{ij} = 0 & (i,j) \in Z \\
& x_{ij} \quad \text{integral} & (i,j) \notin Z,
\end{array}
\tag{11}
$$

where, by definition, $a^+ := \max\{a, 0\}$.

For any fixed value of $\tau > 0$ the problem of finding an apportionment with maximum absolute error at most τ corresponds to finding a feasible flow on $G = (M, N; E)$ with the above capacities. It is well known that the existence of a feasible flow can be established in polynomial time through the solution of a maximum flow problem [1]. A feasible flow x_{ij} satisfies the given capacities and, by the well-known Integrality Theorem of Network Flows [1], if there is a feasible flow x there is also an integral flow since the capacities are integers for all (i,j). Hence, the optimal value τ^* can be obtained by finding the minimum τ such that a feasible flow with error at most τ exists. Serafini and Simeone [28] note that, due to the integrality of the x_{ij}'s, in this procedure only a finite set of values must be considered for τ (*relevant errors*). In fact, the relevant errors are those values of τ such that either $q_{ij} - \tau$ or $q_{ij} + \tau$ is an integer in the interval $[0, r_i]$ for some $(i,j) \notin Z$. Thus, the optimal value τ^* can be obtained by applying the following algorithm:

1. Perform a binary search on the set of relevant values for τ.
2. At each iteration, for the current value of τ check whether a feasible flow exists through the solution of a maximum flow problem.

Since the relevant errors are at most $n\,(H+m)$, $O(\log(n\,(H+m)))$ maximum flow problems must be solved, so that the overall time complexity is polynomial. Serafini and Simeone also present a more complex algorithm with a strongly polynomial complexity (see, [28]).

They also provide a refinement of the above algorithm based on finding unordered lexicographic minima aimed to minimize also the errors which are less than the maximum error. Under the mild condition that the fractional parts of the target quotas q_{ij} are all different, this guarantees the uniqueness of the solution which is a crucial issue in real electoral applications. In the following, when analyzing the L_∞-error approach, we will always refer to this version of the procedure as "MinMaxLexBest".

At the end of this section, we want to emphasize that, while best approximation through the minimization of the L_1- or L_2-error objectives is handled via the solution of minimum cost flow problems, the L_∞-error model requires the solution of maximum flow problems: although both type of problems are solvable in polynomial time, the latter ones are conceptually simpler than the former and can be solved by more efficient algorithms.

There is another feature of this particular L_∞-error model which makes it an interesting and viable model. Since solving an optimization problem requires an ad-hoc mathematical knowledge, the layman could legitimately doubt that the given solution satisfies the claimed requirement, namely, that it minimizes the stated norm. In principle, all the described methods exhibit some sort of strong duality property which could enable a checking method based on dual variables. However, a procedure of this type would in general require again a non trivial mathematical knowledge. On the contrary, it turns out that the L_∞-error model, being based on the max flow–min cut theorem, allows for a checking method which is amenable to the layman, so that this method can be considered as 'transparent' to the voter. For details about how this kind of 'certificate' can be provided to and handled by the layman see [29].

9 Relative error minimization

All methods described so far are concerned with absolute errors $|q_{ij} - x_{ij}|$. Since a difference of one seat has a stronger impact on a cell with a little number of votes than on a cell with many votes, it makes sense to consider the relative error $|1 - x_{ij}/q_{ij}|$ instead of the absolute error. Indeed Balinski and Young ([7] p. 129) observe that "it can be argued that staying within the quota is not really compatible with the idea of proportionality at all, since it allows a much greater variance in the per capita representation of smaller states than it does for larger states"(by "staying within the quotas" the authors mean $\lfloor q_{ij} \rfloor \le x_{ij} \le \lceil q_{ij} \rceil$). We may note that divisor methods by their nature are more affine to measuring relative distances rather than absolute distances.

However, little attention has been devoted in the literature to the minimization of the relative error. In [28] this case is considered at length for the L_∞ norm and a weakly polynomial algorithm is described based on the same ideas as for the absolute error minimization. Indeed the same flow model can be used with the difference that the flow capacity bounds in (11) become

$$\lceil q_{ij}\,(1-\tau)\rceil^{+} \leq x_{ij} \leq \lfloor q_{ij}\,(1+\tau)\rfloor\,,$$

and the minimum value of τ (relative error in this case) has to be found such that (11) (with the new bounds) is feasible. Since the number of relevant errors is not so nicely bounded as for the absolute error only a weakly polynomial algorithm has been proposed in [28].

Concerning the norms L_1 and L_2 we may extend the previous results by noting that the function $|1 - x_{ij}/q_{ij}|$ coincides, on the integral points, with the function $g_{ij}(x_{ij})/q_{ij}$, where $g_{ij}(x_{ij})$ has been defined in (7). Similarly the function $(1 - x_{ij}/q_{ij})^2$ can be also approximated by a piecewise linear function with equal values on the integral points. Therefore the previous observations can be extended to the relative error case without altering the algorithmic ideas of the absolute error minimization.

It is possible to show that also for the relative error the optimal solution can be outside the quota roundings, a fact that is quite plausible for the relative error. The same Example 1 can be used to show this fact. The apportionment X^1 is optimal for all three norms L_1, L_2 and L_∞, if $n \geq 3$.

10 Comparing the apportionments of the different best approximation models

In this section we provide a comparative analysis between the seat apportionments produced by the best approximation models and methods discussed in the previous sections, and also w.r.t. the solution provided by the Balinski and Demange's Tie and Transfer algorithm. In particular, we shall analyze the data of the Italian Elections of the Chamber of Deputies of 2008. First of all, Table 1 shows the seat apportionments obtained as optimal solutions of the best approximation models minimizing the L_∞-, L_1-, and L_2- error objective functions – and we denote these models by MinMax (or MinMaxLexBest), MinL1, and MinL2, respectively – together with the fair share quotas and seat allocation obtained by applying the TT algorithm (first and second column for every party, respectively).

For these experiments we use the fair shares as target quotas to get a uniform comparison among the methods. In Table 1 the rows refer to the 26 Italian constituencies for the Election of the Chamber of Deputies, while the columns refer to the parties. We did not report the allocation for the "Südtiroler Volkspartei", a local party competing only in the constituency "Trentino Alto-Adige" (TA, row 6). Actually, once the total number of seats has been established for this party at the national level, under the zero vote-zero seat condition, every apportionment method assigns Südtiroler Volkspartei the same number of seats (equal to the national one). Since in 2008 Südtiroler Volkspartei got 3 seats in TA, in Table 1 we avoided reporting the Südtiroler Volkspartei column and, consequently, we updated both the house size and the total number of seats in TA by subtracting these three seats. There are also other two local parties, namely, "Lega Nord" (LN) and "Movimento Per le Autonomie" (MPA), which compete only in the northern and in the southern constituencies, respectively. More precisely, LN presents its lists only in the first 14 constituencies of Table 1, while MPA competes only in the 12 remaining ones. To make the table more compact, we therefore decided to report the results of these two parties in a single column, using the hyphen "–" to separate the names and the national seats of these two parties. Consequently, the second column of Table 1 reports the seats for LN in rows 1-14, and those for MPA in rows 15-26.

Still referring to the Italian Elections of 2008, in Table 2 we report a comparative analysis among the different apportionments similar to the one already provided in [28]; here we add the apportionment obtained with the methods minimizing the norms L_1, L_2. The corresponding outputs are compared both between themselves and with the fair shares[1]. For each region-party pair (i, j) we have taken the value

[1]The small differences that the reader may notice between these numbers and the corresponding ones in [28] are due to slight differences between the data sets used in the two studies.

	PDL					LN – MPA					PD					IDV					UDC					r_i
P1	8.66	9	9	9	9	2.53	3	2	2	2	9.85	9	10	10	10	1.62	2	2	2	2	1.32	1	1	1	1	24
P2	8.62	9	9	9	9	4.33	4	4	4	4	6.84	7	7	7	7	0.95	1	1	1	1	1.24	1	1	1	1	22
L1	15.4	15	15	15	15	7.55	8	8	8	8	13.52	13	14	13	13	1.99	2	2	2	2	1.48	2	1	2	2	40
L2	14.1	14	14	14	14	13.75	14	14	14	14	11.20	11	11	11	11	1.75	2	2	2	2	2.14	2	2	2	2	43
L3	5.44	5	6	5	5	3.24	3	3	3	3	5.06	5	5	5	5	0.52	1	0	1	1	0.71	1	1	1	1	15
TA	2.30	2	2	2	2	1.13	1	1	1	1	2.73	3	3	3	3	0.37	0	0	0	0	0.45	1	1	1	1	7
V1	8.38	8	8	8	8	9.53	10	10	10	10	8.02	8	8	8	8	1.23	1	1	1	1	1.81	2	2	2	2	29
V2	5.92	6	6	6	6	5.93	6	6	6	6	6.03	6	6	6	6	1.06	1	1	1	1	1.04	1	1	1	1	20
FV	4.95	5	5	5	5	2.03	2	2	2	2	4.53	4	4	4	4	0.62	1	1	1	1	0.84	1	1	1	1	13
LI	6.85	7	7	7	7	1.39	1	1	1	1	7.11	7	7	7	7	0.93	1	1	1	1	0.69	1	1	1	1	17
ER	13.37	13	13	13	13	3.97	4	4	4	4	21.67	22	22	22	22	2.01	2	2	2	2	1.96	2	2	2	2	43
TO	13.49	14	14	14	14	0.95	1	1	1	1	20.28	20	20	20	20	1.52	1	1	1	1	1.74	2	2	2	2	38
UM	3.49	4	3	4	4	0.18	0	0	0	0	4.55	5	5	5	5	0.31	0	0	0	0	0.45	0	1	0	0	9
MA	6.22	6	6	6	6	0.42	0	1	1	1	7.46	8	7	7	7	0.81	1	1	1	1	1.06	1	1	1	1	16
La1	18.28	18	18	18	18	0.11	0	0	0	0	17.65	18	18	18	18	2.07	2	2	2	2	1.87	2	2	2	2	40
La2	8.24	8	8	8	8	0.06	0	0	0	0	5.19	5	5	5	5	0.50	1	1	1	1	0.99	1	1	1	1	15
AB	6.46	7	7	7	7	0.24	0	0	0	0	5.28	5	5	5	5	1.11	1	1	1	1	0.89	1	1	1	1	14
MO	1.16	1	1	1	1	0.16	0	0	0	0	0.58	1	1	1	1	0.90	1	1	1	1	0.18	0	0	0	0	3
C1	17.40	17	17	17	17	0.92	1	1	1	1	10.89	11	11	11	11	1.83	2	2	2	2	1.94	2	2	2	2	33
C2	15.59	16	16	16	16	0.62	1	0	1	1	9.02	9	9	9	9	1.42	1	2	1	1	2.33	2	2	2	2	29
PU	22.01	22	22	22	22	0.81	1	1	1	1	15.15	15	15	15	15	2.23	2	2	2	2	3.77	4	4	4	4	44
BA	2.46	3	3	3	3	0.04	0	0	0	0	2.62	3	3	3	3	0.40	0	0	0	0	0.45	0	0	0	0	6
CA	10.24	10	10	10	10	0.61	1	1	1	1	8.20	8	8	8	8	0.91	1	1	1	1	2.01	2	2	2	2	22
S1	12.88	13	13	13	13	1.49	1	2	1	1	7.35	8	7	8	8	1.11	1	1	1	1	3.15	3	3	3	3	26
S2	14.33	14	14	14	14	2.76	3	3	3	3	7.71	8	8	8	8	0.91	1	1	1	1	2.26	2	2	2	2	28
SA	8.55	9	9	9	9	0.12	0	0	0	0	7.39	7	7	7	7	0.81	1	1	1	1	1.10	1	1	1	1	18
c_i	255					57 – 8					226					30					38					614

Table 1: Seat allocations obtained with different models (Electoral data: Italy 2008). For each party, the first column refers to the Fair Shares; columns from the second to the fourth correspond to the solution provided by Tie and Transfer, MinMaxLexBest, MinL1, and MinL2 methods, respectively. Column 2 reports seats for LN party in rows 1-14 and seats for MPA party in rows 15-26.

$|a_{ij} - b_{ij}|$, with a the seat apportionment given by one method and b the one given by another method (a fractional apportionment for FS); we have considered both the maximum difference $\max_{ij} |a_{ij} - b_{ij}|$ (upper-right entries in bold face) and the average difference $\sum_{ij} |a_{ij} - b_{ij}|/|E|$ (lower-left entries in normal face), with $|E|$ the number of region-party pairs not in Z (or, equivalently, the number of arcs in the associated bipartite network).

From the analysis of the tables one can see that the seat allocations obtained by the best approximation methods do not differ too much from each other and they are also very close to the fair shares and the allocation provided by TT. In particular, we observe that for the 2008 data the MinL1 and the MinL2 models provide the same apportionment. We notice here that this is not a general result since it was not observed for all the other electoral data sets. It is also noteworthy that all the apportionments obtained via best approximation methods are closer to the ideal fair share matrix than the TT one according to the maximum and the average error measure shown in Table 2.

11 Balinski and Demange's axioms of proportionality

In this section we perform an evaluation of the best approximation methods within the theoretical framework for dealing with BAP provided by Balinski and Demange [5]. We recall that in this framework a seat assignment is considered "right" if it satisfies a specific set of axioms. In particular, Balinski and Demange [5] introduce six axioms for integral proportionality and show that they are always satisfied by the divisor based methods for BAP such as the TT algorithm.

A different and legitimate overall criterion is to consider "right" a seat assignment minimizing some

ITALY 2008	FS	MinMaxLexBest	MinL1	MinL2	TT
FS	—	**0.6221**	**0.6472**	**0.6472**	**0.8502**
MinMaxLexBest	0.2512	—	**1**	**1**	**1**
MinL1	0.2492	0.0763	—	**0**	**1**
MinL2	0.2492	0.0763	0	—	**1**
TT	0.2535	0.1069	0.0305	0.0305	—

Table 2: Maximum (bold) and average (normal) errors for each pair of methods.

measure of the deviations of the seats from given quotas. In principle there is no meta-criterion specifying which methodology should be chosen in designing a BAP method, whether the axiom satisfaction criterion or the norm minimization one. We believe that the choice between these different approaches is a matter of subjective preference of the legislator

It is therefore important to realize how much a seat assignment computed by minimizing the deviations fails to satisfy the axioms, and, conversely, how much, methods designed to satisfy the axioms fail to minimize the deviations.

We have already shown in the previous section (see in particular Table 2) the kind of error displayed by the TT algorithm (i.e., an algorithm satisfying the axioms) compared with methods designed to minimize the error, by using data from the Italian political elections 2008. Of course the analysis could be carried out with many other data sets, but we think that this comparison is sufficient to give an idea of the performance of the TT algorithm with respect to the goal of minimizing the error.

Now we discuss which axioms are satisfied by the best approximation methods presented in this paper. In the following we state each axiom and briefly discuss the results w.r.t. the best approximation methods for BAP. We notice that the axiom of Relevance is related to a more general BAP problem considered by Balinski and Demange in which row- and column- sums must lie in an interval of values; thus, it is not meaningful in our formulations of BAP in which row- and column- sums are fixed. The axioms of Exactness and of Homogeneity, stated w.r.t. the fair share quotas, can be easily analyzed.

Axiom of Exactness: *If the fair share matrix is integer in all components then it must be the unique apportionment.* ∎

It is easy to check that Exactness holds for all the best approximation models.

Axiom of Homogeneity: *If some row i or column j of V is scaled by a factor s, and the corresponding sum of the seats r_i or c_j remains unchanged, then the allocation X in that row or column must not change.* ∎

Since the fair share matrix is invariant for arbitrary row and column scaling by positive factors, the Homogeneity axiom is satisfied if the target quotas are the fair shares. For regional quotas homogeneity is limited to row scalings.

The remaining axioms of Uniformity, Monotonicity and Completeness are analyzed in detail in the following. Uniformity, also referred to as Consistency, requires that any part of an apportionment must be itself an apportionment. It was originally introduced in [6] through two statements, the second of which refers to the particular case of multiple apportionments. Here we focus only on the first statement which encompasses the main idea of the axiom.

Model	2008	2006	2001	1996	1994
MinMaxLexBest	0.9695	0.9779	0.9807	0.9806	0.9623
MinMax	0.9651	0.9774	0.9783	0.9749	0.9608
MinL1	0.3157	0.3861	0.3179	0.4351	0.4729
MinL2	0.3171	0.3822	0.3179	0.3591	0.4729

Table 3: Uniformity index computed over five data sets of Italian elections.

Axiom of Uniformity or Consistency: *Let X be a $m \times n$ apportionment for the BAP instance (V, r, c), I and J two subsets of M and N, respectively. The apportionment method that produced X is uniform if for any apportionment Y of $(V_{I\times J}, r_I, c_J)$ one has $Y = X_{I\times J}$.* ∎

There is no guarantee that Uniformity holds for best approximation methods. In fact, extracting rows and columns from the vote matrix produces new quotas and the best error w.r.t. to the new quotas can be obtained with a different seat assignment. Consider for instance the following counterexample where fair share quotas have been used. It easy to verify that for this example one gets the same and unique optimal assignment for all norms L_1, L_2 and L_∞. Similar counterexamples w.r.t. regional quotas could be also provided.

Example 3

$$V = \begin{pmatrix} 65 & 30 & 82 \\ 43 & 32 & 98 \end{pmatrix}, \qquad r = \begin{pmatrix} 15 \\ 15 \end{pmatrix}, \qquad c = \begin{pmatrix} 9 & 5 & 16 \end{pmatrix},$$

which gives the fair share matrix

$$Q = \begin{pmatrix} 5.38007 & 2.39821 & 7.22171 \\ 3.61993 & 2.60179 & 8.77829 \end{pmatrix}.$$

The optimal assignment for the L_1, L_2 and L_∞ norms is

$$X = \begin{pmatrix} 5 & 3 & 7 \\ 4 & 2 & 9 \end{pmatrix}.$$

If we restrict the problem to the second and third column, we have the new data

$$V' = \begin{pmatrix} 30 & 82 \\ 32 & 98 \end{pmatrix}, \qquad r' = \begin{pmatrix} 10 \\ 11 \end{pmatrix}, \qquad c' = \begin{pmatrix} 5 & 16 \end{pmatrix},$$

with fair share matrix

$$Q' = \begin{pmatrix} 2.4891 & 7.5109 \\ 2.5109 & 8.4891 \end{pmatrix},$$

and optimal assignment for the L_1, L_2 and L_∞ norms

$$X' = \begin{pmatrix} 2 & 8 \\ 3 & 8 \end{pmatrix}.$$

∎

We introduced a uniformity index in order to understand "how much" uniformity is satisfied by best approximation methods in real cases. We have used as data set the last five Italian elections of the Chamber of Deputies (1994, 1996, 2001, 2006, 2008) with the fair shares as target quotas. Clearly testing Uniformity on all possible sub-matrices is out of question. So we have decided to test the method only on all 2×2 sub-matrices, with the idea in mind that small sub-matrices are more prone to exhibit a non uniform behavior.

The uniformity index is computed as the percentage of cases in which a success was observed. A value equal to 1 for the uniformity index indicates a uniform behavior of the method (at least for that vote matrix). In Table 3 we report the values obtained for such an index.

For the MinL1 and MinL2 models the value of the uniformity index is quite poor, while MinMax and MinMaxLexBest methods show a much better performance. Although none of the best approximation methods guarantees uniformity, the MinMax approach provide uniformity index values very close to 1, so that they can be considered "quasi-uniform" methods.

This result should be no surprise. The error for L_1 and L_2 is computed by summing over all entries of the matrix and a discrepancy between quota and seats in one entry can be compensated by a discrepancy in another entry. In other words, there is a strong interaction among the matrix entries. By reducing the matrix, the interaction decreases and different solutions may be produced.

Axiom of Monotonicity: *Let X be a $m \times n$ apportionment for the BAP instance (V, r, c) and X' be an apportionment of (V', r, c), with $v'_{hk} > v_{hk}$ and $v'_{ij} = v_{ij}$ for $(i, j) \neq (h, k)$. The apportionment method that produced X and X' satisfies monotonicity if $x'_{hk} \geq x_{hk}$.*

We observed a different performance of the best approximation methods w.r.t. Monotonicity. Changing only one entry in the vote matrix affects all entries of the fair share matrix and it is not so evident that Monotonicity should hold in any case by minimizing the error. Indeed we have found the following counterexample showing a non monotonic behavior (it uses the same vote matrix of Example 2 in Section 6).

Example 4

$$V = \begin{pmatrix} 796 & 965 & 352 & 12 & 923 \\ 883 & 132 & 95 & 61 & 880 \\ 614 & 972 & 710 & 658 & 433 \end{pmatrix}, \qquad r = \begin{pmatrix} 29 \\ 20 \\ 16 \end{pmatrix}, \qquad c = \begin{pmatrix} 18 & 12 & 14 & 12 & 9 \end{pmatrix},$$

which gives the fair share matrix

$$Q = \begin{pmatrix} 7.62544 & 8.63784 & 7.88650 & 0.60945 & 4.24074 \\ 8.94641 & 1.24965 & 2.25114 & 3.27658 & 4.27622 \\ 1.42815 & 2.11251 & 3.86237 & 8.11398 & 0.48304 \end{pmatrix}.$$

The apportionment X minimizing $\|X - Q\|_\infty$ (and incidentally minimizing also $\|X - Q\|_1$, see Section 6) is

$$\begin{pmatrix} 8 & 9 & 8 & 0 & 4 \\ 9 & 1 & 2 & 4 & 4 \\ 1 & 2 & 4 & 8 & 1 \end{pmatrix},$$

with error $\tau = 0.72342$ achieved at the entry $(2, 4)$. Let V' be obtained from V by changing only v_{11}.

$$V' = \begin{pmatrix} 797 & 965 & 352 & 12 & 923 \\ 883 & 132 & 95 & 61 & 880 \\ 614 & 972 & 710 & 658 & 433 \end{pmatrix}.$$

The new fair shares are

$$Q' = \begin{pmatrix} 7.62890 & 8.63694 & 7.88521 & 0.60924 & 4.23968 \\ 8.94365 & 1.25013 & 2.25188 & 3.27707 & 4.27725 \\ 1.42744 & 2.11293 & 3.86292 & 8.11369 & 0.48306 \end{pmatrix}.$$

The apportionment X' minimizing $\|X' - Q'\|_\infty$ (but not $\|X - Q\|_1$) is

$$\begin{pmatrix} 7 & 9 & 8 & 1 & 4 \\ 9 & 1 & 2 & 3 & 5 \\ 2 & 2 & 4 & 8 & 0 \end{pmatrix},$$

with error $\tau' = 0.72275$ achieved at the entry $(2,5)$. Note that

$$\tau' < \|X - Q'\|_\infty = |x_{24} - q'_{24}| = 0.72293, \qquad \tau < \|X' - Q\|_\infty = |x'_{25} - q_{25}| = 0.72378.$$

For $v_{11} \leq 821$ one always obtains $x_{11} = 7$. Only for $v_{11} = 822$ one obtains again $x_{11} = 8$. ∎

We tested Monotonicity on the previous five data sets of Italian elections by systematically modifying the votes in any given cell of V, and checking the corresponding apportionment. The test has been carried out by considering as target both the fair shares and the regional quotas.

In all cases our experiments report a monotonic behavior. We may conclude that all best-approximation methods are 'quasi-monotonic'. In particular, we conjecture that Monotonicity does hold for the MinMax method w.r.t. regional quotas (whilst the previous Example 4 shows that this is not true w.r.t. fair share quotas).

Axiom of Completeness: *Let $(\hat V, \hat r, \hat c)$ be a given instance of BAP, and let (V^k, r^k, c^k) be a sequence of (real) matrices and vectors such that:*

- *for every k, $\bar X$ is an apportionment of (V^k, r^k, c^k)*
- *$(V^k, r^k, c^k) \to (\hat V, \hat r, \hat c)$, when $k \to \infty$.*

Completeness holds if $\bar X$ is also an apportionment of $(\hat V, \hat r, \hat c)$.

Let $Q(V, r, c)$ be the target quotas for (V, r, c). In particular let $Q^k := Q(V^k, r^k, c^k)$ and $\hat Q := Q(\hat V, \hat r, \hat c)$. Let $X(V, r, c)$ be the apportionment obtained from (V, r, c). Let

$$\tau_k = \|\bar X - Q^k\|, \qquad \hat\tau = \|\hat X - \hat Q\|.$$

We investigate under which conditions the Axiom holds. Since $X(V, r, c)$ is obtained by composing $Q(V, r, c)$ and $X(q, r, c)$, these conditions involve the quotas as well. Let Z^k be the set of pairs with zero votes for V^k and $\hat Z$ for $\hat V$.

Clearly continuity of the target quotas w.r.t. the vote matrix is an essential requirement in having the Completeness Axion satisfied. The regional quotas are obviously continuous. The fair share quotas are continuous under additional assumptions. It has been proved in [5] that continuity holds if there are no zero entries in the vote matrix. This assumption can be restricted to the invariance of the zero set for all possible vote matrices as proved in [28]. The non invariance of the zero set can lead to anomalous results as shown in the following two examples, where quotas and votes can be identified and therefore continuity of the quotas w.r.t. the votes is granted.

Example 5

Consider the following matrix of regional quotas (parties A–G and regions 1–5). In this example the L_∞ norm is used.

	A	B	C	D	E	F	G
1	0.992	0.870	0.170	0.994	0.988	0.986−ε	ε
2	0.460	0.580	0.991	0.993	0.989	0.987	0
3	0	0	0	0.990	0	0.010	0
4	0	0	0	0.441	0	0.559	0
5	0	0	0	0.140	0.860−ε	0	1+ε

with $r = (\,5 \quad 5 \quad 1 \quad 1 \quad 2\,)$, $c = (\,1 \quad 1 \quad 1 \quad 4 \quad 3 \quad 3 \quad 1\,)$ (invariant w.r.t. ε). For $0 < \varepsilon < 0.006$ the solution $\bar{X}$ is

	A	B	C	D	E	F	G
1	1	0	0	1	1	1	1
2	0	1	1	1	1	1	0
3	0	0	0	1	0	0	0
4	0	0	0	0	0	1	0
5	0	0	0	1	1	0	0

with L_∞-error $\tau_\varepsilon = 1 + \varepsilon$, for the pair (5,G). But for $\varepsilon = 0$ the solution $\hat{X}$ is

	A	B	C	D	E	F	G
1	1	0	0	2	1	1	0
2	0	1	1	1	1	1	0
3	0	0	0	1	0	0	0
4	0	0	0	0	0	1	0
5	0	0	0	0	1	0	1

with L_∞-error $\hat{\tau} = 1.006$ for the pair (1,D). Note that $\hat{X}$ is optimal also for $0.006 < \varepsilon < 0.86$. ∎

Example 6

Let Q be the following $(n+1) \times (n+1)$ matrix of fair share quotas

$$Q = \begin{pmatrix} \varepsilon & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} - \varepsilon \\ \frac{1}{n} & \frac{n-1}{n} & 0 & \cdots & 0 & 0 \\ \frac{1}{n} & 0 & \frac{n-1}{n} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n} & 0 & 0 & \cdots & \frac{n-1}{n} & 0 \\ \frac{1}{n} - \varepsilon & 0 & 0 & \cdots & 0 & \frac{n-1}{n} + \varepsilon \end{pmatrix}, \qquad r = (\,1 \quad 1 \quad \cdots \quad 1\,), \qquad c = (\,1 \quad 1 \quad \cdots \quad 1\,).$$

There are essentially three apportionments up to permutation of the indices $\{2, \ldots, n\}$ for the second one, namely

$$X^1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \multicolumn{6}{c}{\dotfill} \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \quad X^2 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \multicolumn{6}{c}{\dotfill} \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \quad X^3 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \multicolumn{6}{c}{\dotfill} \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

for which we have

$$\|X^1 - Q\|_1 = 4 - 4\,\varepsilon, \qquad \|X^2 - Q\|_1 = 6 - \frac{6}{n} - 2\,\varepsilon, \qquad \|X^3 - Q\|_1 = 6 - \frac{6}{n} + 4\,\varepsilon.$$

For $n \geq 3$ and $\varepsilon > 0$ the optimum is $\bar{X} = X^1$. But for $\varepsilon = 0$, X^1 is not feasible and the optima are X^2 and X^3. The same result holds also for the L_2-norm for which we have

$$\begin{aligned} \|X^1 - Q\|_2^2 &= (1-\varepsilon)^2 + 3\,(\varepsilon - \frac{1}{n})^2 + 3\,\frac{n-1}{n^2} \\ \|X^2 - Q\|_2^2 &= 3\,(1 - \frac{1}{n})^2 + \varepsilon^2 + 3\,(\varepsilon - \frac{1}{n})^2 + 3\,\frac{n-2}{n^2} \\ \|X^3 - Q\|_2^2 &= 3\,(1 + \varepsilon - \frac{1}{n})^2 + \varepsilon^2 + 3\,\frac{n-1}{n^2}, \end{aligned}$$

but not for the L_∞-norm for which X^2 is always the optimal apportionment (provided $\varepsilon < 1/n$). ∎

Apparently, the critical issue in both examples is the presence of quotas $q_{ij}^k \to 0$ whilst $x_{ij} = 1$. In this case the apportionment $\bar{X}$ is not feasible for the limit vote matrix and the Completeness Axiom cannot hold. Let us first state a useful simple lemma.

Lemma 1 *For sufficiently large k, $Z^k \subset \hat{Z}$ and every apportionment that is feasible for $\hat{V}$ is also feasible for V^k.* ∎

The next lemma shows that the Completeness Axiom holds for any type of norm if the quota matrix is continuous w.r.t. the vote matrix and the limiting seat matrix is feasible for the limiting vote matrix.

Lemma 2 *Assume that $\bar{X}$ is a feasible apportionment for $\hat{V}$, $\hat{X}$ is optimal for $\hat{V}$ w.r.t. a given norm, and $Q(V)$ is continuous. Then $\bar{X} = \hat{X}$ and $\tau^k \to \hat{\tau}$.*

Proof: By the continuity of $Q(V)$ and the continuity of the norm we have that

$$\lim_k \tau_k = \lim_k \|Q^k - \bar{X}\| = \|\hat{Q} - \bar{X}\| =: \tau'.$$

We claim that $\tau' = \hat{\tau}$. Note that $\tau' < \hat{\tau}$ cannot hold because $\bar{X}$ is assumed feasible for $\hat{V}$ and therefore $\hat{X}$ could not be optimal w.r.t. $\hat{Q}$ (in Counterexamples 5 and 6 it is indeed $\tau' < \hat{\tau}$ because the feasibility assumption for $\bar{X}$ is not satisfied). Hence let us assume $\hat{\tau} < \tau'$. Let $\varepsilon = \tau' - \hat{\tau}$. Then

$$\|\hat{Q} - \hat{X}\| = \|\hat{Q} - \bar{X}\| - \varepsilon = \tau' - \varepsilon.$$

There exists K such that $\|\hat{Q} - Q^k\| < \varepsilon/2$ and $|\tau^k - \tau'| < \varepsilon/2$ for $k > K$. For any $k > K$ we have

$$\|Q^k - \hat{X}\| = \|Q^k - \hat{Q} + \hat{Q} - \hat{X}\| \leq \|Q^k - \hat{Q}\| + \|\hat{Q} - \hat{X}\| < \frac{\varepsilon}{2} + \tau' - \varepsilon =$$

$$\frac{\varepsilon}{2} + \tau' - \tau^k + \tau^k - \varepsilon < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \tau^k - \varepsilon = \tau^k = \|Q^k - \bar{X}\|.$$

Since, by Lemma 1, $\hat{X}$ is feasible for V^k, the stated inequality contradicts optimality of $\bar{X}$ w.r.t. Q^k for $k > K$. Hence $\tau' = \hat{\tau}$ and $\bar{X}$ is optimal for $\hat{V}$. ∎

The critical assumption in Lemma 2 is the feasibility of $\bar{X}$ also for the limit vote matrix. This in turn is linked to the existence of quotas $q_{ij}^k \to 0$ with $\bar{x}_{ij} = 1$, i.e., to the possibility of having the set Z of zero votes varying w.r.t. the vote matrices. However, as pointed out in [28], it may be safely assumed that the presence of zeroes is due to structural reasons (a party does not present any list of candidates in a region) rather than to accidental ones (a party does not get any vote in a region in spite of its list in that region being nonempty). For these reasons we consider the Restricted Completeness Axiom as defined in [28], which makes sense in a practical context. For the sake of clarity we restate here the conditions of the Restricted Completeness Axiom.

Let $\mathbf{V}$ be the set of all real-valued nonnegative matrices whose smallest positive entry is at least 1 and whose set of null entries is a given subset $Z \subset M \times N$. Notice that the set $\mathbf{V}$ is closed. Then the Restricted Completeness Axiom considers only matrices V^k in $\mathbf{V}$.

Then the following Lemma is clearly true.

Lemma 3 *The apportionment $\bar{X}$ is feasible for $\hat{V}$ under the conditions of the Restricted Completeness Axiom.* ∎

The following lemma is Theorem 9 in [28].

Lemma 4 *The fair share quotas $Q(V)$ are continuous w.r.t. V under the conditions of the Restricted Completeness Axiom.* ∎

Hence, by putting together the previous lemmas we may state the following result.

Theorem 1 *If the target quotas are the fair shares, then the Restricted Axiom of Completeness holds for any type of norm.* ∎

Moreover, since the regional quotas are clearly continuous, we also have

Theorem 2 *If the target quotas are the regional quotas, then the Restricted Axiom of Completeness holds for any type of norm.*

For the L_∞-norm minimization and the fair share quotas we may relax the conditions of the Restricted Axiom of Completeness and ask only for continuity of the fair shares (see Example 6).

Theorem 3 *If the target quotas are the fair shares and are continuous and the error is measured by the L_∞-norm, then the Axiom of Completeness holds.*

Proof: By Lemma 2, it remains to prove that $\bar{X}$ is feasible also for $\hat{V}$. For $\hat{X}$, the optimal apportionment for $\hat{V}$, we have

$$\hat{q}_{ij} - \hat{\tau} \leq \hat{x}_{ij} \leq \hat{q}_{ij} + \hat{\tau},$$

with $\hat{\tau} < 1$ by the properties of fair shares. Let $\varepsilon > 0$ s.t. $\hat{\tau} + \varepsilon < 1$. Then there is K such that $|q_{ij}^k - \hat{q}_{ij}| < \varepsilon$ for all (i,j) and all $k > K$, so that

$$q_{ij}^k - (\hat{\tau} + \varepsilon) < \hat{q}_{ij} + \varepsilon - (\hat{\tau} + \varepsilon) \leq \hat{x}_{ij} \leq \hat{q}_{ij} - \varepsilon + (\hat{\tau} + \varepsilon) < q_{ij}^k + (\hat{\tau} + \varepsilon),$$

and $\hat{X}$ is an apportionment with error not worse than $\hat{\tau} + \varepsilon$ for V^k. By Lemma 1, $\hat{X}$ is feasible for V^k for sufficiently large k. If $\bar{X}$ is not feasible for $\hat{V}$ then there is a pair (i,j) with $q_{ij}^k \to 0$ and $\bar{x}_{ij} = 1$. This implies $\tau^k \to 1$ and $\bar{X}$ cannot be optimal for V^k. ∎

12 Conclusions

A consistent line of research has addressed the biproportional apportionment problem as an optimization problem with the goal of minimizing some measure of deviation from given quotas, i.e., rational numbers representing an ideal fractional apportionment. The focus of this research is not on the quota definition, but on the idea of rounding the quotas via mathematical programming techniques.

In this paper we have surveyed the most important results concerning this approach. The choice of the norm to be used in measuring the deviation from the quotas leads to different methods. The unifying feature of all methods stems from the underlying structure of the problem, which can be formulated as a transportation problem on a bipartite graph. This allows for polynomial algorithms in spite of the fact that the solution must be integer. Hence the algorithms can yield a solution very quickly. In addition, being mostly based on linear programming, they can be easily implemented and could represent viable methods to compute parliament seats.

These important features make the error minimization methods a valuable alternative to the traditional axiomatic approach proposed by Balinski and Demange in [6]. Actually, the two approaches are completely different in nature and they are characterized by different properties. For example, the Controlled Rounding method is able to guarantee the quota satisfaction, and this may be a key element in the decision process for the choice of the "best method" for a country. In addition, while, on the one hand, the Tie and Transfer is designed to satisfy a specific set of axioms, on the other hand, the error minimization methods follow the paradigm of simplicity, a widely accepted tenet in science[2].

In any case, models MinL1, MinL2, and MinMax must be addressed if one wishes to obtain the smallest possible error between the apportionment and the (ideal) fractional quotas. As discussed in Section 3, a critical point is the possible occurrence of multiple optima for MinL1 and for MinMax. MinL2 is not expected to have multiple optima. As clear from the Example 2 in Section 6 the possibility of multiple optima cannot be accepted at all for BAP.

This serious problem is overcome if the fair share quotas are used for MinL1 and if MinMax is refined by MinMaxLexBest which always provides a unique optimal solution. Actually, in the light of the results presented in this paper, from a theoretical viewpoint MinMaxLexBest could be considered the "best" among the error minimization methods. Several reasons support this conclusion: besides the uniqueness of the solution, it has the best theoretical complexity; it satisfies three out of the five axioms of Balinski and Demange that we analyzed (even if this depends on the specific choice of the target quotas); it also showed good values for our uniformity index and, according to our conjecture, when the regional quotas are considered the possibility that also Monotonicity is satisfied is still an open issue. An additional advantage of MinMaxLexBest is the existence of a simple certificate of optimality which can be used by the layman - both the legislator and the voter - to verify the actual quality of the apportionment.

Besides the theoretical evaluation of models and methods discussed in this paper, it must be always taken into account that in the political context of BAP decisions about the apportionment method to choose strictly depends on specific institutional requirements and considerations related to the history and political tradition of each country. According to the aim of the present paper, the new class of error minimization methods is intended as an alternative to the already existent methods for BAP. The idea is to provide the legislator a variety of methods for BAP, each inspired by its own principle, but all equally rigorous and mathematically correct. We believe that the final decision of choosing the method that best fits a country goes beyond the task of the mathematician.

[2]We recall here that simplicity is one of the main issues to design fair electoral systems, as stated in the "The Erice Decalogue", a document that collects the main conclusions of the International Workshop on Mathematics and Democracy: Voting Systems and Collective Choice, Erice, September, 18-23, 2005 [30].

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