Definition A1. Let $(\mathcal{L}, \Delta, S)$ be a locality, set $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$, and write M_P for $N_{\mathcal{L}}(P)$ (for $P \in \Delta$). Then $(\mathcal{L}, \Delta, S)$ is a *reduced* if the following condition hold.

(R) $C_{M_P}(O_p(M_P)) \leq O_p(M_P)$ for all $P \in \Delta$.

Definition A.2. Let $(\mathcal{L}, \Delta, S)$ be a reduced locality. An object $P \in \Delta$ is *Alperin-Goldschmidt essential* in $\mathcal{L}$ if either $P = S$ or:

(1) $C_{\mathcal{L}}(P) \leq P$,
(2) $N_S(P) \in Syl_p(N_{\mathcal{L}}(P))$, and
(3) $N_{\mathcal{L}}(P)/P$ has a strongly p-embedded subgroup.

Write $\mathbf{A}(\mathcal{L})$ for the set of all $P \leq S$ such that P is Alperin-Goldschmidt essential in $\mathcal{L}$.

Definition A.3. Let $(\mathcal{L}, \Delta, S)$ be a reduced locality and let $f \in \mathcal{L}$. Then f is $\mathbf{A}(\mathcal{L})$-*decomposable* if there exists $w \in \mathbf{D}$ and a sequence σ of members of $\mathbf{A}(\mathcal{L})$:

$$w = (f_1, \cdots f_n), \quad \sigma = (P_1, \cdots, P_n),$$

such that the following hold.

(1) $S_f = S_w$ and $f = \Pi(w)$.
(2) $P_i = S_{f_i}$ for all i.
(3) For all i: either $f_i \in O^{p'}(N_{\mathcal{L}}(P_i))$ or $P_i = S$.

We also say that (w, σ) is an $\mathbf{A}(\mathcal{L})$-*decomposition* of f.

The reader may have noticed that condition (2) in A.3 implies that the sequence σ is determined by w. Thus there is some redundancy in the definition. For that reason we shall also speak of the $\mathbf{A}(\mathcal{L})$-decomposition w and its *auxiliary sequence* σ.

Lemma A.4. *Let $\mathcal{L} = (\mathcal{L}, \Delta, S))$ be a reduced locality. Then every element of $\mathcal{L}$ has an $\mathbf{A}(\mathcal{L})$-decomposition.*

Proof. Set $\mathbf{A} = \mathbf{A}(\mathcal{L})$. Among all $f \in \mathcal{L}$ such that f has no $\mathbf{A}$-decomposition, choose f with $P := S_f$ as large as possible. If $P = S$ then (f) is an $\mathbf{A}$-decomposition of f (with auxiliary sequence (S)). Thus, $P \neq S$. Set $P' = P^f$, and set $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$.

By 2.9 there exists an $\mathcal{L}$-conjugate Q of P (and hence also of P') such that both Q and $Q \cap T$ are fully normalized in $\mathcal{F}$. As $P, P' \in \Delta$ there are then elements $g, h \in \mathcal{L}$ with $Q = P^g = (P')^h$. Since $N_S(Q) \in Syl_p(N_{\mathcal{L}}(Q))$ it follows from 2.3(b) and Sylow's theorem that g and h may be chosen so that $N_S(Q)$ contains both $N_S(P)^g$ and $N_S(P')^h$. The maximality of P then implies that g and h possess $\mathbf{A}$-decompositions. The same is then true of g^{-1} and h^{-1} via the inverses of the words (and the reversals of the sequences of subgroups of S) which yield $\mathbf{A}$-decomposability for g and h.

Set $f' = g^{-1}fh$, $M = N_{\mathcal{L}}(Q)$, and $R = N_S(Q)$. Then $f' \in M$, $u := (g, f', h^{-1}) \in \mathbf{D}$ via Q, and $\Pi(u) = f$. If f' has an $\mathbf{A}$-decomposition then so does f, and thus we may assume that $f = f'$ and $P = Q = P'$. Morever, $P = O_p(M)$ since we now have $f \in M$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX and $O_p(M) \leq S_f$. Applying the Alperin-Goldschmidt Theorem [Gold] to M, we find that the conjugation automorphism $c_f \in Aut(P)$ is a composition $c_f = c_{x_1} \circ \cdots \circ c_{x_n}$ with $x_i \in N_M(E_i)$ for some $E_i \in \mathbf{A}(M)$, and where $\mathbf{A}(M)$ denotes the Alperin-Goldschmidt conjugation family. Thus $N_M(E_i)/E_i$ has a strongly p-embedded subgroup, and hence $Q \leq E_i$ for all i.

Set $x = x_1 \cdots x_n$ and suppose that x has a $\mathbf{A}$-decomposition. Set $z = fx^{-1}$. Then $z \in C_M(P)$, and since $\mathcal{L}$ is reduced (by hypothesis), we obtain hence $z \in O_p(M)$. Thus $z \in P$. Let (w, σ) be an $\mathbf{A}$-decomposition of x. Then $((z) \circ w, (S) \circ \sigma)$ is an $\mathbf{A}$-decomposition of f. We conclude that

(*) x has no $\mathbf{A}$-decomposition.

There is then an index k such x_k has no $\mathbf{A}$-decomposition. Then $Q = E_k$ by the maximality of Q, and hence M/Q has a strongly p-embedded subgroup. Thus $Q \in \mathbf{A}$. By the Frattini Lemma (for groups) we may write $x = ab$ where $a \in O^{p'}(M)$ and where $b \in N_M(N_S(Q))$. Then b has an $\mathbf{A}$-decomposition by the maximality of Q, while (a, Q) is itself a $\mathbf{A}$-decomposition for a. This shows that x has a $\mathbf{A}$-decomposition, contrary to (*), and completing the proof. □

The proof of A.4 can be altered to yield a proof of:

Lemma A5. *Let $(\mathcal{L}, \Delta, S)$ be a locality, and set $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$. Then $\mathcal{F}$ is generated by its fusion subsystems $\mathcal{F}_{N_S(P)}(M_P)$, as P varies over objects $P \in \Delta$ such that P is fully normalized in $\mathcal{F}$, and such that either $P = S$ or M_P/P has a strongly p-embedded subgroup.*

Proof. [Exercise: Just follow along with the proof of A4, using $\mathcal{F}$ instead of $\mathcal{L}$.] □

Remark. With A5 we now have the result (stated as 2.10.5 in the lectures) that if $(\mathcal{L}, \Delta, S)$ is a locality then its fusion system $\mathcal{F} := \mathcal{F}_S(\mathcal{L})$ is Δ-saturated. That is, (A) every $P \in \Delta$ has an $\mathcal{L}$-conjugate which is fully normalized in $\mathcal{F}$; (B) each $M_P := N_{\mathcal{F}}(P)$ for P a fully normalized object satisfies the condition that $\mathcal{F}_{N_S(P)}(M_P)$ is equal to $N_{\mathcal{F}}(P)$; and(C) $\mathcal{F}$ is generated by the fusion subsystems $N_{\mathcal{F}}(P)$ for $P \in \Delta \cap \mathcal{F}^c$ with P fully normalized.

In the case that $\mathcal{F}^c \subseteq \Delta$, these conditions suffice to guarantee that $\mathcal{F}$ is in fact saturated. The proof of this can be found in David Craven's book on fusion systems (somewhere in the middle: sorry for the imprecise reference). The proof is lengthy - and one wonders if it can be simplified. (It's based on the proof in [BCGLO1].) But A4 is so much nicer than A5 that one is led to make the following definitions.

Definition A6. Let $\mathcal{F}$ be a saturated fusion system on S, and let $P \leq S$ be a subgroup of S. Then P is *radical* in $\mathcal{F}$ if $Inn(P) = O_p(Aut_{\mathcal{F}}(P)$. (Write $\mathcal{F}^{cr}$ for the set of all $P \leq S$ such that P is both centric and radical in $\mathcal{F}$.) We say that the subgroup $P \leq S$ is *subcentric* in $\mathcal{F}$ if P has a fully normalized $\mathcal{F}$-conjugate such that the group $A_P := Aut_{\mathcal{F}}(Q)$ satisfies the condition: $C_{A_P}(O_p(A_P)) \leq O_p(A_P)$. (Write $\mathcal{F}^s$ for the set of all $P \leq S$ such that P is subcentric in $\mathcal{F}$. Terminology and notation due to Ellen Henke.)

Definition A7. Let $(\mathcal{L}, \Delta, S)$ be a locality, and set $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$. Then $(\mathcal{L}, \Delta, S)$ is a Δ-*linking system* if the following conditions hold.

(LS1) $\mathcal{F}$ is saturated.
(LS2) $\mathcal{F}^{cr} \subseteq \Delta \subseteq \mathcal{F}^s$.
(LS3) $C_{\mathcal{L}}(O_p(N_{\mathcal{L}}(P)) \leq O_p(N_{\mathcal{L}}(P))$ for all $P \in \Delta$.