Star Partitions on Graphs

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Abstract. Given an undirected graph, a star partition is a partition of the nodes into subsets with at least two nodes so that the subgraph induced by each subset has a spanning star. Star partitions are related to well-known problems concerning domination in graphs and edge covering. We focus on the Constrained Star Partition Problem (CSP) that asks for finding a star partition of given cardinality. The problem is new and presents interesting peculiarities. We explore the relation between the cardinalities of star partitions and domatic bipartitions, showing that there are star partitions of any cardinality between minimum and maximum values, and that a similar but weaker result holds for domatic bipartitions. We study the computational complexity of different versions of star partition and domatic bipartition problems, proving that most of them, in particular CSP, constrained domatic bipartition and balanced domatic bipartition, are NP-complete. We also show that star partition problems are polynomial on trees and, more generally, on bounded treewidth graphs. We introduce an integer linear programming formulation that defines a polytope containing all the star partitions of a graph, showing that its vertices have only integral components for trees, which implies that linear programming can be used to solve weighted star partition problems on trees.

1 Introduction

A *star* is a tree that has at most one node of degree greater than one. We say that a star is *proper* if it contains at least two nodes. In a star with three or more nodes, we say that the node with degree greater than one is the *center* of the star. The *center* of a star with two nodes is any arbitrarily chosen node.

Given an undirected graph $G = (V,E)$, a *star partition* is a partition of V into subsets so that the subgraph induced by each subset has a spanning proper star. In the present paper we study the cardinality of star partitions, defined as the number of parts in the partition of V. In particular, we focus on the Constrained Star Partition Problem (CSP) defined as follows: given an undirected graph $G = (V,E)$ and a positive integer s, find, if it exists, a star partition of cardinality s. Graphs considered in this paper are assumed to be undirected and without isolated nodes, if not differently stated, since no star partition exists for graphs with isolated nodes. Star partitions and their cardinalities have been brought to our attention while working on the problem of finding an optimal shift scheduling of pharmacies (see e.g. Andreatta et al. (2015)).

There is a vast literature devoted to graph partitioning: see, for instance, the recent book by Bichot and Siarry Bichot and Siarry (2011) and the references therein, and the review by Andreatta et al Andreatta et al. (2016a). However, to the best of our knowledge, a few papers in the literature deal with node partitioning based on stars. In Baidari et al. (2012) the subgraph induced by each part of the node partition has to be a star (rather than to have a spanning star) and, moreover, singletons can take part in the partition (whereas we just allow proper stars). Maravalle et al Maravalle et al. (1997) consider the problem of optimally partitioning a tree into a prescribed number of subtrees.

Star partitions have also relations to other well-known problems concerning domination in graphs, in particular dominating sets and domatic bipartitions, and minimal edge covers. In Section 2, we will discuss these relations and highlight the differences with respect to CSP. In Section 3, we will prove that, given a graph, there are star partitions of any cardinality between the minimum and the maximum, and that a similar but weaker property holds for domatic bipartitions. In Section 4, we show that the decision version of CSP is NP-complete and we state other complexity results for related problems concerning dominating sets and domatic bipartitions. As a corollary, we will solve a question about the complexity of Balanced Location Partitioning problem, left open in Andreatta et al. (2015). Section 5 focuses on bounded treewidth graphs and, in particular, on trees, showing that finding star partitions for any cardinality is polynomial. Section 6 introduces an Integer Linear Programming formulation to describe the set of all the star partitions of a graph, and proves that its associated polyhedron has integral vertices for trees. The concluding Section 7 gives some final remarks.

2 Relations and differences with other known problems

In this section, we show that finding a star partition of minimum or maximum cardinality is related to well studied problems concerning edge covering and domination in graphs, whereas CSP has peculiarities coming from the cardinality constraint.

A subset $E' \subseteq E$ is an *edge cover* of G if E' spans all the nodes in V. A *minimal edge cover* is an edge cover that is minimal with respect to inclusion. The *edge cover number* is the size of a minimum cardinality edge cover of G and it is usually denoted by $\beta'(G)$ West (2001).

Proposition 1 *Given a graph $G = (V,E)$, finding a maximum cardinality star partition is equivalent to finding a minimum cardinality edge cover.*

Proof. The connected components of any minimal edge cover are proper stars, since paths of length three or more are not allowed by minimality. Moreover, given a star partition, there are one or more spanning proper stars induced by each part. Therefore, a star partition induces one or more spanning forests whose connected components are proper stars. Hence, there is a one-to-one correspondence between these spanning forests and the minimal edge covers of the graph. For any minimal edge cover $C \subset E$ and related star partition Σ, we have $|C| = |V| - |\Sigma|$. Therefore, maximizing $|\Sigma|$ is equivalent to minimizing $|C|$, and we can obtain a maximum cardinality star partition from a minimum cardinality edge cover, and vice versa. □

A subset $V' \subseteq V$ is *dominating* if each node of G is either in V' or it is adjacent to some node in V'. The *domination number* is the minimum size of a dominating set of G and is denoted by $\gamma(G)$ West (2001).

Lemma 2 *For any graph G without isolated nodes, the minimum cardinality of a star partition is equal to the domination number $\gamma(G)$.*

Proof. Given a graph $G = (V,E)$, a subset $S \subset V$ and a node $v \in S$, we say that v has an *external private neighbor* with respect to S if there exists a node $u \in V \setminus S$ such that $(u,v) \in E$ and $(u,w) \notin E$, for any $w \in S \setminus \{v\}$. As shown in Bollobas and Cockayne (1979), for any graph without isolated nodes there exists a minimum cardinality dominating set D every node of which has an external private neighbor. In other words, any node $v \in D$ dominates at least one node u that is not dominated by any other node in D. Therefore, we can build a star partition Σ of G by considering, for each $v \in D$, the proper star centered in v and including the external private neighbors of v. Remaining dominated nodes can be associated to any arbitrarily chosen adjacent node in D. We have $|\Sigma| = |D| = \gamma(G)$ and, hence, $\gamma(G)$ is an upper bound for the minimum cardinality of a star partition of G. Furthermore, $\gamma(G)$ is also a lower bound since, given any star partition of G and a spanning star for each of its parts, the set of the star centers is a dominating set of the same cardinality as the star partition. □

Proposition 3 *Given a graph $G = (V,E)$ without isolated nodes, finding a minimum cardinality star partition is equivalent to finding a minimum cardinality dominating set.*

Proof. Given a minimum cardinality star partition, consider a spanning star for each part of the partition: the set containing the star centers is dominating and, by Lemma 2, it has minimum cardinality.

Vice versa, let D be any minimum cardinality dominating set of G. The following procedure builds a minimum cardinality star partition of G.

Step 1: Define a partition of V such that each part contains exactly one dominating node in D and dominated nodes are associated to any arbitrarily chosen adjacent node in D.

Step 2: If each part contains at least two nodes, then the partition of V is a star partition. Otherwise, consider a part of the partition of V containing exactly one node, say d. Node d belongs to set D and, by hypothesis, it is not isolated in

G. Node d is not adjacent to any dominating node in D, otherwise $D \setminus \{d\}$ would be a dominating set, contradicting the minimality of D. Furthermore, node d is not adjacent to a node $v \in V \setminus D$ belonging to a part containing only two nodes, say v and u, otherwise the set $D \setminus \{d,u\} \cup \{v\}$ would be a dominating set with cardinality strictly smaller than $|D|$. Therefore, node d must be adjacent to a node $v \in V \setminus D$ belonging to a part that contains at least three nodes. By moving node v from its part to the part of d, we get a new partition of V with same cardinality $|D|$. The number of parts containing exactly one node is decreased by one.
Step 3: repeat Step 2 on the new partition until each part contains at least two nodes, i.e. the partition of V is a star partition.
The cardinality of the obtained star partition is $|D| = \gamma(G)$ and, by Lemma 2, it is minimum. □

Proposition 1 establishes relations between the cardinalities of minimal edge covers and star partitions. While minimum size edge covers are well studied, no literature is devoted, to the best of our knowledge, to a constrained version of the problem and, hence, to CSP.

Finding a minimum size dominating set is also a well-known problem. However, Proposition 3 exploits properties of minimum cardinality dominating sets and relates to minimum cardinality star partitions, whereas it does not apply to star partitions of arbitrary cardinality and, hence, it cannot be directly used to solve CSP.

The study of star partitions with arbitrary cardinality has strong connections with domatic bipartitions. A *domatic bipartition* is a partition of V into two dominating sets B and W (see e.g. Cockayne and Hedetniemi (1977); Garey and Johnson (1979)). We define the *cardinality* of a domatic bipartition as $\min\{|B|,|W|\}$.

Given a graph $G = (V,E)$ and an integer number p, the Constrained Domatic Bipartition Problem asks to find a domatic bipartition, if it exists, of cardinality p. Andreatta et al. in Andreatta et al. (2016) fully investigated this problem on trees.

Given a star partition Σ of a graph $G = (V,E)$, it is straightforward to associate one or more domatic bipartitions, whose cardinalities can be related to Σ as follows. Let η_i be the number of nodes in the i-th part of Σ and consider, for each part, a spanning proper star. We color each star such that it has a white center and black leaves (white star), or a black center and white leaves (black star). Let B (resp. W) be the set of black (resp. white) nodes: it is easy to see that B and W configure a domatic bipartition of G having cardinality $\min\{|B|,|W|\}$.

Let S_b and S_w be the set of indexes corresponding to, respectively, black and white stars, and let n_b be the number of black nodes. The following relation holds:

$$n_b = |S_b| + \sum_{i \in S_w} (\eta_i - 1) = |S_b| + |S_w| + \sum_{i \in S_w} (\eta_i - 2)$$

and so

$$n_b = |\Sigma| + \sum_{i \in S_w} (\eta_i - 2). \tag{1}$$

By choosing $S_w = \emptyset$ (i.e. all stars are black) we get $n_b = |\Sigma|$, and, by choosing $S_b = \emptyset$ (i.e. all stars are white) we get $n_b = |V| - |\Sigma|$ and in both cases we have a domatic bipartition of cardinality $|\Sigma|$. Between these two extremes we can get those values n_b that result from (1) by all possible choices of S_b and S_w. We observe that not all integer values between $|\Sigma|$ and $|V| - |\Sigma|$ are feasible for n_b in general and therefore all feasible domatic bipartitions cardinality values are in general not contiguous integer numbers (we will come back to this important point in Section 3). For instance, consider the graph in Figure 1 where there is only one star partition: it has two parts and each part includes four nodes. The values $\eta_i - 2$ are $\{2,2\}$. Hence, from (1), we can color $n_b \in \{2,4,6\}$ black nodes and obtain domatic bipartitions of cardinality 2, 4 and 2 respectively. In the figure the cases $n_b = 2$ and $n_b = 4$ are shown, the case $n_b = 6$ can be obtained from $n_b = 2$ by switching colors.

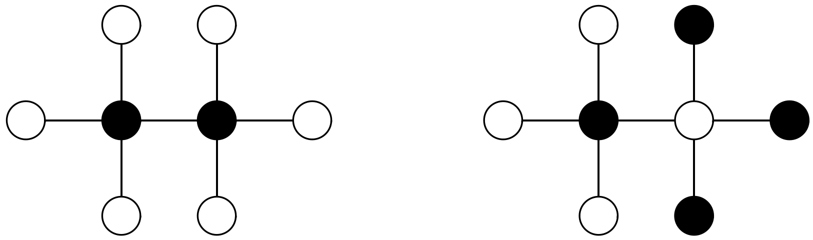

Fig. 1. A graph and two domatic bipartitions with two and four black nodes resp.

Let us now start from a domatic bipartition with $|B| = n_b$. By choosing all the edges (u,v) such that $u \in B$ and $v \in W$, we have an edge cover, that can be made minimal, and therefore we find a star partition Σ. Since in each part there is at least one black node, we have $|\Sigma| \leq n_b$.

Notice that this construction provides us with a procedure to build a minimum cardinality star partition from a minimum cardinality dominating set D, as an alternative to the one described in the proof of Proposition 3. Indeed, since the complement of a minimal (with respect to inclusion) dominating set is dominating as well Ore (1962), D and $V \setminus D$ configure a domatic bipartition of minimum cardinality $\gamma(G)$. The star partition Σ obtained by applying the above construction to this domatic bipartition has $|\Sigma| \leq |D| = \gamma(G)$ and, by Lemma 2, Σ is a minimum cardinality star partition.

Notice also that Lemma 2 can be obtained from the relations between the number of black nodes in a domatic bipartition and the cardinality of a star partition discussed above.

3 The contiguity property

Given a graph $G = (V,E)$, recall that the minimum cardinality of a star partition is equal to the domination number $\gamma(G)$. We denote by $\mu(G)$ the maximum cardinality of a star partition of G. By the proof of Proposition 1, $\mu(G) = |V| - \beta'(G)$, where $\beta'(G)$ is the edge cover number. We will prove that G has a star partition of cardinality s, for any integer s between $\gamma(G)$ and $\mu(G)$.

Given a set $V' \subseteq V$, denote by $G(V')$ the subgraph of G induced by V'. Observe that, for any part $P \subseteq V$ of a star partition of G, the choice of a spanning star for $G(P)$ may be not unique: if $v \in P$ is the center of any such spanning star, we say that v is a *feasible center* of $G(P)$.

Observation 4 *Given a graph $G = (V,E)$ and a spanning subgraph $H = (V,E')$, $E' \subseteq E$, any star partition of H is also a star partition of G.*

Lemma 5 *Given a graph $G = (V,E)$ and an edge $e \in E$ such that the graph $G^- = (V, E \setminus \{e\})$ has no isolated nodes, we have $\gamma(G) \leq \gamma(G^-) \leq \gamma(G) + 1$.*

Proof. Since, by Observation 4, any star partition of G^- is also a star partition of G, $\gamma(G) \leq \gamma(G^-)$.
Let Σ be a minimum cardinality star partition of G and $e = (u,v)$. If u and v belong to different parts of Σ, then Σ is also a star partition of G^- and $\gamma(G) = \gamma(G^-)$. Otherwise, let P be the part of Σ containing both u and v.
If there is at least one spanning star S of $G(P)$ that does not contain e, S is also a spanning star of $G^-(P)$ and $\gamma(G) = \gamma(G^-)$. Otherwise, e belongs to all the spanning stars of $G(P)$ and we distinguish the following three cases.

Case 1: $P = \{u,v\}$.

We recall that u and v are not isolated in G^- by hypothesis, that is, there exist $t,w \in V \setminus P$ such that $(u,t),(v,w) \in E \setminus \{e\}$ (possibly $t = w$). Let P_t and P_w be the parts of Σ containing, respectively, t and w. We consider the following subcases.

Case 1.1: both t and w are feasible centers of, respectively, $G(P_t)$ and $G(P_w)$.
We have $t \neq w$, as otherwise a star partition of G with cardinality $|\Sigma|-1$ could be obtained by moving u and v to part $P_t = P_w$ and removing part P, contradicting the minimality of Σ in G. Moreover, $P_t = P_w$, as otherwise a star partition of G with cardinality $|\Sigma|-1$ could be obtained by moving u to part P_t and v to part P_w and removing part P, contradicting the minimality of Σ in G. Hence, a star partition Σ^- of G^- can be obtained from Σ by moving u to $P_t = P_w$ and w to P. Since $|\Sigma^-| = |\Sigma|$, we have $\gamma(G^-) = \gamma(G)$.

Case 1.2: one node between t and w, let it be t without loss of generality, is a feasible center of $G(P_t)$, and w is not a feasible center of $G(P_w)$ (possibly $P_t = P_w$).
Notice that $|P_w| \geq 3$, as otherwise w would be a feasible center of $G(P_w)$. A star partition Σ^- of G^- can be obtained from Σ as follows: move u from P to P_t; move w from P_w to P. Since $|\Sigma^-| = |\Sigma|$, we have $\gamma(G^-) = \gamma(G)$.

Case 1.3: neither t nor w are feasible centers of $G(P_t)$ and $G(P_w)$ respectively.
As observed above, we have $|P_t| \geq 3$ and $|P_w| \geq 3$.
If $t = w$, then a star partition Σ^- of G^- can be obtained from Σ by moving $t = w$ to P, so that $|\Sigma^-| = |\Sigma|$ and we have $\gamma(G^-) = \gamma(G)$.
If $t \neq w$ and $P_t \neq P_w$, a star partition Σ^- of G^- can be obtained from Σ as follows: remove part P, remove t from P_t and w from P_w; create a new part containing u and t; create a new part containing v and w. Since $|\Sigma^-| = |\Sigma|+1$, we have $\gamma(G^-) \leq \gamma(G)+1$.
The same argument holds if $t \neq w$, $P_t = P_w$ and $|P_t| \geq 4$.
If $t \neq w$, $P_t = P_w = P'$ and $|P'| = 3$, a star partition Σ^- of G^- can be obtained from Σ by moving t from P' to P and v from P to P'. In this case, $|\Sigma^-| = |\Sigma|$, and $\gamma(G^-) = \gamma(G)$.
This completes the proof of Case 1.

Case 2: $|P| \geq 3$ and $G(P)$ has exactly one spanning star S.
Notice that the center of S is either u or v. Without loss of generality, let u be the center and recall that v is not isolated in G^- by hypothesis: let $(v,w) \in E \setminus \{e\}$, and P_w the part of Σ containing w.
We start by considering the subcase $P_w \neq P$. If w is a feasible center of $G(P_w)$, then we can move v from P to P_w and obtain, from Σ, a star partition of G^- of cardinality $\gamma(G)$, that is $\gamma(G^-) = \gamma(G)$. Otherwise, as observed above, $|P_w| \geq 3$, and we can obtain from Σ a star partition for G^- by moving v and w to a new star, so that $\gamma(G^-) \leq \gamma(G)+1$.
If $P_w = P$, then $|P| \geq 4$, as otherwise w would be adjacent in G to all the nodes of P, configuring a spanning star for $G(P)$ that does not contain e, whereas we are considering the case where e belongs to all the spanning stars of $G(P)$. Hence, we can move v and w to a new part, obtaining from Σ a star partition for G^- showing that $\gamma(G^-) \leq \gamma(G)+1$.

Case 3: $|P| \geq 3$ and $G(P)$ has at least two spanning stars.
Since e belongs to both stars, either u is a leaf and v the center, or vice versa. One star must thus be centered in u (star S_u) and the other in v (star S_v). Notice that $|P| \geq 4$: indeed if $P = \{u,v,w\}$, then w is adjacent in G to both u and v. Hence the star centered in w spans $G(P)$ and does not contain e, which contradicts the hypothesis that e belongs to all the spanning stars of $G(P)$. In particular, there are at least two distinct nodes a and b in $P \setminus \{u,v\}$, adjacent to both u and v. We can thus obtain a star partition Σ^- of G^- from Σ as follows: remove u and a from P; create the new part $\{u,a\}$. Since $|\Sigma^-| = |\Sigma|+1$, we have $\gamma(G^-) \leq \gamma(G)+1$. □

Theorem 6 (Contiguity Property for star partitions) *Any graph $G = (V,E)$ without isolated nodes has star partitions of any cardinality between $\gamma(G)$ and $\mu(G)$.*

Proof. Let F be a spanning forest of G obtained by considering a maximum cardinality star partition of G and, for each part P, picking a spanning star of $G(P)$. Since the components of F are proper stars, F has no isolated nodes and a unique star partition, and therefore $\gamma(F) = \mu(F) = \mu(G)$. The forest F can be obtained from G by a sequence of single edge removals, giving rise to a sequence of graphs $G = G_0,\ G_1,\ \ldots,\ G_k = F$, where, for each $i = 1,\ldots,k$, G_i is obtained from G_{i-1} by removing one edge. We observe that any graph G_i spans the node set of the original graph G and has no isolated nodes. For each graph G_i, consider a minimum cardinality star partition Σ_i and obtain a sequence $\sigma = (\Sigma_0,\ldots,\Sigma_k)$ of star partitions, with $|\Sigma_0| = \gamma(G)$ and $|\Sigma_k| = \gamma(F) = \mu(G)$. By Observation 4, star partitions in σ are also star partitions of G, and, by Lemma 5, $|\Sigma_{i-1}| \leq |\Sigma_i| \leq |\Sigma_{i-1}| + 1$, for every $i = 1,\ldots,k$. Therefore, σ contains star partitions of any cardinality between $\gamma(G)$ and $\mu(G)$. □

We have already observed that the minimum cardinality of a domatic bipartition is $\gamma(G)$ and, by definition, the maximum cardinality is at most $\lfloor |V|/2 \rfloor$. Given the close connection between star partitions and domatic bipartitions, we may wonder whether the contiguity property also holds for domatic bipartitions, that is, whether there exists a domatic bipartition of any cardinality between the minimum and the maximum values.

The answer is negative in general as shown by the counterexample in Figure 1: the graph has domatic bipartitions of cardinality 2 and 4 but it is easy to check that no domatic bipartition of cardinality 3 exists.

From the contiguity property for star partitions (Theorem 6), and observing that, from (1) with $S_w = \emptyset$, a domatic bipartition with the same cardinality of a star partition always exists, we obtain the following corollary.

Corollary 1. *Any graph $G = (V,E)$ without isolated nodes has domatic bipartitions of any cardinality between $\gamma(G)$ and $\mu(G)$.*

The corollary allows us to identify two cases for which the contiguity property for domatic bipartitions holds.

Proposition 7 *If $G = (V,E)$ admits a perfect matching, then the maximum cardinality of a domatic bipartition of G is $|V|/2$ and the contiguity property for domatic bipartitions holds for G.*

Proof. A perfect matching of G corresponds to a star partition of cardinality $|V|/2$. Therefore $\mu(G) = |V|/2$ and the statement directly follows from Corollary 1. □

Proposition 8 *If $G = (V,E)$ admits an edge cover whose connected components have at most two edges, then the maximum cardinality of a domatic bipartition of G is $\lfloor |V|/2 \rfloor$ and the contiguity property for domatic bipartitions holds for G.*

Proof. Let $C \subseteq E$ be an edge cover according to the hypothesis. C is clearly minimal and corresponds to a star partition Σ, with $\gamma(G) \leq |\Sigma| \leq \mu(G)$. Each part includes two or three nodes. If there are no parts having three nodes, then C is a perfect matching and the thesis follows from Proposition 7. Otherwise, we apply (1) to obtain domatic bipartitions of different cardinalities. To this end, let us consider, for each part, its spanning star and the related center. We can color each star either black or white: a black star has black center and white leaves and vice versa for a white star. Let S_b and S_w be the sets of, respectively black and white stars. We recall that η_i is the number of nodes of the i-th star of Σ and we observe that, for stars with three nodes, $\eta_i - 2 = 1$. Hence domatic bipartitions of any cardinality between $|\Sigma|$ and $\lfloor |V|/2 \rfloor$ can be obtained by coloring white a suitable number of three-nodes stars. □

4 Computational complexity issues

We have seen in the previous section that star partitions satisfy the contiguity property. Clearly, deciding whether a graph admits a star partition of cardinality s has an obvious negative answer if $s > \mu(G)$ and an obvious positive answer if $s = \mu(G)$. Since $\mu(G)$ can be computed in polynomial time, the case $s \geq \mu(G)$ is polynomial and we are left with assessing the computational complexity for the case $s < \mu(G)$. Because of the contiguity property there is not much difference in deciding whether there exists a star partition of cardinality $s < K$ or $s = K$ for an assigned constant K.

The question is different for domatic bipartitions of given cardinality p. We have seen that there are graphs for which the contiguity property for domatic bipartitions holds for any value between $\gamma(G)$ and $\lfloor |V|/2 \rfloor$. However, in general, we do not know whether there are domatic bipartitions of cardinality p for $\mu(G) < p \leq \lfloor |V|/2 \rfloor$. For the particular value $p = \mu(G)$ the problem of deciding whether there exists a domatic bipartition of cardinality p can be solved in polynomial time whereas, as we show below, the same problem is hard both for values lower than $\mu(G)$ and for values higher than $\mu(G)$. This difference makes it necessary to analyze the problem of finding domatic bipartitions of given cardinality by specifying the range for p.

Let us formally state the problems we are considering in this paper as the following decision problems.

Minimum Star Partition (MSP): Given a graph $G = (V, E)$ and an integer s, does there exist a star partition of G with cardinality at most s?

Constrained Star Partition (CSP): Given a graph $G = (V, E)$ and an integer s, does there exist a star partition of G with cardinality exactly s?

Note that CSP as stated above is the decision version of the search problem stated in Section 1. We use the same acronym for both problems. This should raise no ambiguity because we use the decision version only in this section.

Minimum Dominating Set (MDS): Given a graph $G = (V, E)$ and an integer K, does there exist a dominating set $D \subset V$ of cardinality at most K?

Minimum Domatic Bipartition (MDB): Given a graph $G = (V, E)$ and an integer p, does there exist a domatic bipartition of G of cardinality at most p?

Constrained Domatic Bipartition (CDB): Given a graph $G = (V, E)$ and an integer p, does there exist a domatic bipartition of G of cardinality p?

Balanced Domatic Bipartition (BDB): Given a graph $G = (V, E)$ and an integer $K \geq 0$, does there exist a domatic bipartition (B, W) of V such that $||W| - |B|| \leq K$?

Notice that BDB is equivalent to deciding whether there exists a domatic bipartition having cardinality at least $(|V| - K)/2$.

It is known that MDS is NP-complete (Garey and Johnson (1979) p. 190). Since a minimum dominating set and its complement constitute a domatic bipartition Ore (1962), MDB is also NP-complete.

Theorem 9 *MSP is NP-complete.*

Proof. The thesis is an obvious consequence of Proposition 3, stating the equivalence between MSP and MDS. □

Theorem 10 *CSP is NP-complete for $s < \mu(G)$.*

Proof. We transform MSP to CSP. If there exists a star partition of cardinality at most $s < \mu(G)$, by the contiguity property there exists also a star partition of cardinality exactly s. The converse implication is trivial. □

Theorem 11 *CDB is NP-complete for $p < \mu(G)$.*

Proof. We transform MDB to CDB. If there exists a domatic bipartition of cardinality at most $p < \mu(G)$ in G, by Corollary 1 there exists also a domatic bipartition of cardinality exactly p. The converse implication is trivial. □

Theorem 12 *BDB is NP-complete.*

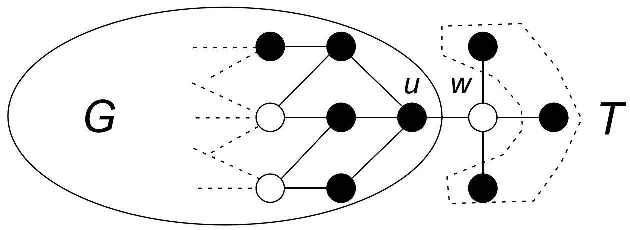

Fig. 2. The graph $\mathscr{G}$ with the critical domatic bipartition

Proof. We transform MDB for a graph G with constant $p < \mu(G)$ into BDB on a graph $\mathscr{G}$ with constant K equal to p. Let n be the number of nodes in G. The graph $\mathscr{G}$ consists of G, a node w appended to an arbitrary node u of G and a set T of $n-p+1$ nodes appended to w (refer to Fig. 2). The graph $\mathscr{G}$ has $n+1+n-p+1 = 2(n+1)-p$ nodes.

($\Longrightarrow$) Assume that there exists a domatic bipartition (B,W) in G of cardinality at most p, i.e., with $1 \le |B| \le p$ and $n-1 \ge |W| \ge n-p$. The bipartition $(B',W') = (B \cup T, W \cup \{w\})$ in $\mathscr{G}$ is domatic and $n-p+2 \le |B'| \le n+1$, $n \ge |W'| \ge n-p+1$, so that $2-p \le |B'|-|W'| \le p$ and $||B'|-|W'|| \le p$.

($\Longleftarrow$) Now assume there exists a domatic bipartition (B',W') in $\mathscr{G}$ such that $-p \le |W'|-|B'| \le p$. Necessarily either $T \subset B'$ and $w \in W'$ or $T \subset W'$ and $w \in B'$. Without loss of generality, we may assume the former case. The bipartition (B',W') in $\mathscr{G}$ induces a bipartition $(B' \cap V, W' \cap V) = (B,W)$ in G.
The bipartition (B,W) is not necessarily domatic. It is not domatic only in the critical case in which u and w are in different subsets of the bipartition of $\mathscr{G}$ and all neighbor nodes of u in G are in the same subset as u. The critical case is shown in Fig. 2.
We have $|B| = |B'|-n+p-1$ and $|W| = |W'|-1$ so that

$$|W|-|B| = |W'|-1-|B'|+n-p+1 \ge n-2p.$$

Since $|W|+|B| = n$ we get $|B| \le p$.
If (B,W) is domatic, we have found a domatic bipartition of cardinality at most p. If it is not domatic, then, as observed, $u \in B$ and all neighbors of u in G are in B. Hence the bipartition $(B",W") = (B \setminus \{u\}, W \cup \{u\})$ obtained by switching u from B to W is domatic. In this case $|B"| \le p-1$ and we have found a domatic bipartition of cardinality at most p. $\square$

We note that the proof of NP-completeness of problem BDB implicitly considers only $K \ge 1$. We wonder whether the same complexity result holds also for $K = 0$, i.e., for a perfectly balanced bipartition. The answer is affirmative as we next show but another proof is required. This proof makes use of the same trick of the previous proof but with different parameters and needs the contiguity property.

Theorem 13 *BDB for $K = 0$ is NP-complete.*

Proof. We transform CDB for a graph G with constant $p < \mu(G)$ and n nodes into BDB by building a graph $\mathscr{G}$ that consists of G, a node w appended to an arbitrary node u of G and a set T of $n-2p+1$ nodes appended to w (refer again to Fig. 2). The graph $\mathscr{G}$ has $n+1+n-2p+1 = 2(n-p+1)$ nodes.
Assume that there exists a domatic bipartition (B,W) in G of cardinality p, i.e., with $|B| = p$ and $|W| = n-p$. Now the bipartition $(B \cup T, W \cup \{w\})$ in $\mathscr{G}$ is domatic and $|B \cup T| = |W \cup \{w\}| = n-p+1$, i.e., it is perfectly balanced.
Now assume there exists a domatic bipartition (B',W') in $\mathscr{G}$ with $|B'| = |W'|$. Let, without loss of generality, $T \subset B'$ and $w \in W'$. This bipartition in $\mathscr{G}$ induces a bipartition $(B' \cap V, W' \cap V) = (B,W)$ in G with $|B| = p$ and $|W| = n-p$. As

in the previous proof, this bipartition is not necessarily domatic. If it is domatic we have found a domatic bipartition in G with cardinality p. If it is not domatic then the new bipartition $(B \setminus \{u\}, W \cup \{u\})$ obtained by switching u from B to W is domatic. Now $|B \setminus \{u\}| = p-1$. By the contiguity property there exists a domatic bipartition of cardinality p. □

Theorem 14 *CDB is NP-hard for $p > \mu(G)$.*

Proof. BDB can be solved by calling an algorithm for CDB a polynomial number of times, with p taking values from $\lceil (|V|-K)/2 \rceil$ to $\lfloor |V|/2 \rfloor$. Since this is a Turing reduction from BDB to CDB and BDB is NP-complete (Theorem 12), it follows that CDB is NP-hard. □

4.1 Results on Balanced Location Partitioning problem

The result of NP-completeness for BDB allows to finalize the discussion on Balanced Location Partitioning problem defined in Andreatta et al. (2015), closing the only missing tile with regard to the computational complexity of the various location partitioning problems discussed in that paper. We recall the following definitions:

Location Partitioning (LP): Given a set F of facilities and a set C of customers, distances $d_{ij} \geq 0$ for all $i \in C$ and $j \in F$, an integer $H \leq |F|$, a number L, is there a partition of F into subsets $J_1, J_2, \ldots, J_H$, such that $\sum_{1 \leq h \leq H} \sum_{i \in C} \min_{j \in J_h} d_{ij} \leq L$?

Balanced Location Partitioning (BLP): Given a set F of facilities and a set C of customers, distances $d_{ij} \geq 0$ for all $i \in C$ and $j \in F$, an integer $H \leq |F|$, a number L, is there a partition of F into subsets $J_1, J_2, \ldots, J_H$, such that $|J_h| \in \{\lfloor |F|/H \rfloor, \lceil |F|/H \rceil\}$ for any $h \in \{1, \ldots, H\}$, and $\sum_{1 \leq h \leq H} \sum_{i \in C} \min_{j \in J_h} d_{ij} \leq L$?

Both problems can be further specialized according to the assumptions on the distances d_{ij} and on the customer and facility sets. Andreatta et al Andreatta et al. (2015) considered the following three versions of the problems, in the order of decreasing generality:

(a) the distances d_{ij} are nonnegative;
(b) the customers locations and facilities are two subsets (not necessarily disjoint) of the nodes of a given graph and the d_{ij} are measured on the shortest paths with respect to nonnegative edge lengths;
(c) as in (b) but the two subsets coincide and include all the nodes of the graph.

The computational complexity of these problems has been extensively studied in Andreatta et al. (2015), where the following result has been proven.

Proposition 15 (Andreatta et al. (2015)) *Both LP and BLP are NP-complete for all three versions if $H \geq 3$. Both LP and BLP are NP-complete for versions (a) and (b) if $H = 2$. LP, version (c), is polynomial if $H = 2$.*

The only open case in Andreatta et al. (2015) is BLP, version (c), with $H = 2$. For ease of the reader, we reformulate the problem as follows.

Balanced Location Partitioning, version (c), $H = 2$ (BLPc2): Given a graph $G = (V, E)$ with nonnegative edge lengths that define distances $d_{ij} \geq 0$, $i, j \in V$, and a number L, is there a bipartition of V into two subsets J and $V \setminus J$, such that $|J| = \lfloor |V|/2 \rfloor$ and $\sum_{i \in V} (\min_{j \in J} d_{ij} + \min_{j \notin J} d_{ij}) \leq L$?

Theorem 16 *BLPc2 is NP-complete.*

Proof. We transform BDB, with constant $K = 0$ if $|V|$ is even and $K = 1$ if $|V|$ is odd, into BLPc2 by using the same graph, taking unitary edge lengths and setting $L = |V|$.
For any balanced domatic bipartition $(J, V \setminus J)$ we have (actually this is true for any domatic bipartition, even if not balanced)

$$\sum_{i \in V} (\min_{j \in J} d_{ij} + \min_{j \notin J} d_{ij}) = |V| = L.$$

Conversely let $(J, V \setminus J)$ be a balanced bipartition such that

$$\sum_{i \in V} (\min_{j \in J} d_{ij} + \min_{j \notin J} d_{ij}) \leq |V| = L.$$

Since $\min_{j \in J} d_{ij} = 0$ and $\min_{j \notin J} d_{ij} \geq 1$ if $i \in J$ and similarly $\min_{j \notin J} d_{ij} = 0$ and $\min_{j \in J} d_{ij} \geq 1$ if $i \notin J$, we have

$$|V| \geq \sum_{i \in V} (\min_{j \in J} d_{ij} + \min_{j \notin J} d_{ij}) = \sum_{i \notin J} \min_{j \in J} d_{ij} + \sum_{i \in J} \min_{j \notin J} d_{ij} \geq |V|$$

that implies

$$\min_{j \notin J} d_{ij} = 1, \quad i \in J, \qquad \min_{j \in J} d_{ij} = 1, \quad i \notin J,$$

or in other words that each node in J is adjacent to a node in $V \setminus J$ and also each node in $V \setminus J$ is adjacent to a node in J. Hence the bipartition is balanced and domatic. $\square$

5 Star Partition problems on trees and bounded treewidth graphs

The relations between star partitions and other known problems, in particular minimum edge cover and minimum dominating set, allow us to solve star partition problems in polynomial time for trees and, more generally, bounded treewidth graphs. We recall that the class of bounded treewidth graphs includes trees, forests, series parallel graphs, pseudoforests, cactus graphs, outerplanar graphs, Halin graphs etc. Some NP-complete problems can be solved in polynomial time when restricted to bounded treewidth graphs. For the definition and properties of bounded treewidth graphs see e.g. Diestel (2016).

Lemma 17 *A minimum cardinality star partition can be found in linear time on bounded treewidth graphs.*

Proof. Finding a minimum dominating set on a bounded treewidth graph $G = (V, E)$ can be done in $O(|V|)$ Alber and Niedermeier (2002). By applying the procedure in the proof of Proposition 3, a minimum dominating set is transformed into a minimum cardinality star partition in $O(|V|)$. $\square$

Lemma 18 *A maximum cardinality star partition can be found in polynomial time on any graph and in linear time on trees.*

Proof. Finding a maximum matching on any graph $G = (V, E)$ can be done in $O(|V|^{1/2}|E|)$ in general Micali and Vazirani (1980) and in linear time on trees Savage (1980). The maximum matching can be transformed into a minimum edge cover in $O(|V|)$, by adding an edge for each unmatched node. By Proposition 1, a minimum edge cover corresponds to a maximum cardinality star partition. $\square$

Corollary 2. *The decision version of CSP can be solved in polynomial time on bounded treewidth graphs and in linear time on trees.*

Proof. The result comes directly from Lemma 17, Lemma 18 and the contiguity property for star partitions stated in Theorem 6. □

Theorem 19 *The problem of finding a star partition of any cardinality can be solved in polynomial time on bounded treewidth graphs $G = (V,E)$, and in $O(|V|^2)$ on trees.*

Proof. Following the proof of Theorem 6, a star partition of any possible cardinality can be found through the following steps. First, we determine a maximum cardinality star partition of G and a related spanning forest $F = (V,E')$, which can be done in $O(|V|^{1/2}|E|)$ by Lemma 18. We then find a minimum cardinality star partition in $O(|V|)$ by Lemma 17. If the cardinalities of the maximum and minimum star partitions are equal, then the procedure stops. Otherwise we update G by removing one edge in $E \setminus E'$, we find a minimum cardinality star partition of G, and we iterate until $G = F$. We observe that if the treewidth of the original graph is bounded by constant k, then the treewidth of any graph considered at each iteration is bounded by the same k, since they are obtained by removing edges Diestel (2016). Therefore, each iteration takes $O(|V|)$ time by Lemma 17. Since the number of iterations is $O(|E|-|E'|) = O(|E|)$, the overall procedure is $O(|V||E|)$ and, in particular, $O(|V|^2)$ on trees. □

6 An Integer Linear Programming formulation for star partitions

In this section we will introduce an Integer Linear Programming (ILP) formulation to describe the set of all star partitions of a graph, define the star partition polytope associated to the formulation and investigate its integrality. We say that a polytope is *integral* if all its vertices have only integral components.

Let us consider an undirected graph $G = (V,E)$ and, for every node $i \in V$, let $N(i)$ be the set of nodes adjacent to node i.

We associate a binary variable y_i to each node $i \in V$, with y_i equal to 1 whenever node i is a star center, and a binary variable x_{ij} to each ordered pair $i \in V$ and $j \in N(i)$, with x_{ij} equal to 1 whenever node i is a leaf of a star with center in node j. Then the following ILP formulation represents all star partitions of G:

$$\sum_{j\in N(i)} x_{ij} + y_i = 1 \qquad i \in V, \tag{2}$$

$$x_{ij} \le y_j \qquad j \in N(i) \text{ and } i \in V, \tag{3}$$

$$y_i \le \sum_{j\in N(i)} x_{ji} \qquad i \in V, \tag{4}$$

$$x_{ij} \ge 0 \qquad j \in N(i) \text{ and } i \in V, \tag{5}$$

$$x_{ij} \in \mathbb{Z} \qquad j \in N(i) \text{ and } i \in V. \tag{6}$$

Equations (2) state that each node i is either a star center or a leaf of some star, inequalities (3) state that node j must be a star center if node i is a leaf of j, and inequalities (4) compel each star to be proper. Notice that $y_j \ge 0$ follows by (3) and (5), and conditions (2) and (6) imply that all variables are less than or equal to 1 and that y_i are integral.

We already noticed that, for any given star partition of G, there are one or more spanning forests such that each connected component is a star and spans a part of it. There is a one-to-one correspondence between the set of all these spanning forests of G and the set of all the solutions of (2–6). As a consequence, any star partition of G corresponds to one or more solutions of the proposed model, and each feasible solution defines a star partition of G.

The equality

$$\sum_{i\in V} y_i = s \tag{7}$$

can be included in the formulation to ask for star partitions with exactly s stars.

Through equalities (2) restated as

$$y_i = 1 - \sum_{j \in N(i)} x_{ij} \qquad \text{for all } \; i \in V, \tag{8}$$

the model can be refined obtaining the following alternative formulation, where only variables x_{ij} appear:

$$x_{ij} + \sum_{k \in N(j)} x_{jk} \leq 1 \qquad j \in N(i) \text{ and } i \in V, \tag{9}$$

$$\sum_{j \in N(i)} (x_{ij} + x_{ji}) \geq 1 \qquad i \in V, \tag{10}$$

$$x_{ij} \geq 0 \qquad j \in N(i) \text{ and } i \in V, \tag{11}$$

$$x_{ij} \in \mathbb{Z} \qquad j \in N(i) \text{ and } i \in V. \tag{12}$$

The cardinality constraint (7) turns into

$$\sum_{i \in V} \sum_{j \in N(i)} x_{ij} = |V| - s. \tag{13}$$

We say that each inequality (9) is an *arc-constraint* associated with the ordered pair of nodes ij, and each inequality (10) is a *node-constraint* associated with node i.

The two formulations are equivalent, i.e., they have the same solution set, since one can be obtained from the other through a sequence of pivot operations. From now on, we will focus on the second formulation and we will consider only x_{ij} variables, since the values of y_i variables are fixed by (8).

Given a graph G, let $Q(G)$ be the *star partition polytope* defined by inequalities (9–11). $Q(G)$ is the projection of the polytope defined by (2–5) onto the space of variables x_{ij}. Given an integer s, we also define the polytope $Q_s(G)$ as the intersection of $Q(G)$ with the set of solutions of the cardinality constraint (13).

6.1 Integrality of the star partition polytopes on paths

The matrix associated to the linear relaxation of formulation (2–7) is a $0, \pm 1$ matrix, that is, all its entries are in $\{0, +1, -1\}$, and the matrix associated to the linear relaxation of (8–13) is a $0, 1$ matrix. A $0, \pm 1$ matrix is *totally unimodular* if every square submatrix has determinant equal to 0, $+1$ or -1. We recall that a $0, 1$ matrix has the *consecutive ones property* if, after a possible permutation of the columns, the ones appear consecutively in each row. It is well known that consecutive ones property implies total unimodularity.

Lemma 20 *If the graph G is a path, then the constraint matrix of formulation (9–11) and (13) is totally unimodular.*

Proof. Since the graph is a path, the matrix associated to constraints (9–11) and (13) satisfies the consecutive ones property: this can be seen by associating indexes from 1 to $|V|$ to the nodes in the path such that node i is adjacent to node $i+1$ ($i \in \{1, \ldots, |V|-1\}$), and by lexicographically ordering variables x_{ij}. Since consecutive ones property implies total unimodularity, the thesis follows. □

Proposition 21 *If the graph G is a path, then both the polytopes $Q(G)$ and $Q_s(G)$ are integral.*

Proof. The assert directly follows from Lemma 20 and the integrality of the right-hand-sides of (9–11) and (13). □

The integrality property also holds for the linear relaxation of formulation (2–7).

Proposition 22 *If the graph G is a path, both the polytope associated to (2–5) and the one associated to (2–5) and (7) are integral.*

Proof. By the same arguments as for Lemma 20, the matrix associated to constraints (8–11) and (13) is totally unimodular and can be transformed into the matrix of constraints (2–5) and (7) by pivoting operations. Since pivoting preserves totally unimodularity Schrijver (1986), the assert follows. □

6.2 Integrality of the star partition polytopes on trees

Lemma 20 does not hold for trees, as shown by a star with three leaves: the corresponding constraint matrix of formulation (9–11) is not totally unimodular. Moreover, the polytope $Q_s(G)$ is not necessarily integral if the graph G is a tree. For instance, consider the tree T' obtained from a path of length 8 by appending a node adjacent to the central node. The tree T' admits star partitions having cardinality equal to 3, 4 and 5. The polyhedron $Q_4(T')$ has 16 fractional vertices and 8 integral vertices. Nevertheless, in the following we will prove that the star partition polytope $Q(T)$ of any tree T is integral.

First notice that formulation (9–11) for polytope $Q(G)$ is a linear system of the following type:

$$\begin{cases} \mathbf{A_1 x} \leq \mathbf{1} \\ \mathbf{A_2 x} \geq \mathbf{1} \\ \mathbf{x} \geq \mathbf{0} \end{cases}$$

where both $\mathbf{A_1}$ and $\mathbf{A_2}$ are $0,1$ matrices, $\mathbf{A_1 x} \leq \mathbf{1}$ are set packing inequalities and correspond to arc-constraints (9), $\mathbf{A_2 x} \geq \mathbf{1}$ are set covering inequalities and correspond to node-constraints (10).

A *circulant* is a $0,1$ square matrix with two ones per row and per column, that does not contain any proper submatrix with the same property. A circulant is odd if its order (i.e., the number of rows and columns) is odd. A $0,1$ matrix is *balanced* if it does not contain any odd circulant. This notion was first introduced by Berge Berge (1970). It is known that the family of $0,1$ balanced matrices strictly contains all the $0,1$ totally unimodular matrices. The following result is due to Fulkerson, Hoffman and Oppenheim Fulkerson et al. (1974).

Proposition 23 (Fulkerson et al. (1974)) *The polytope defined by the linear system*

$$\begin{cases} \mathbf{A_1 x} \leq \mathbf{1} \\ \mathbf{A_2 x} \geq \mathbf{1} \\ \mathbf{A_3 x} = \mathbf{1} \\ \mathbf{x} \geq \mathbf{0} \end{cases}$$

is integral if $\begin{pmatrix} \mathbf{A_1} \\ \mathbf{A_2} \\ \mathbf{A_3} \end{pmatrix}$ *is a $0,1$ balanced matrix.*

We are going to prove that, if the graph is a tree T, then the coefficient matrix $\mathbf{A}$ of formulation (9–11) is balanced and therefore the polytope $Q(T)$ is integral by Proposition 23.

Let $T = (V,E)$ be a tree. For convenience, we define a directed graph $\mathscr{G} = (V, E_{\mathscr{G}})$ with the same node set as T, and two arcs ij and ji in $E_{\mathscr{G}}$ for each edge $(i,j) \in E$. Each arc $ij \in E_{\mathscr{G}}$ corresponds to a variable x_{ij} and to a column in matrix $\mathbf{A}$.

Let $\mathbf{A_C}$ be any circulant in matrix $\mathbf{A}$. Rows and columns of $\mathbf{A}$ can be permuted so that the circulant $\mathbf{A_C}$ takes the following form:

$$\begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ 1 & & & & 1 \end{pmatrix}.$$

Let m be the order of $\mathbf{A_C}$. The columns in $\mathbf{A_C}$ correspond to an ordered set of arcs $(\alpha_1, \ldots, \alpha_m)$, with the property that both arcs α_i and α_{i+1} (for any $i = 1, \ldots, m-1$) appear in a same constraint, and therefore they share at least one common extreme node. This property also links arcs α_m and α_1.

By ignoring the orientations, the arcs $(\alpha_1, \ldots, \alpha_m, \alpha_1)$ induce an edge-circulant, that is an ordered set of edges in T, with the property that consecutive edges share at least one node. Let T_{supp} be the support graph of the edge-circulant induced by $\mathbf{A_C}$. Graph T_{supp} is a subgraph of T and is connected, hence, it is a subtree.

Lemma 24 *The graph T_{supp} is a path.*

Proof. We prove that T_{supp} does not contain any node of degree 3. For nodes with higher degree, the proof follows similarly.
By contradiction, assume that T_{supp} has three edges $e_1 = (v_1, v_4)$, $e_2 = (v_2, v_4)$ and $e_3 = (v_3, v_4)$, all incident in node v_4 of degree three.
For each pair e_i, e_j of edges in $\{e_1, e_2, e_3\}$ $(i \neq j)$, there is a corresponding row in $\mathbf{A_C}$ that contains two 1, one in a column corresponding to an arc associated with e_i, the other in a column corresponding to an arc associated with e_j. These three rows correspond to three arc-constraints, because node-constraint given by node v_4 would imply the presence of three 1 in the corresponding row of $\mathbf{A_C}$, and any other node-constraint given by a generic node $v \neq v_4$ could not have two 1 in the columns corresponding to e_1, e_2 and e_3, since any node (except for v_4) is incident in at most one of the edges e_1, e_2 and e_3.
Each edge e_i may correspond to arc $v_i v_4$ or to arc $v_4 v_i$. We distinguish two cases, whether there is at least one edge e_i corresponding to arc $v_i v_4$, or each edge e_i corresponds to arc $v_4 v_i$.

Case 1: there is at least one edge e_i corresponding to arc $v_i v_4$.
Without loss of generality let edge e_1 correspond to arc $v_1 v_4$. The constraint relating edges e_1 and e_2 is an arc-constraint. Therefore, edge e_2 must correspond to arc $v_4 v_2$, and arcs $v_4 v_3$ and $v_4 v_1$ cannot be in the columns of $\mathbf{A_C}$. Hence, e_3 corresponds to arc $v_3 v_4$. Then, the constraint relating edges e_1 and e_3 cannot exist, since it would be an arc-constraint relating two arcs both entering in node v_4.

Case 2: each edge e_i corresponds to arc $v_4 v_i$.
The constraint linking arcs $v_4 v_1$ and $v_4 v_2$ is an arc-constraint induced by an arc $u v_4$, with node u not belonging to T_{supp}. Then, this constraint should also involve arc $v_4 v_3$, and the corresponding row in $\mathbf{A_C}$ should have three entries equal to 1. □

Theorem 25 *Given a tree $T = (V, E)$, the corresponding coefficient matrix* $\mathbf{A}$ *in formulation (9–11) is balanced.*

Proof. We will prove that any circulant in matrix $\mathbf{A}$ is of order two. Therefore $\mathbf{A}$ does not contain any odd circulant and it is balanced.
Let $\mathbf{A_C}$ be a circulant in matrix $\mathbf{A}$ and T_{supp} the related support path. If T_{supp} contains only one edge (i, j), then the columns of $\mathbf{A_C}$ correspond to arcs ij and ji and $\mathbf{A_C}$ is a circulant of order two. Otherwise, let i be a leaf of T_{supp} and let (i, j) and (j, k) be edges in T_{supp}.

In the following, we consider all the arc-constraints and node-constraints where variables x_{ij} or x_{ji} appear. These constraints are subdivided in six classes, according to the variables they involve. The constraints (rows of $\mathbf{A}$) are referred to by indicating the arcs (columns of $\mathbf{A}$) having a 1 in that row. Only arcs related to edges in T_{supp} are indicated. The list of classes follows:

(c1) ij, ji, jk (arc-constraint given by arc ij)
(c2) ij, ji (arc-constraint given by arc ji or node-constraint given by node i)
(c3) ij, ji, jk, kj (node-constraint given by node j)
(c4) ji, jk, kj (arc-constraint given by arc kj)
(c5) ij (arc-constraints given by any arc hi, for all $h \notin V(T_{supp})$)
(c6) ji, jk (arc-constraints given by any arc hj, for all $h \notin V(T_{supp})$).

Observe that class (c2) contains exactly two constraints, and that the listed constraints (c5) cannot appear in the circulant, since the corresponding rows would have a single 1. Each of classes (c1), (c3) and (c4) contains a single constraint. If two constraints of the same class ((c2) or (c6)) correspond to rows of $\mathbf{A_C}$, then $\mathbf{A_C}$ has two equal rows, yielding a circulant of order two, which, by definition of circulant, must coincide with $\mathbf{A_C}$. Hence, in the following, we may assume that the set of constraints corresponding to rows of the circulant $\mathbf{A_C}$ contains at most one element in each class.
Let us distinguish two cases, according to whether arc ij is in the circulant or not.

Case a: arc ij is in the circulant $\mathbf{A_C}$.
Then exactly two constraints among the ones listed in classes (c1), (c2) and (c3) must be in the circulant.
If also arc ji is in $\mathbf{A_C}$, then both these two constraints contain a 1 in column ij and a 1 in column ji, yielding a circulant of order two which, by definition of circulant, must coincide with $\mathbf{A_C}$.
If arc ji is not in $\mathbf{A_C}$, then the listed constraints (c1) and (c3) appear in $\mathbf{A_C}$, arc jk must belong to the circulant, and these two constraints in columns ij and jk yield a circulant of order two as above.
Therefore, in this case, only circulants of order two are possible, either corresponding to arcs ij and ji or corresponding to arcs ij and jk.

End of Case a

Case b: arc ij is not in the circulant $\mathbf{A_C}$.
Therefore, arc ji must be in $\mathbf{A_C}$ and the listed constraints (c2) cannot appear in the circulant, since the corresponding rows would have a single 1. Exactly two among the listed constraints (c1), (c3), (c4) and (c6) must be in the circulant.
If the circulant has order larger than two, then other two variables besides ji are in the circulant such that one appears in one of these two constraints and the other does not. These two variables could only be jk and kj, but then at least one row in $\mathbf{A_C}$ would contain at least three 1, since the only constraints containing arc kj are in (c3) and (c4), and they both contain arcs jk and ji.
Therefore, also in this case, only circulants of order two are possible, either corresponding to arcs ji and jk or corresponding to arcs ji and kj.

End of Case b

Summarizing, only circulants of order two are possible. Therefore, the matrix $\mathbf{A}$ does not contain any odd circulant, and hence it is balanced. □

Theorem 26 *Given any tree T, the star partition polytope $Q(T)$ is integral.*

Proof. The proof directly follows from Proposition 23 and Theorem 25. □

An alternative proof of Theorem 26 is available in Andreatta et al. (2016b). It is by induction on the number of nodes of the tree T and uses the properties of vertices.

We may wonder if Theorem 26 can be extended to arbitrary graphs. Unfortunately, the answer is negative, as follows from the following remark.

Remark 1. Theorem 26 does not hold on arbitrary graphs, not even on bounded treewidth graphs. A counterexample is the graph corresponding to a circuit with four nodes. The graph belongs to the class of series-parallel graphs, a subfamily of bounded treewidth graphs with treewidth equal to 2. The star partition polytope of a circuit with four nodes has, among others, the fractional vertex defined by $x_{ij} = \frac{1}{3}$ for all $i \in V$ and $j \in N(i)$.

Remark 2. Given a weighted graph $G = (V,E)$, let w_j be a weight associated with node $j \in V$ and w_{ij} be a weight associated with the ordered pair (i, j), for $j \in V$ and $i \in N(j)$. We can associate a weight to any star partition Σ of G as follows. For each part P of Σ, consider one of its spanning stars and let j be its center. The weight of this star can be set equal to the sum of the center weight w_j and the weights w_{ij} of all the ordered pairs (i, j) with $i \in P \setminus \{j\}$. The weight of the part P can be defined as the minimum among the weights of all its spanning stars, and the weight of a star partition is the sum of the weights of its parts. By Theorem 26, finding a minimum weight star partition is solvable in polynomial time on trees.

7 Conclusions

We have defined the star partition of an undirected graph and we have considered problems related to its cardinality. Although finding a star partition of minimum or maximum cardinality is strongly related to well-known problems in graph theory, namely minimum dominating set and minimum edge cover, the constrained version presents interesting peculiarities and relations to domatic bipartitions. Hence, we have focused on CSP, where a star partition of prescribed cardinality has to be found. We have seen that star partitions can be obtained from domatic bipartitions and vice versa. Starting from a given star partition (resp. domatic bipartition), the analyzed transformation allows obtaining domatic bipartitions (resp. star partitions) of different cardinalities: we have investigated the relations among them and, in particular, we have characterized the possible cardinalities of domatic bipartitions resulting from a given star partition.

CSP satisfies the contiguity property, that is, a graph has always star partitions of any cardinality between the minimum and the maximum values. Some examples show that the same property does not hold in its full form for domatic bipartitions. However the transformation cited above and the contiguity property for CSP guarantee that domatic bipartitions exist for any cardinality between the minimum value and the maximum star partition cardinality, whereas, for larger values, the same property does not necessarily hold.

The contiguity property has also important consequences on the computational complexity of the problems considered in this paper. In particular, it allows proving that CSP is NP-complete, as a direct consequence of the complexity of the minimum dominating set problem. In addition, Constrained Domatic Bipartition problem turns out to be NP-complete for any cardinality lower than the maximum star partition cardinality. Furthermore, we have proven that the balanced version of the domatic bipartition problem (where the difference between the cardinalities of the two dominating subsets is small) is also NP-complete. As a by-product, we have closed the complexity analysis of another partitioning problem on graphs, the Location Partitioning problem (Andreatta et al., 2015), showing that this problem is also NP-complete for the balanced case with two parts. We have noticed that the star partition problems are polynomial on trees, as one may expect from the complexity of domination on this class of graphs, and also on the larger class of bounded treewidth graphs.

Towards the study of more general star partition problems, we have characterized the star partitions of a graph through an integer linear programming formulation. We have shown that the vertices of the related polytope have only integral components in case of trees and that the same property holds for CSP if we further restrict to paths. These results imply that linear programming can be used to efficiently find star partitions having minimum weight on trees with node and edge weights. The solution of weighted star partition problems on more general graphs is the object of future research.

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