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Rethinking structural balance in signed social networks

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ABSTRACT

Signed social networks account for the situation in which friendship, trust, alliance – represented by positive edges – coexist with relations of dislike, distrust, conflict — represented by negative edges. According to Heider's hypothesis of balance certain combinations of positive and negative edges in a network confer stability-balance-to the networks, while others – imbalanced networks – are unstable and tend towards balance. Here, we review the concept of structural balance starting from "first principles" based on the original formulation of Heider. Based on Harary's "tendency towards completeness" hypothesis we formulate that a network is balanced if it can give rise to a balanced complete signed graph. We then propose an algorithm for edge completion of networks and prove several results concerning the balance of cycles, and graphs. We then consider balance as a dynamic process in which the entities of the system try to make agreements by pairs to reach consensus or "agreed upon dissensus". At this point we arrive at a few general conditions that any index quantifying the degree of balance – or degree of imbalance – has to have. We then analyze a degree of balance index proposed by Estrada and Benzi (2014) and show that it fulfills all these requirements. In contrast, we show how some other approaches to quantify the degree of balance are incomplete and we provide examples of the difficulties found with their use. Using all these developed conceptions about the level of balance in signed social networks we proceed to the study of the international relation among countries in the world for the period 1938–2008. We conclude that the system of international relations is highly imbalanced for the whole period with no trend to increasing balance with the passage of time. Finally, we consider the use of degree of balance indices relative to null models and clarify that they do not account for the degree of balance of a signed network but for the "effort" needed to create such degree of balance among a large set of randomly shuffled networks.

1. Introduction

In social systems, where the entities are individuals, institutions or countries, it is frequent to distinguish two kinds of relations [20,51]. Positive relations refer to friendship, collaboration or alliances between the entities of the system, and negative ties represent enmities, opposite view about a topic, or fights. Such systems are representable by signed networks in which the nodes represent the entities and signed edges – positive or negative – represent the relations

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between them. In this context it is interesting to consider how "stable" the system is in its current state of positive and negative interactions. The state often believed to be the most stable, i.e., the one to which the system should converge with time, is known as a balanced state. The first attempt to formalize social balance was made by Heider in his seminal paper of 1946, "*Attitudes and cognitive organization*" [29], where he stated that:

(a) *In the case of two entities, a balanced state exists if the relation between them is positive (or negative) in all respect.*

(b) *In the case of three entities, a balanced state exists if all three relations are positive in all respects, or if two are positive and one negative.*

In his further extensive account of social balance published in his book "*The Psychology of Interpersonal Relations*" [30] Heider included the cases in which there are three entities with one negative and two positive relations as well as the one with the three negative relations. These two systems are recognized as imbalanced. Heider referred there to balanced states as harmonious and as preferred to those which are imbalanced. In other words, Heider paved the way to the search for how far from balance a given social system is, as it will be progressing to a balanced state with time. However, in the same paragraph Heider also claimed that [30]:

> "*On the other hand, there may also be a tendency to leave the comfortable equilibrium to seek the new and adventurous. The tension produced by unbalanced situations often has a pleasing effect on our thinking and aesthetic feelings. Balanced situations can have a boring obviousness and a finality of superficial self-evidence. Unbalanced situations stimulate us to further thinking; they have the character of interesting puzzles, problems which make us suspect a depth of interesting background. Sometimes they evoke, like other patterns with unsolved ambiguities, powerful aesthetic forces of a tragic or comic nature.*"

This second statement seems to be completely ignored by workers in this area who continuously focus only in the expectation that social systems should be balanced. This general trend could be somehow influenced by the strong mathematical results obtained by Harary in 1953–54 [25]. The general idea of this paper was then contextualized by Cartwright and Harary in their seminal paper published in 1956 "*Structural balance: A generalization of Heider's theory*" [10]. This work opened the doors to the search for graph-theoretic invariants that characterize the degree of balance of a signed network [16,19,21,22,24,33,35,38,39,42,44,48,49]. They elegantly proved that a balanced system can be split into two disjoint sets of agents, such that all positive connections are among agents in the same groups, and negative connections only occur between groups. Their work not only translates Heider's theory into a graph-theoretic language, but also expanded Heider's small-scale idea into a very general theory, which for the first time went beyond the use of triads and so it opens the arsenal of mathematical tools to the analysis of this theory. These theoretical findings of Harary [25] (see also Cartwright and Harary [10]) are formulated into the following four rules about combinations of friendship/enmities in signed social networks [47]:

1. A friend of a friend will be a friend;
2. A friend of an enemy will be an enemy;
3. An enemy of a friend will be an enemy;
4. An enemy of an enemy will be a friend.

The implicit hypothesis of balance theory is that agents in a social system move towards balance, which is plausible due to the fact that in many occasions agents need only local information to know their balance state [15]. It is then believed, and many algorithms have been implemented towards this belief [7,8,13,14,31,32,50,52], that in an imbalanced system a pair of agents tend to change the sign of their relation in order to move towards the balance of the system. Apart from observations in relatively small groups, including also a few contrary to the balance hypothesis (see [43,46] for examples and discussions), there is no foundation for this assertion except the belief that "*imbalanced triples creates 'tension' for an actor and, with it, psychological discomfort*" [14]. Then, to reduce such discomfort the agents try to balance their triples. Notice for instance that counter-arguments against the balance hypothesis also exist. For instance, Berger et al. [9] considered the following situation: in certain rural communities there exists a norm that men should drink some alcohol, but that women should refuse alcohol. Now assume a man loves his wife and he likes alcohol, whereas his wife dislikes alcohol. This situation is imbalanced and should leads to tension according to balance hypothesis. However, in fact this situation is completely in accordance with existing social norms and likely not tension producing. As remarked by Opp [46] social situations which are imbalanced but accord with social norms, in fact neither elicit tension nor change and, consequently, falsify the balance hypothesis. Doreian [14] has also pointed out that the life of actors is complicated when they share many different triangles. For instance, let us consider two actors A and B who have a negative relation between them. Let us consider that they both have positive relations with an actor C. On the other hand, A has a negative relation with D while B has a positive relation with that actor. In closing, the triangle ABC is imbalanced while ABD is balanced. Thus, there is a dilemma for actors A and B about whether to change their relation or not. Changing it to positive will balance the triangle ABC but imbalance ABD. Assuming that the decision can be taken by either C or D implies that they have knowledge not only about their first order relations – those with A and B – but also second order ones, which is not necessarily the case in general social systems. Then, the process resembles more an agreement by pairs, giving rise to a global consensus, than the simplistic algorithm of changing the sign of relations.

The finding of Harary [25] (see also Cartwright and Harary [10]) that a balanced system can be split into two mutually disjoint sets with positive relations inside the sets and negative ones between them is certainly appealing. We can think about the formation of two factions of friends which are mutual enemies. As put forward by Leinhardt [34] "*if Cartwright and Harary's interpretation of balance theory is correct and balance theory is an important structural rule, then the mathematics of graph theory lead us to expect groups to dichotomize into two distinct and opposed factions. Unfortunately, the enormous experience that social scientists have had with sociometric data simply does not confirm this implication*". Thus, why two factions in particular? Why not "three factions", "four factions" and so forth. This was already noticed by Davis in 1967 [12] who extended the result of Harary [25] (see also Cartwright and Harary [10]) of polarization of a network into two factions to the polarization into any number of clusters (with positive edges inside the sets and negative between them). He proved that a graph is clusterable if and only if it contains no cycle with exactly one negative edge. Although all balanced graphs are clusterable, not all clusterable graphs are balanced. The problem of determining whether a graph is clusterable or not is again a black-and-white version of the problem, similar to the balanced/not balanced dichotomy, which does not quantify the "balance" of a network. The fact that a graph having only one negative edge in some triangle is negative is not clusterable may introduce some problems in order to quantify a "degree of clusterability".

There is no doubt that "balance theory" is an important theory in social sciences. The recent explosion of studies in social networks has boomed the use of balance in signed networks again. Since the beginning of the use of the concept of balance in social sciences there have been discussions about its validity. There have been social systems which have been found to tend naturally to a balanced state while others not. This prompted Anderson [6] to claim that structural balance theory "*should be seen not as laws of behavior, but rather as rules-of-thumb that persons sometimes use when they construe their life-worlds*" and that its propositions "*have erroneously been viewed as candidates for the status of behavior laws. Instead, they should be seen as a set of rules for forming social relationships, used indexically by persons in some contexts*". But the current problem is more dangerous as there are a few examples in which exactly the same system has been found to be balanced by some authors and imbalanced by others. In similar lines we found that some results claim a trend to balance in the international relations since the beginning of the XX century, while others found the opposite trend.

The balance hypothesis has been adopted by the realism school in studying international relations (see [40] for details). According to their paradigm "*states ally against common enemies and thus states sharing common enemies should not fight each other*" [40]. Here again the idea that the "enemy of my enemy is my friend" is the driving paradigm of these studies. These studies started with a paper published by Harary [26] in 1961 focused on the Middle East in 1956 where he observed some sequence moving first toward balance and then moving away from it. Healy and Stein [28] studied the Bismarckian system in Europe from 1870 to 1881 claiming some support for structural balance theory. However, these studies are very limited in geopolitical terms and Maoz et al. [41] has recently recognized that few empirical studies have examined whether (global) international relations tend to balanced states or not. In this direction, Doreian and Mrvar [18] studied in 2015 the international system of nations from 1946 through 1999 and concluded that regardless of the type of degree of balance measure used "*the results provided decisive evidence contradicting the balance theoretic hypothesis of signed networks moving towards balanced states*". In one of the most exhaustive analyses of the balance hypothesis in international relations over 186 years Maoz et al. [40,41] concluded that these systems "*exhibit significant relational imbalance: states that share the same enemies and allies are disproportionally likely to be both allies and enemies at the same time*". More recently, however, Kirkley, Cantwell and Newman [33] analyzed the signed network of international relations for the period 1938–2008 and concluded that it is "significantly balanced" compared to an appropriate null model.

In this work we develop a somehow pedagogical approach to the study of the degree of balance in social systems. We remark that our interest here is constrained to social systems, and that the applications of any findings in this work is not necessarily true for biological, chemical or other type of systems. Thus, we advice the reader to try to justify from first principles their hypothesis in applying these approaches beyond social systems. We start from the statement of three general axioms for balance based solely in those stated by Heider in 1946 [29]. We then consider a dynamic approach to balance based on global consensus dynamics and following the finding of Altafini [4,5]. At this point we arrive at a few general characteristics that an index of degree of balance should have to have. We concluded that any index based only on signed triangles is incomplete, that any index based only on signed cycles is incomplete, and that any index which does not take into account backtracking walks is also incomplete. Then, we analyze an index previously proposed by Estrada and Benzi [19] that fulfills these conditions to be a complete description of the degree of balance. Using this index we analyze the international relations between countries in the world for the period 1938–2008 and concluded that (i) the international relations are significantly imbalanced in agreement with the theory and findings of Maoz et al. [40,41]. We also found that such level of imbalance is slightly increasing from 1938 to 2008. We are able to identify using the previously mentioned degree of balance index some major historic events in the world. That is, the levels of imbalance in the global international relations are significantly below the mean for the periods of the World War II, the Korean war, the Vietnam war, the Iraq–Iran war, the Gulf war and the (third) Yugoslav wars. Finally, we clarify some recent finding by Kirkley, Cantwell and Newman [33] concerning the degree of balance relative to a null model.

The goal of this work is to establish some solid grounds for the quantification of the degree of balance of any signed network. In doing so we follow the advice of Norman and Roberts [45] who remarked that what was needed "*is a balance measure which was not ad hoc, but rather which could be derived from a simple set of assumptions or axioms.*" Using one of such indices we will (re)analyze the dataset of international relations from 1938 to 2008 to measure its degree of balance and consider the causes of contradictory results obtained in the literature.

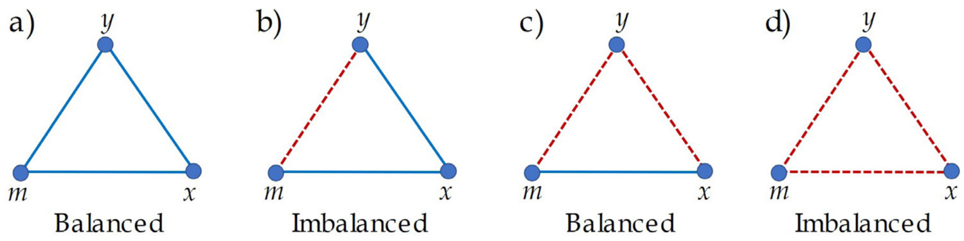

Fig. 2.1. The four signed triangles and their balance. Continuous (blue) lines represent positive (friendship) relations and broken (red) lines represent negative (enmities) relations. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

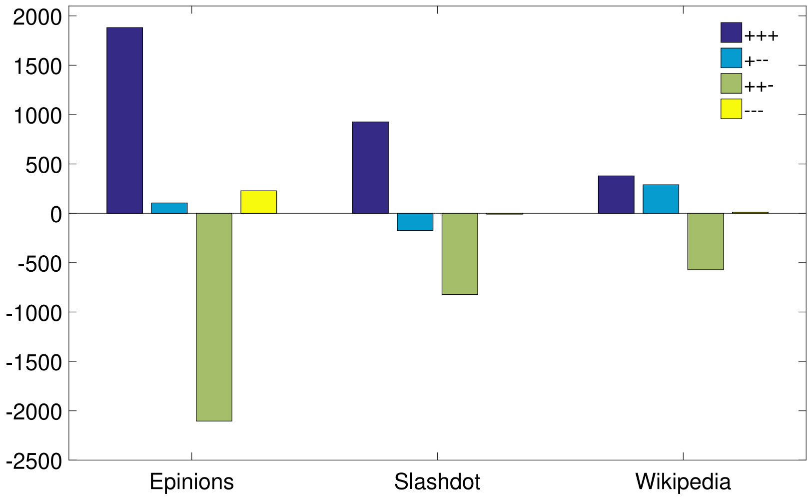

Fig. 2.2. Surprise index for the four signed triangles present in three online social networks according to [36].

2. Balance based on triads only

We can represent all the friendship–enmity relations in a social system by a signed graph, which is defined below.

Definition 1. A signed graph is the 4-tuple $G = (V, E, \Sigma, \varphi)$, where $V = \{v_1, \ldots, v_n\}$ is the set of nodes or vertices representing individual social entities, $E \subseteq V \times V$ is the set of edges formed by (ordered or unordered) pairs of nodes, $\Sigma = \{+, -\}$ is a set of signs, positive for friendship relations and negative for enmity ones, and $\varphi : E \to \Sigma$ is a mapping assigning one sign to each edge.

For excellent compendiums about signed graphs the reader is referred to the work of Zaslavsky [53,54]. According to Heider axioms of balance for triads we have the following.

Definition 2. *A triangle is balanced if all three relations are positive, or if two are positive and one negative.*

The balance/lack-of-balance of triangles are displayed in Fig. 2.1.

Before proceeding we center for a moment on the analysis of these triads. They are very important, particularly because many authors consider only them in their analysis of the degree of balance (see further critique in this work). An exhaustive counting of these four triads in real-world social networks was performed by Leskovec et al. in 2010 [36] for the signed networks representing: (i) the trust network of the Epinions product review Web site, where users can indicate their trust or distrust of the reviews of others; (ii) the social network of the blog Slashdot, where a signed link indicates that one user likes or dislikes the comments of another; and (iii) the voting network of Wikipedia, where a signed link indicates a positive or negative vote by one user on the promotion to admin status of another. The results of Leskovec et al. [36] are shown graphically in Fig. 2.2 in the form of the "surprise" index, which accounts for the number of standard deviations by which the actual number of a given triad differs from the expected number under a random-shuffling model.

According to what we should expect from the balance theory the positive triads $(+ + +)$ and $(+ - -)$ should be overrepresented in the datasets because they contribute to increasing the degree of balance. At the same time the negative triads $(+ + -)$ and $(- - -)$ should be underrepresented because they contribute to decreasing the degree of balance. The

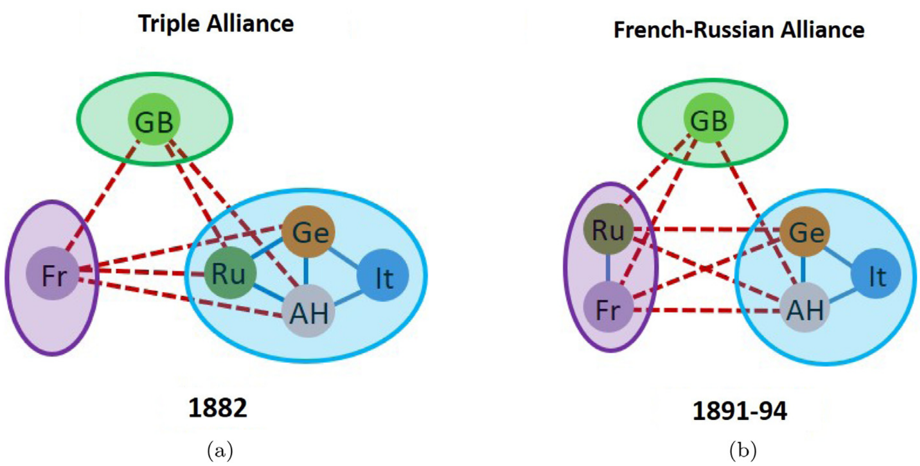

Fig. 2.3. Illustration of the tripartite structure formed by European powers during two periods of the XIX century. The countries are: Russia (Ru), France (Fr), Great Britain (GB), Germany (Ge), (Austro–Hungarian Empire (AH) and Italy (It). Continuous (blue) lines represent positive (friendship) relations and broken (red) lines represent negative (enmities) relations.

data published by Leskovec et al. clearly shows that the positive triads are overrepresented in almost all datasets (the exception is $(+--)$ in Slashdot) and the negative triad $(++-)$ is underrepresented in the three datasets, as expected from balance theory. However, the negative triad $(---)$ is surprisingly overrepresented in the Epinions and Wikipedia datasets.

Now we are interested in analyzing why the negative triad $(---)$ is overrepresented in certain networks. Before proceeding we state one of the first mathematical results obtained by Harary [25] (see also Cartwright and Harary [10]) in their analysis of balance in signed networks, which is known as the "structure theorem".

Theorem 3. *A signed graph is balanced if and only if nodes can be separated into two mutually disjoint sets of nodes, such that positive edges joint nodes only inside the subsets and negative edges joint nodes from different subsets.*

This means that a balanced network can be represented as certain type of bipartite graph in which the negative edges connect the nodes between the two disjoint subsets of nodes connected by positive edges. In other words, in a balanced network there are two factions or coalitions of friends who are confronted to another coalition of friends. This clearly guarantees that every existing triangle has either the three edges positive or exactly two edges negative. This idea is very appealing from a sociological point of view: we have balance when the forces are clearly separable into two factions, such that there are no enmities inside the factions, but only among them. The question that immediately emerges is: Why two factions? Why not three, four, etc.? We are not referring to examples like the ones displayed by Cartwright and Harary [10] in their Fig.8, where apparently there are more than two factions, but in which the graphs are of the class of bipartite networks mentioned before. We are thinking in the case of certain classes of tripartite, tetrapartite, or in general multipartite graphs in which groups of nodes with positive edges between them are separated from each other by negative edges.

To illustrate the connection between the negative triads $(---)$ and the class of tripartite graphs mentioned before we show a couple of examples from the real-world. They correspond to the signed networks representing the alliances-enmities between six European powers at the end of the XIX century. The first example (Fig. 2.3(a)) represents the relations between these powers during the Triple Alliance in 1882. In this case the vertices of the graph can be split into three disjoint subsets $V_1 = \{\text{GB}\}$, $V_2 = \{\text{Fr}\}$, and $V_2 = \{\text{Ru, Ge, AH, It}\}$. Notice that there are negative relations between every pair of subsets. Thus the graph is not bipartite but tripartite. The second example (Fig. 2.3(b)) represents the relations between these powers during the French–Russian Alliance in the period 1891–94. In this case the vertices of the graph can be split into three disjoint subsets $V_1 = \{\text{GB}\}$, $V_2 = \{\text{Fr, Ru}\}$, and $V_2 = \{\text{Ge, AH, It}\}$. Here again there are negative relations between the three pairs of subsets. In these cases there are negative triads of the type $(---)$ that connect three countries, each of them in a different coalition. For instance, Fr–GB–Ru, Fr–GB–AH in Fig. 2.3(a) and GB–Ru–AH, GB–Fr–AH in Fig. 2.3(b). We claim here that these triads are not necessarily destabilizing of the system, but as in the case of balanced networks, they may contribute to keep groups of alliances separated by negative relations only. This could be the reason why they appear so unexpectedly overrepresented in the datasets analyzed by Leskovec et al. [36] and possibly in many others. Also, their presence could be one of the reasons why some networks are not as balanced as expected from the balance theory.

Another important lesson learned from the analysis of the type of multipartite separation of nodes in a balanced signed network mentioned before is the following. Let us use an example for the sake of clarity. Consider a network formed by 4 nodes a, b, c, d with $\{(a, b), (a, c)\} \in E^-$ and $\{(b, d), (c, d)\} \in E^+$. In this case it is obvious that we can split the network into two parts, one formed by $\{a\}$ and the other by $\{b, c, d\}$, such that inside each of the groups there are no interactions or positive interactions only, and between the groups there are negative interactions only. That is, we have formed the same kind of bipartite graph mentioned before. But, this network contains no triangle, it is a signed square. A similar situation emerges if $\{(a, b), (c, d)\} \in E^-$ and $\{(b, d), (a, c)\} \in E^+$. This observation clearly indicates that there is nothing special about considering triangles-only for determining whether a network is balanced or not. Therefore, we will extend the Heider idea of balance beyond triangles and extend it to cycles of any length.

3. Balance beyond triads

The necessity of analyzing balance beyond triads was put forward by Cartwright and Harary [10] when they wrote that in a signed graph representing some psychological situation "*it may happen that only cycles of length 3 and 4 are important for the purpose of determining balance*", such that "*it will not matter at all to the group as a whole whether a cycle of length 100, say, is positive*". We will return to the graduate value of the importance of cycles of different lengths to the degree of balance, but in this section we want to show the following.

It seems from Cartwright and Harary [10] that the extension of balance to higher order cycles is somehow ad hoc. That is, if balanced/imbalanced relations exist at triad level why not at tetrads and so forth. Although the results presented in this section have been already proved by other ways we develop an approach here that allows us to extend the concept of balanced triangles to any other cycle, thus providing a foundation for this extension. A triangle is a complete graph which arises from closing a triad of nodes. That is, for a triad a–b–c a triangle is formed by adding an edge between a and c. In the case that we have a structure like a–b–c–d we have triads a–b–c and b–c–d. Thus, the "natural" evolution of this system is to close those triads by adding the edges a–c and b–d. However, at this point of the evolution we have created two new unclosed triads, i.e., a–c–d and a–b–d. Thus, closing them means to add the edge a–d. The result of this iterative, evolutionary process is a complete graph. We have removed the signs of the edges for the sake of simplicity but the situation is similar if we add them. This hypothetical tendency towards completing all possible social relations in a group was put forward first by Harary under the following statement [27]:

Tendency Toward Completeness (Harary). *A group structure will tend toward completeness, that is, if two entities are not yet interrelated in the structure, then a bond will tend to be induced to appear between them.*

Independently on whether real-world networks follow such tendency or not we can see it as an ideal state to which a social group may tend. In this way we can consider that any signed graph, different from a complete signed graph, is in a transition state towards such ideal situation of full-connectivity. It is evident from Heiders axioms of balance based on triads that:

Definition 4. A complete signed graph is balanced if all its triangles are balanced.

Thus, we can consider that any signed graph, which is just a transition state towards a complete signed graph is balanced/unbalanced depending on its capacity to accommodate new signed edges that produces balanced/unbalanced complete signed graphs. That is:

Definition 5. *A non-complete signed network $G \neq K_n$ is balanced if by adding signed edges to G a complete signed network $G' = K_n$ with all its triangles balanced can be built. Otherwise the network is unbalanced.*

We then develop a simple algorithm for completing the edges of a non-complete signed graph, which will be useful to prove our main result.

Algorithm 1 Edge completion algorithm

1. Label the nodes of the cycle C_n as $v_1, v_2 \ldots, v_n$.
2. Pick a node v_j. If $\left(v_{j+1}, v_{j+2}\right) \in E^+$ add the edge v_j, v_{j+2} with the same sign as the edge v_j, v_{j+1}. If $\left(v_{j+1}, v_{j+2}\right) \in E^-$ add the edge v_j, v_{j+2} with the opposite sign as the edge v_j, v_{j+1}.
3. Repeat until all the edges v_j, v_k for all $k \neq j$ exists.
4. Repeat the process for all nodes $v_j, j = 1, \ldots n$.

The main result of this section is given in the following. Let us use the following notation. By $E^+ \left(E^-\right)$ we designate the sets of positive (negative) edges in a signed graph. Also by $T^+ \left(T^-\right)$ we designate the sets of positive (negative) triangles in a signed graph.

Theorem 6. *Let C_n be a signed cycle with n nodes. Then, C_n is balanced if and only if the number of negative edges in C_n is even. Otherwise the cycle C_n is unbalanced.*

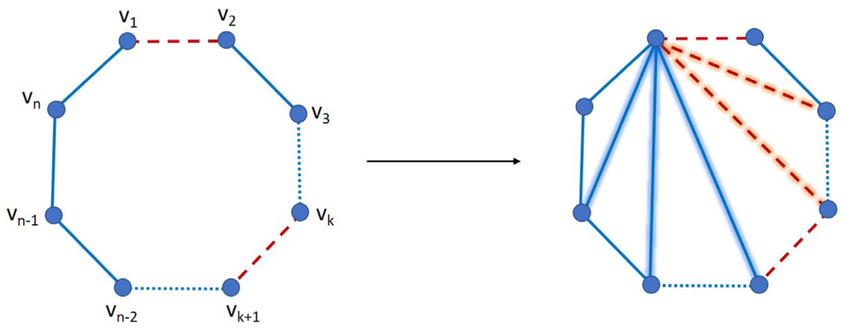

Fig. 3.1. Illustration of the first step of the edge completion algorithm applied to a signed cycle with an even number of negative edges (two in this case) and starting by the node labeled as v_1.

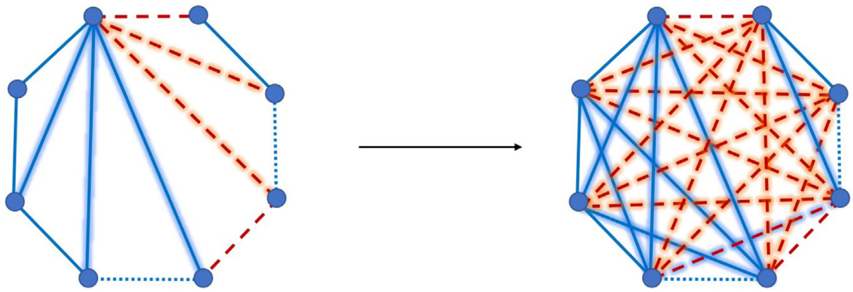

Fig. 3.2. Illustration of the full application of the edge completion algorithm on a signed cycle with an even number of negative edges (two in this case) which has started at the node labeled as v_1(left).

Proof. We will use the edge completion algorithm to show whether or not a balanced complete signed graph can be obtained from a given signed graph. Let C_n be a cycle with n nodes labeled clockwise as $v_1, v_2 \ldots, v_n$. Let us first consider that the cycle has an even number of negative edges. Without any loss of generality let us fix this number by two. Let $(v_1, v_2) \in E^-$ and $(v_k, v_{k+1}) \in E^-$ and the rest of edges be positive. Let us consider that $v_k \neq v_2$. Then, according to the edge completion algorithm we need to add edges $\left(v_1, v_j\right) \in E^-$ for all $j = 2, \ldots, k$. The triangles v_1, v_j, v_{j+1} have two negative edges, i.e., $\left(v_1, v_j\right) \in E^-$ and $\left(v_1, v_{j+1}\right) \in E^-$, and one positive edge, i.e., $\left(v_j, v_{j+1}\right) \in E^+$. Consequently, $\left(v_1, v_j, v_{j+1}\right) \in T^+$. As the edge $(v_k, v_{k+1}) \in E^-$ we need to add $(v_1, v_{k+1}) \in E^+$. Then, the triangle $(v_1, v_k, v_{k+1}) \in T^+$ because it has two negative edges, i.e., $(v_1, v_k) \in E^-$ and $(v_k, v_{k+1}) \in E^-$, and one positive edge, i.e., $(v_1, v_{k+1}) \in E^+$. We now continue adding more positive edges $(v_1, v_r) \in E^-$ for all $n - 1 \leq r \leq k + 1$. Consequently all the triangles $(v_1, v_r, v_{r+1}) \in T^+$ because their three edges are positive. We illustrate this part of the process in Fig. 3.1.

The process is now repeated for every node v_s for all $s = 2, \ldots, n$. This gives rise to a signed complete graph in which all the triangles of the type $\left(v_s, v_q, v_{q+1}\right) \in T^+$ for all $s = 2, \ldots, n$ and for all $q = 3, \ldots, n$ (see Fig. 3.2 for the example based on node 2).

Let (v_a, v_b, v_c) be a triangle where neither two nodes are adjacent in the cycle C_n. We need to prove that all these triangles are positive to prove this part of the result. Let us consider that (v_a, v_b) and (v_a, v_c) have the same sign, either both are positive or both are negative. This means that the path $v_b, v_{b+1}, \ldots, v_{c-1}, v_c$ contains either no negative edges or an even number of negative edges. This is a consequence of the fact that the signs of the edges built from v_a to the nodes in the path $v_b, v_{b+1}, \ldots, v_{c-1}, v_c$ have either not changed or have had a positive number of changes. Then, if $(v_b, v_{b+1}) \in E^+$ there will be an even number of sign changes along the path, which will end up with $(v_b, v_c) \in E^+$. Thus, $(v_a, v_b, v_c) \in T^+$.

If $(v_b, v_{b+1}) \in E^-$ and the other negative edge is $(v_d, v_{d+1}) \in E^-$, the edge $(v_b, v_d) \in E^-$, therefore the edge $(v_b, v_{d+1}) \in E^+$ as well as all other edges from v_b to any node in the path between v_{d+1} and v_c. Consequently, $(v_b, v_c) \in E^+$ and the triangle $(v_a, v_b, v_c) \in T^+$.

Let us consider now that $(v_a, v_b) \in E^-$ and $(v_a, v_c) \in E^+$. This implies that there is an odd number of negative edges in the path $v_b, v_{b+1}, \ldots, v_{c-1}, v_c$. Let us consider without any loss of generality that such a number of negative edges is one (otherwise you can apply the previous result to the first $2h - 1$ negative edges in the path and consider the remaining one as the example). Let (v_d, v_{d+1}) be such negative edge in the path between v_b and v_c. Then, if $(v_b, v_{b+1}) \in E^+$, the edge (v_b, v_{d+1}) will be negative, as well as every edge from v_b to any node in the path $v_{d+1}, v_{d+2}, \ldots, v_{c-1}, v_c$. Consequently, $(v_b, v_c) \in E^-$ and the triangle $(v_a, v_b, v_c) \in T^+$.

If $(v_b, v_{b+1}) \in E^-$, all the edges from v_b to any node in the path $v_{d+1}, v_{d+2}, \ldots, v_{c-1}, v_c$ are negative. Consequently, $(v_b, v_c) \in E^-$. In both cases the triangle $(v_a, v_b, v_c) \in T^+$. Extending the result to any even number of negative edges is straightforward.

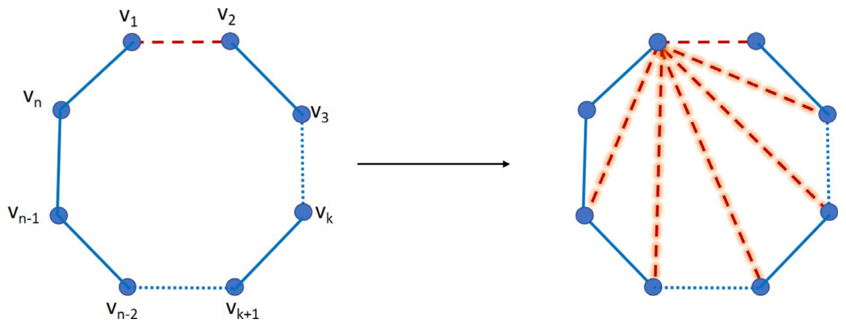

Fig. 3.3. Illustration of the first step of the edge completion algorithm applied to a signed cycle with an odd number of negative edges (one in this case) and starting by the node labeled as v_1.

Let us now consider the case in which the number of negative edges in the cycle C_n is odd. Without any loss of generality let us consider such number equal to one. Let us label the nodes of the cycle as $v_1, v_2 \ldots, v_n$, such that $(v_1, v_2) \in E^-$. Then, because there are no other negative edge in the path $v_3, v_4, \ldots, v_{n-1}, v_n$ all the edges $\left(v_1, v_j\right)$ for all $j = 3, \ldots, n$ to be added to the cycle must be negative. Consequently, the triangle $(v_1, v_{n-1}, v_n) \in T^-$ because it has two edges positive, i.e., $(v_1, v_n) \in E^+$ and $(v_{n-1}, v_n) \in E^+$, and one edge is negative, i.e., $(v_1, v_{n-1}) \in E^-$ (see Fig. 3.3). Therefore, the cycle is unbalanced. The extension to any number of odd negative edges is straightforward. □

Example 7. As a matter of example let us analyze all the signed squares. In the case of the four edges of the square being positive it is straightforward to realize that we can add the two diagonals as positive edges giving rise to a balanced complete signed graph. Let us then consider the case in which there is a single negative edge. As can be seen in Fig. 3.4(a) adding any of the two main diagonals to his square immediately reveals that the resulting graph is unbalanced. Therefore, any way of completing the graph will give rise to an unbalanced complete signed graph. The same happens for the square having three negative edges (see Fig. 3.4(b)). However, for the three graphs in which the number of negative edges is even we can build complete signed graphs with 4 nodes which are balanced (see Fig. 3.4(c)).

In Fig. 3.5 we illustrate the six possible signed squares remarking the conclusion of whether they are balanced or not on the basis of Theorem 6.

So far we have deal only with signed cycles. What is the contribution of an acyclic subgraph to the balanced of a graph? Suppose that you have a signed tree $T_n = (V, E, \Sigma, \varphi)$ with n nodes. Then, pick a pair of nodes v_i and v_j such that $\left(v_i, v_j\right) \notin E$. Then, because T_n is a tree there is a unique shortest path $P_{ij} = v_i, v_{i+1}, \ldots, v_{j-1}, v_j$ between v_i and v_j. Let the number of negative edges in P_1 be even (odd). Then, by adding the positive (negative) edge v_i, v_j we create a cycle with an even number of negative edges. By Theorem 6 we can see that this cycle is balanced. Thus, the acyclic structure is also balanced. As we can repeat this operation for every pair of nonadjacent nodes in the tree we can see that it is balanced. The main consequence of this results is the following.

Corollary 8. *A signed graph is balanced if and only if all its cycles are balanced.*

Once we have proved the conditions for balance in undirected signed networks we will use them to analyze the balance of signed directed graphs. We recall that a directed signed graph $G = (V, E)$ with adjacency matrix A is strongly connected if there is a directed path connecting every pair of nodes in G. Also a directed signed graph is digon sign-symmetric if $A_{ij}A_{ji} > 0$ for every pair of edges sharing the same nodes. That is in a bidirected edge both directions have the same sign. Also, we remind that the underlying undirected graph of G is the graph with adjacency matrix $A_u = \left(A + A^T\right)/2$. Then, we will adopt a result from Altafini [5] to consider balance on directed signed graphs (see [5] for more equivalent conditions).

Lemma 9. *Let G be a strongly connected, digon sign-symmetric signed digraph with adjacency matrix A. Then, G is structurally balanced if and only if any of the following equivalent conditions holds:*

(1) The underlying undirected graph is structurally balanced;

(2) all directed cycles of G are positive.

4. Social balance as the result of a consensus process

It can be argued that networks are the framework that facilitate the propagation of 'information' among the entities of a complex system. Information should be understood here in a wide generic way. In the case of signed networks they describe the situation where there is a polarization into two factions, which is characteristic [5] "*in many antagonistic*

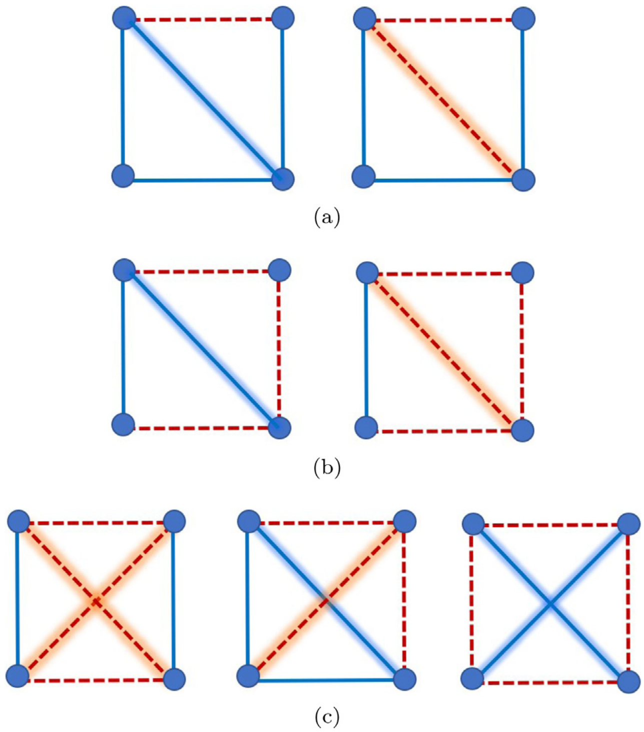

Fig. 3.4. Illustration of the edge completion algorithm to determine whether signed squares are structurally balanced or not. (a) The first step in the case of a signed square with only one negative edge. (b) The first step in the case of a signed square with 3 negative edges. (c) Complete signed graphs for the squares with 2 and 4 negative edges.

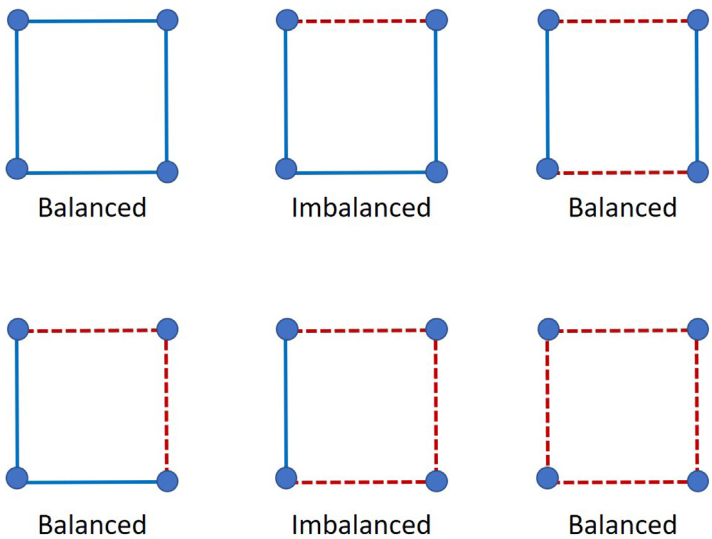

Fig. 3.5. Representation of signed squares and their balance. Continuous (blue) lines represent positive (friendship) relations and broken (red) lines represent negative (enmities) relations.

systems describing bimodal coalitions, like two-party political systems, duopolistic markets, rival business cartels, competing international alliances'' [5]. Then, if information is flowing through these networks, and the agents are trying to reach a consensus about their opinions, how structural balance, or the lack of it, influences this process? In particular, under which conditions the agents can achieve an equilibrium state where one group has a consensus on one opinion and the other group has a consensus on another one. This type of "agreed upon dissensus" has been solved by Altafini [5] by using

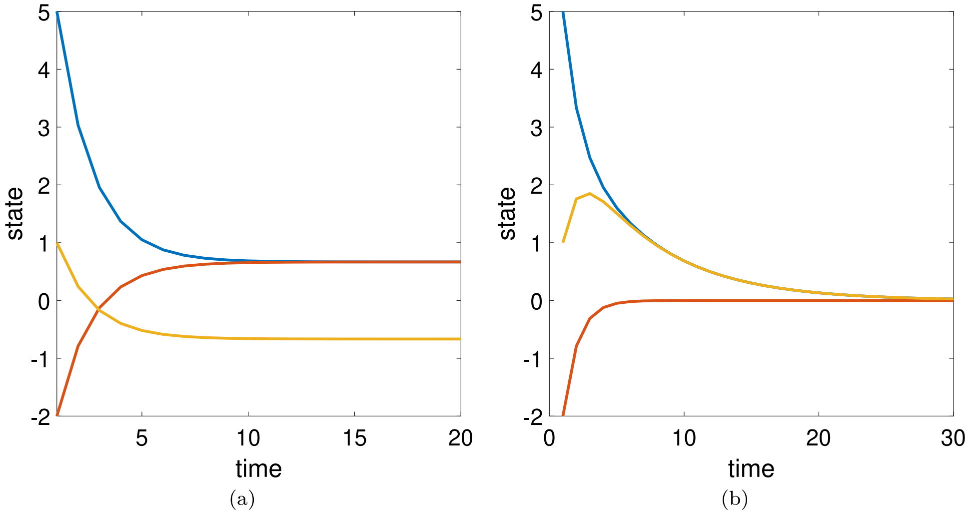

Fig. 4.1. Time evolution of the global consensus dynamics in two triangles: $(+--)$ (a) and $(++-)$ (b).

the diffusion (consensus) model:

$$\dot{u}(t) = -\mathcal{L}u(t); u(0) = u_0, \tag{4.1}$$

where $u_0 \neq 0$ and $\mathcal{L}$ is the Laplacian operator on the signed graph, which is defined as

$$\mathcal{L}_{ij} = \begin{cases} \sum_{j=1}^{n} |A_{ij}| & i = j \\ -A_{ij} & (i,j) \in E \\ 0 & otherwise, \end{cases} \tag{4.2}$$

In this way the time evolution of the state of a given node is written as

$$\dot{u}_i(t) = \sum_{(j,i)\in E} |A_{ij}| \left(u_i(t) - \text{sgn}\left(A_{ij}\right) u_j(t)\right). \tag{4.3}$$

The equilibrium state is obtained when $t \to \infty$, for which Altafini [5] has defined the bipartite consensus state as follows.

Definition 10. The system (4.1) admits a bipartite consensus ("*agreed dissensus*") solution if $\lim_{t\to\infty} |u_i(t)| = \alpha > 0$ for all $i = 1, \ldots, n$ and for all $u_0 \in \mathbb{R}^n$.

Then, the following remarkable result was proved by Altafini [5], which will be of great importance for the current work.

Theorem 11. *Let G be a signed graph. The system (4.1) admits a bipartite consensus solution if and only if G is structurally balanced. If instead G is structurally imbalanced then* $\lim_{t\to\infty} u(t) = 0$ *for all* $u_0 \in \mathbb{R}^n$.

Let us give a simple example to illustrate this result. In Fig. 4.1 we illustrate the time evolution of the state of the three nodes in two different triangles. As can be seen, the triangle in (a) is balanced as the system reaches the bipartite consensus, and the triangle in (b) is not balanced as the equilibrium state is $\lim_{t\to\infty} u(t) = 0$.

In the previous section we identify which signed squares are balanced and which ones are not. Then, we study here all the signed squares using the consensus method to observe the coincidence. We should notice that in the case of the all-positive square (in general any all-positive graph) the consensus algorithm converges to a steady state which is the average of the values of each node in the initial state. That is, in this case the consensus is the same as for an unsigned graph. In Fig. 4.2 we illustrate the results for all the signed squares where we can see that the steady state reached for those squares having an even number of negative edges is formed by two groups: "agreed upon dissensus", and that for the squares with an odd number of negative signs is $u(t) = 0$.

The importance of this result is that now we have a functional definition of balance on networks. That is, a network is balanced if there are two opposed groups which are able to reach a bipartite consensus. Otherwise, the network is imbalanced. We can also give an interpretation of why such agreed dissensus is reached in a balanced network. We can think that in a consensus state every group has obtained enough strength as to be competent to confront the other group. If this "balance" is not reached then it is better to make a global – possibly compromising – agreement. We can visualize

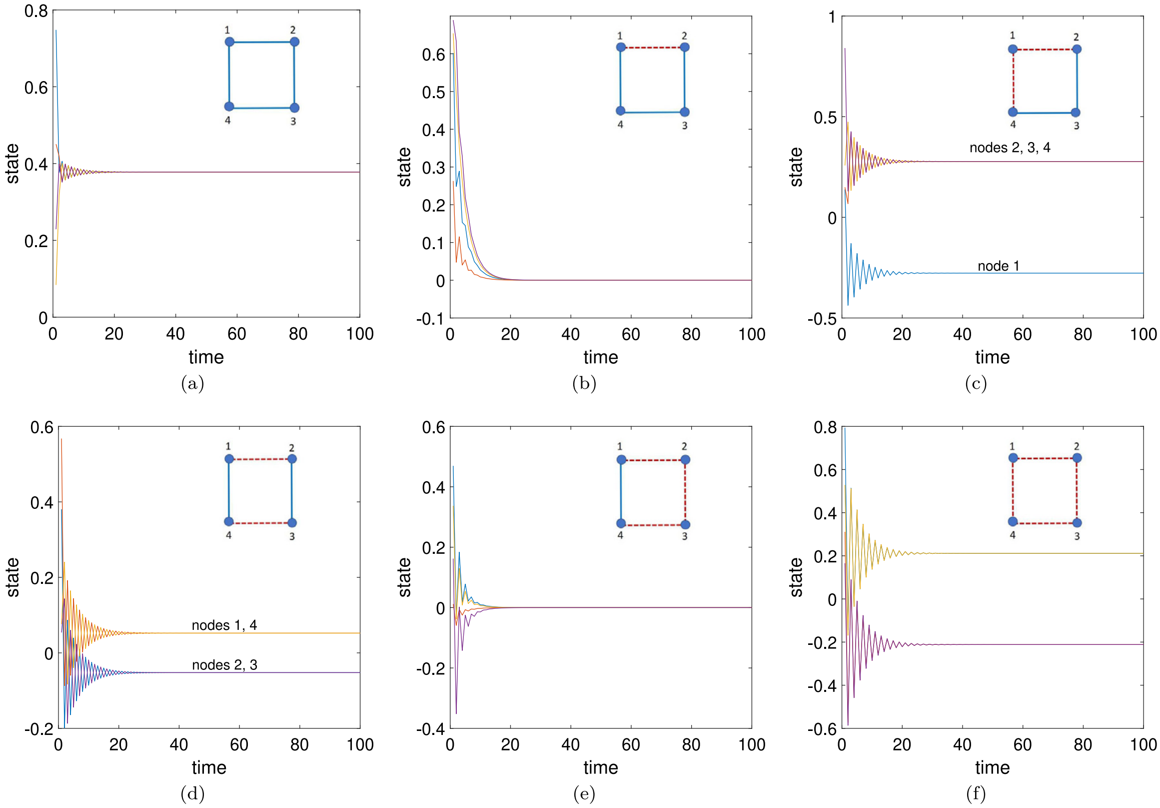

Fig. 4.2. Results of the consensus algorithm for all signed squares using a random initialization state for each node.

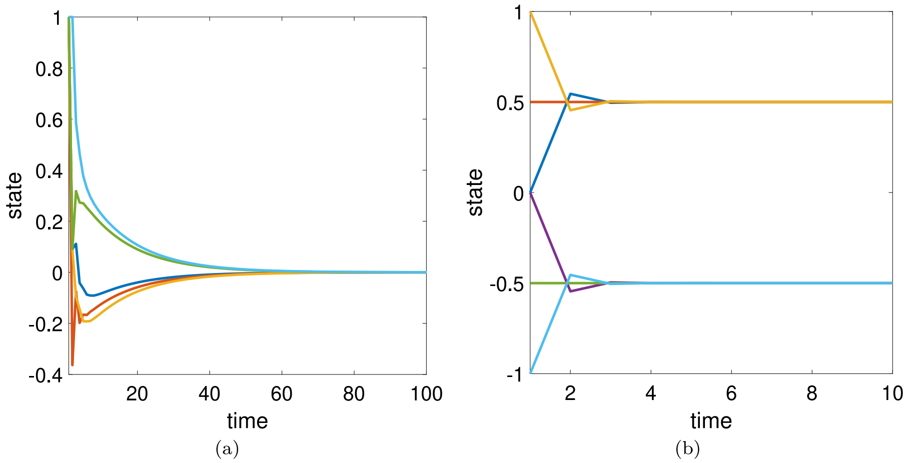

Fig. 4.3. Time evolution of the global consensus state among the six main European powers in 1904, when the Entente Cordiale was created (a), and between the same powers in 1907, when the British–Russian alliance was founded (b).

this situation with what happen with the six main European powers at the beginning of the XX century. As can be seen in Fig. 4.3(a) in 1904, when the Entente Cordiale was created, the six main European powers were in a global imbalanced state, and they were not able to create an "*agreed dissensus*" state between the two factions. However, in 1907, when the British–Russian alliance was founded, the two factions were in strong-enough position as to make agreements inside them and creating a bipartite consensus as observed in Fig. 4.3(b).

But, what can we learn from the "consensus" formulation of balance that we do not understand from the structural theory? Let us take the graph formed by only two nodes connected by a simple negative edge. Is this graph balanced or imbalanced? In the "classical" structural theory the question is not well-posed as there are no cycles in this graph.

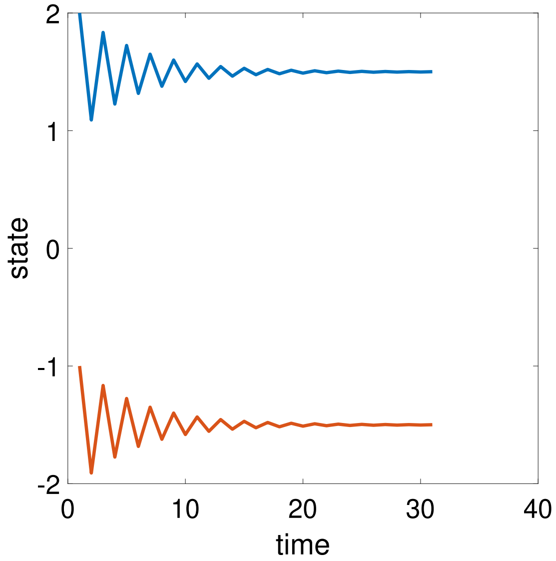

Fig. 4.4. Time evolution of the global consensus dynamics in dyad: (—–).

However, a simple calculation shows in Fig. 4.4 that this system is balanced. This of course coincides with our previous result that every acyclic structure is balanced but here we can learn something more about the causes of this balance. The only way in which we can understand this result is by assuming that in the consensus process the information departing from one node, let say x, and arriving at the other node, let say y, returns to the origin describing a backtracking cycle $x-y-x$, resp. $y-x-y$. This is equivalent to saying that there are two parallel edges between x and y forming a cycle of length 2. That is, x is an enemy of y, which is an enemy of x, thus x is a friend of itself. This situation may seem of little practical utility but encloses a very important message. Namely, it means that when quantifying the degree of balance on networks, the backtracking of information back and forth between the nodes describes "*artificial*" cycles – sometimes named "trivial" walks – which are important to capture the global level of balance in the system.

Mathematically, this situation is explained by the fact that the solution of the system (4.1) is given by

$$u\left(t\right)=\exp\left(-t\mathcal{L}\right)u_0, \tag{4.4}$$

where we can expand the semigroup of the Laplacian using a Taylor series

$$u\left(t\right)=\left(I-t\mathcal{L}+\frac{\left(t\mathcal{L}\right)^2}{2!}+\cdots+\frac{\left(-t\mathcal{L}\right)^s}{s!}+\cdots\right)u_0. \tag{4.5}$$

It is known that $\mathcal{L}=\Delta-A$, where Δ is the diagonal matrix of node degrees. Then, the following result has been proved by Aguilar and Gharesifard [2]:

Lemma 12 (*Powers of the Laplacian Matrix*)**.** *Let $G=(V,E)$ be a connected graph with vertex set $V=1,2,\ldots,n$, and let $r\in\mathbb{N}$. Then for $u,v\in V$ such that $r\leq d(u,v)$ we have that*

$$\left(\mathcal{L}^k\right)_{uv}=\begin{cases}0 & 0\leq k<r\\ (-1)^k\left(A^k\right)_{uv} & k=r,\end{cases} \tag{4.6}$$

where $d(u,v)$ is the length of a shortest path connecting both nodes.

It is known that $\left(A^k\right)_{uv}$ counts the number of *walks* of length k between the two nodes, where a *walk of length k* in G is a set of nodes $i_1,i_2,\ldots,i_k,i_{k+1}$ such that for all $1\leq l\leq k$, $(i_l,i_{l+1})\in E$. A closed walk is a walk that starts and ends at the same node. The previous Lemma indicates that the higher contribution to each entry of the powers of the Laplacian matrix, which appear in the solution of the consensus dynamics, is given by the number of walks of length $r=d(i,j)$. Higher powers of the Laplacian necessarily contain contributions from backtracking walks, i.e., those that go back and forth through the same nodes in the graph. Then, the existence or not of bipartite consensus depends not only on the graph theoretic cycles existing in the graph, but also on those backtracking walks created by the back-and-forth flow of information in the consensus process. In order to understand the role of these backtracking walks on the degree of balance of a given structure we start by considering simple cycles like the signed triangles. Let us consider the triangle $\triangle xyz$ in which the edge xy is negative and the two others are positive. Let us consider a consensus process taken place on this triangle in which two choices exist, let say a or b. If a node selects an option, say a, and the next node is in a negative relation with it, this node will select option b. Otherwise it will select a. Then, if the process starts with x selecting a and the process evolves in the sequence x–y–z–x we have a–b–b–b. Then, there is a tension as x has started with choice a and she has received the offer b after one cycle. However, let us consider a two-rounds sequence x–y–z–x–y–z–x, such that we have: a–b–b–b–a–a–a. Thus, with a second round consensus we have an agreement. The first round of consensus is given by a non-backtracking cycle, while the second round consensus is given by a backtracking cycle of length 6. Obviously, the first round consensus is more "important" than the second round one, and the consensus protocol weights them accordingly. This means that in a dynamic perspective, an unbalanced subgraph is not as balanced as expected due to the fact that some backtracking rounds of consensus can give rise to balanced structures. Such backtracking walks can be very important in quantifying the degree of balance in certain structures as we will see in the next section of this work.

5. How to quantify the degree of social balance after all?

Here again we have to start from the pioneering work of Cartwright and Harary [10] who were the first to propose the quantification of the degree of balance of a signed network. According to them: "*it is intuitively clear that some imbalanced signed graphs are "more balanced" than others!*" They then proposed "*the introduction of some scale of balance, along which the "amount" of balance possessed by an imbalanced signed graph*". Their quantification naturally relays on "*the ratio of the number of positive cycles to the total number of cycles*". One ingredient missing in this quantification is the following. On page 202 of his book Heider [30] clearly stated the conditions of balance, which as the first statement includes the following: "*A dyad is balanced if the relations between the two entities are all positive or all negative. Disharmony results when relations of different sign character exist*". Thus, the measure of Cartwright and Harary [10], which start its quantification of balance from cycles of length 3, is incomplete. In general, we conclude that:

(i) every measure of the degree of balance based only on signed triangles is incomplete;
(ii) every measure of the degree of balance which is based only on signed cycles is incomplete;
(iii) every measure of the degree of balance which does not take into account backtracking walks is incomplete.

The test for the lack of completeness of any of these measures can be carried out using triangle-free graphs, or signed trees, to mention just two examples.

In order to find a measure that captures all the important ingredients of the degree of balance it is appropriate to consider walks on graphs. In the first place, the term $\frac{(A)^s}{s!}$ which appear in Lemma 12 contains information about all walks in a graph. Second, the terms $(A^s)_{ii}$ count all the closed walks of length s in the graph. Third, the terms $(A^s)_{ii}$ account for acyclic structures which include dyads as a particular case. Fourth, the sum of all weighted walks in a graph of the form $\sum_{s=0}^{\infty}\frac{(A^s)_{ii}}{s!}$ always converge to $\exp(A)$, and it gives more weight to dyads, then triads, squares, etc.

Thus, it seems naturally to propose a measure quantifying the ratio, not of cycles as proposed by Cartwright and Harary [10], but of signed walks to unsigned walks. That is, for a signed directed (undirected) network its underlying unsigned network consists of the same set of nodes and edges, but in which all edges have positive sign. The sign of a walk is the product of the signs of all the edges involved in it. Let A be the adjacency matrix of the signed graph and let D be the adjacency matrix of its underlying unsigned graph. Then, if λ_j and μ_j are the eigenvalues of A and D, respectively, the network degree of balance index proposed by Estrada and Benzi is defined as

$$K=\frac{\sum_{s=0}^{\infty} tr\left(A^s\right)/s!}{\sum_{s=0}^{\infty} tr\left(D^s\right)/s!}=\frac{tr\left(\exp\left(A\right)\right)}{tr\left(\exp\left(D\right)\right)}=\frac{\sum_{j=1}^{n}\exp\left(\lambda_j\right)}{\sum_{j=1}^{n}\exp\left(\mu_j\right)}. \tag{5.1}$$

The upper bound for this index is $K=1$, which is obtained for any balanced graph, and it is a consequence of the following result proved by Acharya in 1980 [1].

Theorem 13. *For any signed graph, the matrices A and D are isospectral if and only if the signed graph is balanced.*

The lower bound of this index is zero, which can be asymptotically obtained for several types of highly imbalanced graphs (see [19] for an example). The previous result also immediately implies that every tree is balanced, which is trivial if we consider that a tree can always be split into two disjoint sets of nodes with positive edges inside the subsets of the partition and negative ones between them. It is worth remarking that other measures for quantifying global degree of balance have been proposed recently by using walks on graphs. They mainly rest on the use of the resolvent of the adjacency matrix. Consequently, they contain a parameter which is bounded between zero and the inverse of the largest eigenvalue of the adjacency matrix in their definition. The current approach is nonparametric.

We can now illustrate the importance of backtracking walks discussed in the previous section for quantifying the degree of balance of a signed graph. Let us consider the signed graphs consisting of two highly imbalanced cliques connected by a chain of positive or negative links (see Fig. 5.1). When the two cliques are connected directly to each other, i.e, $n=0$, the degree of balance is $K\approx 0.399$. As soon as we start to separate the two cliques, the degree of balance growth and when $n=4$ the graph is significantly more balanced with $K\approx 0.664$. Because the two cliques are connected by an acyclic chain, i.e., the two cliques are not forming part of a common cycle, all the effects observed here are due to the

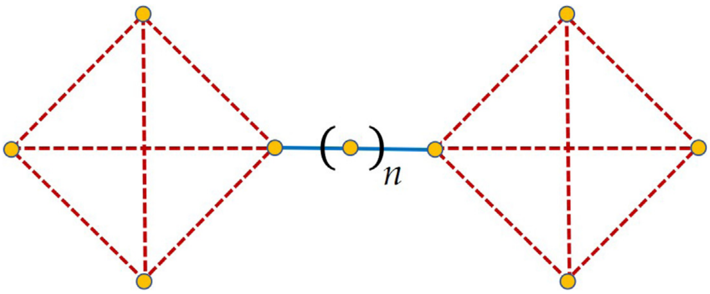

Fig. 5.1. Illustration of two imbalanced cliques separated by a chain of edges having $n = 0, 1, \ldots$ nodes.

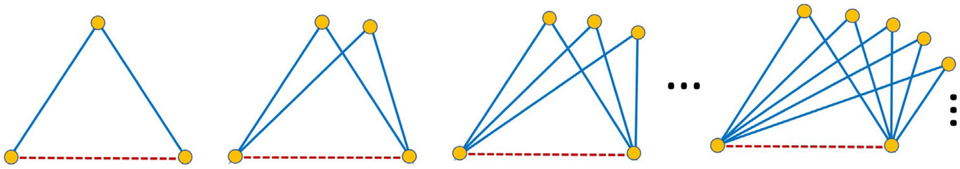

Fig. 5.2. Illustration of the failure of the sign reversal method in accounting for the degree of balance in simple systems. All the signed graphs illustrated have sign reversal index equal to one, while the number of imbalanced triangles are one, two, ..., many.

influence of backtracking walks, i.e., trivial cycles including acyclic structures. Notice that none of the approaches based on counting cycles (of any length), nor of those counting edges to be reversed to transform the graph into balanced, nor any other approach not including acyclic subgraphs, account for these effects. This effect is very appealing from a sociological point of view. Basically it tells us that the degree of imbalance is not only influenced by the number of imbalanced cycles, but also by their proximity. The closest the imbalanced clusters are, the larger the imbalance of the whole system. It is intuitive to think that two groups of imbalanced relations which are close to each other generates more tension in the system than such groups largely distant from each other. This is exactly what the numbers tell us when we use the K index or its similar. We should remark that this is not an effect produced by including more edges into the system as we have tested that these results are reproduced for graphs having exactly the same number of positive and negative edges (results not shown).

In the next section we exploit this finding to quantify the degree of balance of any network.

5.1. Some remarks about other approaches to quantify the degree of balance

An alternative approach to quantify the degree of balance of a signed graph is by considering the smallest number of edges whose reversal of sign leads to a balanced network — line index of the degree of balance [27]. Alternatively it can be the number of signed edges whose removal gives rise to a balanced network [3]. Doreian has called this method "*more useful*" than those based on counting cycles [14]. If we agree, as usually it is done in the study of social balance, that every triad of imbalanced relations produce tension in the system then we will be in trouble when applying the sign reversal indices to account for the degree of balance. In Fig. 5.2 we illustrate several simple systems in which there is only one negative edge which is "gluing" several triangles. In the first case the number of triangles is one, in the second it is two, in the next this number is three and so forth. All these triangles are imbalanced as they have one negative relation and two positive ones. We should suppose that each of these triangles contributes to the tensions in the system. Thus, the degree of balance decreases from left to right, but the sign reversal index is the same for all of them: just one.

This situation can be illustrated by comparing the first two systems. Let us consider that they are either three or four pupils in a classroom. In the first case there will be problems due to the imbalance of one triad of pupils, but in the second case there will be problems from two triads, which obviously will generate more tension in the classroom than the first case. However, for the teacher, the solution of both cases requires exactly the same effort, which is that of transforming the negative relation between two pupils into a positive one. Therefore, the sign reversal indices account not for the degree of balance, but for the "energy" or "effort" needed to transform a given system into a balanced one.

Another strategy used to quantify the degree of balance of a signed network is to consider the solution of clustering problems on signed graphs [11,16,17,23,37]. That is, these approaches first find a partition of the signed graph into clusters such that the sum of the number of positive edges between clusters and the number of negative signs inside clusters is minimum. Apart from the conceptual and computational complexities of graph clustering, the first questioning to these methods is the fact that they give the same value to negative edges inside clusters than to positive edges between them. From a social science perspective is doubtful that having an enmity relation inside your coalition has the same "value" as having a friendship relation between different factions. From a mathematical point of view it is not difficult to find

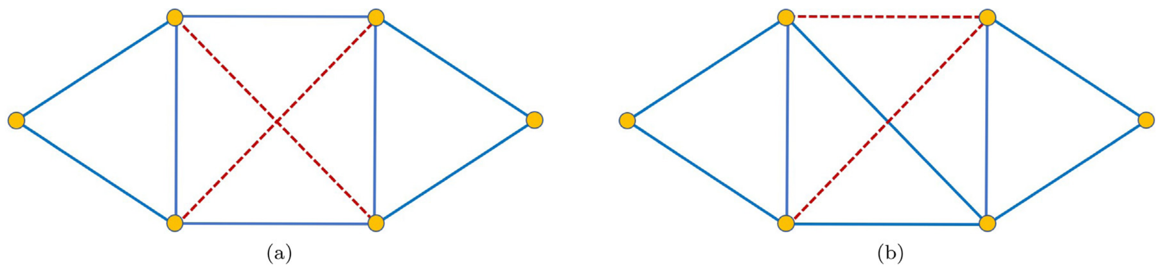

Fig. 5.3. Illustration of the failure of the partitioning method in accounting for the degree of balance in simple systems. The graph in (a) has $K \approx 0.464$ and the one in (b) has $K \approx 0.636$.

examples of networks in which the expected degree of balance – even according to the number of signed triangles only – is significantly different and for which these methods provide the same level of balance. For instance, according to the method proposed by Levorato et al. [37] in which the number of "frustrating" edges as defined before is counted, the two graphs in Fig. 5.3 display the same degree of balance. That is they have two positive edges between the two clusters, which represent 20% of imbalance in both graphs. However, the index K clearly identifies the first graph as significantly less balanced than the second one. Here, the problem is that identifying perfect balance with a bipartition in which there are no frustrating edges does not imply that certain level of balance implies an "optimal" partition understood in terms of minimizing the number of frustrations.

6. Balance of international relations

As a testing scenario of the ideas developed in this work we select the alliances and conflicts that have existed among different countries along the history. The reason why we prefer the use of this type of signed networks instead of other representing interpersonal social relations is the following. In the international relations among countries we have a historic account of events that have taken place at different periods of time. Thus, we can consider the consequences of the balanced or imbalanced situations found in a realistic historic scenario. In the case of interpersonal relations we lack such "historicity" and it would be difficult to analyze whether a balanced situation in a group is a sort of stable or unstable situation for the social entity they represent.

To warming up we present the situation of alliances and enmities between the six major players on European international relations at the end of XIX century and the beginning of the XX century. In Fig. 6.1 we show the alliances (blue lines) between France (Fr), Great Britain (GB), Germany (Ge), Russia (Ru), the Austro–Hungarian Empire (AH) and Italy (It) at six different times. With red dotted lines we represent the enmities among these countries. In the figure the bars represent the degree of balance K for each of these situations. For instance, the "Three Emperor's League" produced a degree of balance of $K \approx 0.467$, while the "British–Russian Alliance" produces a perfect degree of balance $K = 1.000$. If we consider the general trend between 1872 to 1907 we can conclude that there was an increasing tendency in the degree of balance between European forces. Indeed, the Pearson correlation coefficient of the degree of balance as a function of the years in which the alliances are formed is 0.66. However, another possible interpretation of these results exists. Namely that the degree of balance among European forces was almost always about $K \approx 0.5$ (the average of K between the years 1872 to 1904 is $\langle K \rangle \approx 0.513$) and that something extraordinary happened in 1907, where the degree of balance dramatically jump to $K = 1.000$. We should not forget that in 1907 – three years after the "British–Russian Alliance" was signed – Europe erupted into the First World War. No doubts that we are dealing with very limited information in this case. Thus, we propose to study a longer period of time and a larger number of conflicts and alliances among World countries that allow us to extract more solid conclusions.

We now consider a series of networks of international relations, representing positive (alliances) and negative (wars) ties between countries for the period between 1938 and 2008. Each of the 70 signed networks represents a particular year, where nodes represent countries and two countries are connected by a positive tie if they have a formal alliance in that year and a negative tie if there is a militarized dispute between them. The data was provided by the authors of [33] who considered that in those cases where countries have both an alliance and a war in the same year the corresponding edge is taken to be negative. We then calculated the degree of balance index K for each of the signed networks and report these values as the degree of balance of the international relations in the World for that year. The results of these calculations are displayed graphically in Fig. 6.2 where we have remarked some major historic events in different colors.

The first interesting result is that the global trend of the degree of balance in international relations for this period is slightly negative, almost neutral (see broken line in Fig. 6.2). That is, the Pearson correlation coefficient of the degree of balance versus the years is $r \approx -0.095$. This means that the general trend in this period of time is not to increase the global degree of balance. The second interesting result is the oscillatory behavior of the degree of balance between periods of major wars. For instance, degree of balance drops drastically after 1939 to the smallest degree of balance of the 70 years analyzed, coinciding with the period of the second world war (1939–1945). Immediately after WWII the

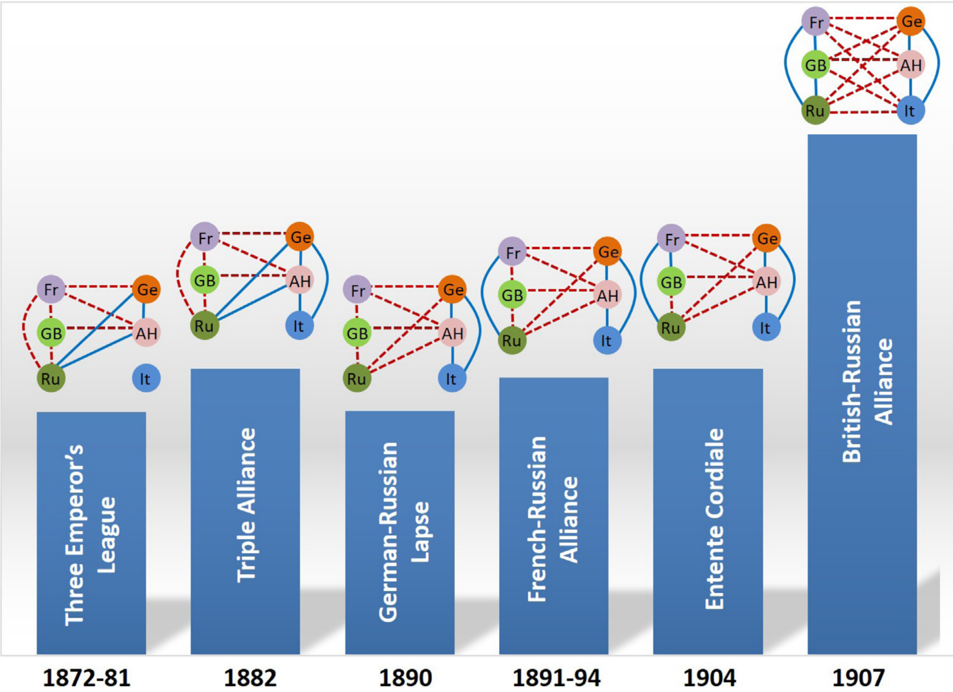

Fig. 6.1. Degree of balance K among the six main European powers at six different times between 1872 and 1907.

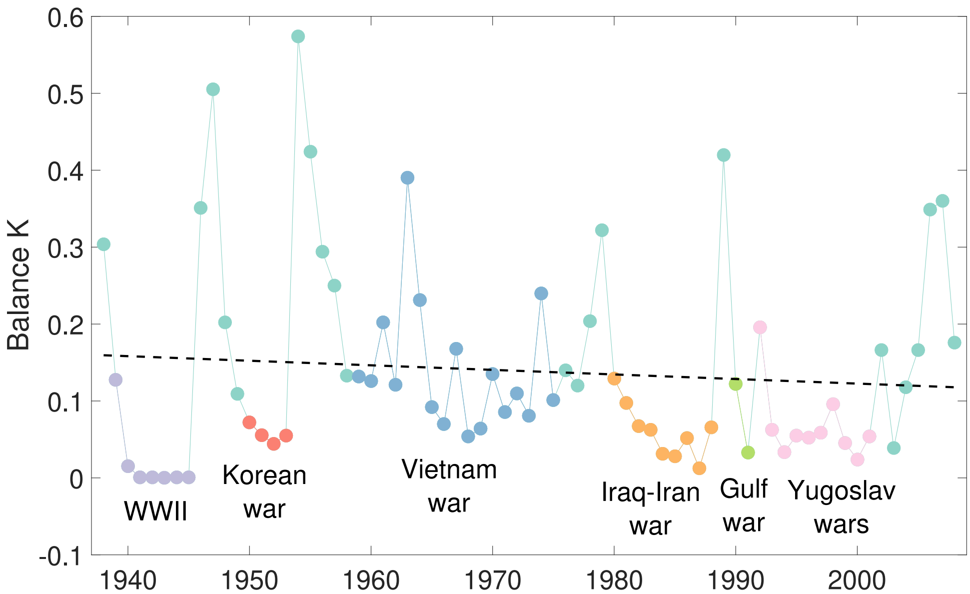

Fig. 6.2. Degree of balance K among the world countries for a period of 70 years between 1938 and 2008. The broken line represents the main trend of the global degree of balance K. We have colored in different colors some of the main international wars during this period. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

degree of balance grows up to values about 0.5 in 1947, to start decreasing again to values below 0.1 for the period of the Korean war, 1950–1953. Again, after the Korean war the world degree of balance increases to its highest value in the whole period. That is, in 1954 the global degree of balance is about 0.6, but then consistently drops to values averaging 0.141 for the period of the Vietnam war. Notice that the average global degree of balance during the Vietnam war is higher than that during the Korean war and that the one during WWII. Also, the degree of balance during this period, 1959–1975 is not so regular and contains some peaks of higher degree of balance like in year 1963 where the degree of

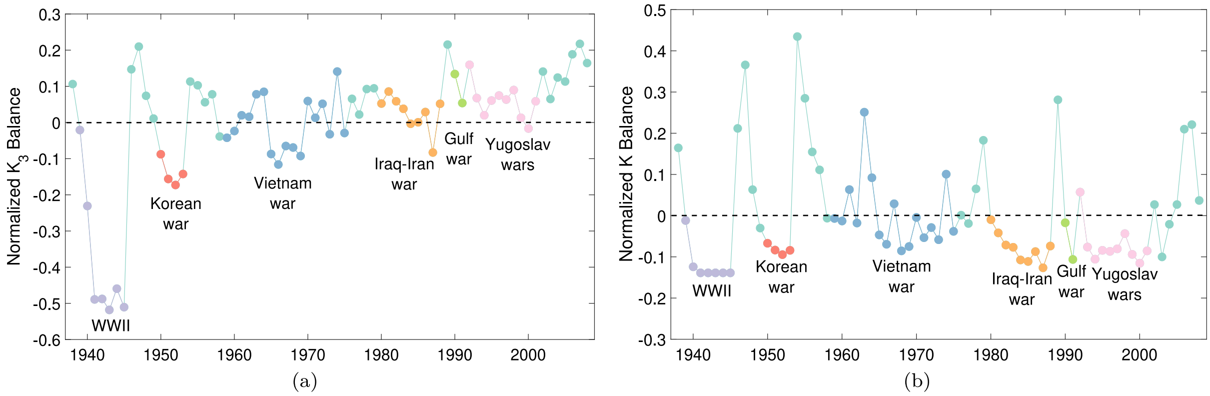

Fig. 6.3. Detrended degree of balance based on triangles only K_3 (a) and on all walks K (b) among the world countries for a period of 70 years between 1938 and 2008. The broken lines represent the zero level. Below this line we have that the world is less balanced than the average for this 70 years period and over this line it means that the world is more balanced than the average. We have colored in different colors some of the main international wars during this period. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

balance is about 0.4. It should be noticed that the Vietnam war is typically divided into the periods before 1964 and after this year, where American intervention was more massive. The continuous "ups and downs" of the degree of balance for the periods of peace and war is observed until the end of the Yugoslav wars in 2001.

The question now is whether such interesting historical trends can also be observed by using only signed triads. To answer this question we study the index K_3 which counts the ratio of signed triads to total number of triads for the world international relation in the same period of time. The first remarkable observation is that indeed this index displays a strong positive correlation with the years, $r \approx 0.551$. That is, according to this index there is a tendency to increase global degree of balance in the international relations for this period of 70 years. However, when we analyze the details in the historic revelations from this index we notice that it does not reflect very well the historic reality. In Fig. 6.3(a) we illustrate the normalized values of this index for all the years studied, where the normalization consists in subtracting the mean degree of balance of the period to the degree of balance of every year. The same results can be observed without normalization but the normalization allows us to have positive and negative values, which are easier to interpret. We clearly see that the triad degree of balance for the years of the WWII are extremely low in comparison with the mean, clearly identifying this historic period as far from mean degree of balance. The same happens for the period of the Korean war, which is clearly identifiable as a lack of degree of balance period. For the period of the Vietnam war this index shows that 9 years are below the mean degree of balance and 8 years are over this mean. The situation is then drastically worsen for the periods of the Iraq–Iran war, only one year below mean degree of balance, the Gulf war, no year below the mean, and the Yugoslav wars, only one year below the mean degree of balance. In total this index is able to identify only 46.9% of years of extreme conflicts in this period of 70 years. In contrast, the index K, which accounts for degree of balance in a global way, identifies 87.8% of the years in extreme conflicts in this period. It clearly identifies 100% of the years in the WWII and the Korean war, 12 out of 17 of the years on the Vietnam war, the whole periods of the Iraq–Iran war and the Gulf war, as well as 9 out of the 10 years of the Yugoslav wars. This experiment clearly teaches us that the use of incomplete information, e.g., triads-only measure, to account for global degree of balance in signed networks is not appropriate. Also, that the "balance hypothesis" telling that all social systems tend to a global degree of balance state is not necessarily true when the appropriate data and indices are used.

The problem of determining whether the periods of high degree of balance are the causes of the subsequent periods of wars, or if the periods of low degree of balance during wars trigger an increase of degree of balance for the subsequent periods of peace, is more than risky, and resemble the chicken–egg problem. In any case, the analysis of the cause–effects of these oscillations in the world degree of balance for war–peace periods is very risky with the data considered here and should be the work of specialists in international relations. However, what is perfectly clear from the numbers is that:

(i) the global trend in the world international relations for the period 1938–2008 is to slightly decrease degree of balance,
(ii) in no case the global degree of balance is very high with values always below 0.6,
(iii) periods of peace are characterized by relatively high degree of balance and periods of major wars are characterized by relatively low degree of balance,
(iv) there is an alternation between periods of relatively high degree of balance with those of relatively low one, which coincide with those periods of major international conflicts, alternated with peace periods.

According to Maoz et al. [41] international relations show "*significant imbalances that are inconsistent with the expectations of balance*" hypothesis due to the fact that "*states sharing common enemies confront each other (...) much more often than is to be expected by chance alone*". That is, the all-negative triangles are more frequently found in international relations that it should be expected by chance. We have seen exactly this conclusion from the paper of Leskovec et al. [36] for a completely different kind of social scenario. As reasoned by Maoz et al. [41] it is astonishing that "*allies that share common enemies share both common interests and common enmities*". We should add to this conclusion that the presence of totally negative triangles could be the consequence of complete separation of three factions in the form of a tripartite graphs, where positive relations are inside the disjoint subsets and negative ones among them.

7. Network degree of balance relative to a null model

In a recent manuscript Kirkley, Cantwell and Newman (KCN) have concluded [33], using exactly the same dataset of international relations studied in the previous section, that such real-world signed network is "*significantly balanced*". In order to arrive to this conclusion KCN modified some known degree of balance indices, including the one defined by Estrada and Benzi, by normalizing them with the corresponding index for an appropriate null model. That is, let B be a degree of balance index, such as the index K studied here. Let $\langle B_{null} \rangle$ be the same index averaged for a large number of random shuffling of the signs in the network studies. Then, KCN defined the following index [33]:

$$\eta\left(B\right) = \frac{B}{\langle B_{null} \rangle}. \tag{7.1}$$

Their motivation is that "*measures of imbalance are difficult to employ on their own because we lack a scale on which to calibrate their values*" [33]. They ask for instance whether $B = 0.5$ for a particular network is large or small. However, the problem is whether $\eta\left(B\right)$ is a measure of the degree of balance of a network or something else. The answer can start to emerge if we consider the 4 signed triangles illustrated in Fig. 2.1. This is a paradigmatic example of balance/imbalance in simple structures. The calculation of the η index will clearly indicates that: $\eta\left(+++\right) = \eta\left(++-\right) = \eta\left(+--\right) = \eta\left(---\right) = 1$. That is, the four triangles have exactly the same value $\eta \equiv 1$. Then, saying that the triangles (b) and (d) are balanced is wrong. Saying that they are as balanced as the triangles (a) and (c) is also wrong. What is correct to say is that the triangles (b) and (d) are as *imbalanced* as we would expect from their topology, and that the triangles (a) and (c) are as *balanced* as we would expect from their topology. The index $\kappa\left(B\right) = 1 - \eta\left(B\right)$ quantifies the "surprise" we receive when we have a given value of the degree of balance index for a graph. In the case of the four triangles mentioned before this surprise is null, i.e., it is obvious that they have to be either balanced (a) and (c) or imbalanced (b) and (d). Notice that the index $\eta\left(B\right)$ has a lower bound (zero), but it has no upper bound. For $0 < \eta\left(B\right) < 1$ we have that the corresponding graph is less balanced than expected, i.e., $\kappa\left(B\right) > 0$. When, $\eta\left(B\right) > 1$ the graph is more balanced than expected, i.e., $\kappa\left(B\right) < 0$. However, this index is not considering the degree of balance in its original definition. Although the example of the four signed triangles can be illuminating we can think about any pair of graphs having, for instance degree of balance $B\left(G_1\right) = 0.10$ and $B\left(G_2\right) = 0.90$. The first one is clearly imbalanced as it has only 10% of the balance that a graph can have due to the fact that $0 \leq B \leq 1$. The second one is balanced as it has 90% of the total degree of balance that it can have. Then, if $\langle B_{null} \rangle \approx 0.05$ for the first graph and $\langle B_{null} \rangle \approx 0.99$ for the second one, we have that $\eta\left(B\right)$ for the first graph is larger than that of the second one, indicating a bigger surprise but not a bigger degree of balance.

Another interesting characteristic of the $\eta\left(B\right)$ indices is that they can change nonmonotonically even in cases when the corresponding degree of balance index changes monotonically. To illustrate this we select here two simple graphs, which are illustrated in Fig. 7.1(a) and (b). The first graph is obtained from a circulant graph of degree 4 in which we change the sign of the edges connecting a node to its first nearest neighbors. The second is built from a complete graph in which we do the same. The dotted (red) lines represent links with negative sign (-1) and the continuous (blue) lines represent edges with positive sign (+1). There are many other possible constructions of these kinds of graphs and these are selected just for the sake of exemplification. Now, we study how the index K changes with the number of nodes n of these two graphs for $6 \leq n \leq 30$. The results are plotted in Fig. 7.1(c). It can be seen that graph (a) increases its degree of balance with n up to an asymptotic value (this value can be obtained analytically but it does not matter for the discussion here). On the other hand, the graph (b) decreases its degree of balance up to an asymptotic value. Then, for any value of n the graph (a) is more balanced than the graph (b). In Fig. 7.1(d) we show the values of $\langle K_{null} \rangle$ averaged for 1,000 reshuffling of the signs of the two nodes, where we can see that they follow similar trends as the "real" graphs. Then, in Fig. 7.1(e) we plot the values of $\eta\left(K\right)$ as a function of the number of nodes. The results are shocking: first, the graph (b) is shown to have larger value of η than (a). Then, there is a range of n in which this order is reversed and (a) has larger η than (b), and finally at $n = 25$ the order is swap again. Obviously what η is capturing is not the "degree of balance" of these graphs as the degree of balance of the two graphs changes monotonically, one increasing and the other decreasing to asymptotic values.

In order to understand what is happening in these graphs we should start by considering that the index η is a ratio of the degree of balance in the "real" network to that obtained by a random reshuffling of signs. This random reshuffling should ideally produce an average of the degree of balance of all possible signed graphs which have the same underlying unsigned graph. In the signed graphs constructed from the circulant graphs the largest imbalance can be created by triangles of the type (−+ +). This is due to the fact that with the topology of these graphs it is combinatorially more probable to create randomly a higher number of these triangles than that of (−−−). The number of these triangles is $t^{-}\left(G_1\right) = n$. In a similar way the most balanced graphs are created by the triangles of the type (−−+), because it is combinatorially more probable to create randomly such triangles than to create (+ + +) in the topology of these graphs.

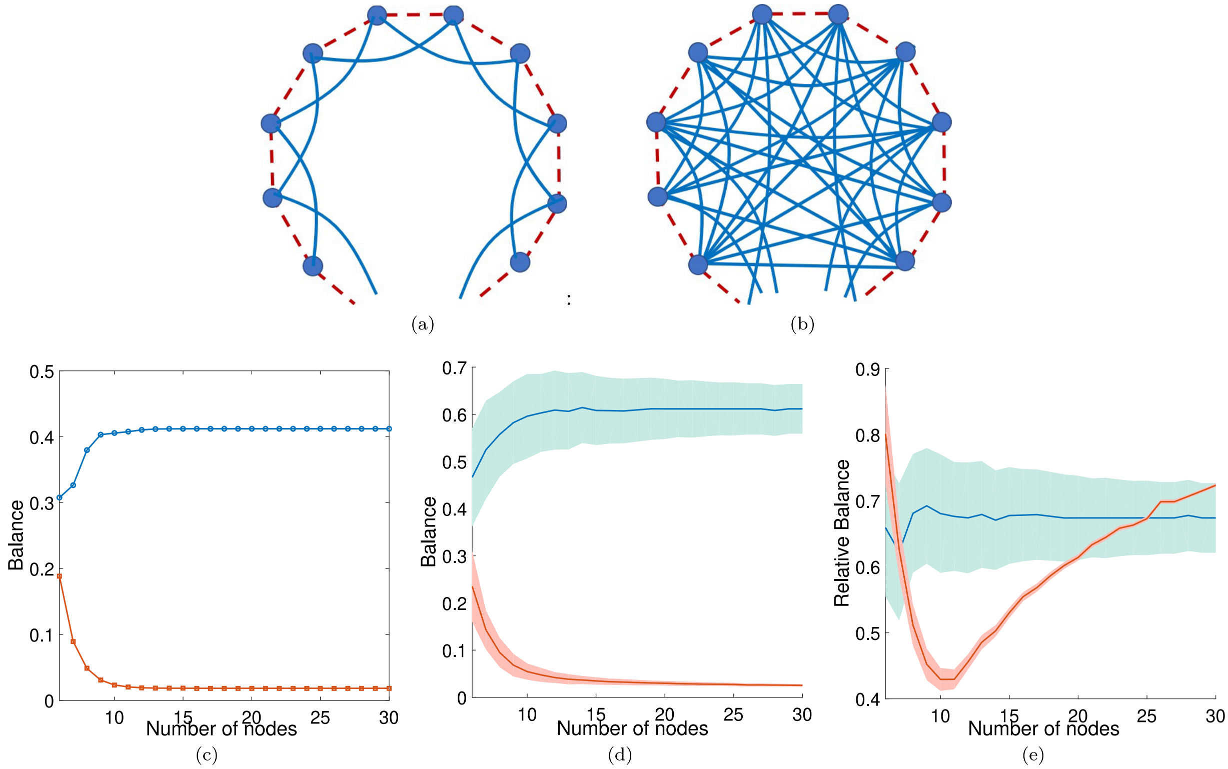

Fig. 7.1. Illustration of two signed graphs constructed, one from circulant graphs (A) and the others from complete graphs (B). The dotted (red) lines represent links with negative sign (-1) and the continuous (blue) lines represent edges with positive sign (+1). (C) Plot of the K indices for the graphs represented in (A) (blue line with circles) and (B) (red line with squares) as a function of the number of nodes. (D) Plot of K indices after sign randomization of the graphs illustrated in (A) and (B) as a function of the number of nodes. The blue line and band correspond to the graphs of type (A) and the red ones to the graphs of type (B). The bands represent the standard deviations. (E) Plot of the relative degree of balance indices η with corresponding standard deviations for the graphs illustrated in (A) and (B) as a function of the number of nodes. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Here, again, the number of these triangles is $t^+(G_1) = n$. This means that the contribution to degree of balance and to imbalance in these graphs is practically the same, which should give rise to a constant relative degree of balance in these graphs. In any case the probability of placing two positive or two negative edges in each triangle of these graphs is about (2/3), which is exactly the constant value around which η moves for these graphs.

In the case of the signed graph constructed from the complete graph, the maximum imbalance can be obtained by the $(+ + -)$ triangles which grow as $t^-(G_2) = n^2 - 4n$. On the other hand, the maximum degree of balance is obtained by the $(+ --)$ and $(+ + +)$ whose number can be obtained by subtracting the total number of triangles and the number of negative ones: $t^+(G_2) = \frac{1}{6}\left(n^3 - 9n^2 + 22n\right)$. As we can see in Fig. 7.2 these two functions intersect at $n = 11$. Thus, for $6 \le n \le 11$ these graphs are dominated by negative triangles, which indicates that the degree of balance will decay. For $n > 11$ these graphs are dominated by the number of positive triangles, which grow cubically. Thus the degree of balance will increase for $n > 11$, exactly as observed in Fig. 7.2. This example teaches us that the relative degree of balance index depends mainly on the combinatorial "richness" for allocating the signs of the original graph in a topologically equivalent structure. If such richness is quite big, like in the case of the graph derived from the complete graph previously studied, the index $\eta(B)$ can change in a very wide range of values having even nonmonotonic behavior as the one observed in the example considered here.

In closing, the conclusion obtained by KCN [33] should not be that the international relations for the period 1938–2008 are balanced — as claimed in [33]. It should be that the international relations in this period show different levels of imbalanced, and that those levels of balance and imbalance are lower than expected. This means that the "effort" made by the global international relation system to reach different levels of balance/imbalance is approximately the same for every year in this period.

8. Conclusions

Balance as understood by Heider [29] is a "yes" or "not" (balanced/imbalanced) approach in which a system displaying balance was thought to be more stable and desired than an imbalanced one. The mathematical formalization of this

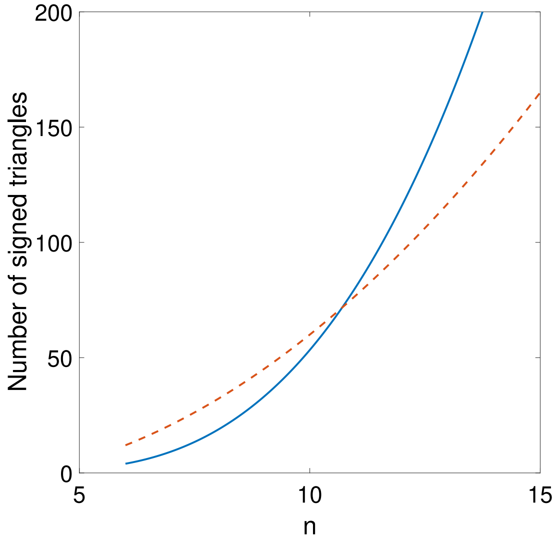

Fig. 7.2. Plot of the number of negative triangles (broken red line) and of positive triangles (solid blue line) for the signed graphs constructed from the complete graph as described in the main text.

approach by Cartwright and Harary [10] immediately implied the existence of a gray scale of degree of balance, in which a system is balanced, or it has certain degree of balance. This gray scale of balance should indicate that a balanced network has the largest value in the scale, and the departure from such state should be reflected by a decay in the position of the network in that scale. Thus, it is not correct that imbalanced networks occupy the same position in the scale than a balanced one, which could be the case of the consideration of scales relative to a null model. The scales of degree of balance relative to null models quantify the "effort" needed for finding the current degree of balance of a network in a random search. We concluded in this work that a "complete" scale of balance should: (i) account for all cyclic and acyclic structures in the signed network, and (ii) give more weight to the smaller signed structures in the network than to the larger ones. Techniques based on counting the number of edges whose removal or sign reversal transform the network into a balanced one are not appropriate as they do not give rise to gray scales of degree of balance as indicated in this work. In this context, the use of matrix functions which counts all the signed subgraphs (cyclic and acyclic) in a network appears to be more appropriate than other measures. In particular the exponential of the adjacency matrix of the signed network seems a natural election due to the fact that it is related not only to the structural definition of degree of balance but also to the dynamic one based on a global consensus between groups in the signed network.

On the other hand, the belief according to the balance hypothesis, that the social structure of the groups tends towards balance through time, should be analyzed individually for each specific scenario and with appropriate degree of balance measures as the one discussed here. In particular, results should not be biased towards this belief without strong empirical or first principles evidence that the system should tend towards such state. In particular, it is needed to take into account that not all "imbalanced" structures, such as certain types of fully-negative triangles, contribute to unstable states of the system. As we have shown here in several real-world signed networks these fully negative triangles are overrepresented in relation to the random expectation. In certain cases these triangles separate groups of mutual enemies forming some classes of multipartite networks where the edges inside the subsets are positive and those between them are negative. This is, of course, a generalization of Cartwright–Harary [10] theory of separation into two groups, but also a constraint to Davis theory [12] in which all fully-negative triangles are considered balanced. In the particular case of the international relations in the world for long periods of time we have confirmed previous results about the high imbalance of the system. In addition we have observed that for the period 1938–2008 the degree of balance has slightly decayed, thus there is no indication of a trend towards a balanced state in the global international relations.

We just want to finish here by quoting Maoz et al. [41] with the most important advice you can receive if you work with signed social networks, particularly for international relations: "*we should not dismiss the significant occurrence of imbalanced relations as a typical feature of fluid international relations*". Those who have ears please hear!

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