Parametric maximum flow methods for minimax approximation of target quotas in biproportional apportionment

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Abstract

In this paper we study the biproportional apportionment problem, which deals with the assignment of seats to parties within regions. We consider the minimization of both the maximum absolute error and the maximum relative error of the apportioned seats with respect to target quotas. We show that this can be done polynomially through a reduction to a parametric maximum flow problem. Moreover, the maximum absolute error can be minimized in strongly polynomial time. More generally, our method can be used for computing ℓ_∞ projections onto a flow polytope. We also address the issue of uniqueness of the solution, proposing a method based on finding unordered lexicographic minima. Our procedure is compared to other well-known ones available in the literature. Finally we apply our procedures to the data of the 2008 Italian political elections, for which the procedure stated by the law produced an inconsistent assignment of seats.

Key words: electoral systems, biproportional apportionment, parametric maximum flow, min-max optimization, lexicographic optimization.

1 Introduction

In many proportional electoral systems where several parties are in competition and the territory is subdivided into constituencies or regions, it is usually not enough to compute the total number of seats that go to the different parties and to the different regions, but it is important to know how the seats are allocated to the parties within the regions. For example, the list of candidates of a certain party may differ from region to region, and thus the above seat allocation may significantly affect the "rostering" of the Parliament, that is, the very set of elected representatives.

The problem of seat allocation to the parties within the regions arises in several countries, e.g. Germany, Italy, Mexico, Switzerland, Belgium, Denmark, Iceland, Faroe Islands, etc. It may become of paramount importance, for obvious reasons, in the European Parliament elections, where the "regions" correspond to member countries of the EU.

A well-known quantitative formulation of the above seat allocation problem (see, e.g., Balinski [2]) is as follows. The law, before the election takes place, pre-determines the total number of seats to be apportioned to each region. In general, this number is roughly proportional to the region's population, in agreement with the well-known One Person-One Vote Principle. Then, on the basis of the vote outcome, each party is granted a certain number of seats, often - but not always - roughly proportional to the total number of votes obtained by the party. Here is the formulation.

Given:

(a) a matrix of votes obtained by each party in each region;
(b) the total number of seats apportioned to each region;
(c) the total number of seats granted to each party;

find a matrix of seats allocated to each party within in each region. Such matrix must satisfy the following conditions:

(i) (INTEGRALITY and NONNEGATIVITY) all of its entries must be nonnegative integers;
(ii) (REGION SUMS) the sum of the seats assigned to all parties within a given region must be equal to the number of seats actually at stake in the region;
(iii) (PARTY SUMS) the sum of the seats awarded to a given party in all regions must be equal to the total number of seats it was awarded.
(iv) (ZERO VOTE - ZERO SEAT) If a party receives no vote in some region, it cannot receive seats in that region;
(v) (PROPORTIONALITY) the matrix of seats should be "as proportional as possible" to the matrix of votes (precise alternative meanings will be given to this statement in Sec. 3)

A matrix satisfying conditions (i)–(iv) is called an *apportionment*; if the integrality condition is relaxed, it will be called a *fractional apportionment*. The problem defined by the above conditions (i)-(v) is called a *biproportional apportionment problem* (BAP). Different versions of it differ by the way condition (v) is formally stated.

The electoral law of some countries includes unsound and self-contradictory procedures for solving BAP. Balinski and Ramírez [7] discovered a bug in the Mexican electoral law. After a while, the law was changed. A similar bug turns out to occur in the current Italian law, as well as in the previous one, for the Chamber elections, cf. Pennisi [22], Pennisi et al. [23]. The Italian case deserves some discussion. For five consecutive political elections (1994, 1996, 2001, 2006, and 2008) the Italians have voted according to laws including incorrect procedures for BAP. As a result, condition (ii) was not met in 6 regions in 1996, in 2 regions in 2006, in 4 regions in 2008. This resulted in a violation of the very Constitutional Law, which dictates (art. 56) that the seats apportioned to the regions should be determined from their populations according to the so-called "Largest Remainders" or Hare proportional formula (cf., e.g., Grilli di Cortona et al. [15] p. 79). Particularly stunning was the case of the 2006 elections, where the region of Molise ended up with 2 seats in the Chamber of Deputies instead of the 3 it was entitled to, while the Trentino–Alto Adige region got 11 seats instead of 10. As a consequence, the Trentino–Alto Adige region got one seat per 85,400 inhabitants, while the Molise one got one seat per 160,300 inhabitants. So, the appropriate motto there would have been: One Person - Half Vote !

As mentioned above, these anomalies are caused by an incorrect procedure for BAP. Such procedure proceeds, in a first stage, region by region. In each region one computes the (generally fractional) exact quotas of seats to be granted to the individual parties in the region under the assumption of perfect proportionality between votes and seats[1]. These quotas are then rounded either down or up according to

[1]Actually this statement is not quite correct because of the majority bonus embodied in the law. However, if the matrix of votes is preliminarily scaled so as to suitably inflate the majority votes and deflate the minority ones, then the statement becomes correct

the Largest Remainders formula. At the end of the first stage the PARTY SUMS constraints (iii) might not be met. The lawmaker is aware of this possibility and thus at this point a second stage starts, whereby seats are transferred one by one from parties with a surplus of seats to ones with a deficit of seats; however, this rebalancing procedure does not guarantee that eventually (ii) or (iii) are met. The underlying (wrong) assumption is that one can always obtain an apportionment by rounding either down or up, in a suitable way, the above region quotas. Realistic counterexamples show that this is not always possible.

The point is that a BAP is a non trivial mathematical problem, and simplistic solution approaches fall short of reaching their target. On the other hand, mathematically sound procedures do exist. In two seminal papers, Balinski and Demange [3, 4] characterized proportionality between real or integral matrices in terms of certain reasonable axioms; furthermore, they showed that, for any given rounding rule, there is a unique apportionment satisfying the stated proportionality axioms, and that such an apportionment can be obtained by "Scale and Round"; finally, they described a "Tie and Transfer" (TT) algorithm (a sort of out-of-kilter one) for actually computing the scaling factors and the required apportionment. The most common rounding rules are: round to the lower, to the upper, or to the closer integer. Accordingly, in the single-region case one re-obtains the well-known divisor methods of Jefferson (or D'Hondt), Adams, and Webster (or Sainte Laguë), respectively. In general, Balinski and Demange's Scale and Round procedure may be regarded as a two-dimensional extension of those vector apportionment methods known as Divisor ones (cf. Grilli di Cortona et al. [15]). Much later, Zachariasen [34] pointed out that in general the complexity of TT is pseudopolynomial and described an enhanced version of the algorithm that runs provably fast (technically, in weakly polynomial time).

Subsequently, Pukelsheim and his collaborators at the University of Augsburg on the one hand proposed conceptually simpler Discrete Alternating Scaling (DAS) algorithms, (although some finiteness and complexity issues are still unsettled); on the other hand they developed BAZI, a public domain software, written in JAVA, which includes also some sound procedures for BAP (Pukelsheim [25], Maier [19]). This software was successfully used in the 2006 elections of the Zurich Canton (Pukelsheim [26]) and more recently (2009) in those of the Aargau and the Schaffhausen Cantons.

Several authors have advocated the use of sound biproportional apportionment algorithms to distribute seats to parties within districts in countries such as Mexico (Balinski and Ramírez [7]), France (Balinski [2]), Switzerland (Pukelsheim and Schuhmacher [27], Pukelsheim [26], Balinski and Pukelsheim [5, 6]), Italy (Pennisi et al. [23]), Spain (Ramírez et al. [29]), Faroe Islands (Zachariasen [34]). Balinski ([2] p. 215) discusses also the application of such algorithms to the election of French representatives in the European Parliament.

Since Balinski and Demange's algorithm is provably sound, why not to transplant it into the Italian legislation? This idea looks very appealing, but in practice there is one basic obstacle: the above algorithm is inherently related to Divisor Methods, whereas throughout the history of the Italian Republic (1946 - to date) the method traditionally adopted for the Chamber elections has been the Largest Remainders one.

In the present paper, we propose an alternative optimization approach having the following features:

- It always correctly outputs an apportionment, if one exists;
- It is provably fast (that is, it runs in polynomial and even in strongly polynomial time);
- It is adherent to the Italian electoral traditions;
- Although it has been specifically designed for the Italian system, it is very flexible and it may be adapted to a variety of other contexts;
- Its most basic implementation is very simple and it requires only a max flow routine.
- Although it shares with the other available algorithms for BAP a certain degree of mathematical sophistication, for any given instance of BAP one can attach to its output a "certificate of optimality"

that allows anybody to easily check feasibility and optimality.

The last item is discussed at length in [32]. Here we just sketch the basic idea. As remarked above, BAP is a non trivial mathematical problem, and all the currently available algorithms for its solution exhibit some mathematically advanced features. Can they be translated into an actual law? Citizens demand simple, easy to understand, voting systems. One possible way out of this dilemma is: leave to a mathematically sophisticated algorithm the task of *producing* an optimal apportionment, but attach to it a "certificate of optimality", that is, describe a simple procedure whereby even a layman can *check*, through certain elementary operations and without any knowledge of specialized mathematical theorems, that the seat allocation output by the algorithm indeed satisfies all the requirements for an apportionment, and that it is "as proportional as possible" to the vote matrix.

The general idea of the algorithm is the following. As seen above, the current institutional procedure first computes party quotas row-wise. Rounding them arbitrarily down or up may fail to produce an apportionment. By the way, for the same reason the "controlled rounding" procedure of Cox and Ernst [10] starting from the above quotas might not work, as already noticed by Gassner [14] with reference to the Belgian system. Following the customary approach of "enlarging the solution space when a problem has no solution", we shall look for an apportionment that is "as close as possible" – in a well-defined sense – to the above matrix of quotas. Our approach bears some resemblance with Cox and Ernst's one, but it makes use of the ℓ_∞ norm rather than the ℓ_1 norm. Unlike the other known methods, ours does not rely on rounding. The techniques we use in order to handle the above least distance problem rely on the solution of a sequence of maximum network flow ones. These are computationally easier to solve than the minimum cost network flow problems to which divisor-based biproportional apportionment ones have been reduced in several papers (M. Zachariasen [34], Rote and M. Zachariasen [30], Gaffke and Pukelsheim [12, 13]). It is worth pointing out here that also the above quoted "controlled rounding" method, as well as the more general minimization of the ℓ_1 error over all the apportionments, require the solution of minimum cost flow problems [10, 16, 20, 21].

The paper is organized as follows. In Sec. 2 we set up the notation and state precise mathematical formulations of the biproportional apportionment problem. In Sec. 3 we define suitable error measures which may be used in the formulation of the "best approximation" problem. Then in Sec. 4 we describe parametric maximum flow models to find the minimum error. In Sec. 5 we provide some error bounds which are also useful in designing algorithms. We present some algorithms in Sec. 6. These algorithms feature increasingly complex descriptions, but at the same time increasing computational complexity performances. We show that in general the overall procedure runs in strongly polynomial time based on the solution of a continuous parametric maximum flow problem. In Sec. 7 we show how to deal with the crucial issue of the uniqueness of the optimal apportionment. A comparison with other well-known BAP methods is pursued in Sec. 8. Finally, in the Appendix we apply our procedures to the data of the 2008 Italian political elections.

2 Biproportional apportionments

The data are: a set M of electoral regions, integer numbers r_i (seats assigned to region i), with $H := \sum_i r_i$ (house size), a set N of parties, integer numbers v_{ij} (votes to party j in region i), with $v_{iN} := \sum_{j\in N} v_{ij}$ (votes in region i), $v_{Mj} := \sum_{i\in M} v_{ij}$ (national votes to party j) and $v_{MN} := \sum_{ij} v_{ij}$ (total votes). Let V be the matrix $[v_{ij}]$. Let $Z := \{(i,j) : v_{ij} = 0,\ i \in M,\ j \in N\}$.

We assume that the numbers r_i are fixed before elections (as customary). We also assume that we are given positive integer numbers p_j, such that $\sum_{j\in N} p_j = H$, which have been computed from the vector

$v_M = (v_{M1}, \ldots, v_{Mn})$ according to a rule $p := \pi(v_M, H)$ ($p_j = \pi_j(v_M, H)$). These are the seats assigned in the house to each party. In most systems the rule π producing the seats p_j is of proportional type, i.e., the numbers p_j are proportional to the numbers v_{Mj} as much as possible. This is not the case if, as it happens in the current Italian electoral system, the majority coalition (defined on the basis of the votes v_{ij}) is granted at least a certain number of seats. We briefly comment on this at the end of this section.

One well-known rule $\pi(v_M, H)$ is the so called *Largest Remainders* one, which computes the seats as follows:

(i) initially, one computes the *national quotas* q_j as

$$q_j := \frac{v_{Mj}}{v_{MN}}\, H;$$

(ii) then one sets $p_j := \lfloor q_j \rfloor$, $j = 1, \ldots, n$;

(iii) to make the sum of the components of p equal to H, one increases by one unit the components of p whose fractional parts (or "remainders") $q_j - \lfloor q_j \rfloor$ are largest.

The resulting vector p satisfies the *Hare Property*

$$\lfloor q_j \rfloor \le p_j \le \lceil q_j \rceil, \qquad j = 1, \ldots, n.$$

The BAP consists in computing from the above data nonnegative integer numbers x_{ij} such that

$$x_{Mj} := \sum_{i \in M} x_{ij} = p_j, \quad j \in N, \qquad x_{iN} := \sum_{j \in N} x_{ij} = r_i, \quad i \in M, \qquad x_{ij} = 0, \quad (i,j) \in Z. \tag{1}$$

In this paper we assume the existence of *target quotas* q_{ij} to which the numbers x_{ij} should be as close as possible. The target quotas must satisfy the following requirements $\sum_{ij} q_{ij} = H$ and $q_{ij} = 0$ if $(ij) \in Z$.

In actual electoral systems and in the literature two types of quotas have been mainly considered: the *regional quotas* and the *fair shares*. The underlying idea behind regional quotas (used in the Italian system) is to enforce as long as possible proportionality, region by region, of the vectors x_{iN} to the vectors v_{iN}. To this purpose the regional quotas are defined as the rational numbers:

$$q'_{ij} := \frac{v_{ij}}{v_{iN}}\, r_i, \qquad i \in M,\ j \in N. \tag{2}$$

By definition, $\sum_j q'_{ij} = r_i$ and so $\sum_{ij} q'_{ij} = \sum_i r_i = \sum_j p_j = H$. Also $q'_{ij} = 0$ if $(i,j) \in Z$.

The fair shares are defined as the rational numbers

$$q''_{ij} := \lambda_i\, v_{ij}\, \mu_j, \qquad i \in M,\ j \in N,$$

where λ_i are positive row multipliers and μ_j are positive column multipliers such that $\sum_i \lambda_i\, v_{ij}\, \mu_j = p_j$ and $\sum_j \lambda_i\, v_{ij}\, \mu_j = r_i$. Clearly $\sum_{ij} q''_{ij} = H$ and $q''_{ij} = 0$ if $(i,j) \in Z$. The multipliers are easily found by alternately scaling rows and columns to meet the row-sum and the column-sum requirements. This iterative procedure, known under the name of RAS or Iterative Proportional Fitting (IPF), quickly converges to the fair shares. In fact, Kalantari et al. [17] analyzed the complexity of RAS, giving a pseudopolynomial upper bound on the number of iterations needed to achieve any desired precision, see Sec. 8 for details. Actually, one can compute the fair shares in weakly polynomial time by convex network flow techniques [18], exploiting the fact that they minimize a certain convex separable entropy function over all the fractional apportionments.

Thus we may choose as target quotas either the regional shares or the fair shares, but possibly also other numbers, provided that the two stated requirements are satisfied. The way the target quotas are defined does not affect our subsequent results, unless explicitly stated.

A peculiarity of the present Italian legislation consists in a majority bonus to be granted to the majority coalition. In order to take this fact into account, the votes of the majority may be inflated, and those of the minority deflated, by preliminarily multiplying them by a suitable majority and minority factor, respectively (although the actual law follows a more tortuous route).

3 Error measures

In the present section we indicate how to measure the errors w.r.t. the target quotas q_{ij} in assigning the actual seats. We consider two kinds of error measures. The first one is the *absolute error*, defined by

$$\tau_{ij} := |x_{ij} - q_{ij}|.$$

A measure like τ_{ij} considers an error both the quantity $(q_{ij} - x_{ij})$, if $q_{ij} > x_{ij}$, which corresponds to an assignment perceived as unfair by party j, and the quantity $(x_{ij} - q_{ij})$, if $x_{ij} > q_{ij}$, which corresponds to an assignment perceived as unfair by the other parties.

The second measure is the relative error, defined by

$$\sigma_{ij} := \frac{|x_{ij} - q_{ij}|}{q_{ij}}.$$

In this paper we mostly deal with the minimization of the maximum error. Considering that an assignment affects many individuals who should be equally treated and therefore a large error for a pair (i, j) cannot be compensated by many small errors for other pairs, we think that minimizing the maximum error is a proper way to assign seats. Therefore, if we consider the absolute error, we want to minimize

$$\tau := \max_{ij} |x_{ij} - q_{ij}|. \tag{3}$$

subject to (1). Let τ^* be the minimum maximum absolute error. If we consider the relative error, we want to minimize

$$\sigma := \max_{(ij)\notin Z} \frac{|x_{ij} - q_{ij}|}{q_{ij}}, \tag{4}$$

subject to (1). Let σ^* be the minimum maximum relative error. Note that σ can be expressed as

$$\sigma := \max\left\{\sigma^+ - 1\,;\, 1 - \sigma^-\right\}, \qquad \sigma^+ := \max_{(ij)\notin Z} \frac{x_{ij}}{q_{ij}}, \qquad \sigma^- := \min_{(ij)\notin Z} \frac{x_{ij}}{q_{ij}}. \tag{5}$$

The two error measures σ^+ and σ^- take care of assignments that give more and less seats, respectively, than expected. We note that $\sigma \geq 1$ for any assignment such that $x_{ij} = 0$ for at least one pair (i, j). However, if $0 < q_{ij} < 1$, $x_{ij} = 0$ is an acceptable assignment that does not deserve such a large penalty measure. Therefore it is convenient to redefine σ^- in (5) as

$$\sigma^- := \min_{(ij): q_{ij} \geq 1} \frac{x_{ij}}{q_{ij}}.$$

Divisor rules which assign seats so as to minimize the maximum ratio q_{ij}/x_{ij} correspond to maximizing σ^-. In the literature of electoral systems, other min-max optimization problems have been investigated, see e.g. [9], Chap. 5.

Let us also remark that, while our methods minimize ℓ_∞ norms (3) over *all* the apportionments, Controlled Rounding ones might not do the same with respect to ℓ_1 norms. The latter methods [10] are known to minimize absolute errors w.r.t the ℓ_1 norm under the assumption that the optimum seat assignments are obtained by rounding either up or down the quotas. However, this assumption may not be satisfied in general. Consider the following example with q an $(n+2)\times(n+2)$ matrix of fair share quotas

$$q = \begin{pmatrix} \frac{n-1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{n-1}{n} & 0 & \cdots & 0 \\ \frac{1}{n} & 0 & \frac{n-1}{n} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n} & 0 & 0 & \cdots & \frac{n-1}{n} \end{pmatrix}, \qquad r = (\,2 \quad 1 \quad 1 \quad \cdots \quad 1\,), \qquad p = (\,2 \quad 1 \quad 1 \quad \cdots \quad 1\,)$$

The best ℓ_1 apportionment is given by $x = \operatorname{diag}\{2,1,1,\ldots,1\}$ with error $e_1(n) = 4+4/n$, with x_{11} outside the range $\{0,1\}$. If we restrict the apportionments to $\{0,1\}$, then the best ℓ_1 apportionment is given by $x_{11} = x_{12} = x_{21} = 1$, $x_{ii} = 1$ for $i \geq 3$, otherwise $x_{ij} = 0$. Its error is $e_2(n) = 6-2/n$. Hence $e_2(n) > e_1(n)$ for $n > 4$. Incidentally, the first apportionment is optimal also w.r.t the ℓ_2 norm.

4 Minimizing the maximum error

Given a bound $\tau > 0$ on the absolute error or a bound $\sigma > 0$ on the relative error, the problem of finding an apportionment within the stated bound can be modeled as a feasible flow problem on a suitable bipartite network $(M, N; E)$: there is a set of source nodes M (regions), a set of sink nodes N (parties). The arcs E correspond to the pairs $(i,j) \notin Z$. From each source i there is an outgoing flow equal to r_i and into each sink j there is an incoming flow equal to p_j.

In case we measure the absolute error, each arc (i,j) has a capacity interval

$$[c^-_{ij}\ ,\ c^+_{ij}] := [\lceil q_{ij} - \tau \rceil^+\ ,\ \lfloor q_{ij} + \tau \rfloor], \tag{6}$$

where, by definition $a^+ := \max\{a, 0\}$. In case we measure the relative error, each arc (i,j) has a capacity interval

$$[c^-_{ij}\ ,\ c^+_{ij}] := \begin{cases} [\lceil (1-\sigma)\, q_{ij} \rceil^+ , \lfloor (1+\sigma)\, q_{ij} \rfloor] & \text{if } q_{ij} \geq 1 \\ [0, \lfloor (1+\sigma)\, q_{ij} \rfloor] & \text{if } q_{ij} < 1. \end{cases} \tag{7}$$

A feasible flow x_{ij} satisfies $c^-_{ij} \leq x_{ij} \leq c^+_{ij}$ and, by the well-known Integrality Theorem for Flows [1], if there is a feasible flow x there is also an integral flow since the capacity values are integers. Hence τ^* and σ^* are the minimum bounds, in the respective cases, such that a feasible flow exists.

The existence of a feasible flow can be established through the solution of the following max-flow problem [1]: a source node s and a sink node t are added to the bipartite graph; each arc (i,j) is given the capacity interval $[0, c^+_{ij} - c^-_{ij}]$; then, for each i, if $b_i := \sum_j c^-_{ij} < r_i$ the arc (s,i) is added with capacity interval $[0, r_i - b_i]$ (if $b_i > r_i$ there is no feasible flow) and, for each j, if $d_j := \sum_i c^-_{ij} < p_j$ the arc (j,t) is added with capacity interval $[0, p_j - d_j]$ (if $d_j > p_j$ there is no feasible flow). Let $\hat{x}$ be the maximum flow on this network. There exists a feasible flow if and only if all arcs (s,i) are saturated. If they are saturated the flow on each arc (i,j) is $\hat{x}_{ij} + c^-_{ij}$ (indeed $\sum_j \hat{x}_{ij} + c^-_{ij} = r_i - b_i + b_i = r_i$ and similarly for each node j).

Note that, due to the integrality of the x_{ij}'s, the error of the seat assignment x_{ij} may assume values only in a finite set. In more detail, we call *relevant* (for the absolute error minimization) those values for τ such that either $q_{ij} - \tau$ or $q_{ij} + \tau$ is an integer in the interval $[0, r_i]$ for some $(i,j) \notin Z$. Note that, for any two consecutive relevant $\tau_1 < \tau_2$, the capacity values (6) are unchanged for any τ such that $\tau_1 \le \tau < \tau_2$ and this explains why we may simply consider relevant values.

Similarly, we call *relevant* (for the relative error minimization) those values for σ such that either $(1-\sigma)\,q_{ij}$ or $(1+\sigma)\,q_{ij}$ is an integer in the interval $[0, r_i]$ for some $(i,j) \notin Z$. Similarly, for any two consecutive relevant $\sigma_1 < \sigma_2$, the capacity values (7) are unchanged for any σ such that $\sigma_1 \le \sigma < \sigma_2$.

By definition, the relevant values are at most $r_i + 1$ for each pair (i,j), both for τ and σ. By summing over all regions for a fixed j we get at most $H + m$ relevant values and so the total number of relevant errors is at most $n\,(H+m)$.

There are several strategies to find the optimum error among the relevant values. For the minimization of both the absolute and the relative error there are natural MILP formulations [23], namely

$$
\begin{array}{lll}
\min \ \tau & & \\
& \displaystyle\sum_i x_{ij} = p_j & j \in N \\
& \displaystyle\sum_j x_{ij} = r_i & i \in M \\
& q_{ij} - \tau \le x_{ij} \le q_{ij} + \tau & (i,j) \notin Z \\
& x_{ij} \ge 0 \text{ and integral} & (i,j) \notin Z \\
& x_{ij} = 0 & (i,j) \in Z
\end{array}
$$

for the absolute error minimization and

$$
\begin{array}{lll}
\min \ \sigma & & \\
& \displaystyle\sum_i x_{ij} = p_j & j \in N \\
& \displaystyle\sum_j x_{ij} = r_i & i \in M \\
& q_{ij} - \sigma\, q_{ij} \le x_{ij} \le q_{ij} + \sigma\, q_{ij} & (i,j) \notin Z, \ q_{ij} \ge 1 \\
& 0 \le x_{ij} \le q_{ij} + \sigma\, q_{ij} & (i,j) \notin Z, \ q_{ij} < 1 \\
& x_{ij} \ge 0 \text{ and integral} & (i,j) \notin Z \\
& x_{ij} = 0 & (i,j) \in Z
\end{array}
$$

for the relative error minimization.

However, the special structure of the problems allows for their solution in polynomial time via parametric network flow techniques. We postpone the presentation of the algorithms to Sec. 6 after the definitions of some useful error bounds.

5 Error bounds

Before stating the bounds, we recall that the *fractional* BAP consists in finding nonnegative rational numbers $\bar{x}_{ij}$ such that (1) is satisfied and either τ or σ is minimized. Let $\bar{\tau}$ and $\bar{\sigma}$ be the respective minima. We also consider, only for the absolute error, the value t^*, defined as the minimum nonnegative *integer* such that there is an apportionment $\hat{x}$ with

$$(\lfloor q_{ij} \rfloor - t^*)^+ \le \hat{x}_{ij} \le \lceil q_{ij} \rceil + t^*.$$

This corresponds to rounding down and up the quotas and then possibly extending the interval in both directions by the least integer such that an apportionment exists. If the target shares are the fair shares, then $\bar{\tau} = 0$, $\bar{\sigma} = 0$ and $t^* = 0$.

– **Rounding bound**. Note that for each pair (i, j) the error cannot be less than

$$\min \{ \lceil q_{ij} \rceil - q_{ij} \, , \, q_{ij} - \lfloor q_{ij} \rfloor \} \, .$$

Therefore, since we consider the maximum error, we must have

$$\tau^* \geq \tau_{min} := \max_{ij} \min \{ \lceil q_{ij} \rceil - q_{ij} \, , \, q_{ij} - \lfloor q_{ij} \rfloor \} \, .$$

If $\tau < \tau_{min}$ at least one capacity interval is empty. Similarly,

$$\sigma^* \geq \sigma_{min} := \max_{ij : q_{ij} \geq 1} \min \left\{ \frac{q_{ij} - \lfloor q_{ij} \rfloor}{q_{ij}} \; ; \; \frac{\lceil q_{ij} \rceil - q_{ij}}{q_{ij}} \right\} .$$

– **Proximity bound**. Let $\bar{x}_{ij}$ be the best (fractional) apportionment of the fractional BAP. So, in case we look for the minimum absolute error,

$$(q_{ij} - \bar{\tau})^+ \leq \bar{x}_{ij} \leq q_{ij} + \bar{\tau}$$

and the fractional BAP with bounds reset as

$$\lfloor q_{ij} - \bar{\tau} \rfloor^+ \leq x_{ij} \leq \lceil q_{ij} + \bar{\tau} \rceil \tag{8}$$

is feasible. By integrality of the capacity bounds there exists an integral apportionment satisfying (8). Therefore,

$$q_{ij} + \tau^* \leq \lceil q_{ij} + \bar{\tau} \rceil \leq q_{ij} + \bar{\tau} + 1, \qquad (q_{ij} - \bar{\tau} - 1)^+ \leq \lfloor q_{ij} - \bar{\tau} \rfloor^+ \leq q_{ij} - \tau^*,$$

from which

$$\bar{\tau} \leq \tau^* \leq \bar{\tau} + 1.$$

– **Fractional bound**. Let $t_j := \sum_i q_{ij}$ and $\Delta_j := p_j - t_j$. Note that $\sum_j \Delta_j = 0$. From

$$-\tau \leq x_{ij} - q_{ij} \leq \tau$$

for any fractional apportionment x with error τ, we get, by summing over i for each index j,

$$-\tau \, m \leq p_j - t_j \leq \tau \, m \quad \Longrightarrow \quad \tau \geq \frac{|\Delta_j|}{m},$$

which implies

$$\bar{\tau} \geq \frac{|\Delta_j|}{m}.$$

The inequality turns into an equality if all quotas are positive and not "too small" for parties with $\Delta_j < 0$. Indeed, if

$$q_{ij} > 0, \qquad m \, q_{ij} + \Delta_j \geq 0, \qquad \sum_j q_{ij} = r_i \tag{9}$$

(the last assumption is satisfied for regional quotas), then the apportionment

$$x_{ij} := q_{ij} + \frac{\Delta_j}{m}$$

is feasible and has error $\tau = \max_j |\Delta_j|/m$. Therefore it is optimal and, in this case,

$$\bar{\tau} = \frac{|\Delta_j|}{m}.$$

Therefore, if (9) is satisfied, we have, by using the proximity and the rounding bound

$$\max\left\{\tau_{min}\,;\,\max_j \frac{|\Delta_j|}{m}\right\} \leq \tau^* \leq \max_j \frac{|\Delta_j|}{m} + 1.$$

– **Floor bound**. We now relate the optimal values t^* and τ^* as shown in the following theorem.

Theorem 7. $t^* = \lfloor \tau^* \rfloor$.

Proof. Let x^* be any optimal apportionment, i.e., the maximum absolute error of x^* is equal to τ^*. For any (i,j), one has

$$\lceil q_{ij} - \tau^* \rceil^+ \leq x^*_{ij} \leq \lfloor q_{ij} + \tau^* \rfloor\,, \tag{10}$$

whence

$$(\lfloor q_{ij} \rfloor - \lfloor \tau^* \rfloor)^+ \leq \lceil q_{ij} - \tau^* \rceil^+ \leq x^*_{ij} \leq \lfloor q_{ij} + \tau^* \rfloor \leq \lceil q_{ij} \rceil + \lfloor \tau^* \rfloor\,,$$

so that

$$t^* \leq \lfloor \tau^* \rfloor \leq \tau^*. \tag{11}$$

On the other hand, if $\hat{x}$ is an apportionment such that

$$\lfloor q_{ij} \rfloor - t^* \leq \hat{x}_{ij} \leq \lceil q_{ij} \rceil + t^*, \tag{12}$$

then one has

$$q_{ij} - t^* - 1 < \lfloor q_{ij} \rfloor - t^* \leq \hat{x}_{ij} \leq \lceil q_{ij} \rceil) + t^* < q_{ij} + t^* + 1. \tag{13}$$

Hence

$$\tau^* < t^* + 1. \tag{14}$$

Inequalities (11) and (14) imply the thesis. ∎

Hence the theorem states the following bounds

$$t^* \leq \tau^* < t^* + 1$$

and implies the following corollary, where $< q_{ij} >$ is the fractional part of q_{ij}.

Corollary 1. *τ^* is among the relevant errors of the form either $t^* + < q_{ij} >$ or $t^* + 1 - < q_{ij} >$.* ∎

Corollary 2. *If q_{ij} are the fair shares then τ^* must be smaller than 1 and it must be among the relevant errors of the form either $< q_{ij} >$ or $1 - < q_{ij} >$.* ∎

– **Absolute bound**. We have shown that if the q_{ij} are the fair shares then there exists an absolute bound equal to 1 on τ^*. Now we show that, under other assumptions, there exists an absolute bound on τ^* and that this bound is between 1 and 2. Let p^N be the vector of seats assigned by the Largest Remainders Rule starting from the vector v_M of national votes, and let p^R be the vector of seats assigned by the Largest Remainders Rule starting from the vector q_M of column-sums q_{Mj} of the regional quotas.

We introduce now the following property which may hold between target quotas and seats p_j. It is like the Hare property, which however is relative to national quotas.

Definition 1. *The* Rounding property *is satisfied if, for quotas q_{ij}, the seats p_j are such that*

$$p_j \in \Big\{ \Big\lfloor \sum_i q_{ij} \Big\rfloor , \Big\lceil \sum_i q_{ij} \Big\rceil \Big\}$$

∎

Note that the rounding property is equivalent to assume $|p_j - \sum_i q_{ij}| < 1$. Clearly the rounding property implies $\sum_i \lfloor q_{ij} \rfloor \le p_j \le \sum_i \lceil q_{ij} \rceil$.

Theorem 8. *If the Rounding property holds and the quotas q_{ij} are positive and satisfy $\sum_j q_{ij} = r_i$ for each i, then $\tau^* \le 2$.*

Proof. An apportionment has maximum absolute error $\tau^* \le 2$ if and only if there exists a feasible flow in the capacitated network $(N, M; N \times M)$ with N set of source nodes with outgoing flow p_j, M set of sink nodes with incoming flow r_i and capacity intervals $[(\lfloor q_{ij} \rfloor - 1)^+, \lceil q_{ij} \rceil + 1]$ for each (i, j). By Gale's theorem ([11] and [1] p. 196) there is a feasible flow if, for each $M^* \subset M$ and $N^* \subset N$, we have

$$\sum_{j \in N^*} p_j - \sum_{i \in M^*} r_i \le \sum_{i \notin M^*} \sum_{j \in N^*} (\lceil q_{ij} \rceil + 1) - \sum_{i \in M^*} \sum_{j \notin N^*} (\lfloor q_{ij} \rfloor - 1)^+ \tag{15}$$

Let $\alpha_j := p_j - \sum_i q_{ij}$. By hypothesis $|\alpha_j| < 1$ and $\sum_j \alpha_j = 0$. If $M^* = M$, (15) is

$$\sum_{j \in N^*} p_j - \sum_{i \in M} r_i = - \sum_{j \notin N^*} p_j \le - \sum_{i \in M^*} \sum_{j \notin N^*} (\lfloor q_{ij} \rfloor - 1)^+$$

i.e.

$$\sum_{j \notin N^*} (p_j - \sum_{i \in M^*} (\lfloor q_{ij} \rfloor - 1)^+) \ge 0$$

implied by the Rounding property. If $M^* \ne M$, we derive

$$\sum_{j \in N^*} p_j - \sum_{i \in M^*} r_i = \sum_{j \in N^*} (\alpha_j + \sum_{i \notin M^*} q_{ij}) - \sum_{i \in M^*} \sum_{j \notin N^*} q_{ij} \le$$

$$\sum_{j \in N^*} (1 + \sum_{i \notin M^*} q_{ij}) - \sum_{i \in M^*} \sum_{j \notin N^*} q_{ij} \le \sum_{j \in N^*} \sum_{i \notin M^*} (q_{ij} + 1) - \sum_{i \in M^*} \sum_{j \notin N^*} q_{ij} \le$$

$$\sum_{i \notin M^*} \sum_{j \in N^*} (\lceil q_{ij} \rceil + 1) - \sum_{i \in M^*} \sum_{j \notin N^*} (\lfloor q_{ij} \rfloor - 1)^+$$

were the first equality is obtained by simple substitutions, the first inequality is implied by $|\alpha_j| < 1$, the second inequality is implied by $|N^*| \, |M \setminus M^*| \ge |N^*|$ and the third inequality is trivial. ∎

Corollary 3. *If $p = p^R$ and q_{ij} are positive regional quotas, then $\tau^* \leq 2$*

Proof. The Rounding property is satisfied and $\sum_j q_{ij} = r_i$ for the regional quotas and p^R. ∎

Definition 2. *A vector p has the* weak Hare property *if*

$$\lfloor q_j \rfloor - 1 \leq p_j \leq \lceil q_j \rceil + 1, \qquad j = 1, \dots, n.$$

∎

Then Corollary 3 can be restated as

Corollary 4. *If $p = p^R$ and q_{ij} are positive regional quotas, then every optimal apportionment x^* has the row-wise weak Hare Property, that is,*

$$\lfloor q_{ij} \rfloor - 1 \leq x^*_{ij} \leq \lceil q_{ij} \rceil + 1, \qquad \text{for all} \quad i, j.$$

The above Corollary 4 should be contrasted with the negative result (Pennisi et al. [23]) that, when the vector of total seats granted to the parties is p^N, there might be no apportionment at all satisfying the row-wise Hare Property. In view of these considerations, an attractive suggestion for lawmakers would be to define the vector of total seats granted to the parties to be p^R rather than p^N.

Are there cases when the Rounding property holds with $p = p^N$? The next result provides an interesting sufficient condition for this to happen.

Corollary 5. *If $r_i/v_{iN} = H/v_{MN}$ for all i, $p = p^N$ and q_{ij} are positive regional quotas, then $\tau^* \leq 2$ and thus every optimal apportionment has the weak Hare property.*

Proof. It follows from the hypothesis that $q_{ij} = H\, v_{ij}/v_{MN}$ and so $\sum_i q_{ij} = H\, v_{Mj}/v_{MN}$. By the rule of Largest Remainders the Rounding property is satisfied. Now the thesis follows from Theorem 8. ∎

Since the number of seats r_i at stake in region i is roughly proportional to the population of the region, the meaning of the condition $r_i/v_{iN} = H/v_{MN}$, for all i, in Corollary 5, is that the rate of absenteeism is roughly the same in all the regions.

Now we show that there are instances with $\tau^* > 1$ even if the Rounding property is satisfied. Consider the following instance with parties A–F, regions 1–5,

$$r = (\,5 \quad 5 \quad 1 \quad 1 \quad 1\,), \qquad\qquad p = (\,1 \quad 1 \quad 1 \quad 4 \quad 3 \quad 3\,),$$

and q as in the table

	A	B	C	D	E	F
1	0.992	0.870	0.170	0.994	0.988	0.986
2	0.460	0.580	0.991	0.993	0.989	0.987
3	0.001	0.001	0.001	0.986	0.001	0.010
4	0.001	0.001	0.001	0.440	0.001	0.556
5	0.001	0.001	0.001	0.140	0.856	0.001

with

$$\sum_i q_{ij} = (\,1.455 \quad 1.453 \quad 1.164 \quad 3.553 \quad 2.835 \quad 2.540\,)\,.$$

Rounding down the regional quotas in the above matrix, one always gets 0 and rounding them up one always gets 1. One can check that there is no way to assign 0 or 1 seats to each pair (i,j). Indeed the parties D, E and F would receive at most 6 seats altogether in the regions 1 and 2. Hence the parties A, B and C would receive at least 10-6=4 seats in the same regions 1 and 2. But these parties are allotted 3 seats in total! Hence at least one party among D, E and F should receive two seats either in region 1 or 2. For a minimax solution this has to be for party D in region 1 with optimal error equal to 1.006. One of these minimax solutions is given by the seat assignment:

	A	B	C	D	E	F
1	0	0	1	2	1	1
2	1	1	0	1	1	1
3	0	0	0	1	0	0
4	0	0	0	0	0	1
5	0	0	0	0	1	0

6 Algorithms for optimal apportionment

Now we address the issue of designing algorithms to find the minimum error, both absolute and relative. There are two aspects to be considered. Clearly the computational complexity of an algorithm must be taken into account. On the other hand, the size of the problems to be solved is never very large. On the contrary, given the power of modern computers, these instances can be considered small and even a naive algorithm can run in a satisfactory amount of time. Hence we think that it is also important that an algorithm be sufficiently simple to be implemented easily and described in a law.

For these reasons we present three algorithms for the absolute error minimization. The first one is simple but runs in pseudopolynomial time. The second one is slightly more complex since it requires binary search and runs in weakly polynomial time. Finally, the third algorithm is definitely more complex but runs in strongly polynomial time. We also describe a fourth weakly polynomial algorithm for relative error minimization.

Algorithm 1 – The first algorithm works in two phases: first we compute t^*, by simply trying all values $t^* = 0, 1, \ldots$ in ascending order until there is an apportionment $\hat{x}$ feasible for the bounds

$$(\lfloor q_{ij} \rfloor - t^*)^+ \leq \hat{x}_{ij} \leq \lceil q_{ij} \rceil + t^*.$$

Once t^* is known we apply the result of Cor. 1 and we try all $2\,m\,n$ relevant errors $\tau = t^* + < q_{ij} >$ and $\tau = t^* + 1 - < q_{ij} >$, keeping the minimum τ such that a feasible apportionment x_{ij} exists with

$$\lceil q_{ij} - \tau \rceil^+ \leq x_{ij} \leq \lfloor q_{ij} + \tau \rfloor .$$

As stated in Sec. 4, the existence of a feasible apportionment is checked through the solution of a maximum flow problem. Let $\mathrm{MF}(h,k)$ be the complexity of a max flow routine on a graph with h nodes and k arcs. For generic target quotas q_{ij} the optimal t^* can be as large as $\hat{r} := \max_i r_i - 1$ and therefore the two phases have complexity $O(\hat{r}\,\mathrm{MF}(n+m, n\,m))$ and $O(n\,m\,\mathrm{MF}(n+m, n\,m))$ respectively. Clearly the first phase is pseudopolynomial depending on the number $\hat{r}$. In practice t^* almost never exceeds 2 and therefore in almost all cases at most 3 calls to the max flow routine are enough in the first phase. The second phase can be somewhat longer since the number $2\,m\,n$ can be larger than one thousand in some cases and a few seconds may be necessary to run all these max flow routines. Yet for practical purposes a few seconds is an acceptable amount of time and the running time trades off favorably with the ease of implementation (provided that a max flow routine is available).

We note that in case the target quotas are the fair shares this algorithm is strongly polynomial, since $t^* = 0$. The algorithm is also strongly polynomial if the assumptions stated in Theorem 8 are met, since in this case $t^* \leq 2$.

Algorithm 2 – In order to overcome the pseudopolynomial complexity of the previous algorithm we design a second algorithm which uses binary search over the range $[0, \hat{r}]$ to find the optimal t^*. Then we can compute and sort all relevant values $\tau = t^* + < q_{ij} >$ and $\tau = t^* + 1 - < q_{ij} >$ with complexity $O(m\,n \log(m\,n))$ and use binary search again on these relevant values. This way the complexity becomes $O((\log(\hat{r}) + \log(m\,n))\,\mathrm{MF}(n+m, n\,m))$ (assuming $O(m\,n) \subset O(\mathrm{MF}(n+m, n\,m))$) and is polynomial - although only weakly polynomial due to the dependancy on $\log \hat{r}$. Since the network is bipartite and generally one has $n < m$, one can achieve an overall complexity $O(n^2\, m \log(m\,n\,\hat{r}))$ ([1] p. 255).

Algorithm 3 – Now we develop a strongly polynomial algorithm for minimizing the maximum absolute error in the general case. The algorithm consists of three phases. In the first phase we locate by binary search an interval including the optimal value of the fractional BAP. In the second phase we compute this solution exploiting a result by Radzik [28]. Finally, on the basis of the proximity bound of Sec. 5, we apply binary search again to find an integer solution of the BAP. Each phase is strongly polynomial.

In the following we are concerned with the fractional BAP. In order to solve this problem, the arc capacity intervals must be set as

$$[(q_{ij} - \tau)^+ \ , \ q_{ij} + \tau].$$

Let $\mathcal{F}(\tau)$ be the feasible set of all feasible flows with the value τ defining the capacities. Alternatively $\bar{\tau}$ is the smallest value such that $\mathcal{F}(\tau) \neq \emptyset$.

First phase: For the sake of notation let us denote

$$q_{(i,j)_0} := 0, \qquad q_{(i,j)_{|E|+1}} := +\infty.$$

Let us assume that the q_{ij} values are all different. This assumption can be normally met in practice. Hence we may sort these values as

$$q_{(i,j)_0} < q_{(i,j)_1} < q_{(i,j)_2} < \ldots < q_{(i,j)_{|E|}} < q_{(i,j)_{|E|+1}} \ .$$

By binary search on these $|E| + 2$ values we find that $\tau = q_{(i,j)_h}$ such that no feasible flow exists for $\tau = q_{(i,j)_h}$ and feasible for $\tau = q_{(i,j)_{h+1}}$. This can be accomplished by $\log |E|$ calls to a max flow routine, hence in strongly polynomial time. Then we know that $q_{(i,j)_h} < \bar{\tau} \leq q_{(i,j)_{h+1}}$.

Second phase: let us partition the arcs in E as $E_0 := \{(ij) \in E : q_{ij} \leq q_{(i,j)_h}\}$ and $E_1 := E \setminus E_0$. Note that for all values $q_{(i,j)_h} < \tau \leq q_{(i,j)_{h+1}}$ the arc capacities are set as

$$[0 \ , \ q_{ij} + \tau], \quad (ij) \in E_0, \qquad [q_{ij} - \tau \ , \ q_{ij} + \tau], \quad (ij) \in E_1 \ .$$

The problem of finding the smallest value $\bar{\tau} \in [q_{(i,j)_h}, q_{(i,j)_{h+1}}]$ such that there exists a feasible flow is a parametric flow problem which can be solved by using the the same approach as in Radzik [28] particularized to our case. Let $S \subset N \cup M$ be a subset of nodes and $\bar{S}$ its complement. For each S let

$$\delta(S)^+ := \{(i,j) : i \in S \cap M \, , \, j \in \bar{S} \cap N\},$$

$$\delta(S)^- := \{(i,j) : i \in \bar{S} \cap M \, , \, j \in S \cap N\}$$

and

$$\delta(S) = \delta(S)^+ \cup \delta(S)^-.$$

The capacity of the cut induced by S is defined as

$$c(S,\tau) = \sum_{(i,j)\in\delta(S)^+} (q_{ij}+\tau) - \sum_{(i,j)\in\delta(S)^-} (q_{ij}-\tau)^+ = \sum_{(i,j)\in\delta(S)^+} (q_{ij}+\tau) - \sum_{(i,j)\in\delta(S)^-\cap E_1} (q_{ij}-\tau) =$$

$$\sum_{(i,j)\in\delta(S)^+} q_{ij} - \sum_{(i,j)\in\delta(S)^-\cap E_1} q_{ij} + (|\delta(S)^+| + |\delta(S)^- \cap E_1|\,\tau =: c_0(S) + c_1(S)\,\tau.$$

Note that $c_1(S)$ is integer-valued and that $c_1(S) = |\delta(S)| - |\delta(S)^- \cap E_0| \le m\,n$. Let

$$b(S) = \sum_{i\in S\cap M} r_i - \sum_{j\in S\cap N} p_j$$

be the net flow out of the nodes in S. Define

$$f(S,\tau) := b(S) - c(S,\tau) = b(S) - c_0(S) - c_1(S)\,\tau.$$

If $f(S,\tau) \le 0$ for all S, then by Gale's Theorem [1, 11] there exists a feasible flow. Hence we define the function

$$h(\tau) := \max_S f(S,\tau) = \max_S (b(S) - c_0(S) - c_1(S)\,\tau)$$

and are interested in finding $\bar\tau$ such that $h(\bar\tau) = 0$. Let

$$\mathcal{S}_k := \{S : c_1(S) = k\}\,.$$

Then

$$h(\tau) = \max_k \left\{-k\,\tau + \max_{S\in\mathcal{S}_k}(b(S) - c_0(S))\right\}.$$

Let $h_k := \max_{S\in\mathcal{S}_k}(b(S) - c_0(S))$. So

$$h(\tau) = \max_k \{-k\,\tau + h_k\}\,.$$

A geometrical interpretation is as follows. The function $h(\tau)$ is the pointwise maximum of a finite number of linear functions $b(S) - c_0(S) - c_1(S)\,\tau$ with slope $-c_1(S)$ and intercept $b(S) - c_0(S)$. As seen above, the slope can take at most $m\,n$ values. For each of them, say k, the straight line with slope k and maximum intercept lies above the other lines with the same slope: hence in the pointwise maximum expression we need to retain only such highest lines. Hence the function $h(\tau)$ is convex and piecewise linear with at most $m\,n$ breakpoints.

Consider the following iteration (Newton's method in Radzik [28]): given τ, compute S such that $h(\tau) = f(S,\tau)$ (via max flow-min cut); then, given S, compute τ such that $f(S,\tau) = 0$, i.e.

$$\tau = \frac{b(S) - c_0(S)}{c_1(S)}$$

and repeat until $h(\tau) = 0$. The number of iterations is bounded above by the number of breakpoins of $h(\tau)$, hence by $m\,n$.

Third phase: once we have determined $\bar\tau$ we make use of the proximity bound and find by binary search τ^* within the interval $[\bar\tau, \bar\tau+1]$. There are at most $2\,|E|$ relevant values in this interval, hence the binary search over the relevant values requires at most $\log(2\,|E|)$ steps, i.e. a strongly polynomial number of iterations.

The argument used in phase two can be employed in other more general settings. This fact deserves to be outlined as follows. Consider a Linear Fractional Combinatorial Optimization Problem (LFCO)

$$\max_{x \in X} \{a(x) - b(x)\,\tau\} \tag{16}$$

in which X is a finite set of combinatorial objects (e.g. cuts, spanning trees, etc.), a and b are additive functions, $b(x)$ integer, $0 < b(x) < P(n)$, $x \in X$, where n is a size parameter (e.g. number of nodes of the underlying graph) and $P(n)$ is a polynomial in n. Then the optimal value $h(\tau)$ of (16) is a convex piecewise linear function of τ with at most $P(n)$ breakpoints. It follows that, if one can compute $h(\tau)$ in strongly polynomial time for each given τ, then one can solve $h(\tau) = 0$ in strongly polynomial time.

It is also worth pointing out that both the weakly polynomial algorithm and the strongly polynomial one work with a similar complexity analysis for the following general problem. Consider a network with n nodes and m arcs with lower capacities and upper capacities bounded above by H. Let F be the polytope of all feasible flows. Let q be any point of the simplex $S^{(m)} = \left\{q \geq 0 : \sum_{ij} q_{ij} \leq H\right\}$. Find an integral point of F at minimum ℓ_∞ distance from q.

Algorithm 4 – In order to find the minimum relative error we present a (weakly) polynomial-time algorithm. We recall that the relevant errors for the relative error minimization are either of the form $(1 - z/q_{ij})$ for $0 \leq z \leq \lfloor q_{ij} \rfloor$ or of the form $(z/q_{ij} - 1)$ for $\lceil q_{ij} \rceil \leq z \leq r_i$. It would be tempting to compute and sort all relevant values and then apply binary search. However, the time needed for the sheer calculation of the $n\,(H+m)$ relevant values is linearly dependant on H and therefore is pseudopolynomial. We still want to carry out a binary search, but we have to do it without computing and sorting all relevant errors.

The idea is to apply the binary search over a real interval and to compute the two relevant errors that are closest (one from the left and one from the right) to the middle value of the interval. Once the interval is sufficiently small we may start a second binary search over a small number of relevant values. The crucial observation is that within any interval of length $1/q_{ij}$ there are at most two relevant errors for the pair (i, j). Hence within any interval of length $1/(\max_{ij} q_{ij})$ there are at most two relevant errors for any pair (i, j). Then the first binary search stops as soon as the interval length is less than $1/(\max_{ij} q_{ij})$.

The maximum relevant error is bound by $\max_i r_i / \min_{ij} q_{ij}$. The binary search starts with the left bound $\hat{\ell} := 0$ and right bound $\hat{r} := \max_i r_i / \min_{ij} q_{ij}$. At each iteration it computes the middle value $\mu := (\hat{\ell} + \hat{r})/2$ and the closest relevant error to μ from the left σ^-, and the closest one from the right σ^+. To compute σ^+ we need to compute

$$\operatorname*{argmin}_{\substack{0 \leq z \leq \lfloor q_{ij} \rfloor \\ q_{ij} - z - \mu\, q_{ij} \geq 0}} \frac{q_{ij} - z - \mu\, q_{ij}}{q_{ij}} = \lfloor q_{ij} - \mu\, q_{ij} \rfloor\,,$$

$$\operatorname*{argmin}_{\substack{\lceil q_{ij} \rceil \leq z \leq r_i \\ z - q_{ij} - \mu\, q_{ij} \geq 0}} \frac{z - q_{ij} - \mu\, q_{ij}}{q_{ij}} = \lceil q_{ij} + \mu\, q_{ij} \rceil\,,$$

by tacitly assuming that the argument is empty (or equivalently the minimum is $+\infty$) if $\lfloor q_{ij} - \mu\, q_{ij} \rfloor < 0$ in the first case and $\lceil q_{ij} + \mu\, q_{ij} \rceil > r_i$ in the second case. The expressions yield

$$\sigma^+ := \min_{ij} \left\{ \min \left\{ 1 - \frac{\lfloor q_{ij} - \mu\, q_{ij} \rfloor}{q_{ij}}\ ;\ \frac{\lceil q_{ij} + \mu\, q_{ij} \rceil}{q_{ij}} - 1 \right\} \right\}.$$

For the closest error σ^- from the left we have to compute

$$\operatorname*{argmax}_{\substack{0 \leq z \leq \lfloor q_{ij} \rfloor \\ q_{ij} - z - \mu\, q_{ij} \leq 0}} \frac{q_{ij} - z - \mu\, q_{ij}}{q_{ij}} = \max \{\lceil q_{ij} - \mu\, q_{ij} \rceil\ , 0\}\,,$$

$$\operatorname*{argmax}_{\substack{\lceil q_{ij} \rceil \le z \le r_i \\ z - q_{ij} - \mu\, q_{ij} \le 0}} \frac{z - q_{ij} - \mu\, q_{ij}}{q_{ij}} = \min\left\{ \lfloor q_{ij} + \mu\, q_{ij} \rfloor \ , \ r_i \right\},$$

by assuming again that the argument is empty (or equivalently the maximum is $-\infty$) if $\lceil q_{ij} - \mu\, q_{ij} \rceil > \lfloor q_{ij} \rfloor$ in the first case and $\lfloor q_{ij} - \mu\, q_{ij} \rfloor < \lceil q_{ij} \rceil$ in the second case. The expressions yield

$$\sigma^- := \max_{ij} \left\{ \max \left\{ 1 - \frac{\max\left\{ \lceil q_{ij} - \mu\, q_{ij} \rceil \ , \ 0 \right\}}{q_{ij}} \ ; \ \frac{\min\left\{ \lfloor q_{ij} + \mu\, q_{ij} \rfloor \ , \ r_i \right\}}{q_{ij}} - 1 \right\} \right\}.$$

The existence of a feasible flow is checked for both σ^- and σ^+. If the problem is infeasible for σ^- and feasible for σ^+ then the binary search can be stopped because the optimal error has been detected as $\sigma^* = \sigma^+$. If the problem is infeasible for σ^+ then the left interval bound is reset as $\hat{\ell} := \sigma^+$. If the problem is feasible for σ^- then the right interval bound is reset as $\hat{r} := \sigma^-$. In both cases the interval is at least halved.

This binary search is carried out until $\hat{r} - \hat{\ell} < 1/(\max_{ij} q_{ij})$ (and clearly the optimum value has not been found yet). Hence this phase requires at most $\log(\max_i r_i) - \log(\min_{ij} q_{ij}) + \log(\max_{ij} q_{ij})$ computations of two max flow routines and is therefore weakly polynomial.

The next phase is based on the previous observation that for each pair (i,j) there are at most two relevant errors within any interval of length $1/(\max_{ij} q_{ij})$. Hence the two relevant errors closest to $\hat{\ell}$ and greater than $\hat{\ell}$ are

$$1 - \frac{\lfloor q_{ij} - \hat{\ell}\, q_{ij} \rfloor}{q_{ij}} \qquad \text{and} \qquad \frac{\lceil q_{ij} + \hat{\ell}\, q_{ij} \rceil}{q_{ij}} - 1.$$

The computation of these values for all pairs (i,j) requires $O(n\,m)$ time, and the subsequent sorting requires $O(n\,m \log(n\,m))$. The next step is the binary search over these values which has complexity $O(\mathrm{MF}(n+m, n\,m) \log(n\,m))$.

In conclusion the computational complexity of finding the min-max relative error is

$$O(\mathrm{MF}(n+m, n\,m)\,(\log(\max_i r_i) - \log(\min_{ij} q_{ij}) + \log(\max_{ij} q_{ij}) + \log(n\,m))).$$

7 Uniqueness of solutions

Any sound seat assignment method, taking as input the votes, must output a unique assignment. On the other hand, optimization problems usually admit multiple optimal solutions. Therefore, if one wishes to follow an optimization approach for producing a seat assignment, it is crucial to develop a method that outputs a unique assignment.

In the previous sections we have pursued the search of seat assignments minimizing the maximum error. However, minimax solutions are insensitive to errors which are less than the maximum error and therefore often exhibit many alternative optima. Therefore we need to define a stronger form of optimality to refine the choice among the optima.

For a given apportionment x^* let $\tau^*_{hk} := |q_{hk} - x^*_{hk}|$ be the error for the pair (h,k). Let

$$L(h,k) := \left\{ (i,j) \neq (h,k) : \tau^*_{ij} \le \tau^*_{hk} \right\}, \qquad U(h,k) := \left\{ (i,j) : \tau^*_{ij} > \tau^*_{hk} \right\}.$$

$L(h,k)$ is the set of pairs with error not larger than τ^*_{hk} and $U(h,k)$ is the complement set, excluding (h,k) itself. Then we say that the apportionment x^* is *strongly optimal* if, for any pair (h,k), there is no apportionment with error $\tau_{hk} < \tau^*_{hk}$, $\tau_{ij} \le \tau^*_{hk}$ for $(i,j) \in L(h,k)$ and $\tau_{ij} \le \tau^*_{ij}$ for $(i,j) \in U(h,k)$. Clearly strong optimality implies optimality.

The definition takes care of the fact that, while trying to improve the error for some pair (h, k), it is not allowed to worsen those pairs which have a larger error than (h, k). On the contrary it is allowed to worsen those pairs whose error is less than or equal to (h, k) up to the 'threshold' τ^*_{hk}. Note also that if another pair exhibits the same error as (h, k), one is allowed to try to decrease the error for (h, k) without being compelled to decrease also the error for the other pair.

A strongly optimal solution is robust with respect to any criticism and complaint parties might raise in order to gain more seats for themselves or to reduce seats for the other parties. However, even strongly optimal solutions are not unique. In order to find a unique strongly optimal solution we resort to the concept of unordered lexico minima.

An *unordered lexico minimum* is defined in the following way: given a vector $a \in R^n$ let $\theta(a) \in R^n$ be the vector obtained from a by permuting its entries so that the entries in $\theta(a)$ are arranged in non-increasing order (in case of equal entries break the tie in any fixed way). Then given two vectors $a, b \in R^n$ we say that a is *Unordered Lexico better* than b if $\theta(a)$ is lexicographically smaller than $\theta(b)$, i.e. there exists an integer k such that $\theta_i(a) = \theta_i(b)$ for $i < k$ and $\theta_k(a) < \theta_k(b)$. This definition is taken from [31] where it has been applied in a similar context. A vector $a \in A \subset R^n$ is an Unordered Lexico Optimum in A if there is no $b \in A$ which is Unordered Lexico better than a. Like the Lexicographic ordering also the Unordered Lexicographic ordering is a total order and thus it has a unique minimum within a finite set.

In our case the vectors consist of the relevant errors for all pairs (i, j). In order to find unordered lexico minima, once a minimax solution has been found with relevant error τ_k for the *blocking pair* $(i, j)_k$ (k is the index of the relevant τ in an ascending ordering of the relevant errors), we want to find a solution minimizing

$$\max_{(ij) \neq (i,j)_k} |x_{ij} - q_{ij}|.$$

This can be done as before with the only difference that the capacity for the pair $(i, j)_k$ is no longer changed. Once we have found a second solution with error τ_h ($h < k$) and blocking pair $(i, j)_h$, we proceed recursively by fixing the capacities of the blocking pairs one at a time. In this procedure we have to use relevant error that are less than τ_{min} or σ_{min}. If for the current relevant τ we have $\tau < \min\{q_{ij} - \lfloor q_{ij} \rfloor ; \lceil q_{ij} \rceil - q_{ij}\} \leq 1/2$ we simply fix the capacity for the arc (i, j) to $[\bar{q}_{ij}, \bar{q}_{ij}]$ with $\bar{q}_{ij}$ equal to q_{ij} rounded to the nearest integer and the computation is finished because there cannot be any better error. Hence it turns out that the blocking pairs are those for which the absolute error is larger than one half.

It is immediate to see that unordered lexico minimal errors are also strongly optimal. If the relevant τ values are all different (which is very likely) the procedure is straightforward. If some τ's are equal then we have to choose the blocking pairs among those achieving the maximum error. This can be done by finding all arcs belonging to minimal cuts in a max flow problem and this can be carried out according to a result by Picard and Queyranne [24]. We proceed similarly for σ.

By applying this technique to the instance in Sec. 5, we first identify the blocking pair $(1, D)$ that, by receiving two seats for a target quota 0.994, exhibits the error 1.006. It is impossible to reduce this error for all pairs. According to our procedure, we try to decrease the errors for all pairs except for the blocking pair $(1, D)$. Hence the capacity of this pair remains fixed at the interval $[0, 2]$. Now we decrease τ until we find the blocking pair $(1, B)$ with error 0.87 (0 seats for a target quota 0.87). Now we fix the capacity for the pair $(1, B)$ at $[0, 1]$. At this point there are no more pairs with error larger than $1/2$ and the procedure stops. This is the final seat assignment, which is much more satisfactory than the previous one:

	A	B	C	D	E	F
1	1	0	0	2	1	1
2	0	1	1	1	1	1
3	0	0	0	1	0	0
4	0	0	0	0	0	1
5	0	0	0	0	1	0

Remark. In the single-region case (vector apportionment) both the min-max and the Unordered lexico-min apportionments coincide with the Largest Remainders one. As a matter of fact, in this case one has $\tau^* < 1$; hence any min-max apportionment satisfies the Hare Property. Therefore, any party j gets $\lfloor q_j \rfloor$ seats, plus possibly an extra one. An exchange argument shows that, in order to minimize the maximum error, the extra seats must be given to the parties whose fractional parts $q_j - \lfloor q_j \rfloor$ are largest.

8 Comparison with other methods

The aim of the present section is to compare our method with other well-known procedures for biproportional apportionment available in the literature. We shall pursue this comparison from a threefold perspective. First, we discuss which of the Balinski and Demange's six proportionality axioms are satisfied by our method. Second, some empirical indications of the "distance" between our apportionment and others are provided for three real-life elections. Third, the performances of the methods under consideration are discussed with respect to some general – theoretical and practical – criteria.

As mentioned in the Introduction, the fundamental work of Balinski and Demange provided both a theoretical and computational framework for dealing with BAP. The reader is referred to [3] for an actual list – omitted here for lack of space – of their six axioms for integral proportionality.

Since scaling all entries of the vote matrix by a common positive factor leaves both the regional quotas and the fair shares unchanged, the Exactness axiom holds. More generally, the fair share matrix is invariant for arbitrary row and column scaling by positive factors, while for regional quotas this is true only for row scalings. Hence the Homogeneity axiom is satisfied if the target quotas are fair shares, and half-satisfied if they are regional quotas. The Relevance Axiom is meaningful only in the case of inequality constraints, so it is irrelevant in our context.

The Monotonicity Axiom is seen to hold both in the $1 \times n$ case (for arbitrary positive n) and in the 2×2 one. We conjecture that it holds also for general matrices at least for strongly optimal solutions w.r.t regional quotas.

There is no guarantee that the Uniformity (or Consistency) Axiom holds: in fact, we have remarked at the end of the previous section that in the single-region case the (generally unique) min-max apportionment coincides with the Largest Remainders one; and the latter is well known [8] to be not consistent in general.

We recall that the Completeness Axiom deals with a sequence of (real-valued) vote matrices $V^k \to \hat{V}$ such that the apportionment $x(V^k) =: \bar{x}$ is invariant for all k and there exists $\hat{x} := x(\hat{V})$. The Completeness Axiom requires $\hat{x} = \bar{x}$. For the Axiom to hold continuity of the target quotas w.r.t. the vote matrix is an essential requirement. However, it may happen in general that there are quotas $q_{ij}^k \to 0$ with $\bar{x}_{ij} = 1$. In this case $\bar{x}$ is not feasible for $\hat{V}$ and the Axiom cannot hold. Even the continuity of the target quotas can be affected by this circumstance.

The possibility of quotas $q_{ij}^k \to 0$ with $\bar{x}_{ij} = 1$ is linked to the possibility of having the set Z of zero votes varying w.r.t. the vote matrices. However, it may be safely assumed that the presence of zeroes is due to structural reasons (a party does not present any list of candidates in a region) rather than to accidental ones (a party does not get any vote in a region in spite of its list in that region being nonempty). For these reasons we introduce the following Restricted Completeness Axiom which makes sense in a practical context.

Let $\mathbf{V}$ be the set of all real-valued nonnegative matrices whose smallest positive entry is at least 1 and whose set of null entries is a given subset $Z \subset M \times N$. Notice that the set $\mathbf{V}$ is closed. Then the Restricted Completeness Axiom considers only matrices V_k in $\mathbf{V}$.

Then the apportionment minimizing the maximum absolute error satisfies the Restricted Completeness Axiom if the target quotas $q(V)$ are continuous w.r.t. the vote matrix V. Indeed, by taking the limit in the finite system of inequalities

$$|\bar{x}_{ij} - q_{ij}(V_k)| \leq |y_{ij} - q_{ij}(V_k)|, \qquad \text{for each apportionment } y,$$

one gets the required inequalities

$$|\bar{x}_{ij} - q_{ij}(\hat{V})| \leq |y_{ij} - q_{ij}(\hat{V})|, \qquad \text{for each apportionment } y.$$

Continuity certainly holds true without any additional assumption when the $q_{ij}(V)$'s are regional quotas. It remains true also for fair shares under the assumptions of the Restricted Completeness Axiom. We recall that the continuity of the fair share matrix in case the set Z is empty has been proven in [4]. Next we extend this result to the case of a non empty set Z invariant for all matrices in the sequence whose limit is the vote matrix. To prove the result we make use of the upper bound given in Theorem 5.2 of [17] on the number K_ε of iterations required by RAS to achieve an ℓ_2-error smaller than a given ε:

$$K_\varepsilon \leq C\, \frac{\rho\, H^2}{\varepsilon^2}\, \ln \frac{\rho}{\nu} \tag{17}$$

where C is a fixed constant, $\rho := \max\{\max_i r_i; \max_j p_j\}$, $\nu := \min_{(ij)\notin Z} v_{ij}$ and H is the house size.

Theorem 9. *Let (V, r, p) be a vote matrix together with given vectors r and p for which the fair share matrix exists. Let V_j be its RAS iterates with $F = \lim_j V_j$. Let $(V^k, r^k, p^k) \to (V, r, p)$ be any sequence and let V_j^k be the RAS iterates of (V^k, r^k, p^k) with $F^k = \lim_j V_j^k$. Suppose $Z^k = Z$ for all k. Then $F^k \to F$, i.e., the fair share matrix is continuously dependent on the data.*

Proof. Since $(V^k, r^k, p^k) \to (V, r, p)$ we may redefine in (17)

$$\rho := \max\{\max_i\{r_i; \sup_k r_i^k\}; \max_j\{p_j; \sup_k p_j^k\}\},$$

$$\nu := \min\{\min_{(ij)\notin Z} v_{ij}; \inf_k \min_{(ij)\notin Z} v_{ij}^k\}, \quad H := \max\{\sum_i r_i; \sup_k \sum_i r_i^k\},$$

having $\rho < \infty$, $H < \infty$ and $\nu > 0$. In particular $\nu > 0$ is guaranteed by the invariance of the set Z. Hence, there exists a fixed constant $\overline{C}$, such that for any $\varepsilon > 0$, there exists an index $j = \overline{C}/\varepsilon^2$ such that

$$\|V_j^k - F^k\| \leq \varepsilon, \quad \forall k, \qquad \|V_j - F\| \leq \varepsilon.$$

Now

$$\|F^k - F\| = \|F^k - V_j^k + V_j^k - V_j + V_j - F\| \leq 2\,\varepsilon + \|V_j^k - V_j\|.$$

Since each RAS iteration is continuous, $V_j^k \to V_j$, i.e., for the same ε, there exists K, possibly dependent on j, such that for $k > K$

$$\|V_j^k - V_j\| < \varepsilon.$$

Summing up, for any $\varepsilon > 0$ there exists K such that, for $k > K$

$$\|F^k - F\| \leq 3\,\varepsilon.$$

Therefore $F_k \to F$ and this implies the continuity of the fair share matrix. ∎

ITALY	FS	PMF0	PMF	CR	TT
FS		**0.660928**	**0.660928**	**0.660928**	**0.833503**
PMF0	0.256604		**1**	**1**	**1**
PMF	0.256816	0.0763359		**1**	**1**
CR	0.253796	0.0305344	0.0458015		**1**
TT	0.263785	0.1068700	0.0610687	0.0763359	

SPAIN	FS	PMF0	PMF	CR	TT
FS		**0.898518**	**0.898518**	**0.955904**	**1.08074**
PMF0	0.232432		**1**	**1**	**1**
PMF	0.162368	0.201754		**1**	**1**
CR	0.159719	0.228070	0.0438596		**1**
TT	0.169750	0.210526	0.0526316	0.0526316	

FAROE	FS	PMF0	PMF	CR	TT
FS		**0.687644**	**0.687644**	**0.687644**	**0.733836**
PMF0	0.281439		**1**	**1**	**1**
PMF	0.267512	0.250000		**0**	**1**
CR	0.267512	0.250000	0		**1**
TT	0.275533	0.166667	0.125000	0.125000	

Table 1: Maximum (bold) and average (normal) errors for each pair of methods

Next, we have compared the seat apportionments given by different methods applied to the real polls of three countries, namely, Italy 2008, Spain 2008, and Faroe Islands 2004. For all methods we have used the fair share quotas to get a uniform comparison among the methods, even in the case of Italy, differently from Appendix A where the target quotas were those provided by the Ministry. The methods we have considered are: the parametric maximum flow method for minimizing the maximum absolute error (PMF0) described in this paper, the lexico-min refinement (PMF) described in Sec. 7, the Controlled Rounding (CR) procedure by Ernst and Cox [10] and the Tie and Transfer (TT) procedure of Balinski and Demange [4]. We have compared the corresponding output apportionments with the fair shares (FS) and also between themselves. For each region-party pair (i, j) we have taken the value $|a_{ij} - b_{ij}|$, with a the seat apportionment given by one method and b the one given by another method (a fractional apportionment for FS); we have considered both the maximum difference $\max_{ij} |a_{ij} - b_{ij}|$ and the average difference $\sum_{ij} |a_{ij} - b_{ij}|/|E|$ with $|E|$ the number of region-party pairs not in Z (or the number of arcs in the associated bipartite network). The values of $|E|$ are 131 for Italy, 228 for Spain and 48 for the Faroe Islands.

The maximum difference and the average difference are reported in Table 1 which consists of three tables, one for each country. In each table the upper triangular entries (bold face) refer to the maximum differences and the lower triangular entries (normal face) to the average absolute errors between the respective methods, respectively.

Note that the seat assignments produced by the various methods differ at most by one for each region-party pair. Clearly the best maximum error is achieved by PMF0 which has been designed toward this goal. It is interesting to notice that TT may give a maximum error larger than 1 even w.r.t fair share quotas. Considering the average error, PMF performs much better than PMF0 for obvious reasons. A comparison of the various entries in Table 1 shows that PMF and CR have the best results and produce very close seat assignments (for the Faroe Islands it is indeed the same). The 0.0438596 average error between the seats given by PMF and CR for Spain, multiplied by 228 ($|E|$ for Spain) gives 10. Hence only 10 seats differ (by 1) out of 228 seats.

Admittedly, our sample of 3 countries is too small to draw definitive conclusions. Nevertheless, the above experimental results provide some clues about the mutual closeness of the different integral or fractional apportionments under consideration.

Finally, we have summarized the comparative advantages and disadvantages of our method (PMF) versus the main other available algorithms, namely, (i) Balinski and Demange's Tie-and-Transfer with rounding to the closest integer (TT), (ii) Pukelsheim's Discrete Alternating Scaling (DAS), (iii) Gaffke and Pukelsheim's algorithm, in the Rote and Zachariasen's minimum cost flow implementation, (GFRZ), and (iv) the Cox and Ernst's Controlled Rounding (CR). The first three algorithms yield, for a given rounding method, the same apportionment, namely, the unique one that satisfies all the six proportionality axioms. We took into account the following criteria:

1. Finiteness: the algorithm stops after a finite number of steps;
2. Feasibility: the output seat assignment always yields an apportionment;
3. Soundness: satisfaction of Balinski and Demange's six integral proportionality axioms;
4. Uniqueness: uniqueness of the seat assignment output by the method;
5. Theoretical Complexity: worst-case rate of growth of the number of elementary operations as the instance size increases;
6. Generality: range of applications besides Biproportional Apportionment;
7. Flexibility: dependance on other parameters besides input data;
8. Ease of implementation: no need to write sophisticated ad hoc computer codes;
9. Writability: possibility of translating the procedure into a simple, easy to understand, electoral law;

In the following table, $+$, $\sim$, and $-$ mean that PMF is deemed to exhibit "a better performance than", "the same performance as", and "a worse performance than", respectively, the given alternative method w.r.t. to the given criterion

Criteria\ Method	TT	DAS	GPRZ	CR
Finiteness	$\sim$	$+^{(1)}$	$\sim$	$\sim$
Feasibility	$\sim$	$\sim$	$\sim$	$+^{(2)}$
Soundness	$-$	$-$	$-$	$\sim$
Uniqueness	$\sim$	$\sim$	$\sim$	$+$
Theoretical complexity	$+^{(3)}$	$+^{(4)}$	$+^{(5)}$	$\sim$
Generality	$+$	$+$	$+$	$+$
Flexibility	$\sim^{(6)}$	$\sim^{(6)}$	$\sim^{(6)}$	$\sim^{(6)}$
Ease of implementation	$+$	$-$	$\sim$	$\sim$
Writability $^{(7)}$	$+$	$-$	$+$	$\sim$

Notes:

(1) Rare pathological instances where the method does not converge;

(2) Feasibility not guaranteed when the target quotas are the regional ones; guaranteed when the target quotas are given by any fractional apportionment, e.g., the fair share matrix;
(3) Pseudopolynomial in general; weakly polynomial in Zachariasen's [34] implementation;
(4) Complexity unknown;
(5) Weakly polynomial;
(6) The freedom of choice of the rounding method in TT, DAS, and PGRZ is counter-balanced by the freedom of choice of the target quotas in CR and PMF;
(7) We refer here to the simplest implementation of PMF0 given as Algorithm 1 in Sec.6 .

The above table suggests that the main weakness of PMF is the lack of satisfaction of all the six Balinski and Demange's axioms for integral proportionality, notably, the consistency one. Choosing the fair shares as target quotas mitigates this weakness. The fair share matrix is the unique real matrix that satisfies the five axioms given by the above authors in order to characterize proportionality between real matrices. In other words, fair shares would constitute the ideal apportionment in a "virtual Parliament" where fractional apportionments were allowed. In view of the Integrality Theorem for Network Flows, the apportionment provided by PMF satisfies the Rounding Property w.r.t. the fair shares, which means that such apportionment is close to the ideal one. Table 1 further corroborates this statement. Actually, we recommend the choice of fair shares in conjunction with PMF. The main strengths of PMF lie in its generality (it may be used to solve the general problem of ℓ_∞-projection onto a flow polytope), the availability of an optimality certificate, which enhances its good writability; and its strongly polynomial complexity – provided that the target quotas can be computed in strongly polynomial time.

It is worth emphasizing that the theoretical complexity of any BAP algorithm is not very significant in practice. As it was pointed out in [15], the world of electoral systems is a small one, where the number of parties never exceeds few tens, the number of constituencies few hundreds, and the house size few thousands. As a consequence, all the above algorithms require a very modest running time (few seconds are usually enough). So, a numerical comparison on a real-life benchmark does not make much sense. A satisfactory theoretical complexity may become an asset, though, in other applications where larger instances may occur.

Our method may provide a viable alternative when the acceptance of divisor-based methods, such as TT, DAS, and PGRZ, is problematic because of electoral traditions; and Cox and Ernst's controlled rounding method may fail to yield an apportionment, e.g. when one starts from regional quotas computed by Largest Remainders. Both circumstances occur in the Italian case – actually, this was the initial motivation for our work.

Appendix A - The Italian 2008 Elections

In the latest Italian political elections (April 13-14, 2008) there were 7 parties and 28 regions. One region is the so-called Estero region (for Italians resident in foreign countries) for which there is a separate seat assignment. The region Valle d'Aosta has only one seat whose assignment is done directly. Hence we do not take into account these regions in our analysis and consider the following 26 regions, listed in the tables with the abbreviations:

Piemonte 1 (P1), Piemonte 2 (P2), Lombardia 1 (L1), Lombardia 2 (L2), Lombardia 3 (L3), Trentino-Alto Adige (TA), Veneto 1 (V1), Veneto 2 (V2), Friuli-Venezia Giulia (FV), Liguria (LI), Emilia Romagna (ER), Toscana (TO), Umbria (UM), Marche (MA), Lazio 1 (La1), Lazio 2 (La2), Abruzzo (AB), Molise (MO), Campania 1 (C1), Campania 2 (C2), Puglie (PU), Basilicata (BA), Calabria (CA), Sicilia 1 (S1), Sicilia 2 (S2), Sardegna (SA).

The seven parties and their acronyms are: Popolo delle libertà (PDL), Lega nord (LN), Movimento per le autonomie (MPA), Partito democratico (PD), Italia dei valori (IDV), Unione di centro (UDC), Südtiroler Volkspartei (SVP).

However, the SVP party is present only in the TA region and therefore it is not listed in the tables to simplify the presentation. Furthermore the LN party is present in the regions from P1 up to MA and not in the others while for the party MPA it is just the opposite. Therefore, in order to spare space in the tables, these two parties are reported in the same column.

The seats assigned by the law to the regions, according to the population, before the elections are listed in the last column of both Table 2 and Table 3. The seats granted to the parties at the national level (according to the Largest Remainders Rule) are

$$p = p^N = \{272, 60, 8, 211, 28, 36, 2\}$$

(listed also in the tables with the exception of the two seats of the SVP party). The Table 2 refers to the minimization of the maximum absolute error and the Table 3 to the minimization of the maximum relative error. The regions are listed by rows and the parties by columns. In both tables for each region-party entry there are four numbers. The first number (the same in both table) is the regional quota, as computed by the Ministry, already modified to take into account the majority bonus to the winning coalition PDL-LN-MPA. In the tables the quotas have been rounded to two decimals. The quota for the SVP party in the TA region is 2.64389.

The second number (the same in both tables) is the number of seats assigned by the law (the two seats of SVP in TA are not shown). A closer look at these values reveals a stunning discrepancy. In four regions (P2, TA, V1, S1) the seats eventually assigned by the law *after* the elections are not the ones assigned by the law *before* the election. P2 and V1 have received one seat more and TA and S1 one seat less. This fact, a serious flaw in the Italian electoral law, has been mentioned in the introduction and has been explained in detail in the quoted papers. Furthermore, the regional quotas do not sum up to the r_i values. This is not an effect of rounding the quotas to two decimals. Instead it is another flaw caused by the unnecessarily laborious recomputation of the seat assignment in order to take the majority bonus into account.

In our computation we have used the original r values and the quotas computed by the Ministry (in spite of the fact that they differ a little from the regional quotas).

The Algorithm 2 of Sec. 6 applied to this instance has produced a minimum maximum absolute error of 0.75372 for the pair (V1, UDC) (1 seat against a regional quota of 1.75372). All the other pairs have a smaller error. The corresponding seat assignment is reported in Table 2 as the third figure in each entry. Then we have applied the procedure of Sec. 7 to find the lexicomin solution. This has been obtained by first fixing the capacity of the pair (V1, UDC) to $[1, 2]$ (in order to reduce the error the capacity should be reduced to $[2, 2]$ but we keep it constant to $[1, 2]$) and then lowering the error for all other pairs until we find a blocking pair. Then we fix the corresponding capacities and continue. We have found the following sequence of 12 blocking pairs. Note that all other 118 pairs have error less than one half, i.e. the best possible error.

(TA, SVP):	0.64389;	(S1, MPA):	0.626076;	(S1, PDL):	0.625834;
(P2, PDL):	0.595629;	(P2, LN):	0.595589;	(LI, PDL):	0.58776;
(LI, LN):	0.587678;	(MA, PDL):	0.58536;	(MA, LN):	0.585341;
(L1, PDL):	0.579506;	(L1, LN):	0.579074;	(TA, UDC):	0.540086.

The final unique lexicomin apportionment is the fourth figure in each entry of the Table 2. The figures in bold are those differing from the actual apportionment. For a better reading of the tables we resume below the meaning of the four entries for each region-party cell (for both Table 2 and 3).

	party label			
region label	target quota	actual seats	PMF0 seats	PMF seats

We have also computed the seat assignment by using the relative error. See the third figure in each entry of the Table 3. This solution has a maximum relative error of 0.646726 achieved for the pair (MO, IDV). Since the relative error corresponds to larger fluctuations around the quotas if the quotas are high, the result of minimizing the maximum relative error may produce seat assignments which can be hardly accepted. However, if we refine this assignment by finding the unordered lexico minimum, a new assignment is produced which is much better. See the fourth figure in each entry of the Table 3. In this case we have found a sequence of 34 blocking pairs with last relative error 0.0317565. In this case there are many region-party pairs with a different number of seats than the actual apportionment (in bold). This shows that the Italian system, based on the Largest Remainders Rule with some adjustments, is closer to the idea of minimizing the maximum absolute error.

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	PDL				LN – MPA				PD				IDV				UDC				r_i
P1	9.47	9	9	9	2.53	3	3	3	9.45	9	9	9	1.55	2	1	2	1.28	1	2	1	24
P2	9.60	**10**	9	**9**	4.40	5	5	5	6.15	6	6	6	0.85	1	1	1	1.19	1	1	1	22
L1	16.58	16	16	16	7.42	8	8	8	13.08	13	13	13	1.92	2	2	2	1.43	1	1	1	40
L2	15.35	15	15	15	13.65	14	14	14	10.39	10	11	10	1.61	2	1	2	2.06	2	2	2	43
L3	5.83	6	6	6	3.17	3	3	3	4.53	5	4	5	0.47	0	1	0	0.69	1	1	1	15
TA	2.76	3	3	3	1.24	1	1	1	2.64	3	3	3	0.36	0	1	0	0.46	**0**	0	**1**	10
V1	9.31	9	9	9	9.69	10	10	10	7.81	8	8	8	1.19	1	1	1	1.75	**2**	1	**1**	29
V2	6.26	6	6	6	5.74	6	6	6	5.95	6	6	6	1.05	1	1	1	1.01	1	1	1	20
FV	5.09	5	5	5	1.91	2	2	2	4.40	4	5	4	0.60	1	0	1	0.81	1	1	1	13
LI	7.59	7	7	7	1.41	2	2	2	6.19	6	6	6	0.81	1	1	1	0.67	1	1	1	17
ER	15.03	15	15	15	4.06	4	4	4	20.14	20	20	20	1.86	2	2	2	1.92	2	2	2	43
TO	15.03	15	15	15	0.97	1	1	1	18.61	19	19	19	1.39	1	2	1	1.70	2	1	2	38
UM	3.82	4	4	4	0.18	0	0	0	4.68	5	5	5	0.32	0	0	0	0.44	0	0	0	9
MA	6.58	6	6	6	0.41	1	1	1	7.22	7	7	7	0.78	1	1	1	1.03	1	1	1	16
La1	19.87	20	20	20	0.13	0	0	0	16.12	16	16	16	1.88	2	2	2	1.80	2	2	2	40
La2	8.93	9	9	9	0.07	0	0	0	4.56	5	5	5	0.44	0	0	0	0.94	1	1	1	15
AB	6.74	7	6	7	0.26	0	1	0	4.96	5	5	5	1.04	1	1	1	0.86	1	1	1	14
MO	1.74	2	1	2	0.26	0	1	0	0.39	0	0	0	0.61	1	1	1	0.18	0	0	0	3
C1	18.00	18	18	18	1.00	1	1	1	10.28	10	10	10	1.72	2	2	2	1.84	2	2	2	33
C2	16.32	16	17	16	0.68	1	0	1	8.65	9	9	9	1.35	1	1	1	2.20	2	2	2	29
PU	23.10	23	23	23	0.90	1	1	1	13.95	14	14	14	2.05	2	2	2	3.58	4	4	4	44
BA	2.94	3	3	3	0.06	0	0	0	2.60	3	2	3	0.40	0	0	0	0.44	0	1	0	6
CA	11.29	11	12	11	0.71	1	0	1	7.20	7	7	7	0.80	1	1	1	1.92	2	2	2	22
S1	13.37	**13**	14	**14**	1.63	1	1	1	6.95	7	7	7	1.05	1	1	1	2.97	3	3	3	26
S2	14.98	15	15	15	3.02	3	3	3	7.16	7	7	7	0.84	1	1	1	2.11	2	2	2	28
SA	8.87	9	9	9	0.13	0	0	0	7.21	7	7	7	0.79	1	1	1	1.06	1	1	1	18
p_i	272				60 − 8				211				28				36				617

Table 2: Italian elections 2008: absolute error minimization

	PDL				LN – MPA				PD				IDV				UDC				r_i
P1	9.47	9	4	9	2.53	3	4	3	9.45	9	14	9	1.55	2	1	2	1.28	1	1	1	24
P2	9.60	**10**	8	**9**	4.40	5	6	5	6.15	6	6	6	0.85	1	1	1	1.19	1	1	1	22
L1	16.58	16	24	16	7.42	8	6	8	13.08	13	5	13	1.92	2	3	2	1.43	1	2	1	40
L2	15.35	**15**	23	**14**	13.65	**14**	12	**15**	10.39	10	4	10	1.61	2	2	2	2.06	2	2	2	43
L3	5.83	**6**	9	**7**	3.17	3	4	3	4.53	5	2	5	0.47	0	0	0	0.69	**1**	0	**0**	15
TA	2.76	**3**	3	**4**	1.24	1	2	1	2.64	3	3	3	0.36	0	0	0	0.46	0	0	0	10
V1	9.31	**9**	15	**8**	9.69	**10**	9	**11**	7.81	**8**	3	**7**	1.19	1	1	1	1.75	2	1	2	29
V2	6.26	6	10	6	5.74	6	5	6	5.95	6	3	6	1.05	1	1	1	1.01	1	1	1	20
FV	5.09	5	8	5	1.91	2	3	2	4.40	**4**	2	**5**	0.60	**1**	0	**0**	0.81	1	0	1	13
LI	7.59	**7**	11	**8**	1.41	**2**	2	**1**	6.19	**6**	3	**7**	0.81	1	1	1	0.67	**1**	0	**0**	17
ER	15.03	15	22	15	4.06	4	6	4	20.14	20	11	20	1.86	2	3	2	1.92	2	1	2	43
TO	15.03	15	24	15	0.97	1	1	1	18.61	**19**	11	**18**	1.39	**1**	1	**2**	1.70	2	1	2	38
UM	3.82	4	6	4	0.18	0	0	0	4.68	5	3	5	0.32	0	0	0	0.44	0	0	0	9
MA	6.58	**6**	10	**7**	0.41	**1**	0	**0**	7.22	7	5	7	0.78	1	0	1	1.03	1	1	1	16
La1	19.87	20	17	20	0.13	0	0	0	16.12	16	21	16	1.88	2	1	2	1.80	2	1	2	40
La2	8.93	9	7	9	0.07	0	0	0	4.56	5	7	5	0.44	0	0	0	0.94	1	1	1	15
AB	6.74	7	4	7	0.26	0	0	0	4.96	5	8	5	1.04	1	1	1	0.86	1	1	1	14
MO	1.74	2	2	2	0.26	0	0	0	0.39	0	0	0	0.61	1	1	1	0.18	0	0	0	3
C1	18.00	18	11	18	1.00	1	1	1	10.28	10	16	10	1.72	2	2	2	1.84	2	3	2	33
C2	16.32	16	9	16	0.68	**1**	1	**0**	8.65	9	14	9	1.35	1	2	1	2.20	**2**	3	**3**	29
PU	23.10	23	13	23	0.90	1	1	1	13.95	14	22	14	2.05	2	3	2	3.58	4	5	4	44
BA	2.94	3	2	3	0.06	0	0	0	2.60	3	4	3	0.40	0	0	0	0.44	0	0	0	6
CA	11.29	**11**	6	**12**	0.71	**1**	1	**0**	7.20	7	11	7	0.80	1	1	1	1.92	2	3	2	22
S1	13.37	**13**	8	**12**	1.63	**1**	2	**2**	6.95	7	11	7	1.05	1	1	1	2.97	3	4	4	26
S2	14.98	**15**	11	**14**	3.02	3	2	4	7.16	7	11	7	0.84	1	1	1	2.11	2	3	2	28
SA	8.87	9	5	9	0.13	0	0	0	7.21	7	11	7	0.79	1	1	1	1.06	1	1	1	18
p_i	272				60 − 8				211				28				36				617

Table 3: Italian elections 2008: relative error minimization