Lecture Notes on Fusion Systems

Sergey Shpectorov

August 27, 2013

Chapter 1

Basic definitions

1.1 Fusion systems

Let p be a prime and S a finite p-group.

Definition 1.1.1 *The* complete injective category *on S is the small category $\mathcal{I} = \mathcal{I}(S)$, where $\mathrm{Ob}(\mathcal{I})$ is the set of all subgroups of S and, for $P, Q \in \mathrm{Ob}(\mathcal{I})$, the morphism set $\mathrm{Hom}_{\mathcal{I}}(P, Q)$ coincides with $\mathrm{Inj}(P, Q)$, the set of all injective homomorphisms from P to Q.*

If a homomorphism is injective then any restriction of this homomorphism is also injective, and so $\mathcal{I}(S)$ is closed with respect to the operation of restriction. Let us give a precise definition and set notation for restriction.

Note that we use the right notation for mappings.

Definition 1.1.2 *Suppose $\phi : P \to Q$ is a group homomorphism. We will view P and Q as attributes of ϕ, P being the* source *group of ϕ and Q the* target *group of ϕ.*

For subgroups $P' \leq P$ and $Q' \leq Q$, such that $P'\phi \leq Q'$, let $\phi_{P',Q'}$ be the homomorphism from P' to Q', such that, for $x \in P'$, we have $x\phi_{P',Q'} = x\phi$. Clearly, P' and Q' are the source and target of $\psi = \phi_{P',Q'}$, respectively.

We will say that ψ is a restriction *of ϕ and that ϕ is an* extension *of ψ.*

If $Q = Q'$ then we will simply write $\psi = \phi_{P'}$ and if $P = P'$ then we will write $\psi = \phi_{,Q'}$.

It is well known that conjugation by $x \in S$, *i.e.*, the mapping $c_x : S \to S$ sending $y \mapsto y^x = x^{-1}yx$, is an automorphism of S, hence an injective homomorphism from S back to S. Therefore, $c_x \in \mathrm{Hom}_{\mathcal{I}}(S, S)$ for all $x \in S$. Note that $c_x = c_y$ if and only if $xy^{-1} \in Z(S)$.

We are now ready to give our main definition.

Definition 1.1.3 *A* fusion system $\mathcal{F}$ *on a p-group* S *is a plain subcategory of* $\mathcal{I} = \mathcal{I}(S)$ *('plain' means that* $\mathrm{Ob}(\mathcal{F}) = \mathrm{Ob}(\mathcal{I})$*), satisfying two further properties:*

- $\mathrm{Hom}_{\mathcal{F}}(S,S)$ *contains all conjugation automorphisms* c_x, $x \in S$;
- $\mathcal{F}$ *is closed with respect to restriction, that is, if* $\phi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ *and if* $P' \leq P$ *and* $Q' \leq Q$ *such that* $P\phi \leq Q$ *then* $\phi_{P',Q'} \in Hom_{\mathcal{F}}(P',Q')$; *and*
- $\mathcal{F}$ *is closed with respect to inversion, that is, if* $\phi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ *is an isomorphism between* P *and* Q *then* $\phi^{-1} \in \mathrm{Hom}_{\mathcal{F}}(Q,P)$.

For example, $\mathcal{F} = \mathcal{I}$ is a fusion system, the largest fusion system on S. Hence at least one fusion system on S exists!

The following general construction can be used to build more fusion systems.

Definition 1.1.4 *Suppose a p-group* S *is a subgroup of a group* G*, which can be finite or infinite. Let* $\mathcal{F} = \mathcal{F}_S(G)$ *be the plain subcategory of* $\mathcal{I}(S)$ *with morphism sets* $\mathrm{Hom}_{\mathcal{F}}(P,Q) = \{(c_x)_{P,Q} \mid x \in G, P^x \leq Q\}$.

Note that, for $P,Q,R \leq S$ and for $\phi = (c_x)_{P,Q} \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ and $\psi = (c_y)_{Q,R} \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$, we have that $\phi\psi = (c_x)_{P,Q}(c_y)_{Q,R} = (c_{xy})_{Q,R} \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$, and so $\mathcal{F}$ is closed for composition. Also, $\iota_{P,P} = (c_1)_{P,P}$ is the identity morphism in $\mathrm{Hom}_{\mathcal{F}}(P,P)$ for every $P \leq S$. Hence $\mathcal{F} = \mathcal{F}_S(G)$ is indeed a category, namely, a plain subcategory of $\mathcal{I}(S)$.

Proposition 1.1.5 *For a group* G *and a p-subgroup* S*, the category* $\mathcal{F}_S(G)$ *is always a fusion system on* S.

Proof. Since $\mathcal{F}$ is a plain subcategory of $\mathcal{I} = \mathcal{I}(S)$, we just need to verify the additional properties.

First of all, since $S \subseteq G$, we see that $(c_x)_{P,P} \in \mathrm{Hom}_{\mathcal{F}}(P,P)$ for all $x \in S$. Hence the first property holds.

Suppose that $\phi = (c_x)_{P,Q} \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ for some $x \in G$. Consider $P' \leq P$ and $Q' \leq Q$ such that $P'\phi \leq Q'$, that is, $(P')^x \leq Q'$. Then $\phi_{P',Q'} = (c_x)_{P',Q'}$, and so $\phi_{P',Q'} \in \mathrm{Hom}_{\mathcal{F}}(P',Q')$, proving that $\mathcal{F}$ is closed for restriction.

Finally, if $\phi = (c_x)_{P,Q}$ is an isomorphism then, clearly, $P^x = Q$, and so $P = Q^{x^{-1}}$. This means that $\phi^{-1} = (c_{c^{-1}})_{Q,P} \in \mathrm{Hom}_{\mathcal{F}}(Q,P)$, and hence $\mathcal{F}$ is closed for inverses. □

For example, taking $G = S$, we build the fusion system $\mathcal{S} = \mathcal{S}(S) = \mathcal{F}_S(S)$, whose morphism sets are $\mathrm{Hom}_{\mathcal{S}}(P,Q) = \{(c_x)_{P,Q} \mid x \in S, P^x \leq Q\}$. Comparing with the definition of fusion systems, we see that $\mathrm{Hom}_{\mathcal{S}}(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q)$ for all fusion systems $\mathcal{F}$ on S. This is because $c_x \in \mathrm{Hom}_{\mathcal{F}}(S,S)$ and $\mathcal{F}$ is closed for restriction. It follows that in addition to the largest fusion system on S, namely, $\mathcal{I} = \mathcal{I}(S)$, we also have the smallest fusion system on S, namely, $\mathcal{S} = \mathcal{S}(S)$. All other fusion systems on S are somewhere between $\mathcal{I}$ and $\mathcal{S}$.

Let us also note the following.

Proposition 1.1.6 *If $\mathcal{F}_i$, $i \in I$, are fusion systems on S then $\cap_{i\in I}\mathcal{F}_i$ is again a fusion system on S.*

Proof. Indeed, all properties of fusion systems are preserved by intersection. □

Hence the structure (poset) of all fusion systems on S is closed for intersection. As usual, this means that we can talk about generation for fusion systems.

Definition 1.1.7 *Suppose X is a set of morphisms from $\mathcal{I} = \mathcal{I}(S)$, that is, X is a subset of $\cup_{P,Q\leq S}\mathrm{Hom}_{\mathcal{I}}(P,Q)$. Then we let $\langle X\rangle$ denote the smallest fusion system on S containing all morphisms from X. We call $\langle X\rangle$ the fusion system on S* generated *by X.*

Of course, $\langle X\rangle$ is simply the intersection of all fusion systems containing X.

In general, since all fusion systems on S have the same set of objects, we may identify them with their set of morphisms, particularly, since by our definition every morphism 'knows' its source and target group.

Exercise 1.1.8 *Work out an explicit description of the morphism set of $\langle X\rangle$.*

Exercise 1.1.9 *Suppose $P \leq S$ has the property that P is not conjugate in S to a subgroup of the source group of any morphism from X. Show that $\mathrm{Hom}_{\langle X\rangle}(P,Q) = \mathrm{Hom}_{\mathcal{S}}(P,Q)$ for all $Q \leq S$.*

Exercise 1.1.10 *Would a similar statement involving* target *groups of morphisms be true?*

1.2 Conjugate fusion systems and automorphisms

The definition of an isomorphism of fusion systems is quite straightforward.

Definition 1.2.1 *Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are fusion systems on p-groups S_1 and S_2, respectively. An* isomorphism *from $\mathcal{F}_1$ onto $\mathcal{F}_2$ is a group isomorphism $\pi : S_1 \to S_2$ such that, for all $P, Q \leq S_1$, we have $\mathrm{Hom}_{cF_2}(P\pi, Q\pi) = \mathrm{Hom}_{\mathcal{F}_1}(P, Q)^\pi = \pi^{-1}\mathrm{Hom}_{\mathcal{F}_1}(P, Q)\pi = \{\pi^{-1}\phi\pi \mid \phi \in \mathrm{Hom}_{\mathcal{F}_1}(P, Q)\}$.*

Clearly, every isomorphism $\pi : S_1 \to S_2$, as above induces an isomorphism functor $\hat{\pi}$ from $\mathcal{F}_1$ onto $\mathcal{F}_2$, sending each $P \leq S_1$ to $P\pi \leq S_2$ and each $\phi \in \mathrm{Hom}_{\mathcal{F}_1}(P, Q)$ to $\phi^\pi = \pi^{-1}\phi\pi \in \mathrm{Hom}_{\mathcal{F}_2}(P\pi, Q\pi)$.

Exercise 1.2.2 *Why cannot we just define an isomorphism of fusion systems as an arbitrary isomorphism functor? Give examples of 'nonsense' isomorphism functors.*

Let us now concentrate on the case $S_1 = S_2 = S$. Then any automorphism $\pi \in \mathrm{Aut}(S)$ transforms (conjugates!) each fusion system $\mathcal{F}$ on S into another (or same) fusion system $\mathcal{F}^\pi$. Clearly, inclusion between fusion systems is preserved. In other words, $\mathrm{Aut}(S)$ acts on the poset of fusion systems on S.

Proposition 1.2.3 *Two fusion systems on S are isomorphic if and only if they are conjugate under the above action.*

Proof. Immediately follows from the definitions. □

Finally, we can define the automorphism group of a fusion system.

Definition 1.2.4 *The autmorphism group $\mathrm{Aut}(\mathcal{F})$ of a fusion system $\mathcal{F}$ on S consists of all automorphisms $\pi \in \mathrm{Aut}(S)$ which preserve $\mathcal{F}$, i.e., such that $\mathcal{F}^\pi = \mathcal{F}$.*

Therefore, $\mathrm{Aut}(\mathcal{F})$ is a subgroup of $\mathrm{Aut}(S)$.

Exercise 1.2.5 *Check that $\mathrm{Aut}(cF)$ can also be defined as the stabilizer of $\mathcal{F}$ in the action of $\mathrm{Aut}(S)$ on the poset of all fusion systems on S.*

We conclude this section with the following observation.

Proposition 1.2.6 *The group $\mathrm{Inn}(S)$ of inner automorphisms of S is contained in $\mathrm{Aut}(\mathcal{F})$ for all fusion systems $\mathcal{F}$ on S.*

Proof. Take $x \in S$ and set $\pi = c_x$. If $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ for $P, Q \leq S$ then $\phi^\pi = (c_{x^{-1}})_{P^x,P}\phi(c_x)_{Q,Q^x} \in \mathrm{Hom}_{\mathcal{F}}(P^x, Q^x)$. Hence $\mathcal{F}^\pi = \mathcal{F}$. □

It follows that the action of $\mathrm{Inn}(S)$ on the poset of all fusion systems is trivial. Hence it is in fact the group of outer automorphisms, $\mathrm{Out}(S) = \mathrm{Aut}(S)/\mathrm{Inn}(S)$, that conjugates fusion systems.

Exercise 1.2.7 *Determine the poset of all fusion systems on $S = C_2$, C_3, $C_2 \times C_2$, D_8, Q_8.*

1.3 Another definition

Our definition of fusion systems is, in fact, not the definition that is widely used. So we will also give that other definition and show equivalence. First, we need the following.

Definition 1.3.1 *For $P, Q \leq S$, let $\iota_{P,Q}$ be the* inclusion *mapping $(c_1)_{P,Q} \in \mathrm{Hom}_{\mathcal{S}}(P, Q)$. If $Q = S$, we simply write ι_P for this inclusion mapping.*

Proposition 1.3.2 *A plain subcategory $\mathcal{F}$ of $\mathcal{I} = \mathcal{I}(S)$ is a fusion system if and only if*

- *$\mathrm{Hom}_{\mathcal{S}}(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q)$ for all $P, Q \leq S$; and*
- *every morphism in $\mathcal{F}$ decomposes as an $\mathcal{F}$-isomorphism followed by an inclusion mapping.*

Proof. Suppose $\mathcal{F}$ is a fusion system on S. Since $\mathcal{F}$ contains all conjugation mappings c_x, $x \in S$, and since $\mathcal{F}$ is closed for restriction, we have that $\mathrm{Hom}_{\mathcal{S}}(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q)$ for all $P, Q \leq S$. In particular, $\iota_{P,Q} = (c_1)_{P,Q} \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ whenever $P \leq Q$.

Furthermore, if $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ is an arbitrary morphisms, then $\psi = \phi_{,P\phi}$ is an isomorphism of groups, since ϕ is injective. As $\mathcal{F}$ is closed for inverses, ψ is an $\mathcal{F}$-isomorphism. Manifestly, $\phi = \psi\iota_{P\phi,Q}$, and so ϕ decomposes as an $\mathcal{F}$-isomorphism followed by an inclusion. □

Let us insert a couple of related exercises.

Exercise 1.3.3 *Verify that the two factors in the above decomposition are unique. That is, if $\phi = \psi_1\iota_1 = \psi_2\iota_2$, where ψ_i are isomorphisms and ι_i are inclusions, then $\psi_1 = \psi_2$ and $\iota_1 = \iota_2$.*

We will refer to $\psi = \phi_{P,P\phi}$ as the *core* of ϕ, *i.e.*, it is the isomorphism hidden in ϕ. We see from Proposition 1.3.2 that the fusion system is fully defined (generated!) by the isomorphisms it contains.

Our proof of Proposition 1.3.2 is in fact incomplete. The second half of the proof is the next exercise.

Exercise 1.3.4 *Complete the proof of Proposition 1.3.2, that is, prove that every plain subcategory of $\mathcal{I}$ satisfying the two conditions from the proposition must be a fusion system.*

1.4 Isomorphisms

We saw in the preceding section that isomorphisms play a special role in each fusion system. In this section we collect some observations concerning isomorphisms.

Proposition 1.4.1 *Suppose $\mathcal{F}$ is a fusion system on S. A morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, for $P, Q \le S$, is an isomorphism if and only if $|P| = |Q|$.*

Proof. Note that P and Q are finite groups. Since ϕ is injective, $P\phi \cong P$ and so $|P\phi| = |P|$. Hence $P\phi = Q$ if and only if $|P| = |Q|$. Therefore ϕ is surjective (and so an isomorphism) if and only if $|P| = |Q|$. $\square$

It follows that if $|P| = |Q|$ then every morphism in $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ got to be an isomorphism, and if $|P| < |Q|$ then none of the morphisms in $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ is an isomorphism. Note also that if $|P| > |Q|$ then $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ must be empty.

Clearly, if $Q = P$ then $|P| = |Q|$. Therefore, we have the following.

Corollary 1.4.2 *For $P \le S$, every morphism in $\mathrm{Hom}_{\mathcal{F}}(P,P)$ is an isomorphism. In particular, $\mathrm{Hom}_{\mathcal{F}}(P,P)$ is a group with respect to composition, namely, a subgroup of $\mathrm{Aut}(P)$.*

In view of this, it is common to write $\mathrm{Aut}_{\mathcal{F}}(P)$ for the group $\mathrm{Hom}_{\mathcal{F}}(P,P)$.

Chapter 2

Saturation

2.1 Extension control

We know that fusion systems is always closed with respect to restriction of morphisms. Extension of morphisms is the operation inverse to restriction. Hence $\psi \in \mathrm{Hom}_{\mathcal{F}}(P', Q')$ is an extension of $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ if $P \leq P'$, $Q \leq Q'$, and $x\phi = x\psi$ for all $x \in P$. In particular, this means that $P\psi \leq Q$.

Our axioms do not promise that extension of ϕ to a pair of overgroups $P' \geq P$ and $Q' \geq Q$ is always possible. This prompts the following natural question: How far can we extend a morphism?

If $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ and $Q \leq Q'$ then we can extend ϕ to $\psi = \phi\iota_{Q,Q'} \in \mathrm{Hom}_{\mathcal{F}}(P, Q')$. That is, we can always increase Q in any way we like, and these extensions are a trivial matter. Hence the real problem is whether we can increase the source group P.

Definition 2.1.1 *A morphism* $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ *is* extendible *if it admits an extension* $\psi \in \mathrm{Hom}_{\mathcal{F}}(P', Q')$ *with* $P < P'$*.*

If a morphism ψ is an extension of ϕ then the core $\psi_{,P'\psi}$ of ψ extends the core $\phi_{,P\phi}$ of ϕ. In particular, if ψ extends ϕ nontrivially, *i.e.*, $P < P'$, them also the extension of cores is nontrivial. In view of this, we restrict ourselves to just extending isomorphisms.

Furthermore, note that if $P < P'$ then also $P < N_{P'}(P)$, since S, and hence also P', is a p-group. Furthermore, $\psi_{N_{P'}(P)}$ is an extension of ϕ. Hence, when deciding whether ϕ is extendible, it suffices to only consider P' in which P is normal, that is, we can assume that $P' \leq N_S(P)$.

For our next step, two further bits of notation are needed.

Definition 2.1.2 *For* $P \leq S$*, let* $\epsilon_P : N_S(P) \to \mathrm{Hom}_{\mathcal{F}}(P, P) = \mathrm{Aut}_{\mathcal{F}}(P)$ *be the homomorphism sending* $x \in N_S(P)$ *to* $(c_x)_{P,P} \in \mathrm{Aut}_{\mathcal{F}}(P)$*.*

Note that the kernel of ϵ_P is $C_S(P)$. Furthermore, checking the relevant definitions, one may notice that the image of the homomorphism ϵ_P coincides with $\mathrm{Hom}_{\mathcal{S}}(P,P) = \mathrm{Aut}_{\mathcal{S}}(P)$, where $\mathcal{S}$ is the minimal fusion system on S. In particular, $\mathrm{Aut}_{\mathcal{S}}(P)$ is a p-group isomorphic to $N_S(P)/C_S(P)$.

Definition 2.1.3 *For an isomorphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, let $\hat{\phi}$ be the induced isomorphism $\mathrm{Aut}_{\mathcal{F}}(P) \to \mathrm{Aut}_{\mathcal{F}}(Q)$ defined by $\alpha \mapsto \aleph^{\phi} = \phi^{-1}\alpha\phi$, where, naturally, $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$.*

We will need the following observation.

Lemma 2.1.4 *Suppose an isomorphism $\psi \in \mathrm{Hom}_{\mathcal{F}}(P',Q')$ extends an isomorphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ and $P' \leq N_S(P)$. Then*

- *$Q' \leq N_S(Q)$; and*
- *$\psi\epsilon_Q = \epsilon_P\hat{\phi}$ on P'.*

Proof. Since ψ is an isomorphism between P' and Q' and since P is normal in P', we have that $P\psi = P\phi = Q$ is normal in $P'\psi = Q'$.

For the second claim, take $x \in P'$ and let $y = x\psi$. To show that $x(\psi\epsilon_Q) = x(\epsilon_P\hat{\phi})$, we need to apply the morphisms in the left and right sides to the same $t \in Q$. Since ϕ is an isomorphism, there is $z \in P$ such that $z\phi = t$.

On the one hand, $t(x(\psi\epsilon_Q)) = t(y\epsilon_Q) = t(c_y)_{Q,Q} = t^y$. On the other hand, $t(x(\epsilon_P\hat{\phi})) = t((c_x)_{P,P}\hat{\phi}) = t\phi^{-1}(c_x)_{P,P}\phi = z(c_x)_{P,P}\phi = (z^x)\phi = (z^x)\psi = t^y$. Here we used that ψ is an extension of ϕ. This means both that $(z^x)\phi = (z^x)\psi$ and that $z\psi = z\phi$.

Since both sides evaluate to the same element t^y of Q, the second claim holds. $\square$

Recall that $\mathrm{Aut}_{\mathcal{S}}(P)$ is a p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$ and, similarly, $\mathrm{Aut}_{\mathcal{S}}(Q)$ is a p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$. Recall further that $\hat{\phi}$ is an isomorphism between $\mathrm{Aut}_{\mathcal{F}}(P)$ and $\mathrm{Aut}_{\mathcal{F}}(Q)$. Hence $\mathrm{Aut}_{\mathcal{S}}(Q)\hat{\phi}^{-1}$ is a second p-subgroup in $\mathrm{Aut}_{\mathcal{F}}(P)$.

Definition 2.1.5 *We call the intersection $C_\phi = \mathrm{Aut}_{\mathcal{S}}(P) \cap \mathrm{Aut}_{\mathcal{S}}(Q)\hat{\phi}^{-1}$ the* control *subgroup of ϕ. Let also N_ϕ be the full preimage of C_ϕ under ϵ_P.*

Note that C_ϕ is a subgroup of $\mathrm{Aut}_{\mathcal{S}}(P)$, which is the image of the homomorphism ϵ_P. In particular, our definition of N_ϕ makes sense.

Let us record the following useful fact.

Proposition 2.1.6 *If $\phi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ then $N_\phi \geq C_S(P)$.*

Proof. Indeed, N_ϕ is a full preimage of C_ϕ under ϵ_P and hence it contains the kernel of ϵ_P, which is none else but $C_S(P)$. $\square$

The following proposition explains our 'control' terminology.

Proposition 2.1.7 *Suppose that $\psi \in \mathrm{Hom}_{\mathcal{F}}(P', Q')$ is an extension of $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ and also that $P' \leq N_S(P)$. Then $P' \leq N_\phi$.*

Proof. Clearly, we may assume that ψ and ϕ are isomorphisms.

Let $x \in P'$. Since $P' \leq N_S(P)$, we have that $\alpha = x\epsilon_P$ is defined and it lies in $\mathrm{Aut}_{\mathcal{S}}(P)$. Let $\beta = \alpha\hat{\phi}$. By the second claim of the lemma, $\beta = x(\epsilon_P\hat{\phi}) = x(\psi\epsilon_Q) = y\epsilon_Q$, where $y = x\psi$. By the first claim of the lemma, $y \in Q' \leq N_S(Q)$ and so $\beta = y\epsilon_Q \in \mathrm{Aut}_{\mathcal{S}}(Q)$. We conclude that $\alpha = \beta\hat{\phi}^{-1}$ also lies in $\mathrm{Aut}_{\mathcal{S}}(P)\hat{\phi}^{-1}$. Therefore, $\alpha \in C_\phi$, and so $x \in N_\phi$. $\square$

Hence the subgroups C_ϕ and N_ϕ control extensions of ϕ. Namely, ϕ cannot be 'normally' extended beyond N_ϕ.

Exercise 2.1.8 *State and prove a corollary for arbitrary (i.e., non-'normal') extensions of ϕ.*

We conclude the present section with the following definition.

Definition 2.1.9 *A morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ is called* fully extendible *if it extends to $\psi \in \mathrm{Hom}_{\mathcal{F}}(P', Q')$ with $P' = N_\phi$.*

2.2 Receptive subgroups

The following is a key concept needed for the definition of saturated fusion systems.

Definition 2.2.1 *A subgroup $Q \leq S$ is* receptive *if every $\mathcal{F}$-isomorphism ϕ with target group Q is fully extendible.*

That is, for each $R \leq S$ that is $\mathcal{F}$-conjugate to Q, each morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(R, Q)$ extends to the corresponding overgroup $N_\phi \geq R$.

Let us compare this concept with the following one:

Definition 2.2.2 *A subgroup $Q \leq S$ is* maximally centralized *if $|C_S(Q)| \geq |C_S(R)|$ for all $R \leq S$ that are $\mathcal{F}$-conjugate to Q.*

Proposition 2.2.3 *If Q is receptive then it is maximally centralized.*

Proof. Take $R \leq S$ that is $\mathcal{F}$ conjugate to Q and $\phi \in \mathrm{Hom}_{\mathcal{F}}(R, Q)$. Let $\psi \in \mathrm{Hom}_{\mathcal{F}}(P', Q')$ be an extension of ϕ with $P' = N_\phi$. Such an extension exists since Q is receptive.

By Proposition 2.1.6, $N_\phi \geq C_S(R)$. Since ψ is a homomorphisms and since $R\psi = R\phi = Q$, we have that $C_S(R)\psi \leq C_S(Q)$. As ψ is injective, this clearly means that $|C_S(R)| \leq |C_S(Q)|$. □

Exercise 2.2.4 *Which $Q \leq S$ could be called* fully centralized*? Give a suitable definition and verify whether receptive subgroups are fully centralized.*

Exercise 2.2.5 *Suppose $G \geq S$ and $\mathcal{F} = \mathcal{F}_S(G)$. Prove that $Q \leq S$ is receptive if $C_S(Q)$ is a Sylow p-subgroup of $C_G(Q)$. Give an example where the converse statement fails.*

Exercise 2.2.6 *Suppose again that $\mathcal{F} = \mathcal{F}_S(G)$ for some $G \geq S$. If S is a Sylow p-subgroup of G. Prove that*

- *every $Q \leq S$ is $\mathcal{F}$-conjugate to a receptive subgroup; and*
- *$Q \leq S$ is receptive if and only if it is maximally centralized.*

2.3 Fully normalized subgroups

We know already that $\mathrm{Aut}_{\mathcal{S}}(P)$ is a p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$ for all $P \leq S$.

Definition 2.3.1 *A subgroup $Q \leq S$ is* fully automized *if* $\mathrm{Aut}_{\mathcal{F}}(Q)$ *is a Sylow p-subgroup of* $\mathrm{Aut}_{\mathcal{F}}(Q)$.

This definition expresses the idea that $\mathrm{Aut}_{\mathcal{S}}(Q)$ is as big as it can be.

Proposition 2.3.2 *If Q is fully automized then, for every R that is $\mathcal{F}$-conjugate to Q, there exists $\phi \in \mathrm{Hom}_{\mathcal{F}}(R, Q)$ with $N_\phi = N_S(R)$.*

Proof. Let us start with an arbitrary morphism $\pi \in \mathrm{Hom}_{\mathcal{F}}(R, Q)$, which exists since R and Q are $\mathcal{F}$-conjugate. Since $\hat{\pi}$ is an isomorphism between $\mathrm{Aut}_{\mathcal{F}}(R)$ and $\mathrm{Aut}_{\mathcal{F}}(Q)$, we have that $\mathrm{Aut}_{\mathcal{S}}(R)\hat{\pi}$ is a p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$. Since Q is fully automized, $\mathrm{Aut}_{\mathcal{S}}(Q)$ is a Sylow p-subgroup of the same group $\mathrm{Aut}_{\mathcal{F}}(Q)$. Hence there exists $\beta \in \mathrm{Aut}_{\mathcal{F}}(Q)$ such that $(\mathrm{Aut}_{\mathcal{S}}(R)\hat{\pi})^\beta \leq \mathrm{Aut}_{\mathcal{S}}(Q)$.

Set $\phi = \pi\beta$. Then $\mathrm{Aut}_{\mathcal{S}}(R)\hat{\phi} = \mathrm{Aut}_{\mathcal{S}}(Q)^{\pi\beta} = (\mathrm{Aut}_{\mathcal{S}}(R)\hat{\pi})^\beta \leq \mathrm{Aut}_{\mathcal{S}}(Q)$. Thus, $\mathrm{Aut}_{\mathcal{S}}(R)\hat{\phi} \leq \mathrm{Aut}_{\mathcal{S}}(Q)$, or equivalently, $\mathrm{Aut}_{\mathcal{S}}(R) \leq \mathrm{Aut}_{\mathcal{S}}(Q)\hat{\phi}^{-1}$.

Clearly, this means that $C_\phi = \mathrm{Aut}_{\mathcal{S}}(R) \cap \mathrm{Aut}_{\mathcal{S}}(Q)\hat{\phi}^{-1} = \mathrm{Aut}_{\mathcal{S}}(R)$, and so $N_\phi = N_S(R)$, because the latter is the full preimage of $\mathrm{Aut}_{\mathcal{S}}(R)$ under ϵ_P. □

Definition 2.3.3 *A subgroup $Q \leq S$ is* fully normalized *if it is fully automized and receptive.*

The following result justifies our terminology.

Proposition 2.3.4 *If Q is fully normalized and R is $\mathcal{F}$-conjugate to Q then there exists a morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(R, Q)$ extendible to a morphism in $\mathrm{Hom}_{\mathcal{F}}(N_S(R), N_S(Q))$. In particular, $\mathrm{Hom}_{\mathcal{F}}(N_S(R), N_S(Q))$ is nonempty.*

Proof. Since Q is fully automized, it follows from Proposition 2.3.2 that there exists $\phi \in \mathrm{Hom}_{\mathcal{F}}(R, Q)$ with $N_\phi = N_S(R)$. Since Q is receptive, ϕ extends to an isomorphism $\psi \in \mathrm{Hom}_{\mathcal{F}}(R', Q')$ with $R' = N_\phi = N_S(R)$. As ψ is a homomorphism and $R\psi = Q$, we have that $Q' = R'\psi \leq N_S(Q)$. In particular, $\psi\iota_{Q',N_S(Q)}$ is the desired extension of ϕ in $\mathrm{Hom}_{\mathcal{F}}(N_S(R), N_S(Q))$.
□

Exercise 2.3.5 *Suppose $\mathcal{F} = \mathcal{F}_S(G)$ for an overgroup G of S. Prove that $Q \leq S$ is fully normalized whenever $N_S(Q)$ is a Sylow p-subgroup of $N_G(Q)$.*

Definition 2.3.6 *A subgroup $Q \leq S$ is maximally normalized if $|N_S(Q)| \geq |N_S(R)|$ for all R that are $\mathcal{F}$-conjugate to Q.*

The following is an immediate consequence of Proposition 2.3.4.

Corollary 2.3.7 *Every fully normalized subgroup is maximally normalized.*

2.4 Saturated fusion systems

We are now in a position to give the following key definition.

Definition 2.4.1 *A fusion system $\mathcal{F}$ on a p-group S is* saturated *if every $Q \leq S$ is $\mathcal{F}$-conjugate to a fully normalized subgroup.*

A wealth of examples of saturated fusion systems can be obtained from the following fundamental (though easy!) result.

Proposition 2.4.2 *If S is a Sylow p-subgroup of a finite group G then $\mathcal{F} = \mathcal{F}_S(G)$ is a saturated fusion system.*

Exercise 2.4.3 *Provide a proof for this proposition.*

Exercise 2.4.4 *How would you define a (finite) Sylow p-subgroup of an infinite group G? With this definition, see that the above proposition generalizes to infinite groups G.*

In the remainder of the section we prove that in the presence of fully normalized conjugates maximally centralized subgroups are receptive and maximally normalized subgroups are fully normalized.

Proposition 2.4.5 *Suppose $Q \leq S$ is $\mathcal{F}$-conjugate to R that is fully normalized. If Q is maximally centralized then it is receptive.*

Proof. Since R is fully normalized, Proposition 2.3.4 implies that there exists a morphism $\pi \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q), N_S(R))$ such that $Q\pi = R$. Let $\rho = \pi_{Q,R}$ be the corresponding restriction.

Consider an isomorphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ for some P $\mathcal{F}$-conjugate to Q. We need to show that ϕ extends to a morphism in $\mathrm{Hom}_{\mathcal{F}}(N_\phi, N_S(Q))$.

Clearly, $\phi\rho \in \mathrm{Hom}_{\mathcal{F}}(P, R)$. We first prove that $N_\phi \leq N_{\phi\rho}$. Since ρ extends to π, whose source is the whole of $N_S(Q)$, we have that $N_\rho = N_S(Q)$, which means that $C_\rho = \mathrm{Aut}_{\mathcal{S}}(Q)$. Therefore, $\mathrm{Aut}_{\mathcal{S}}(Q) \leq \mathrm{Aut}_{\mathcal{S}}(R)\hat{\rho}^{-1}$ because C_ρ is by definition $\mathrm{Aut}_{\mathcal{S}}(Q) \cap \mathrm{Aut}_{\mathcal{S}}(R)\hat{\rho}^{-1}$.

Hence $C_\phi = \mathrm{Aut}_{\mathcal{S}}(P) \cap \mathrm{Aut}_{\mathcal{S}}(Q)\hat{\phi}^{-1} \leq \mathrm{Aut}_{\mathcal{S}}(P) \cap (\mathrm{Aut}_{\mathcal{S}}(R)\hat{\rho}^{-1})\hat{\phi}^{-1} = \mathrm{Aut}_{\mathcal{S}}(P) \cap \mathrm{Aut}_{\mathcal{S}}(R)(\widehat{\phi\rho})^{-1} = C_{\phi\rho}$. Thus, indeed, $N_\phi \leq N_{\phi\rho}$, as these are the full preimages of C_ϕ and $C_{\phi\rho}$ under ϵ_P, respectively.

Since R is receptive, $\phi\rho$ extends to a morphism $\psi \in \mathrm{Hom}_{\mathcal{F}}(N_{\phi\rho}, N_S(R))$. Our next goal is to show that $N_\phi\psi \leq N_S(Q)\pi$. (Here we use that $N_\phi \leq N_{\phi\rho}$, and so ψ can be applied to N_ϕ.)

First of all, by Lemma 2.1.4, $\psi\epsilon_R = \epsilon_P\widehat{\phi\rho}$. Applying this to N_ϕ, we obtain $(N_\phi\psi)\epsilon_R = (N_\phi\epsilon_P)\widehat{\phi\rho} = (C_\phi)\widehat{\phi\rho} = (C_\phi\hat{\phi})\hat{\rho} = (\mathrm{Aut}_{\mathcal{S}}(P)\hat{\phi} \cap \mathrm{Aut}_{\mathcal{S}}(Q))\hat{\rho} \leq \mathrm{Aut}_{\mathcal{S}}(Q)\hat{\rho}$. Thus, $(N_\phi\psi)\epsilon_R \leq \mathrm{Aut}_{\mathcal{S}}(Q)\hat{\rho}$.

Hence we achieve our goal if we show that $N_S(Q)\pi$ is the full preimage of $\mathrm{Aut}_{\mathcal{S}}(Q)\hat{\rho}$ under ϵ_R. Let the latter group be denoted by T. Applying Lemma 2.1.4 to the morphism π, we get $(N_S(Q)\pi)\epsilon_R = (N_S(Q)\epsilon_Q)\hat{\rho} = \mathrm{Aut}_{\mathcal{S}}(Q)\hat{\rho}$. Hence $N_S(Q)\pi$ is contained in T. Let us compare the sizes. Clearly, $|T| = |C_S(R)| \cdot |\mathrm{Aut}_{\mathcal{S}}(Q)\hat{\rho}| = |C_S(R)| \cdot |\mathrm{Aut}_{\mathcal{S}}(Q)|$, since $C_S(R)$ is the kernel of ϵ_R and $\hat{\rho}$ is an isomorphism. Also, $|N_S(Q)\pi| = |N_S(Q)| = |C_S(Q)| \cdot |\mathrm{Aut}_{\mathcal{S}}(Q)|$. This is because π is injective and $C_S(Q)$ is the kernel of ϵ_Q. Since R is fully normalized, it is receptive, and hence maximally centralized. By assumption, Q is also maximally centralized, that is, $|C_S(R)| = |C_S(Q)|$. Clearly, this means that $|N_S(Q)\pi| = |T|$ and so $N_S(Q)\pi = T$, as claimed. Now, since $N_S(Q)\pi$ is the full preimage of $\mathrm{Aut}_{\mathcal{S}}(Q)\hat{\rho}$ under ϵ_R and since $(N_\phi\psi)\epsilon_R \leq \mathrm{Aut}_{\mathcal{S}}(Q)\hat{\rho}$, we get $N_\phi\psi \leq N_S(Q)\pi$.

Finally, the composition of three morphisms, the restriction $\psi_{N_\phi, N_\phi\psi}$, the inclusion $\iota_{N_\phi\psi, N_S(Q)\pi}$ (defined because $N_\phi\psi \leq N_S(Q)\pi$), and the inverse of the restriction $\pi_{,N_S(Q)\pi}$, is a morphism in $\mathrm{Hom}_{\mathcal{F}}(N_\phi, N_S(Q))$ and its restriction to P and Q is ϕ since $(x\psi)\pi^{-1} = (x\phi\rho)\rho^{-1} = x\phi$ for every $x \in P$. □

Proposition 2.4.6 *Suppose $Q \leq S$ is $\mathcal{F}$-conjugate to R that is fully normalized. If Q is maximally normalized then it is fully normalized.*

Proof. Note that R is maximally normalized by Corollary 2.3.7. So if Q is also maximally normalized then $|N_S(Q)| = |N_S(R)|$.

Note that $|C_S(Q)| \leq |C_S(R)$, because R is receptive and hence maximally centralized. Also, $\mathrm{Aut}_{\mathcal{S}}(Q)$ is a p-subgroup in $\mathrm{Aut}_{\mathcal{F}}(Q)$, which is isomorphic to $\mathrm{Aut}_{\mathcal{F}}(R)$ since Q and R are $\mathcal{F}$-conjugate. It follows that $|\mathrm{Aut}_{\mathcal{S}}(Q)| \leq |\mathrm{Aut}_{\mathcal{S}}(R)|$. This is because $\mathrm{Aut}_{\mathcal{S}}(R)$ is a Sylow p-subgroup in $\mathrm{Aut}_{\mathcal{F}}(R)$ since R is fully automized.

Now, $|N_S(Q)| = |C_S(Q)| \cdot |\mathrm{Aut}_{\mathcal{S}}(Q)|$ and, similarly, $|N_S(R)| = |C_S(R)| \cdot |\mathrm{Aut}_{\mathcal{S}}(R)|$. Taking into account that $|N_S(Q)| = |N_S(R)|$ and the above two inequalities, we deduce that $|C_S(Q)| = |C_S(R)|$ and $|\mathrm{Aut}_{\mathcal{S}}(Q)| = |\mathrm{Aut}_{\mathcal{S}}(R)|$. The first equality means that Q is maximally centralized and so it is receptive by Proposition 2.4.5. The second equality means that $\mathrm{Aut}_{\mathcal{S}}(Q)$ is Sylow in $\mathrm{Aut}_{\mathcal{F}}(Q)$, and so Q is maximally automized. Therefore, Q is fully normalized, as claimed. □

The two preceding results have the following implication for saturated fusion systems.

Corollary 2.4.7 *In a saturated fusion system $\mathcal{F}$ on S, a subgroup $P \leq S$ is receptive if and only if it is maximally centralized and P is fully normalized if and only if it is maximally normalized.*

We conclude this chapter with a second definition of the saturation property.

Proposition 2.4.8 *A fusion system $\mathcal{F}$ on a p-group S is saturated if and only if the following conditions hold:*

- *every maximally normalized subgroup of S is maximally centralized and fully automized; and*
- *every maximally centralized subgroup is receptive.*

Proof. We have shown that the two properties above always hold in a saturated fusion system. Conversely, suppose that the two properties hold for $\mathcal{F}$. Every $\mathcal{F}$-conjugacy class certainly contains a maximally normalized subgroup. By the first property, this subgroup is fully automized and maximally centralized. By the second property, the subgroup is receptive, hence it is fully normalized. □

Chapter 3

Essential subgroups

3.1 Conjugation families

In this chapter we discuss conjugation families, Alperin's Theorem, and essential subgroups.

Definition 3.1.1 *Suppose $\mathcal{C}$ is a set of subgroups of S. For a morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ we say that it is* expressible *via $\mathcal{S}$ if, for some $k \geq 0$, there exist subgroups $P_0 = P, P_1, \ldots, P_k = Q$, also there exist elements of $\mathcal{C}$, say, $C_1, \ldots, C_k$, and finally, there exist morphisms $\psi_1 \in \mathrm{Aut}_{\mathcal{F}}(C_1), \ldots, \psi_k \in \mathrm{Aut}_{\mathcal{F}}(C_k)$, such that*

- *$P_{i-1}, P_i \leq C_i$ for $i = 1, \ldots, k$;*
- *$P_{i-1}\psi_i \leq P_i$, for $i = 1, \ldots, k$, so that the restriction $\phi_i = \psi_{Q_{i-1},Q_i}$ is defined; and*
- *$\phi = \phi_1 \cdots \phi_k$.*

Note that $k = 0$ is possible, but only for the identity morphisms: $P = Q$ and $\phi = \iota_{P,P}$.

Definition 3.1.2 *For set of subgroups $\mathcal{C}$, let $X_{\mathcal{C}} = \cup_{C \in \mathcal{C}} \mathrm{Aut}_{\mathcal{F}}(C)$.*

The following is a restatement of Definition 3.1.1.

Proposition 3.1.3 *A morphism ϕ is expressible via $\mathcal{F}$ if and only if, for some $k \geq 0$, we have $\phi = \phi_1 \cdots \phi_k$, where each ϕ_i is a restriction of some morphism from $X_{\mathcal{F}}$*

Note that if an isomorphism ϕ is a restriction of another isomorphism ψ then ϕ^{-1} is a restriction of ψ^{-1}. Also, the conjugations c_x, $x \in S$, are contained in $X_{\mathcal{C}}$ if $S \in \mathcal{C}$. This yields the following.

Proposition 3.1.4 *Assuming that $S \in \mathcal{C}$, a morphism ϕ is expressible via $\mathcal{C}$ if and only if ϕ is contained in the fusion system $\mathcal{F}_{\mathcal{C}} = \langle X_{\mathcal{C}} \rangle$ generated by $X_{\mathcal{F}}$.*

Hence it will be convenient for us to set the following notation.

Definition 3.1.5 *We let $\mathcal{F}_{\mathcal{C}}$ denote the fusion system $\mathcal{F}_{\mathcal{C}} = \langle X_{\mathcal{C}} \rangle$. Clearly, this is a subsystem of $\mathcal{F}$.*

We now turn to the main definition of this chapter.

Definition 3.1.6 *We say that the set $\mathcal{C}$ is a* conjugation family *if every morphism in $\mathcal{F}$ is expressible via $\mathcal{C}$.*

We saw above that it is convenient to have $S \in \mathcal{C}$. It turns out this is almost always true for conjugation families.

Proposition 3.1.7 *If $S \neq 1$ then $S \in \mathcal{C}$ for any conjugation family $\mathcal{C}$.*

Proof. Consider $P = 1$, $Q = S$, and $\phi = \iota_{P,Q}$. Since $\mathcal{C}$ is a conjugation family, ϕ must be expressible via $\mathcal{C}$. Hence, for some $k \geq 1$ (since $S \neq 1$, we have that $P \neq Q$), we have $\phi = \phi_1 \cdots \phi_k$, where each ϕ_i is a restriction of a morphism from $X_{\mathcal{C}}$. Note that the target group of ϕ_k is S, which means that ϕ_k is a restriction of a morphism from $\mathrm{Aut}_{\mathcal{F}}(S)$, and so $S \in \mathcal{C}$. □

Of course, we are not interested in $S = 1$ and so from now on S is a member of any conjugation family. The following is immediate from Proposition 3.1.4.

Proposition 3.1.8 *If $S \in \mathcal{S}$ then $\mathcal{C}$ is a conjugation family if and only if $X = \cup_{C \in \mathcal{C}} \mathrm{Aut}_{\mathcal{F}}(C)$ generates $\mathcal{F}$, that is, $\mathcal{F}_{\mathcal{C}} = \mathcal{F}$.*

Exercise 3.1.9 *Can we omit the 'if' part? That is, can we claim that $\mathcal{C}$ is a conjugation family if and only if $X = \cup_{C \in \mathcal{C}} \mathrm{Aut}_{\mathcal{F}}(C)$ generates $\mathcal{F}$?*

The presence of S in $\mathcal{C}$ means that all inclusion maps are contained in $\mathcal{F}_{\mathcal{C}}$. Since every morphism in $\mathcal{F}$ is a composition of an isomorphism and inclusion, we have the following.

Proposition 3.1.10 *If $S \in \mathcal{C}$ then $\mathcal{C}$ is a conjugation family if and only if every $\mathcal{F}$-isomorphism is contained in $\mathcal{F}_{\mathcal{C}}$.*

Exercise 3.1.11 *Describe fusion systems for which the 'if' clause in Proposition 3.1.10 cannot be omitted.*

3.2 Minimal conjugation family

It will be convenient for us to consider various subsets of $\mathcal{C}$.

Definition 3.2.1 *For $P \leq S$, let $\mathcal{C}_P^+$ be the set of all $C \in \mathcal{C}$ with $|C| > |P|$ and let $\mathcal{C}_P$ consist of $\mathcal{C}_P^+$ and all $C \in \mathcal{C}$ that are $\mathcal{F}$-conjugate with P.*

Let us record the following observation.

Proposition 3.2.2 *A morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ is expressible via $\mathcal{C}$ if and only if it is expressible via $\mathcal{C}_P$.*

Proof. If ϕ is expressible via $\mathcal{C}$ then $\phi = \phi_1 \cdots \phi_k$, for some $k \geq 0$, where each ϕ_i is a restriction of a morphism $\psi_i \in \mathrm{Aut}_{\mathcal{F}}(C_i)$ from $X_{\mathcal{C}}$. Let $\phi_1, \ldots, \phi_s$ are isomorphisms while ϕ_{s+1} is not an isomorphism (or $s = k$).

On the one hand, if $i > s$ then C_i contains the target group of ϕ_i, whose size is greater than $|P|$. Hence $C_i \in \mathcal{C}_P^+$. On the other hand, if $i \leq s$ then both the source and target groups of ϕ are $\mathcal{F}$-conjugate to P. Hence either C_i lies in $\mathcal{C}_P^+$, or C_i is $\mathcal{F}$-conjugate to P. In either case $C_i \in \mathcal{C}_P$. □

In fact, up to a factor, we can just get away with $\mathcal{C}_P^+$.

Proposition 3.2.3 *If $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ is expressible via $\mathcal{C}$ then there exist $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ and $\pi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ such that $\phi = \alpha\pi$ and π is expressible via $\mathcal{C}_P^+$.*

Proof. Again, write $\phi = \phi_1 \cdots \phi_i$, where each $\phi_i \in \mathrm{Hom}_{\mathcal{F}}(P_{i-1}, P_i)$ is a restriction of a morphism $\psi_i \in \mathrm{Aut}_{\mathcal{F}}(C_i)$ from $X_{\mathcal{C}}$.

Let the list $(i_1, \ldots, i_r)$ be obtained from $(1, \ldots, k)$ by removing all i such that C_i is $\mathcal{F}$-conjugate to P. Clearly, all $i > s$ remain on the list, where s be the number of factors ϕ_i in the beginning of the product, that are isomorphisms. For convenience, let us set $i_0 = 0$.

Let $\pi = \phi_{i_1} \cdots \phi_{i_r}$. (If $r = 0$ then π is the identity on $P = Q$.) This product is well defined, since the target of $\phi_{i_{j-1}}$ coincides with the source of ϕ_{i_j}. Indeed, $P_{i_{j-1}} = P_{i_{j-1}+1} = \ldots = P_{i_j - 1}$, we only omitted the i, where $P_{i-1} = P_i = C_i$. For the same reason, π is a morphism from $\mathrm{Hom}_{\mathcal{F}}(P, Q)$.

Also note that $P\phi_1 \cdots \phi_s = P_s = P\phi_{i_1} \cdots \phi_{i_t}$, where t is largest such that $i_t \leq s$ (and so $(i_{t+1}, \ldots, i_r) = (s+1, \ldots, k)$). Hence $P\phi = P\phi_1 \cdots \phi_k = P\phi_{i_1} \cdots \phi_{i_r} = P\pi$.

Setting $\alpha = \phi_{,P\phi}(\pi_{,P\phi})^{-1}$, we have that α lies in $\mathrm{Aut}_{\mathcal{F}}(P)$ and, clearly, $\phi = \alpha\pi$, as claimed. □

In the remainder of the section we consider a saturated fusion system $\mathcal{F}$ on S. Under this assumption, we will show that conjugation families exist and, furthermore, are equivalent in a sense that we will make precise.

Definition 3.2.4 *Let $\mathcal{O}$ be the set of all subgroups of S, that is, $\mathcal{O} = \mathrm{Ob}(\mathcal{F})$.*

Proposition 3.2.5 *If $\mathcal{F}$ is saturated then $\mathcal{O}$ is a conjugation family. In particular, conjugation families in $\mathcal{F}$ exist.*

Proof. Clearly, $S \in \mathcal{O}$, and so $\mathcal{O}$ is a conjugation family if and only if $\mathcal{F}_0 = \mathcal{F}_{\mathcal{O}}$ coincides with $\mathcal{F}$.

By contradiction, suppose that $\mathcal{F} \neq \mathcal{F}_0$. Select a morphism ϕ that is not in $\mathcal{F}_0$, with a largest possible source group P. Since $\mathrm{Aut}_{\mathcal{F}}(S) \subseteq X_{\mathcal{O}}$, we have that $P < S$.

Without loss of generality we may assume that ϕ is an isomorphism, because if it is not, we can substitute ϕ with its core. Let Q be the target group of ϕ. Since ϕ is an isomorphism, P and Q are $\mathcal{F}$-conjugate. Since $\mathcal{F}$ is saturated, there exists R that is $\mathcal{F}$-conjugate to P and Q and fully normalized.

By Proposition 2.3.4, there exists $\pi \in \mathrm{Hom}_{\mathcal{F}}(N_S(P), N_S(R))$ such that $P\pi = R$ and, similarly, there exists $\rho \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q), N_S(R))$ such that $Q\rho = R$. Since both π and ρ have larger target groups than ϕ, we must have that π and ρ lie in $\mathcal{F}_0$. Taking restrictions, we see that $\psi = \pi_{P,R}\rho_{Q,R}^{-1} \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ is also contained in $\mathcal{F}_0$.

Finally, $\alpha = \phi\psi^{-1} \in \mathrm{Aut}_{\mathcal{F}}(P)$ and so it lies in $X_{\mathcal{O}}$, and hence in $\mathcal{F}_0$. This implies that $\phi = \alpha\pi$ is in $\mathcal{F}_0$, a contradiction. □

Hence conjugation families definitely exist in saturated systems. Next, let us see how much flexibility we have in choosing members of conjugation families.

Proposition 3.2.6 *Suppose $\mathcal{C}$ is a conjugation family in $\mathcal{F}$. If $\mathcal{C}$ contains P and Q such that $P \neq Q$ and Q is $\mathcal{F}$-conjugate with P then $\mathcal{C} \setminus \{Q\}$ is also a conjugation family.*

Proof. Let $\mathcal{C}_0 = \mathcal{C} \setminus \{Q\}$ and $\mathcal{F}_0 = \mathcal{F}_{\mathcal{C}_0}$. We know that $\mathcal{F} = \mathcal{F}_{\mathcal{C}}$ and we need to show that $\mathcal{F} = \mathcal{F}_0$. Since $X_{\mathcal{C}}$ and $X_{\mathcal{C}_0}$ differ by $\mathrm{Aut}_{\mathcal{F}}(Q)$, it suffices to show that $\mathrm{Aut}_{\mathcal{F}}(Q)$ is contained in $\mathcal{F}_0$.

Since $\mathcal{C}$ is a conjugation family and since P and Q are $\mathcal{F}$-conjugate, there is a morphism in $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ expressible via $\mathcal{C}$. By Proposition 3.2.3, we must have possibly another morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ such that ϕ is expressible via $\mathcal{C}_P^+$. Clearly, $\mathcal{C}_P^+ \subseteq \mathcal{C}_0$, and so ϕ lies in $\mathcal{F}_0$. However, this means that $\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{Aut}_c F(P)\hat{\phi} = \phi^{-1}\mathrm{Aut}_{\mathcal{F}}(P)\phi$ is contained in $\mathcal{F}_0$, since $\mathrm{Aut}_c F(P)$ is contained in $X_{\mathcal{C}_0}$. Hence $\mathcal{F}_0 = \mathcal{F}$, as claimed. □

This means that we never need more than one member of any given $\mathcal{F}$-conjugacy class. In fact, as the following corollary shows, any representative of that class will do.

Corollary 3.2.7 *Suppose $\mathcal{C}$ is a conjugation family in $\mathcal{F}$, $P \in \mathcal{C}$, and Q is $\mathcal{F}$-conjugate with P. Then $(\mathcal{C} \setminus \{P\}) \cup \{Q\}$ is again a conjugation family.*

Proof. Clearly, we may assume that $Q \neq P$. If $\mathcal{C}$ is a conjugation family, then so is also $\mathcal{C} \cup \{Q\}$. Hence $(\mathcal{C} \setminus \{P\}) \cup \{Q\} = (\mathcal{C} \cup \{Q\}) \setminus \{P\}$ is a conjugation family by the preceding proposition. □

Definition 3.2.8 *We call two sets $\mathcal{C}$ and $\mathcal{C}'$ of subgroups of S* similar *if they meet the same $\mathcal{F}$-conjugacy classes.*

What we proved can now be restated as follows: if $\mathcal{C}$ is a conjugation family then any similar set $\mathcal{C}'$ is also a conjugation family.

Definition 3.2.9 *A set $\mathcal{C} \subset \mathrm{Ob}(\mathcal{F})$ is* full *if it is a union of several $\mathcal{F}$-conjugacy classes.*

Clearly, every $\mathcal{C}$ is similar to a unique full set $\mathcal{C}'$. We note also that any intersection of full sets is again full.

Proposition 3.2.10 *If $\mathcal{C}$ and $\mathcal{C}'$ are two full conjugation families then $\mathcal{C} \cap \mathcal{C}'$ is again a conjugation family.*

Proof. Let $\mathcal{F}_0 = \mathcal{F}_{\mathcal{C} \cap \mathcal{C}'}$. As usual, we need to show that $\mathcal{F}_0 = \mathcal{F}$. Suppose by contradiction that $\mathcal{F}_0 \neq \mathcal{F}$. Then, since $\mathcal{F}_{\mathcal{C}'} = \mathcal{F}$, we have that there exists $P \in \mathcal{C}'$ such that $\mathrm{Aut}_{\mathcal{F}}(P)$ is not contained in $\mathcal{F}_0$. In particular, there exist $P \leq S$ such that $\mathrm{Aut}_{\mathcal{F}}(P)$ is not contained in $\mathcal{F}_0$. Choose such a P with P as large as possible. Clearly, P cannot lie in both $\mathcal{C}$ and $\mathcal{C}'$. Without loss of generality, let us assume that $P \notin \mathcal{C}$.

Consider $\phi \in \mathrm{Aut}_{\mathcal{F}}(P)$. Since $\mathcal{C}$ is a conjugation family, ϕ is expressible via $\mathcal{C}$, and so it is expressible via $\mathcal{C}_P$. However, $\mathcal{C}$ does not contain P and it is full, hence $\mathcal{C}$ contains no $\mathcal{F}$-conjugates of P. This means that $\mathcal{C}_P = \mathcal{C}_P^+$. However, $\langle X_{\mathcal{C}_P^+} \rangle$ is contained in $\mathcal{F}_0$ by our choice of P. This shows that ϕ lies in $\mathcal{F}_0$; this is a contradiction as ϕ is an arbitrary morphism from $\mathrm{Aut}_{\mathcal{F}}(P)$. □

Clearly, we also have the following corollary.

Corollary 3.2.11 *If conjugation families in $\mathcal{F}$ exist then $\mathcal{F}$ has a unique smallest full conjugation family $\mathcal{C}_{min}$. Furthermore, a set $\mathcal{C}$ of subgroups of S is a conjugation family in $\mathcal{F}$ if and only if it meets every $\mathcal{F}$-conjugacy class present in $\mathcal{C}_{min}$.*

This set $\mathcal{C}_{min}$ is just the intersection of all full conjugation families in $\mathcal{F}$. Note that $\mathcal{C}_{min}$ always exists if $\mathcal{F}$ is saturated.

3.3 Essential subgroups

In this section we just deal with fusion systems which admit conjugation families, such as, say, saturated fusion systems. In particular, $\mathcal{C}_{\min}$ exists in $\mathcal{F}$.

We have already discussed that S lies in every conjugation family.

Definition 3.3.1 *The $\mathcal{F}$-conjugacy classes contained in $\mathcal{C}_{min} \setminus \{S\}$ are called* essential*. Also, the subgroups from the essential classes are called* essential subgroups*. The number of essential classes in $\mathcal{F}$ is called the* essential rank *of $\mathcal{F}$.*

Exercise 3.3.2 *Prove that if $\mathcal{F}$ is a saturated fusion system of essential rank zero then there is a finite group $G \geq S$, such that S is a normal Sylow p-subgroup of G and $\mathcal{F} = \mathcal{F}_S(G)$.*

We now want to develop properties of essential subgroups. The following statement provides a useful characterization.

Proposition 3.3.3 *A subgroup $P \leq S$ is essential if and only if $\mathrm{Aut}_{\mathcal{F}}(P)$ is not contained in $\langle X_{\mathcal{O}_P^+} \rangle$.*

The proof is left as an exercise.

Definition 3.3.4 *A subgroup $P \leq S$ is* centric *in $\mathcal{F}$ if $C_S(Q) \leq Q$ for every Q that is $\mathcal{F}$-conjugate with P.*

In a sense, we should be talking about centric $\mathcal{F}$-conjugacy classes instead of subgroups, as the property depends on all subgroups in the class. However, sometimes it can be read off a single subgroup.

Proposition 3.3.5 *Suppose P is $\mathcal{F}$-conjugate to Q which is receptive. Then P is centric if and only if $C_S(Q) \leq Q$.*

Exercise 3.3.6 *Provide a proof for this proposition.*

Proposition 3.3.7 *If $\mathcal{F}$ is a saturated fusion system on S then every essential subgroup in S is centric.*

Proof. According to Proposition 3.3.5, we can take P fully normalized (and hence, receptive) and show that $C_S(P) \le P$.

By contradiction, suppose that $C_S(P) \not\le P$. Then $P' = PC_S(P) > P$. Consider $\phi \in \mathrm{Aut}_{\mathcal{F}}(P)$. Since $N_\phi \ge C_S(P)$, we have that $N_\phi \ge P'$. Since P is receptive, ϕ extends to a morphism ψ with the source group P'. As $P' > P$, $\mathcal{O}_{P'} \subseteq \mathcal{O}_P^+$, and so ψ and ϕ are expressible via $\mathcal{O}_P^+$. Since ϕ was arbitrary in $\mathrm{Aut}_{\mathcal{F}}(P)$, we conclude that $\mathrm{Aut}_{\mathcal{F}}(P)$ is fully contained in $\langle X_{\mathcal{O}_P^+}\rangle$. However, this means that P is not essential in view of Proposition 3.3.3. □

We will also need the following well known concept arising in the classification of finite simple groups.

Definition 3.3.8 *Suppose that G is a group and H is a proper subgroup of G with p dividing $|H|$. We say that H is a* strongly p-embedded *subgroup of G if, for any $x \in G$, either $H = H^x$, or $H \cap H^x$ has a p'-order.*

If H is a strongly p-embedded subgroups then H contains $N_G(Q)$ for all p-subgroups $Q \le H$ with $Q \ne 1$. On the one hand, this implies that H contains a Sylow p subgroup T of G. On the other hand, it also means that H contains $K = \langle N_G(Q) \mid 1 \ne Q \le T\rangle$. It can be shown that K is a strongly p-embeddable subgroup of G as long as $K < G$.

Definition 3.3.9 *If $K = \langle N_G(Q) \mid 1 \ne Q \le T\rangle$ is a proper subgroup of G then we call K the* minimal *strongly p-embeddable subgroup of G.*

Every nontrivial p-subgroup of G is contained in a unique strongly p-embedded subgroup. Furthermore, all minimal strongly p-embedded subgroups of G are conjugate in G.

Proposition 3.3.10 *If $\mathcal{F}$ is a saturated fusion system on S and $P < S$ is essential then* $\mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$ *has a strongly p-embedded subgroup.*

Proof. Let $\mathcal{F}_0 = \mathcal{F}_{\mathcal{O}_P^+}$ be the subsystem generated by $X_{\mathcal{O}_P^+}$. Since P is essential, we have that $\mathrm{Aut}_{\mathcal{F}}(P)$ is not contained in $\mathcal{F}_0$. Hence $H = \mathrm{Aut}_{\mathcal{F}_0}(P)$ is a proper subgroup of $G = \mathrm{Aut}_{\mathcal{F}}(P)$. Let the bar indicate images in the factor group $\bar{G} = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$.

Without loss of generality, let us assume that P is fully normalized. In particular, $T = \mathrm{Aut}_S(P) = N_S(P)\epsilon_P$ is a Sylow p-subgroup of G and $\bar{T}$ is a Sylow p-subgroup of $\bar{G}$. Naturally, T is contained $\mathcal{F}_0$ and so $T \le H$ and $\bar{T} \le \bar{H}$. In particular, since $\mathrm{Inn}(P) \le T \le H$, we conclude that $\bar{H} < \bar{G}$, since $H < G$.

Consider $Q \le T$ with $Q > \mathrm{Inn}(P)$, which is equivalent to $\bar{Q} \ne 1$. We claim that $N_{\bar{G}}(\bar{Q}) \le \bar{H}$. Indeed, suppose that $\phi \in G = \mathrm{Aut}_{\mathcal{F}}(P)$ such that

$\bar{\phi}$ normalizes $\bar{Q}$. Since Q is the full preimage of $\bar{Q}$ in G, ϕ normalizes Q. In particular, $Q \leq T \cap T\hat{\phi} = \mathrm{Aut}_{\mathcal{S}}(P) \cap \mathrm{Aut}_{\mathcal{S}}(P)\hat{\phi} = C_\phi$, the control subgroup of ϕ. Therefore, $R \leq N_\phi$, where R is the full preimage of Q under ϵ_P. As P is fully normalized and hence receptive, ϕ extends to a morphism ψ with the source group R. However, $R > P$ (as $Q > \mathrm{Inn}(P) = P\epsilon_P$) and $\mathcal{F}$ is saturated. Hence ψ is expressible via $\mathcal{O}_R \subseteq \mathcal{O}_P^+$. This shows that ϕ, being a restriction of ψ, lies in $\mathcal{F}_0$, that is, $\phi \in H$.

We have shown that $N_{\hat{G}}(\bar{Q}) \leq \bar{H}$, for all $1 \neq \bar{Q} \leq \bar{T}$. Hence $\langle N_{\hat{G}}(\hat{Q}) \mid 1 \neq \bar{Q} \leq \hat{T}\rangle$ is a proper subgroup of $\hat{G}$, and so indeed $\hat{G}$ has a strongly p-embedded subgroup. □

We have reached an important milestone.

Corollary 3.3.11 (Alperin-Goldschmidt Theorem) *In a saturated fusion system $\mathcal{F}$, the set consisting of S together with all fully normalized centric subgroups $P < S$ such that $\mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$ has a strongly p-embedded subgroup, is a conjugation family.*

Returning to the essential subgroups P, we know that P is centric and $\mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$ has a strongly p-embedded subgroup. In fact, the converse is also true.

Proposition 3.3.12 *Suppose $\mathcal{F}$ is saturated. If $P < S$ satisfies:*

- *P is centric; and*
- *$\mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$ has a strongly p-embedded subgroup,*

then P is essential.

The proof is left as an exercise; in fact, a series of exercises corresponding to the steps of the proof.

Suppose that $P < S$ satisfies the two conditions in the proposition. We adopt some notation from the proof of Proposition 3.3.10. Namely, let $\mathcal{F}_0 = \mathcal{F}_{\mathcal{C}_P^+}$. Also, let $G = \mathrm{Aut}_{\mathcal{F}}(P)$ and $\bar{G} = G/\mathrm{Inn}(P)$. Let $T = \mathrm{Aut}_{\mathcal{S}}(P)$. Note that we do not assume that P is fully normalized, and so T does not have to be Sylow in G.

Exercise 3.3.13 *Show that ϵ_P establishes a bijective correspondences between the set of overgroups of P lying in $N_S(P)$ and the set of overgroups of $\mathrm{Inn}(P)$ lying in T (and so we also get a bijective correspondence with the set of all subgroups of $\bar{T}$).*

In particular, $T > \mathrm{Inn}(P)$ and $\bar{T} \neq 1$. Define $E(P)$ as the full preimage in G of the minimal strongly p-embedded subgroup of $\bar{G}$ containing $\bar{T}$. Note that $E(P) < G$.

Exercise 3.3.14 *Suppose an isomorphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ is a restriction of a morphism with a larger source group. (In particular, $\phi \in \mathcal{F}_0$.) Show that $E(P)\hat{\phi} = E(Q)$.*

From this one can derive the following.

Exercise 3.3.15 *Show that we have $E(P)\hat{\phi} = E(Q)$ for any isomorphism $\phi \in \mathrm{Hom}_{\mathcal{F}_0}(P, Q)$.*

The next exercise is based on the Frattini argument.

Exercise 3.3.16 *Prove that $\bar{E(P)}$ is self-normalized in $\bar{G}$, and so also $E(P)$ is self-normalized in G.*

Finally, it is time to show that P is essential.

Exercise 3.3.17 *Prove that $\mathrm{Aut}_{\mathcal{F}_0}(P) = E(P)$. In particular, $\mathcal{F}_0 \neq \mathcal{F}$, and so P is essential.*

The final exercise characterizes other isomorphisms from $\mathcal{F}_0$.

Exercise 3.3.18 *Show that and isomorphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ lies in $\mathcal{F}_0$ if and only if $E(P)\hat{\phi} = E(Q)$.*

Let us conclude this chapter with an additional property that all essential subgroups possess.

Definition 3.3.19 *A subgroup $P \leq S$ is called* radical *if $\mathrm{Inn}(P)$ coincides with $O_p(\mathrm{Aut}_{\mathcal{F}}(P))$, the largest normal p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$.*

Exercise 3.3.20 *Prove that every essential subgroup in a saturated fusion system is radical.*