THE SUBGROUP STRUCTURE OF FINITE ALTERNATING AND SYMMETRIC GROUPS

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Introduction.

In this course we will be studying the subgroup structure of the finite alternating and symmetric groups. What does the phrase "study the subgroups of symmetric groups" mean? In this introduction I'll suggest an answer to that question, and attempt to convince you that answer has some merit. In the process you'll get some idea of the material we will be covering, and I'll attempt to motivate that material.

First, given a finite group G and a set Ω, define a *permutation representation* of G on Ω to be a homomorphism $\pi : G \to S = Sym(\Omega)$ of G into the symmetric group on Ω. From one point of view, the study of the subgroup structure of S amounts to the study of such representations, since the subgroups of S are precisely the images of these representations. At first glance it is not clear that this restatement of the problem represents any progress, as it would appear to be equally vague. However in awhile we will see that the reformulation does have some advantages.

The theory of permutation representations can be embedded in a much more general representation theory of groups, and the first few sections of the notes discuss that theory. In particular in any representation theory, we will wish to take advantage of two types of reductions: First, reduce the study of the general representation of G to the study of the indecomposable and irreducible representations of G. Second, reduce the study of representations of the general finite group to the study of representations of almost simple groups, where G is *almost simple* if its generalized Fitting subgroup is a nonabelian simple group. (ie. G has a unique minimal normal subgroup, and that subgroup is a nonabelian simple group.)

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In the case of permutation representations, the indecomposables are the transitive representations, and the irreducibles are the primitive representations. Each transitive representation of G is equivalent to a representation of G by right multiplication on the space G/H of cosets of some subgroup H of G; indeed H is determined up to conjugation as the stabilizer G_ω in G of some point ω of Ω. Thus it would appear that to "study the subgroup structure of S", one needs to study transitive permutation representations of *all* finite groups, or at least transitive representations of all finite almost simple groups. Moreover the study of the transitive representations of a group G amounts to a study of the subgroups of G, so it would seem that to study the subgroup structure of S, we must also study the subgroup structure of all almost simple groups. In short, the problem is beginning to look more difficult, rather than easier.

Our representation π is *faithful* if $\ker(\pi) = 1$. Fortunately it turns out that the structure of a finite group G admitting a faithful primitive representation is highly restricted. One of the important results in this course will give a precise description of those finite groups G admitting such a representation, together with a description of the embedding of a point stabilizer in G. This result is often called the O'Nan-Scott Theorem, although a second weaker result also goes by the same name. We will also come to the the weaker result shortly.

The O'Nan-Scott Theorem is the basic tool for reducing questions about general permutation representations of general finite groups to the case where the group is almost simple and the representation primitive. In broad terms the reduction goes as follows:

1. Given a question Q about general permutation representations $\pi : G \to Sym(\Omega)$ of general finite groups G, reduce to the case where π is faithful and primitive.

2. Now appeal to the O'Nan-Scott Theorem to conclude that G has one of several highly restricted structures. In the most difficult case, G will be almost simple. Answer Q in all other cases, hence reducing to the case G almost simple.

3. Use the classification of the finite simple groups to conclude that G is on a list of known almost simple groups. Observe that as π is primitive, a point stabilizer $M = G_\omega$ is a maximal subgroup of G.

4. Develop a theory describing (in some suitable sense) the maximal subgroups of the almost simple groups. Use this theory to generate information about the representation of G on G/M, sufficient to answer question Q.

So at last we seem to be making progress. In order to effectively study the subgroup structure of S, it would appear we need to do two things:

(A) Give a precise description of the structure of finite primitive permutation groups, and

(B) using the classification of the finite simple groups, give a useful description of the maximal subgroups of each almost simple group.

Problem (B) is still a bit vague, but we will deal with that in a moment.

As the alternating and symmetric groups are probably the most accessible of the almost simple groups, the study of their maximal subgroups is clearly of importance, but also provides a good setting for exploring and illustrating how to approach the question for other classes of almost simple groups. Moreover this supplies us with a fairly specific problem on which to focus in this course: Describe the maximal subgroups of the alternating and symmetric groups.

Given an almost simple group G, how can we study its subgroups, and in particular its maximal subgroups? Once again we can hope to use representation theory. Namely we seek an object X in some category $\mathcal{C}$ such that G is essentially the group of automorphisms of X in $\mathcal{C}$, and the representation of G on X can be used to study the subgroups of G. Most particularly, we hope to show that most maximal subgroups of G are the stabilizers of certain structures on X which are natural in the category $\mathcal{C}$. Indeed we try to show that any subgroup which does not stabilize one of these structures is almost simple and irreducible on X.

Let $S = Sym(\Omega)$ be the symmetric group on Ω. In this case we take $\mathcal{C}$ to be the category of sets and Ω to be our object, so indeed S is $Aut(\Omega)$. A second major result in this course describes certain structures on Ω, and shows that if H is a subgroup of S stabilizing none of these structures, then H is almost simple and primitive on Ω. This result is fairly easy to derive from the O'Nan-Scott Theorem, and this weaker corollary is also often called the O'Nan-Scott Theorem. Some of the structures that arise are: proper nonempty subsets of Ω (substructures, coproduct structures); nontrivial partitions of Ω (regular coproduct structures); and regular product structures on Ω. We will spend some time studying these structures and relationships among them, since those relationships translate into relationships among the subgroups of S.

We will also briefly discuss the important theorem of Liebeck, Praeger, and Saxl which tells us when the stabilizer in S of a natural structure *is* actually maximal in S, and gives a list of those almost simple primitive subgroups of S which *are not* maximal. Together with the O'Nan-Scott Theorem, the Liebeck-Praeger-Saxl Theorem gives a weak classification of the maximal subgroups of S, and then also a classification of the maximal subgroups of the alternating group on Ω.

Here is a particular question about subgroup structure of finite groups that we will consider:

Palfy-Pudlak Question. Is every nonempty finite lattice isomorphic to the lattice $\mathcal{O}_G(H)$ of overgroups in some finite group G of some subgroup H of G?

Here an *overgroup* of a subgroup H of a group G is a subgroup of G containing H.

The answer to the Palfy-Pudlak Question is almost certainly negative. We will consider a class $D\Delta$ of lattices, which we conjecture supply a negative answer to the Question. To illustrate the reduction process outlined above, I will state a result which reduces the proof of this conjecture to the case where G is almost simple. (Actually the reduction also requires that one treat another class of sublattices of the lattice of subgroups of G.)

In addition to illustrating the reduction process, the Palfy-Pudlak Question points the way toward larger issues involving subgroup structure. Namely, since the classification of the finite simple groups, the majority of work on permutation groups has focused on primitive groups, and most particularly on the maximal subgroups of almost simple groups. Perhaps however it is now time to look deeper into the lattice Λ of subgroups of a finite group G, beyond the maximal subgroups of G. This is all well and good, but the question remains, what results about Λ are on the one hand useful, and on the other, possible to prove?

The Palfy-Pudlak Question suggests that results about the set $\mathcal{O}_G(H)$ of overgroups of suitable subgroups H of G are important. But which subgroups H should we consider in this context?

There are classical results about the overgroups of permutations in S moving small numbers of points of Ω, such as transpositions. There are also more modern results about the overgroups of root subgroups and maximal tori in groups of Lie type. More generally, work on the Palfy-Pudlak Question suggests it is worthwhile to study $\mathcal{O}_G(D)$, for G almost simple and D a small normal subgroup of some maximal subgroup of G. I will describe a few results of this type. I will also briefly discuss the overgroups of primitive subgroups of S.

Finally, here are two specialized but interesting questions which arise in a natural way when trying to prove the conjecture. If time permits, we will consider these questions near the end of the course.

Define the *depth* of a subgroup H of a group G to be the maximal length of a chain in the poset $\mathcal{O}_G(H)$. Thus the maximal subgroups of G are the subgroups of depth 1.

Our first question asks: what are the subgroups of depth 2 in the almost simple groups, and in particular in the symmetric group S? Note that in considering this question we are led to the following situation: M_1 and M_2 are maximal subgroups of G such that $H = M_1 \cap M_2$ is maximal in both M_1 and M_2. In the proof of the conjecture, one can also assume that M_1 and M_2 are the *only* maximal overgroups of H in G.

Write $\Delta(m)$ for the lattice of all subsets of an m-set, and define a pair (G, H) of finite groups to be a $\Delta(m)$*-pair* if $H \leq G$ and $\mathcal{O}_G(H) \cong \Delta(m)$. Our second question asks: what are the $\Delta(m)$-pairs for $m \geq 2$? Probably this is too ambitious a problem, but perhaps it is feasible when G is almost simple.

Section 1 Representations

In this section $\mathcal{C}$ is a category. Given objects A, B in $\mathcal{C}$, write $Mor(A, B)$ for the set of morphisms from A to B. I'll often write $\alpha : A \to B$ to indicate $\alpha \in Mor(A, B)$. Also I'll compose my morphisms from left to right, so if $\beta \in Mor(B, C)$ then $\alpha\beta \in Mor(A, C)$ denotes the composition of α and β.

Recall an *isomorphism* from A to B is a morphism $\alpha : A \to B$ which possesses an *inverse* $\beta \in Mor(A, B)$ such that $\alpha\beta = 1_A$ is the identity morphism on A, and $\beta\alpha = 1_B$ is the identity morphism on B. Moreover the inverse β is unique and denoted by α^{-1}. Finally an *automorphism* of A is an isomorphism $\alpha : A \to A$. Write $Aut(A)$ for the set of automorphisms of A, and recall $Aut(A)$ forms a group, where the group operation is the composition in the category.

Example 1.1. The category of sets and functions. The objects are the sets, $Mor(A, B)$ consists of all functions from A into B, and composition is ordinary composition of functions. The isomorphisms of sets are the bijections, and given a set A, the automorphisms of A are the permutations of A. Thus the group $Aut(A)$ of automorphisms of A is the *symmetric group* $Sym(A)$ on A: the group of all permutations on A under composition of functions. I will apply my permutations on the right, and hence compose permutations from left to right. Thus for $a \in A$ and $\alpha, \beta \in Sym(A)$, $a\alpha$ denotes the image of A under α, and $a(\alpha\beta) = (a\alpha)\beta$.

(1.2) *Let $\alpha : A \to B$ be an isomorphism in a category $\mathcal{C}$ and define*

$$\begin{aligned}\alpha^* : Mor(A, A) &\to Mor(B, B)\\ \beta &\mapsto \alpha^{-1}\beta\alpha\end{aligned}$$

Then

(1) α^* *is a bijection preserving composition.*

(2) α^* *restricts to a group isomorphism of* $Aut(A)$ *with* $Aut(B)$.

(3) *If* $A \cong B$, *then also* $Aut(A) \cong Aut(B)$.

Proof. α^* "preserves composition" in the sense that $(\beta\gamma)\alpha^* = (\beta\alpha^*) \cdot (\gamma\alpha^*)$.

Example 1.3. Suppose X and Y are sets of the same cardinality. Then $X \cong Y$ so by Lemma 1.2, $Sym(X) \cong Sym(Y)$. Thus we write S_n for the symmetric group on a set of order n.

Definition 1.4. Let G be a group. A *representation* of G in the category $\mathcal{C}$ is a group homomorphism $\pi : G \to Aut(A)$ for some object A in $\mathcal{C}$.

Example 1.5 A *permutation representation* of G is a representation in the category of sets. That is a permutation representation is a group homomorphism $\pi : G \to Sym(A)$, since $Sym(A) = Aut(A)$ in the category of sets.

Example 1.6 Let F be a field. A *linear representation* or *FG-representation* is a representation of G in the category of vector spaces over F. Thus $\pi : G \to GL(A)$, where $GL(A)$ is the *general linear group* on A; that is $GL(A)$ is the group of all invertible linear maps on the vector space A.

Definition 1.7. A representation $\pi : G \to Aut(A)$ is *faithful* if π is an injection.

Two representations $\pi : G \to Aut(A)$ and $\sigma : G \to Aut(B)$ are *equivalent* if there exists an isomorphism $\alpha : A \to B$ such that $\sigma = \pi\alpha^*$, using the "*-notation" introduced in Lemma 1.2. Moreover such an isomorphism α is said to be an *equivalence* of the representations.

Observe that an isomorphism α is an equivalence iff for all $g \in G$, $(g\pi)\alpha = \alpha(g\sigma)$. Moreover equivalence is an equivalence relation.

$$\begin{array}{ccc} A & \xrightarrow{g\pi} & A \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ B & \xrightarrow{g\sigma} & B \end{array}$$

Similarly if $\pi_i : G_i \to Aut(A_i)$, $i = 1, 2$, are representations of groups G_i on objects A_i in $\mathcal{C}$, then π_1 is said to be *quasiequivalent* to π_2 if there exists a group isomorphism

$\beta : G_1 \to G_2$ and an isomorphism $\alpha : A_1 \to A_2$ in the category $\mathcal{C}$ such that $\pi_2 = \beta^{-1}\pi_1\alpha^*$. Quasiequivalence is also an equivalence relation.

$$\begin{array}{ccc} A_1 & \xrightarrow{g\pi_1} & A_1 \\ \alpha\Big\downarrow & & \Big\downarrow\alpha \\ A_2 & \xrightarrow{g\beta\pi_2} & A_2 \end{array}$$

The pair of isomorphisms α, β is said to be a *quasiequivalence.*

Notation 1.8. Let us examine these notions in the category of sets. So let X be a set, G be a group, and $\pi : G \to Sym(X)$ a permutation representation of G on X.

Usually we will suppress the representation π and write xg for $x(g\pi)$, when $x \in X$ and $g \in G$. One feature of this notation is that:

$$x(gh) = (xg)h, \quad x \in X,\ g, h \in G.$$

The relation $\sim$ on X defined by $x \sim y$ if and only if there exists $g \in G$ with $xg = y$ is an equivalence relation on X. The equivalence class of x under this relation is

$$xG = \{xg : g \in G\}$$

and is called the *orbit* of x under G. As the equivalence classes of an equivalence relation partition a set, X is partitioned by the orbits of G on X.

Let Y be a subset of X. We say that G *acts* on Y if Y is a union of orbits of G. Notice G acts on Y precisely when $yg \in Y$ for each $y \in Y$, and each $g \in G$. Further if G acts on Y then for each $g \in G$, the restriction $g_{|Y}$ of g to Y is a permutation of Y, and the restriction map

$$G \to Sym(Y)$$
$$g \mapsto g_{|Y}$$

is a permutation representation of G with kernel

$$G_Y = \{g \in G : yg = y \text{ for all } y \in Y\}.$$

In particular G_Y is a subgroup of G called the *pointwise stabilizer* of Y in G. For $x \in X$, write G_x for $G_{\{x\}}$. Thus G_x is a subgroup of G called the *stabilizer* of x in G.

Set $N_G(Y) = \{g \in G : Yg = Y\}$, and call $N_G(Y)$ the *global stabilizer* of Y in G. Write G^Y for the image of $N_G(Y)$ in $Sym(Y)$ under the restriction map, so that $G^Y \cong N_G(Y)/G_Y$.

Our representation π is *transitive* if G has just one orbit on X; equivalently for each $x, y \in X$ there exists $g \in G$ with $xg = y$. We will also say that G is transitive on X.

Example 1.9. Let $H \leq G$ and consider the coset space $G/H = \{Hx : x \in G\}$ of H in G. Then $\alpha : G \to Sym(G/H)$ is a transitive permutation representation of G on the coset space G/H, where

$$g\alpha : Hx \mapsto Hxg$$

for $g \in G$. We call α the *representation of G on the cosets of H by right multiplication.* Notice H is the stabilizer of the coset H in this representation.

Theorem 1.10. *Let G be transitive on X, $x \in X$, and $H = G_x$. Then*

(1) *The map $xg \mapsto Hg$ is an equivalence of the permutation representation of G on X with the representation of G by right multiplication on G/H.*
(2) *X has cardinality $|G : G_x|$ for each $x \in X$.*
(3) *$G_{xg} = (G_x)^g$ for each $g \in G$.*
(4) *If $\beta : G \to Sym(Y)$ is a transitive permutation representation and $y \in Y$, then π is equivalent to β if and only if G_x is conjugate to G_y in G.*

Proof. See 5.8 and 5.9 in [FGT].

(1.11) *Let $\pi_i : G \to Sym(X_i)$, $i = 1, 2$, be transitive permutation representations of a group G and pick $x_i \in X_i$. Then π_1 and π_2 are quasiequivalent iff there exists $\beta \in Aut(G)$ with $G_{x_1}\beta = G_{x_2}$.*

Proof. This is Exercise 1.1.

Now we return to the general setup where $\mathcal{C}$ is an arbitrary category and G is a group.

Remark 1.12. Let A be an object in $\mathcal{C}$. Given a representation π of G, write $[\pi]$ for the equivalence class of the representation π. Observe that we have a permutation representation of $Aut(G)$ on the set of representations of G on A defined by $\beta : \pi \to \pi \cdot \beta = \beta^{-1}\pi$ for $\beta \in Aut(G)$, where $\beta^{-1}\pi$ is the composition of β^{-1} with π. Moreover this representation induces a permutation representation of $Aut(G)$ on the equivalence class of representations of G via $\beta : [\pi] \mapsto [\pi] \cdot \beta = [\pi \cdot \beta]$, with the orbits of G the quasiequivalence classes.

(1.13) *Let $\pi_i : G \to Aut(A)$, $i = 1, 2$, be a pair of faithful representations. Then*
(1) π_1 is quasiequivalent to π_2 iff $G\pi_1$ is conjugate to $G\pi_2$ in $Aut(A)$.
(2) $Aut_{Aut(A)}(G\pi_1) \cong Aut(G)_{[\pi_1]}$.

Proof. Here if $X \leq Y$ are groups then $Aut_Y(X) = N_Y(X)/C_Y(X)$ is the group of automorphisms induced on X in Y via conjugation. Also $Aut(G)_{[\pi_1]}$ is the stabilizer in $Aut(G)$ of the equivalence class of the representation π_1 with respect to the action defined in Remark 1.12. The proof is Exercise 1.2.

Exercises for Section 1.

1. Prove Lemma 1.11.

2. Prove Lemma 1.13.

Section 2. (Co)product structures and indecomposable and irreducible representations

In this section $\mathcal{C}$ is a category. We will use representations of groups G on objects A in $\mathcal{C}$ to study G and A. Further we wish to define notions of "indecomposable representation" and "irreducible representation" in the category $\mathcal{C}$. To do so, we will need information about certain structures on A, the action of $Aut(A)$ on these structures, and the stabilizer in $Aut(A)$ of each structure. Some of these structures can be defined simultaneously for all categories in terms of coproducts and products.

Let $\mathcal{F} = (A_i : i \in I)$ be a family of objects in $\mathcal{A}$. A *coproduct* for $\mathcal{F}$ in $\mathcal{A}$ is an object $C = \coprod_i A_i$ together with morphisms $\iota_i : A_i \to C$ such that whenever B is an object and $\alpha_i : A_i \to B$, $i \in I$, are morphisms, then there exists a unique morphism $\alpha : C \to B$ such that $\iota_i \alpha = \alpha_i$ for all $i \in I$.

Recall that a coproduct may or may not exist for $\mathcal{F}$, but if it does exist, then it is unique up to isomorphism.

Example 2.1. Let $\mathcal{C}$ be the category of sets. Then C is the disjoint union of the sets A_i with $\iota_i : A_i \to C$ the inclusion map. The map α is defined by $a\alpha = a\alpha_i$ for $a \in A_i$.

Similarly the notion of a product is dual to that of the coproduct; that is the definition of the product is the same as that of the coproduct, except the direction of the arrows is reversed. Hence a *product* for $\mathcal{F}$ is an object P together with morphisms $\pi_i : P \to A_i$ (called the *projection* of P on A_i) such that whenever B is an object and $\beta_i : B \to A_i$, $i \in I$ are morphisms, then there is a unique morphism $\beta : B \to P$ such that $\beta\pi_i = \beta_i$ for each $i \in I$. Again a product for $\mathcal{F}$ may or may not exist, but if it exists it is unique up to isomorphism.

Example 2.2. Let $\mathcal{C}$ be the category of sets. Then P is the set product of the sets A_i. For example assume $I = \{1, \dots, n\}$ is finite. Then

$$P = A_1 \times \cdots \times A_n = \{(a_1, \dots, a_n) : a_i \in A_i\},$$

the ith projection π_i is defined by $(a_1, \dots, a_n)\pi_i = a_i$, and the map β is defined by $b\beta = (b\beta_1, \dots, b\beta_n)$.

Definition 2.3. A category $\mathcal{C}$ is a *category of sets with structure* if

(SS1) the objects of $\mathcal{C}$ are sets A together with some "structure" on A, and

(SS2) given objects A, B, the set $Mor(A, B)$ of morphisms from A to B is the set of all functions from the set A to the set B "preserving structure", and

(SS3) composition in the category is composition of functions.

Of course in any given example, we must define precisely what we mean by "structure" and "preserving structure", and these definitions must imply that

(SS4) the composition of structure preserving functions preserves structure, and the identity function $1_A : A \to A$ preserves structure.

A *subobject* of A is a subset of A which inherits the structure on A, to become an object in $\mathcal{C}$. A *factor object* of A is the set $\tilde{A} = \{\tilde{a} : a \in A\}$ of equivalence classes of a suitable equivalence relation $\sim$ on A, with the structure on $\tilde{A}$ "inherited" from A. We will call such equivalence relations *admissible*.

Obvious examples of categories of sets with structure include the category of sets (where there is no extra structure), and the category of groups, where the structure on a group G is provided by the group operation, and a map α from G to a group H preserves structure if $(xy)\alpha = x\alpha \cdot x\alpha$ for all $x, y \in G$.

In the remainder of the section, we assume that our category $\mathcal{C}$ is a category of sets with structure.

Definition 2.4. Define a *coproduct structure* on an object A in our category of sets with structure, to be an equivalence class of families

$$\iota = (\iota_i : A_i \to A : i \in I)$$

of maps making A into a coproduct. Here two families ι and $\bar{\iota}$ are *equivalent* if $A_i\iota_i = A_i\bar{\iota}_i$ for all $i \in I$. Write $[\iota]$ for the equivalence class of ι.

Remark 2.5. Observe that $H = Aut(A)$ permutes the coproduct structures via $[\iota]h = [(\iota_i h : i \in I)]$, with the stabilizer $H_{[\iota]}$ in H of $[\iota]$ the subgroup acting on $A_i\iota_i$ for all $i \in I$.

(2.6) *Let $\iota = (\iota_i : A_i \to A : i \in I)$ be a coproduct in $\mathcal{A}$ and $G = \prod_i Aut(A_i)$. Then*

(1) For each $\mathbf{g} = (g_1, \dots, g_n) \in G$, there exists a unique $\varphi(\mathbf{g}) \in Aut(A)$ such that $g_i\iota_i = \iota_i\varphi(\mathbf{g})$ for each $i \in I$.

(2) The map $\varphi : G \to Aut(A)$ is a group homomorphism whose image lies in $Aut(A)_{[\iota]}$.

(3) Assume for each i that $A_i\iota_i$ is a subobject of A, $\iota_i : A_i \to A_i\iota_i$ is an isomorphism, and whenever $\alpha \in Aut(A)$ acts on $A_i\iota_i$, then $\alpha : A_i\iota_i \to A_i\iota_i$ is an isomorphism. Then $\varphi : G \to Aut(A)_{[\iota]}$ is an isomorphism.

Proof. Exercise 2.1.

Example 2.7. Let $\mathcal{C}$ be the category of sets and A the coproduct of $A_1, \dots, A_r$. Then A is the disjoint union of the sets A_i and the stabilizer G in $Sym(A)$ of the corresponding coproduct structure is the stabilizer of this partition; that is G is the subgroup of $Sym(A)$ acting on each A_i. Thus by Lemma 2.6,

$$G \cong \prod_i Aut(A_i) \cong \prod_i S_{n_i}$$

More concretely, let $G_i = Sym(A)_{A-A_i}$ be the set of permutations $Sym(A)$ fixing each point in $A - A_i$. Then $G_i \cong Sym(A_i)$ and G is the direct product of the subgroups $G_1, \dots, G_r$. That is $G_i \trianglelefteq G$ and each $g \in G$ can be written uniquely as a product $g = g_1 \cdots g_r$ with $g_i \in G_i$.

Definition 2.8. Define the coproduct structure to be *nontrivial* if $|I| > 1$. Define a representation $\pi : G \to Aut(A)$ to be *decomposable* if $G\pi \leq Aut(A)_{[\iota]}$ for some coproduct structure $[\iota]$, and call π *indecomposable* otherwise.

Example 2.9. We just saw that in the category of sets, a coproduct structure is just a partition of A (or equivalently an equivalence relation) and the stabilizer of the structure is just the subgroup of $Sym(A)$ acting on each block. Thus a permutation representation is indecomposable if and only if it is transitive.

Example 2.10. In the category of vector spaces over a field F, a coproduct structure is a nontrivial direct sum decomposition of a vector space A, and the stabilizer of the structure is the subgroup of $GL(A)$ acting on each summand. Thus we have the usual notion of indecomposability for linear representations: a linear representation of G on a vector space A is indecomposable iff V is not the direct sum of proper G-invariant subspaces.

Our philosophy is that if A is the coproduct of objects $(A_i : i \in I)$ in a category of sets with structure, then (essentially) complete information about A can be recovered from the corresponding information about the A_i. Similarly if π is decomposable then π is determined by the *restrictions* $\pi_i : G \to Aut(A_i)$ obtained from $G\pi \leq N_{Aut(A)}(A_i\iota_i)$. Thus there is usually little loss in assuming a representation is indecomposable.

We can also dualize the notion of coproduct structure to that of a product structure.

Definition 2.10. A *product structure* on A is an equivalence class of families $\pi = (\pi_i : A \to A_i)$ of maps making A into a product, with π equivalent to $\bar{\pi}$ if π and $\bar{\pi}$ have the same set of fibres; that is for all $i \in I$

$$\{\pi_i^{-1}(a_i) : a_i \in A_i\} = \{\bar{\pi}_i^{-1}(a_i) : a_i \in A_i\}$$

Remark 2.12. $H = Aut(A)$ permutes product structures via $g : [\pi] \mapsto [\pi]g = [g^{-1}\pi]$, and $H_{[\pi]}$ is the subgroup of H permuting the set $\{\pi_i^{-1}(a_i) : a_i \in A_i\}$ of fibres of π_i, for each $i \in I$.

(2.13) *Let $\pi = (\pi_i : A \to A_i : i \in I)$ be a product in $\mathcal{C}$ and $G = \prod_i Aut(A_i)$. Then*

(1) For each $\mathbf{g} = (g_1, \ldots, g_n) \in G$, there exists a unique $\varphi(\mathbf{g}) \in Aut(A)$ such that $\pi_i g_i = \varphi(\mathbf{g})\pi_i$ for each $i \in I$.

(2) The map $\varphi : G \to Aut(A)$ is a group homomorphism whose image lies in $Aut(A)_{[\pi]}$.

(3) Assume for each $i \in I$, the fibres of π_i define an equivalence relation $\sim_i$ on A making $A^i = A/\sim_i$ into a factor object, such that the map $\tilde{\pi}_i : A^i \to A_i$ defined by $\tilde{\pi}_i : \tilde{a} \mapsto \widetilde{a\pi_i}$ is an isomorphism, and for each $\alpha \in Aut(A)_{[\pi]}$, $\tilde{\alpha} : A^i \to A^i$ is an isomorphism. Then $\varphi : G \to Aut(A)_{[\pi]}$ is an isomorphism.

Proof. Exercise 2.2.

Example 2.14. In the category of sets, a product structure is an equivalence class of identifications of A with some set product $\prod_i A_i$. This definition is a little different than the categorical point of view we used to define product structures in general categories, but it is easy to see the two definitions are equivalent. It takes advantage of the fact that we have a canonical description of the product in the category of sets.

The subgroup fixing this structure is the set of permutations g which can be represented on each A_i so that g "factors through" our identification via

$$g : (a_1, \dots, a_n) \mapsto (a_1 g_1, \dots , a_n g_n),$$

for a (unique) $g_i \in Sym(A_i)$.

We can also view $\prod_i A_i$ as the set of all functions

$$f : I \to \bigcup_{i \in I} A_i$$

such that $f(i) \in A_i$ for each $i \in I$. The group action is defined by $(f \cdot g)(i) = f(i)g_i$. From this point of view, the projection map π_i is defined by $\pi_i(f) = f(i)$. The two points of view are the same via the correspondence $f \mapsto (f(1), \dots , f(n))$.

Example 2.15. In the category of finite dimensional F-spaces the product and coproduct structures are the same, since products and coproducts are the same. However one also encounters tensor product structures in this category.

Definition 2.16. If all members A_i of our family are isomorphic we say our family is *regular*, and we have a weaker notion of "equivalence of structures" leading to larger stabilizers. Namely two regular coproduct structures ι and $\bar{\iota}$ are *similar* if there exists a permutation σ of I such that

$$A_i \iota_i = A_{i\sigma} \bar{\iota}_{i\sigma},$$

and the stabilizer of a similarity class $\langle \iota \rangle$ is the subgroup permuting the subobjects $(A_i \iota_i : i \in I)$. There is an analogous notion for product structures.

If our category is well behaved, then the stabilizer of a similarity class is the wreath product $Aut(A_1)$ wr S_r when $I = \{1, \dots, r\}$ is of order r. Recall:

Definition 2.17. The *wreath product* $W = L$ wr K of a group L by a group K, represented as a group of permutations on $\{1, \dots , r\}$, is the semidirect product of a normal subgroup D, which is the direct product of r copies of L, with K, where the representation of K on D is defined by

$$k : (x_1, \dots , x_r) \mapsto (x_{1k^{-1}}, \dots , x_{rk^{-1}}).$$

Notice that if we define

$$L_i = \{(x_1, \dots, x_r) \in D : x_j = 1 \text{ for } j \neq i\}$$

then $L \cong L_i$ and $D = L_1 \times \cdots \times L_r$ is the direct product of the subgroups L_i, in the sense that each L_i is a normal subgroup of D and each $x \in D$ can be written uniquely as a product $x = l_1 \cdots l_r$ with $l_i \in L_i$. Further $k \in K$ permutes the subgroups L_i in the same way it permutes I: $L_i^k = L_{ik}$ and (recalling that K_i is the stabilizer in K of i) $K_i = C_K(L_i)$.

Examples 2.18. In the category of sets, a regular coproduct structure is a partition of a set A of order rm into r blocks of the same size m. The stabilizer of this structure is the subgroup permuting the blocks and is the wreath product S_m wr S_r of S_m by S_r. The subgroup L_i is the group of all permutations which fix all points not in the ith block.

Example 2.19. A regular product structure on A of type (m, r) in the category of sets is a similarity class of identifications $\alpha : A \to B^I$, $I = \{1, \dots, r\}$, of A with the set product B^I of r copies of some set B of order m, and $Sym(A)$ permutes such structures via $\langle\alpha\rangle g = \langle g^{-1}\alpha\rangle$. The stabilizer of this structure is the subgroup of all $g \in Sym(A)$ such that $\langle\alpha\rangle g = \langle\alpha\rangle$; equivalently for each $f \in B^I$,

$$(f \cdot (g \cdot \alpha^*))(i^{\sigma(g)}) = f(i)g_i$$

for each $i \in I$, some $g_i \in Sym(B)$, and $\sigma(g) \in Sym(I)$. Put another way, $g \cdot \alpha = \alpha \cdot ((\prod_i g_i)\sigma(g))$, where $\sigma(g)$ and g_i act on B^I via

$$(fg_i)(j) = f(j) \text{ for } j \neq i \text{ and } f(i)g_i \text{ for } j = i$$

and

$$(f\sigma(g))(i) = f(i^{\sigma(g)^{-1}})$$

The stabilizer of this structure is isomorphic to the wreath product S_m wr S_r.

Example 2.20. In the category of finite dimensional F-spaces, a regular coproduct structure is a direct sum decomposition of A into r summands of equal dimension m, with the stabilizer the subgroup permuting these summands. The stabilizer is isomorphic to $GL_m(F)$ wr S_r.

We have defined the notion of an indecomposable representation which coincides with the usual notion of indecomposability for linear representations. We next define the notion of an irreducible representation; this notion will coincide with the usual notion of irreducibility for linear representations. Our philosophy is that while a decomposable representation can be retrieved from its restrictions, only partial information about a reducible representation can be obtained from its restrictions. In essence a representation is irreducible if G preserves no proper nontrivial subobject or factor object. We now formalize this notion.

Definition 2.21. Assume $\mathcal{A}$ is a category with suitable notions of subobjects and factor objects, and that $\pi : G \to Aut(A)$ is a representation of G on some object A in $\mathcal{A}$. A subobject B of A is *G-invariant* if $G\pi \leq N_{Aut(A)}(B)$. Similarly an equivalence relation $\sim$ on A is said to be *G-invariant* if $a \sim b$ implies $ag \sim bg$ for all $a, b \in A$ and $g \in G$. Finally π is *irreducible* if there are no nontrivial G-invariant subobjects or admissible relations.

Example 2.22. In the category of sets every subset is a subobject and every equivalence relation is admissible, so π is irreducible if and only if $G\pi$ is transitive and preserves no nontrivial equivalence relations. This is the definition of a *primitive* permutation group.

Example 2.23. In the category of F-spaces the admissible relations are those of the form $a \sim b$ iff $a \in b + B$ for some subspace B of A, so we have the usual notion of irreducibility.

Exercises for Section 2.

1. Prove Lemma 2.6.
2. Prove Lemma 2.13.

Section 3. The generalized Fitting subgroup.

In this section G is a finite group.

Definition 3.1. Let $L_1, \ldots, L_n$ be groups. A *central product* of the groups $L_1, \ldots, L_n$ is a group D which is the product of subgroups $D_1, \ldots, D_n$ such that for each i, $D_i \cong L_i$, and for each $i \neq j$, $[D_i, D_j] = 1$. Here for subgroups X, Y of a group D, $[X, Y] = \langle [x, y] : x \in X, y \in Y \rangle$ is the *commutator* of X and Y, and $[x, y] = x^{-1}y^{-1}xy$ is the *commutator* of elements $x, y \in D$. Observe $[X, Y] = 1$ iff for all $x \in X$ and $y \in Y$, $xy = yx$.

(3.2) *Let $L_1, \dots, L_n$ be groups and $L = L_1 \times \cdots \times L_n$ the direct product of $L_1, \dots, L_n$. Then*

(1) $Z(L) = Z(L_1) \times \cdots \times Z(L_n)$.

(2) Each central product of $L_1, \dots, L_n$ is isomorphic to L/Z for some some $Z \leq Z(L)$ with $Z \cap L_i = 1$ for each i.

Proof. We identify L_i with the subgroup of elements $(x_1, \dots, x_n) \in L$ with $x_j = 1$ for all $j \neq i$, as in Section 4. Subject to this convention, (2) makes sense. The proof is Exercise 3.1.

Definition 3.3. Define a group X to be *perfect* if $X = [X, X]$; that is X is its own commutator subgroup. The group X is *quasisimple* if X is perfect and $X/Z(X)$ is simple. Further a subgroup X of a group G is *subnormal* in G if there exists a *subnormal series* $X = X_0 \trianglelefteq \cdots \trianglelefteq X_n = G$. That is subnormality is the transitive extension of the normality relation. Finally the *components* of G are its subnormal quasisimple subgroups. Write $E(G)$ for the subgroup of G generated by the components of G.

Given a nonabelian simple group L, there is a *universal covering group* $\tilde{L}$ of L which is the largest quasisimple group G such that $G/Z(G) \cong L$. That is if G is such a group then $G \cong \tilde{L}/Z$ for some $Z \leq Z(\tilde{L})$. The center $Z(\tilde{L})$ of the universal covering group is called the *Schur multiplier* of L. See section 33 in [FGT] for further discussion of such things.

Example 3.4. The identity group is a trivial quasisimple group. Nonabelian simple groups are quasisimple. There are also quasisimple groups which are not simple. For example the groups $SL_2(q)$, q odd, are quasisimple with a center of order 2.

(3.5) *For $H \leq G$ let $\mathcal{C}(H)$ be the set of components of H. Then*

(1) $E(G)$ is a characteristic subgroup of G.

(2) Distinct components of G commute, so $E(G)$ is a central product of the components of G.

(3) Set $G^* = G/Z(E(G))$. Then $Z(E(G)) = \langle Z(L) : L \in \mathcal{C}(G) \rangle$, G^* is the direct product of the groups L^*, $L \in \mathcal{C}(G)$, and for each $L \in \mathcal{C}(G)$, L^* is a nonabelian simple group.

(4) If $L \in \mathcal{C}(G)$ and H is a subnormal subgroup of G, then $\mathcal{C}(H) = \{K \in \mathcal{C}(G) : K \leq H\}$, and either $L \leq H$ or $[L, H] = 1$.

Proof. By definition of $\mathcal{C}(G)$, $Aut(G)$ permutes $\mathcal{C}(G)$, so (1) holds. Part (4) is 31.3 and 31.4 in [FGT]. Then (4) implies distinct components of G commute, so that $E(G)$ is a central product of the components of G, establishing (2). Part (3) follows from (2) and 3.2.

Definition 3.6. The *Fitting subgroup* of G is the largest normal nilpotent subgroup of G. Write $F(G)$ for the Fitting subgroup. Given a prime p, write $O_p(G)$ for the largest normal p-subgroup of G. The latter subgroup exists as the product of normal p-subgroups is a normal p-subgroup. As a finite group is nilpotent iff it is the direct product of its Sylow groups, $F(G)$ exists and:

(3.7) *$F(G)$ is the direct product of the groups $O_p(G)$, p prime.*

We seek a canonically defined characteristic subgroup of the general finite group G whose structure is relatively uncomplicated, and which controls the structure of G. That subgroup is the so-called generalized Fitting subgroup of G. Its definition is due to Helmut Bender, building on earlier work of Wielandt and Gorenstein and Walter.

Definition 3.8. Set $F^*(G) = F(G)E(G)$ and call $F^*(G)$ the *generalized Fitting subgroup* of G.

(3.9) *$F^*(G)$ is the central product of $F(G)$ and $E(G)$.*

Proof. By 3.5.4, each component of G commutes with each solvable normal subgroup of G.

Theorem 3.10. *$C_G(F^*(G)) = Z(F(G))$.*

Proof. See 31.13 in [FGT].

Definition 3.11 A finite group G is *almost simple* if $F^*(G)$ is a nonabelian finite simple group.

Given $H \trianglelefteq G$, let $c : G \to Aut(H)$ be the conjugation map; that is for $g \in G$, $gc : h \mapsto h^g = g^{-1}hg$ for $g \in G$ and $h \in H$. When $G = H$, we write $Inn(G)$ for the image Gc of G in $Aut(G)$, and call $Inn(G)$ the group of *inner automorphisms* of G. Note that $Inn(G) \trianglelefteq Aut(G)$, and the group of *outer automorphisms* of G is $Out(G) = Aut(G)/Inn(G)$.

(3.12) *The following are equivalent:*

(1) G is almost simple.

(2) There exists a nonabelian finite simple group L such that $Inn(L) \trianglelefteq G \leq Aut(L)$.

Proof. Suppose G is almost simple and set $L = F^*(G)$. Then L is a nonabelian simple group, and the conjugation map $c : G \to Aut(L)$ is faithful by 3.10, with $Lc = Inn(L) \trianglelefteq Gc \cong G$. Thus (1) implies (2). Similarly if (2) holds, then as $Aut(L)$ acts faithfully on L, and as $c : L \to Inn(L)$ is an $Aut(L)$-equivariant isomorphism, it follows that $C_{Aut(L)}(Inn(L)) = 1$, and hence also $C_G(Inn(L)) = 1$. Thus $Inn(L) = F^*(G)$ by 3.5.2 and 3.9, so (2) implies (1).

Exercises for Section 3.

1. Prove Lemma 3.2.

2. Assume p is a prime and X is a nontrivial finite p-group acting on a finite group L with $C_L(X) = 1$. Prove L is a p'-group.

3. Let S be a nontrivial 2-group. Prove

(1) The exponent of a Sylow 2-subgroup of $Aut(S)$ is less than $|S|$.

(2) Let $\sigma \in Aut(S)$ be of order 2^a, and for $1 \leq i \leq a$, let $\sigma_i \in \langle\sigma\rangle$ be of order 2^i. Then σ_i centralizes each σ-invariant subgroup of S of order 2^{a+1-i}, so in particular $|C_S(\sigma_i)| \geq 2^{a+1-i}$.

4. Let G be a finite group. Define the *generalized Fitting series* for G recursively by $F_0(G) = 1$, and give $F_k(G)$, $F_{k+1}(G)$ is the preimage in G of $F^*(G/F_k(G))$. The *generalized Fitting length* of G is the smallest integer $l(G)$ such that $G = F_l(G)$. If G is solvable then $F^*(G) = F(G)$ and our series is the *Fitting series* and $l(G)$ is the *Fitting length* of G.

Let X be a subnormal subgroup of G. Prove

(1) $F^*(X)$ is subnormal in $F^*(G)$.

(2) $F_k(X)$ is subnormal in $F_k(G)$ for each k.

(3) $F_k(X) = X \cap F_k(G)$.

(4) $l(X) \leq l(G)$.

(5) If $F_k(G) \leq H \leq G$ and all Sylow subgroups of H are abelian, then $F_k(H) = F_k(G)$ and $l(H) = k + l(H/F_k(H))$.

Section 4. Diagonal subgroups.

In this section L is a group, I is a set of finite order r, and $D = L^I$ is the direct product of r copies of L.

Recall there are various ways to describe D. First

$$D = \{(a_1, \dots , a_r) : a_i \in L\}$$

is the set of ordered r-tuples with entries in L, with the group product defined componentwise, and for $i \in I$, the ith projection map $\pi_i : D \to L$ is defined by

$$\pi_i : (a_1, \dots , a_r) \mapsto a_i$$

Second,

$$D = L^I = \{f : f : I \to L\}$$

is the set L^I of functions from I to L with multiplication defined by

$$(f \cdot g)(i) = f(i)g(i)$$

and with $\pi_i(f) = f(i)$. If $I = \{1, \dots , r\}$ then the correspondence between theses two points of view is given by the bijection $f \mapsto (f(1), \dots , f(r))$.

Third, for $i \in I$, let

$$L_i = \{f \in L^I : f(j) = 1 \text{ for all } i \neq j\}$$

Then $\pi_i : L_i \to L$ is an isomorphism. Let $\iota_i : L \to L_i$ be the inverse of this isomorphism. Now $D = \prod_i L_i$ is the direct product of the subgroups L_i in the sense that $[L_i, L_j] = 1$ for $i \neq j$ and each $d \in D$ can be written uniquely in the form $d = d_1 \dots d_r$ with $d_i \in L_i$, and π_i can be regarded as the map $d \mapsto d_i$.

Write $Aut_I(D)$ for the subgroup of $Aut(D)$ permuting the set

$$\Delta = \{L_i : i \in I\}$$

and let $S = Sym(I) \cong S_r$. Recall from 1.2 that the isomorphism $\iota_i : L \to L_i$ induces an isomorphism

$$\begin{aligned} \iota_i^* : Aut(L) &\to Aut(L_i) \\ \beta &\mapsto \iota_i^{-1}\beta\iota_i \end{aligned}$$

Let $D^* = Aut(L)^I$ be the direct product of r copies of $Aut(L)$. Write $D = \prod_i L_i$ as above. Similarly $D^* = \prod_i L_i^*$ with $L_i^* \cong Aut(L)$. We have a representation of S on D defined by $s : f \mapsto f^s$, where

$$(!) \qquad (f^s)(i) = f(i^{s^{-1}}), \text{ so that } L_i^s = L_{i^s},$$

and a similar representation of S on D^*. The corresponding semidirect products DS and D^*S are, from Definition 2.17, the wreath products L wr S_r and $Aut(L)$ wr S_r, respectively. We record (!) as:

(4.1) *In DS, $L_i^s = L_{i^s}$, while in D^*S, $L_i^{*s} = L_{i^s}^*$.*

Remark 4.2. From Definition 3.11, the conjugation map $c : L \to Aut(L)$ is a homomorphism from L into $Aut(L)$, whose image $Inn(L)$ is the group of inner automorphisms of L, and $Inn(L) \trianglelefteq Aut(L)$, with $Out(L) = Aut(L)/Inn(L)$ the group of outer automorphisms of L. If $Z(L) = 1$ then c is injective, so $L \cong Inn(L)$ and, identifying L with $Inn(L)$ via this isomorphism, we can view L as a normal subgroup of $Aut(L)$, L_i as a normal subgroup of L_i^*, D as a normal subgroup of D^*, and DS as a subgroup of D^*S.

Example 4.3. Assume L is a nonabelian simple group. Then (cf. 3.3) the groups L_i, $i \in I$, are the components of D, and hence are permuted by $Aut(D)$, so that in this case $Aut_I(D) = Aut(D)$. Further $Z(L) = 1$, so by Remark 4.2, $L_i \trianglelefteq L_i^*$.

(4.4) *$Aut_I(D) = D^*S \cong Aut(L)$ wr S_r, where D^*S is embedded in $Aut(D)$ via the representation $(f^{\xi s})(i) = f(i^{s^{-1}})\xi_{i^{s^{-1}}}$, for $f \in D$, $\xi = (\xi_1, \dots, \xi_n) \in D^*$, and $s \in S$.*

Proof. Set $A = Aut_I(D)$. By definition of A, A permutes Δ, so (cf. Notation 1.8)

$$A/A_\Delta \cong A^\Delta \leq Sym(\Delta).$$

Set

$$C_i = C_{A_\Delta}(L_i) \text{ and } A_i = \bigcap_{j \neq i} C_j$$

Define $\varphi : D^*S \to Aut(D)$ by

$$(f^{g\varphi})(i) = f(i^{s^{-1}})\xi_{i^{s^{-1}}} \text{ for } f \in D \text{ and } g = \xi s \text{ with } \xi = (\xi_1, \dots, \xi_n) \in D^* \text{ and } s \in S.$$

Then φ is a faithful representation of D^*S on D such that $L_i^*\varphi \leq A_i$ and, using 4.1, S acts faithfully on Δ via $L_i^{s\varphi} = L_{i^s}$. The latter fact says that S is a complement to A_Δ in A. As A_i is faithfully represented on L_i, the former fact says that $L_i^*\varphi = A_i$, and hence $A_\Delta = D^*\varphi$. Thus $\varphi : D^*S \to A$ is an isomorphism.

Definition 4.5. Given a family

$$\boldsymbol{\alpha} = (\alpha_i : i \in I)$$

of isomorphisms $\alpha_i : L \to L_i$, define the *diagonal* of $\boldsymbol{\alpha}$ to be

$$\mathrm{diag}(\boldsymbol{\alpha}) = \{\prod_{i\in I} a\alpha_i : a \in L\} \leq D.$$

Remark 4.6. Given a family $\boldsymbol{\alpha}$ as in Definition 4.5, the map $\alpha : a \mapsto \prod_i a\alpha_i$ is an isomorphism of L with $\mathrm{diag}(\boldsymbol{\alpha})$ called a *diagonal embedding* of L in D. Further for $i \in I$, $\alpha\pi_i = \alpha_i$, so $\pi_i = \alpha_i\alpha^{-1}$ is the composition of isomorphisms and hence $\pi_i : \mathrm{diag}(\boldsymbol{\alpha}) \to L_i$ is an isomorphism for all $i \in I$.

(4.7) *Let $\boldsymbol{\iota} = (\iota_i : i \in I)$, and for $i \in I$, let $\iota_i^* : Aut(L) \to L_i^*$ be the inverse of the restriction to L_i^* of the ith projection $\pi_i^* : D^* \to Aut(L)$. Set $\boldsymbol{\iota^*} = (\iota_i^* : i \in I)$ and $A = Aut_I(D)$. Then*

$$N_A(\mathrm{diag}(\boldsymbol{\iota})) = \mathrm{diag}(\boldsymbol{\iota}^*) \times S \cong Aut(L) \times S_r,$$

with $S = C_A(\mathrm{diag}(\boldsymbol{\iota}))$.

Proof. Let $B = \mathrm{diag}(\boldsymbol{\iota})$ and $B^* = \mathrm{diag}(\boldsymbol{\iota}^*)$. By Remark 4.6, $\iota : L \to B$ is an isomorphism, where

$$a\iota = \prod_i a\iota_i = (a, \dots, a),$$

and by definition of the action of S on D, $(a\iota)^s = a\iota$ for all $a \in L$ and $s \in S$, so $S \leq C_A(B)$. Similarly we have the isomorphism $\iota^* : Aut(L) \to B^*$ and for $\xi \in Aut(L)$,

$$(a\iota)^{\xi\iota^*} = \prod_i (a\iota_i)^{\xi\iota_i} = \prod_i (a^\xi)\iota_i = (a^\xi)\iota,$$

so $B^* \leq N_A(B)$. Thus $\langle S, B^*\rangle \leq N_A(B)$. Also for $f \in D$ and $i \in I$,

$$f(i)^{s\xi\iota^*} = f(i^{s^{-1}})^{\xi\iota^*} = f(i^{s^{-1}})\xi = F(i)^{\xi\iota^* s},$$

so $[S, B^*] = 1$ and hence $\langle S, B^*\rangle = S \times B^*$.

Next $S^\Delta = Sym(\Delta) = N_A(B)^\Delta$, so $N_A(B) = SN_{A_\Delta}(B)$, and it remains to show $N_{A_\Delta}(B) = B^*$. We saw $(a\iota)^{\xi\iota^*} = (a^\xi)\iota$, so B^* acts faithfully as $Aut(L) = Aut(B)$ on B, and hence $N_{A_\Delta}(B) = B^*C_{A_\Delta}(B)$, and we must show $C_{A_\Delta}(B) = 1$.

Finally for $x \in C_{A_\Delta}(B)$, x centralizes the projection $a\iota\pi_1 = a\iota_i \in L_i$, so as $\pi_i : B \to L_i$ is an isomorphism by Remark 4.6, x centralizes L_i. Therefore $x \in \bigcap_i C_A(L_i) = 1$, as desired.

Definition 4.8. A *diagonal subgroup* of D is a subgroup X of D such that for each $i \in I$, the projection $\pi_i : X \to L_i$ is an injection. A *full diagonal subgroup* is a diagonal subgroup for which each projection is an isomorphism.

(4.9) *Let $i_0 \in I$, $\mathcal{D}$ the set of full diagonal subgroups of D, and $\mathcal{A}$ the set of families $\boldsymbol{\alpha} = (\alpha_i : i \in I)$ with $\alpha_{i_0} = \iota_{i_0}$. Then*

(1) The map $\rho : \boldsymbol{\alpha} \mapsto \mathrm{diag}(\boldsymbol{\alpha})$ is a bijection of $\mathcal{A}$ with $\mathcal{D}$.

(2) For $\xi \in D^*$, write $\xi = \prod_i \xi_i$, where ξ_i is the projection of ξ on L_i^*. Then $C_{D^*}(L_{i_0}) = \prod_{i \in I - \{i_0\}} L_i^*$ acts on $\mathcal{A}$ via $\xi : (\alpha_i : i \in I) \mapsto (\xi_i^{-1}\alpha_i : i \in I)$, ρ is $C_{D^*}(L_{i_0})$-equivariant, and $C_{D^*}(L_{i_0})$ is regular on $\mathcal{D}$.

(3) If L is finite then $|\mathcal{D}| = |Aut(L)|^{r-1}$.

Proof. For $X \in \mathcal{D}$, define $\alpha_i^X = \iota_{i_0}\pi_{i_0}^{-1}\pi_i : L \to L_i$ and $\alpha^X = (\alpha_i^X : i \in I)$, where π_{i_0} is the isomorphism from X to L_{i_0}. Then $\boldsymbol{\alpha}^X \in \mathcal{A}$ with $X = \mathrm{diag}(\boldsymbol{\alpha}^X)$. Conversely if $\boldsymbol{\beta} \in \mathcal{A}$ then $\boldsymbol{\alpha}^{\mathrm{diag}(\boldsymbol{\beta})} = \boldsymbol{\beta}$, so (1) holds.

The first two remarks in (2) are straightforward, and visibly $C_{D^*}(L_{i_0})$ is regular on $\mathcal{A}$, so the first two remarks imply the third. Then as $|C_{D^*}(L_{i_0})| = |Aut(L)|^{r-1}$, (1) and (2) imply (3).

(4.10) *Assume L is a nonabelian finite simple group. Then*

(1) $Aut(D) = D^*S \cong Aut(L)$ wr S_r and $D = F^*(Aut(D))$, subject to the identification of D with $\{f \in D^* : f(i) \in Inn(L)$ for all $i \in I\}$.

(2) D^* is transitive on the full diagonal subgroups of D.

(3) If B is a full diagonal subgroup of D then $N_{Aut(D)}(B) = K \times S$, where $K \cong Aut(L)$ is the kernel of the action of $N_{Aut(D)}(B)$ on the set $\mathcal{L} = \{L_i : i \in I\}$ of components of D, and S acts faithfully as $Sym(\mathcal{L})$ on $\mathcal{L}$ with $N_S(J) = C_S(J)$ for $J \in \mathcal{L}$. Further $B = N_D(B)$.

Proof. Part (1) follows from Example 4.3 and and Lemma 4.4. Part (2) is 4.8.2. Then (2) and 4.7 imply (3).

(4.11) *Assume L is a nonabelian finite simple group and $H \leq D$ such that $\pi_i : H \to L_i$ is a surjection for all $i \in I$. Then there exists a partition P of I such that $H = \prod_{J \in P} H\pi_J$ is the direct product of the full diagonal subgroups $H\pi_J$ of D_J, where $\pi_J : H \to D_J$ is the projection with respect to the direct sum decomposition $D = \prod_{J \in P} D_J$ and $D_J = \prod_{j \in J} L_j$.*

Proof. Let $J \subseteq I$ be minimal subject to $K = H \cap B \neq 1$, where $B = D_J$. By minimality of J, $\pi_i : K \to L_i$ is nontrivial for each $i \in J$. As $B \trianglelefteq D$, $K = H \cap B \trianglelefteq H$, so $1 \neq K\pi_i \trianglelefteq H\pi_i = L_i$. Therefore as L_i is simple, $\pi_i : K \to L_i$ is a surjection. Next

$$\ker(\pi_i) \leq \prod_{i \neq j \in J} L_j$$

so by minimality of J, $\ker(\pi_i) = 1$. Thus π_i is an isomorphism and hence K is a full diagonal subgroup of B.

Let $E = D_{I-J}$, so that $D = B \times E$. Let $\pi : H \to B$ be the projection with respect to this direct sum decomposition; again $K = K\pi \trianglelefteq H\pi$. But by 4.10.3, $K = N_B(K)$, so $H\pi = K$. Thus $H = K \times C_H(K)$ with $C_H(K) = \ker(\pi) = H \cap E$. Finally for $i \in I - J$, $L_i = H\pi_i = (H \cap E)\pi_i$, so the lemma holds by induction on r.

Exercises for Section 4

1. Let D be the direct product of $r > 1$ isomorphic nonabelian finite simple groups, let B be a full diagonal subgroup of D, and assume G is a finite group such that $D \trianglelefteq G$, $G = DN_G(B)$, and G acts primitively on the set of components of D. Prove $N_G(B)$ is a maximal subgroup of G.

Section 5 Posets and lattices

Definition 5.1. A *poset* is a partially ordered set; thus a poset is a set X together with a partial order $\leq$ on X. The category of posets is a category of sets with structure, where the structure is provided by the order relation. A function $\alpha : X \to Y$ from X to a poset Y is a *map of posets* (ie. preserves structure) if $a \leq b$ in X implies $a\alpha \leq b\alpha$ in Y.

Each subset Z of X is a poset under the restriction of the partial order on X to Z. Thus each subset is a subposet.

A *lattice* is a poset Λ with the property that for all $x, y \in \Lambda$, there exists a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$ for x and y in the poset Λ. Observe the elements $x \vee y$ and $x \wedge y$ are unique, and we can regard $\vee$ and $\wedge$ as operations on Λ.

A *sublattice* of Λ is a subposet of Λ closed under the operations $\vee$ and $\wedge$. Given $x \leq y$ in Λ, define the *interval* in Λ determined by x and y to be

$$[x, y] = \{z \in \Lambda : x \leq z \leq y\}.$$

Observe that the interval $[x, y]$ is a sublattice of Λ.

Example 5.2. Let G be a group and Λ the set of subgroups of G, partially ordered by inclusion. Thus Λ is a poset. Indeed Λ is a lattice, where for $H, K \leq G$, $H \vee K = \langle H, K \rangle$ and $H \wedge K = H \cap K$.

Given $H \leq G$, define

$$\mathcal{O}_G(H) = \{K \leq G : H \leq K\}.$$

Call $\mathcal{O}_G(H)$ the set of *overgroups* of H in G. Observe $\mathcal{O}_G(H)$ is the interval $[H, G]$ in Λ, and hence is a sublattice of Λ.

(5.3) *Let Λ be a lattice and $X = \{x_1, \ldots, x_n\}$ a finite subset of Λ. Then*

(1) X has a least upper bound $x_1 \vee \cdots \vee x_n$ and a greatest lower bound $x_1 \wedge \cdots \wedge x_n$.

(2) $(x_1 \vee x_2) \vee x_3 = x_1 \vee x_2 \vee x_3 = x_1 \vee (x_2 \vee x_3)$ and $(x_1 \wedge x_2) \wedge x_3 = x_1 \wedge x_2 \wedge x_3 = x_1 \wedge (x_2 \wedge x_3)$.

(3) $\vee$ and $\wedge$ are associative operations.

(4) If Λ is finite then Λ has a greatest element ∞ and a least element 0.

Proof. Exercise 5.1.

Theorem 5.4. *(Palfy-Pudlak) The following are equivalent:*

(1) Each nonempty finite lattice is isomorphic to an interval in the lattice of subgroups of some finite group.

(2) Every finite lattice is isomorphic to a congruence lattice of a finite algebra.

Proof. The proof appears in [PP]. See the expository article [G] for more discussion of lattices and in particular representations of lattices as congruences lattices of algebras.

The Palfy-Pudlak Theorem suggest the following question:

Palfy-Pudlak Question 5.5. Is each nonempty finite lattice isomorphic to a an overgroup lattice $\mathcal{O}_G(H)$ for some finite group G and subgroup H of G?

The answer to the Palfy-Pudlak Question is almost certainly negative. However the question has remained open for almost 30 years since the paper of Palfy and Pudlak was published. There are however several approaches to proving the question has a negative answer. In each approach one defines a class $\mathcal{C}$ of finite lattices, and attempts to show no lattice in $\mathcal{C}$ is isomorphic to an overgroup lattice $\mathcal{O}_G(H)$ for any finite group G and subgroup H. To do so, one "reduces" the problem to the case where G is almost simple. More precisely, one shows that no lattice in $\mathcal{C}$ is an overgroup lattice, *if* some suitable set of properties of almost simple groups can be verified. Of course one of those properties is that no lattice in $\mathcal{C}$ is of the form $\mathcal{O}_G(H)$ with G almost simple and H a subgroup of G.

We now consider two classes of lattices which are candidates for such an approach.

Definition 5.6. Let Λ be a finite lattice and set $\Lambda' = \Lambda - \{0, \infty\}$. Regard Λ as a graph, where the adjacency relation is the comparability relation on Λ. Define Λ to be *disconnected* if the graph Λ' is disconnected.

Example 5.7 Given a positive integer n, define $\mathbf{M}_n$ to be the lattice Λ such that the graph Λ' has n elements but no edges. An *M-lattice* is a lattice of the form $\mathbf{M}_n$ for some positive integer n. Notice that (with the exception of $\mathbf{M}_1$) M-lattices are disconnected.

Example 5.8 Given a positive integer m, write $\Delta(m)$ for the lattice of all subsets of a set of order m. Define a *$D\Delta$-lattice* to be a lattice such that Λ' has $r > 1$ connected components Λ'_i, $1 \le i \le r$, and for each i, $\Lambda'_i \cong \Delta(m_i)'$ for some integer $m_i > 2$. Again $D\Delta$-lattices are disconnected.

There are an infinite number of M-lattices which are intervals in subgroup lattices. For example if p is a prime and $G \cong E_{p^2}$, then $\mathcal{O}_G(1) \cong \mathbf{M}_{p+1}$. However it seems to be the consensus that the set of integers n such that $\mathbf{M}_n$ is an interval lattice is fairly sparse. Moreover there is a reduction by Baddeley and Luccini in [BL] which shows that for a given n, $\mathbf{M}_n$ is not an interval lattice if four or five questions about almost simple groups have positive answers.

Conjecture 5.9. (Aschbacher-Shareshian) No $D\Delta$-lattice is isomorphic to a lattice $\mathcal{O}_G(H)$ for G a finite group and H a subgroup of G.

Moreover there is a reduction theorem for this conjecture. To describe that reduction, we need to define the notion of a lower signalizer lattice.

Definition 5.10. Let L be a nonabelian finite simple group. Define $\mathcal{T}(L)$ to be the set of triples $\tau = (G, H, I)$ such that:

(T1) G is a finite group and $I \trianglelefteq H \le G$, and

(T2) $F^*(H/I) \cong L$.

Assume $\tau \in \mathcal{T}(L)$ and define $\mathcal{W}$ to consist of those H-invariant subgroups W of G such that $W \cap H = I$ and $W \le IF^*(G)$. Call $\mathcal{W}$ the set of *signalizers* for H in G. Partially order $\mathcal{W}$ by inclusion, and write $\Xi(\tau)$ for the poset obtained by adjoining a greatest element ∞ to $\mathcal{W}$. Observe $\Xi(\tau)$ is a lattice. Call such lattices $\Xi(\tau)$ *lower signalizer lattices.*

Theorem 5.11. *Assume Λ is a $D\Delta$-lattice which is isomorphic to an overgroup lattice in some finite group. Then there exists an almost simple group G such that either:*

(1) $\Lambda \cong \mathcal{O}_G(H)$ for some $H \leq G$, or

(2) There exists a nonabelian finite simple group L and $\tau = (G, H, I) \in \mathcal{T}(L)$ such that $\Lambda \cong \Xi(\tau)$ and $G = \langle \mathcal{W}, H \rangle$.

Proof. The proof appears in [A1] and [A2]. This result is the reduction theorem for Conjecture 5.9.

The class of M-lattices is of interest beyond the context of the Palfy-Pudlak Question.

Definition 5.12. Let G be a finite group. Define a subgroup H of G to be of *depth d* in G if d is the maximal length of a chain in the poset $\mathcal{O}_G(H)$. Observe that the maximal subgroups of G are the subgroups of depth 1, while H is of depth 2 iff $\mathcal{O}_G(H)$ is an M-lattice.

Question 5.13. What are the subgroups H of depth 2 in the almost simple groups, and what are the possible M-lattices $\mathcal{O}_G(H)$ that can occur when G is almost simple? In particular describe the subgroups of depth 2 and the corresponding M-lattices in the alternating and symmetric groups.

Exercises for Section 5

1. Prove Lemma 5.3.

Section 6. Primitive permutation groups.

In this section Ω is a finite set of order n and $S = Sym(\Omega)$ is the symmetric group on X.

We use the cycle notation for describing elements of S. In particular recall a *transposition* is a permutation with one cycle of length 2 and $n-1$ cycles of length 1.

Notation 6.1. For $g \in S$ we write $Fix(g)$ for the set of fixed points of g on Ω, and $Mov(g)$ for the set of points moved by g. Here g fixes $\omega \in \Omega$ if $\omega g = \omega$ and g moves ω if $\omega g \neq \omega$. More generally if g acts on $\Delta \subseteq \Omega$, let $Fix_\Delta(g)$ and $Mov_\Delta(g)$ be the set of fixed points of g on Δ and the set of points of Δ moved by g, respectively. Observe that if $h, g \in S$ with $Mov(g) \cap Mov(h) = \varnothing$, then $gh = hg$.

Recall S is is generated by its transpositions. Further a permutation is an *even* permutation if it is the product of an even number of transpositions, and an *odd* permutation if it is the product of an odd number of transpositions. Recall also that a permutation can't be both even an odd (cf. 15.5 in [FGT]), so that the set $Alt(\Omega)$ of all even permutations is a subgroup of S of index 2. This subgroup is the *alternating group* of degree n, and is normal as subgroups of index 2 are always normal. In this section we write A for $Alt(\Omega)$.

Moreover it turns out:

(6.2) *A permutation is even iff it has an even number of cycles of even length.*

The *cycle structure* of g is the function $Cyc_g : \mathbf{N} \to \mathbf{N}$ counting the number of cycles of G of length m for each natural number m. It is an easy exercise to show

(6.3) *Two permutations are conjugate in S iff they have the same cycle structure.*

Let $\pi : G \to S$ be a permutation representation of G on Ω. We recall:

(6.4) *If $\pi : G \to Sym(\Omega)$ is a transitive representation then* $\ker(\pi) = \ker_H(G)$ *is the largest normal subgroup of G contained in H.*

Proof. See 5.7 in [FGT].

Definition 6.5. The representation π is *semiregular* if $Fix(g) = \varnothing$ for each $g \in G$. Equivalently the restriction of G to each of its orbits is equivalent to the *regular representation* of G: the representation of G by right multiplication on itself. We say G is *regular* if it is transitive and semiregular. A *regular normal subgroup* of G is a normal subgroup of G which is regular on X.

(6.6) *(1) If $K \trianglelefteq G$ and H is a complement to K in G then K is a regular normal subgroup in the representation of G on G/H.*

(2) If K is a regular normal subgroup of G in its action on Ω then for $\omega \in \Omega$, G_ω is a complement to K in G and the map $\alpha : K \to \Omega$ defined by $\alpha : k \to \omega k$ is an equivalence of the representation of G_ω on K via conjugation with its representation on Ω.

Proof. Assume the hypotheses of (2). As K is regular on Ω, the map α is a bijection. For $h \in G_\omega$ and $k \in K$,

$$(k^h)\alpha = (\omega k^h) = \omega h^{-1} k h = \omega k h = (\omega\alpha)h$$

so the map α is a permutation equivalence.

Recall the partitions of Ω are in 1-1 correspondence with the equivalence relations on Ω, with an equivalence relation corresponding to the partition defined by the equivalence classes of the relation. The equivalence relation (or partition) is *trivial* if either all elements of Ω are equivalent or all equivalence classes are of order 1.

Definition 6.7. Our group G is said to be *primitive* on Ω if G is transitive and it preserves no nontrivial equivalence relation. That is there is no nontrivial partition of G such that G permutes the blocks of the partition.

Remark 6.8. Recall from Example 2.2 in Section 2 that the primitive representations are the irreducible permutation representations. Namely the subobjects in the category of sets are the subsets, so G is transitive iff it preserves no proper nontrivial subobjects, and all equivalence relations on X are admissible in the category of sets so G is irreducible iff G is primitive.

(6.9) *Let G be transitive on Ω with $n > 1$ and $\omega \in \Omega$. Then*

(1) G is primitive iff G_ω is a maximal subgroup of G.

(2) If G preserves a nontrivial partition P of Ω, then $n = |P| \cdot |B|$, where $B \in P$ is a block in P.

(3) If n is prime then G is primitive.

Proof. Under the hypotheses of (2), as G is transitive on Ω and permutes the blocks in P, G is transitive on P, so all blocks of P are of order $|B|$. Thus (2) holds and of course (2) implies (3). Further ω is contained in some $B \in P$, and for $g \in G_\omega$, $\omega = \omega g \in B \cap Bg$, so as G permutes P, $B = Bg$ and hence $G_\omega \leq N_G(B)$. Further the same argument shows $N_G(B)$ is transitive on B, so as $B \neq \Omega$, $G_\omega \neq N_G(B)$. Also

$$1 < |P| = |G : N_G(B)|$$

so $G \neq N_G(B)$ and hence G_ω is not maximal in G.

Conversely if G_ω is not maximal in G then $G_\omega < H < G$. Let $B = \omega H$ and $P = \{Bg : g \in G\}$. To complete the proof we show P is a nontrivial G-invariant partition of Ω. First as G is transitive on X, Ω is transitive on P and $\Omega = \bigcup_{g\in G} Bg$. Suppose $\beta \in B \cap Bg$; to show P is a partition we must show $B = Bg$. But as H is transitive on B, $\beta h = \omega$ for some $h \in H$, so

$$\omega = \beta h \in Bh \cap Bgh = B \cap Bgh$$

so replacing g by gh and β by ω, we may assume $\omega = \beta \in Bg$. Thus $\omega = \omega kg$ for some $k \in H$. But then $kg \in G_\omega \leq H$, so $g \in k^{-1}H = H$ and hence $Bg = B$, as desired. This also shows $N_G(B) = H$. Finally

$$|B| = |H : G_\omega| > 1$$

and

$$|P| = |G : N_G(H)| = |G : H| > 1$$

so P is nontrivial.

(6.10) *Let $H \trianglelefteq G$. Then*

(1) If M is a maximal subgroup of G and $H \not\leq M$ then $G = HM$.

(2) If G is faithful and primitive on X and $H \neq 1$ then $H \not\leq G_\omega$ and H is transitive on X.

Proof. (1) As $H \trianglelefteq G$, $HM \leq G$. Then as $H \not\leq M$, $M < HM$, so $HM = G$ by maximality of M.

(2) Let $M = G_\omega$; as G is primitive, M is a maximal subgroup of G of 6.9.1. As G is faithful and transitive, M contains no nontrivial normal subgroup of G. Thus as $H \neq 1$, $H \not\leq M$ and $G = MH$ by (1). But then H is transitive; ie. $\Omega = \omega G = \omega MH = \omega H$.

Definition 6.11. Recall G is *2-transitive* on Ω if G is transitive on ordered pairs of distinct points of Ω. Recall also (cf. 15.14 in [FGT]) that 2-transitive groups are primitive.

Theorem 6.12. *(Jordon) Assume G is primitive on Ω and Δ is a subset of Ω such that $0 < |\Delta| < n - 1$ and G_Δ is transitive on $\Omega - \Delta$. Then*

(1) G is 2-transitive on Ω.

(2) If G_Δ is primitive on $\Omega - \Delta$ then for $\omega \in \Omega$, G_ω is primitive on $\Omega - \{\omega\}$.

Proof. See 15.17 in [FGT].

(6.13) *Assume G is primitive on Ω. Then*

(1) If G contains a transposition then $G = S$.

(2) If G contains a 3-cycle then $G = A$ or S.

Proof. These are consequences of Jordon's Theorem 6.12. For example if $t = (\alpha, \beta)$ is a transposition in G, then apply 6.12 with $\Delta = \Omega - \{\alpha, \beta\}$. By 6.12.1, G is 2-transitive on Ω, so as G contains one transposition, it contains all transpositions. Then as S is generated by its transpositions, (1) holds. Part (2) is Exercise 5.6.2 in [FGT].

Exercises for Section 6

1. Let G be a transitive permutation group on a finite set Ω and $\omega \in \Omega$. Prove:

(1) $N_G(G_\omega)$ is transitive on $Fix(G_\omega)$.

(2) Assume G is k-transitive on Ω and let Δ be a k-subset of Ω. Then $N_G(G_\Delta)$ is k-transitive on $Fix(G_\Delta)$.

2. Let Ω be a finite set and D a regular abelian subgroup of $S = Sym(\Omega)$. Prove $D = C_S(D)$.

3. Let Ω be a finite set of order $n \geq 5$, and H a subgroup of $S = Sym(\Omega)$ such that H has two orbits θ and $\Gamma = \Omega - \theta$ on Ω, and $|\theta| = 2$. Assume $G \in \mathcal{O}_S(H)$ is transitive on Ω. Then one of the following holds:

(1) $\{\theta g : g \in G\}$ is a G-invariant partition of Ω.

(2) G is 3-transitive on Ω.

(3) H_θ has two orbits Γ_1 and Γ_2 on Γ, interchanged by H, and either:

(i) G is 2-transitive on Ω and G_θ acts on Γ_1 and Γ_2, or

(ii) setting $\theta = \{\alpha_1, \alpha_2\}$ and $\Omega_i = \{\alpha_i\} \cup \Gamma_i$, $P = \{\Omega_1, \Omega_2\}$ is a G-invariant partition of Ω such that $N_G(\Omega_i)$ is 2-transitive on Ω_i for $i = 1, 2$.

(Hint: You may use the theory of rank 3 permutation groups in section 16 of [FGT].)

Section 7. Partitions, equivalence relations, and chamber systems

In this section we assume that Ω is a finite set and let $S = Sym(\Omega)$ be the symmetric group on Ω and $A = Alt(\Omega)$ the alternating group on Ω.

We begin our study of the subgroup structure of S and A by investigating the subgroups of G stabilizing suitable relations or families of relations on Ω. In particular such subgroups are candidates for maximal subgroups of S and/or A. This leads us to the following definition:

Definition 7.1. Let R be an m-ary relation on Ω; that is R is a subset of the set product of m copies of Ω. Observe that S permutes the m-ary relations on Ω via $s : R \mapsto Rs$ for $s \in S$. The *stabilizer* $N_S(R)$ of R in S is the subgroup of all $g \in S$ such that $Rg = R$.

Definition 7.2. Write $\mathcal{P} = \mathcal{P}(\Omega)$ for the set of partitions of Ω. Each $P \in \mathcal{P}$ determines an equivalence relation $\sim_P$ on Ω, whose equivalence classes are the blocks of P. Of course in the other direction, P is also determined by $\sim_P$ as the set of equivalence classes of the relation.

The *stabilizer* $N_S(P)$ of P in S is the stabilizer in S of the relation $\sim_P$. For $B \in P$, set $\kappa_B = A_{\Omega - B}$ and $\kappa(P) = \langle \kappa_B : B \in P \rangle$. Observe κ_B acts faithfully as $Alt(B)$ on B, and $\kappa(P) = \prod_{B \in P} \kappa_B$ is a direct product.

A partition P is a *regular* (m,k)*-partition* if P has k blocks, each of size m. An equivalence relation is *regular* if its partition is regular.

Define a partial order on $\mathcal{P}$ by $P \leq Q$ if Q is a refinement of P. Equivalently, if $\alpha, \beta \in \Omega$ and $\alpha \sim_Q \beta$ then also $\alpha \sim_P \beta$.

Write 0 for the member of $\mathcal{P}$ with a unique block Ω, and set

$$\infty = \{\{\omega\} : \omega \in \Omega\} \in \mathcal{P}.$$

Thus 0 is the least element and ∞ the greatest element of the poset $\mathcal{P}$. Set $\mathcal{P}' = \mathcal{P}'(\Omega) = \mathcal{P} - \{0, \infty\}$. Thus $\mathcal{P}'$ is the set of nontrivial partitions of Ω.

For $G \leq S$, set $\mathcal{P}(G) = \{P \in \mathcal{P}' : G \leq N_S(P)\}$. Thus $\mathcal{P}(G)$ is the set of nontrivial G-invariant partitions of Ω.

If $Q \leq P$ and $B \in Q$, set $P_B = \{C \in P : C \subseteq B\}$ and observe that $P_B \in \mathcal{P}(B)$, and $Q/P = \{P_B : B \in Q\} \in \mathcal{P}(P)$.

(7.3) *$\mathcal{P}$ is a lattice.*

Proof. For $P, Q \in \mathcal{P}$,

$$P \vee Q = \{A \cap B : A \in P,\ B \in Q, \text{ and } A \cap B \neq \varnothing\},$$

while $P \wedge Q$ is the partition such that $\sim_{P \wedge Q}$ is the equivalence relation generated by $\sim_P$ and $\sim_Q$.

Definition 7.4. Let I be a finite set. A *chamber system* X on Ω over I is a collection $X = \{\sim_i : i \in I\}$ of equivalence relations on Ω. The members of Ω are called the *chambers* of X. A *morphism* $\alpha : X \to X'$ of chamber systems over I is a function $\alpha : \Omega \to \Omega'$ preserving all the relations; that is for each $i \in I$, if $\omega, \lambda \in \Omega$ with $\omega \sim_i \lambda$ then $\omega\alpha \sim_i \lambda\alpha$.

The *stabilizer* of X is the subgroup

$$N_S(X) = \bigcap_{i \in I} N_S(\sim_i).$$

The *rank* of the chamber system X is the order of I. The definitions are due to Tits; cf. section 2.1 in [T]).

In the terminology of [T], X is *connected* if $\sim_1 \wedge \cdots \wedge \sim_r = 0$, where $I = \{1, \ldots, r\}$; ie. the equivalence relation generated by the relations $\sim_i$, $i \in I$, is the trivial relation 0. Define X to be *injective* if $\sim_1 \vee \cdots \vee \sim_r = \infty$; that is for each $\omega \in \Omega$, if $[\omega]_i$ is the equivalence class of $\sim_i$ containing ω, then $[\omega]_1 \cap \cdots \cap [\omega]_r = \{\omega\}$.

Define X to be *regular* if all the partitions $\sim_i$, $i \in I$, are regular.

Define a *CIR-chamber system* to be a connected, injective, regular rank 2 chamber system on Ω.

For $J \subseteq I$, write $\sim_J$ for the equivalance relation $\wedge_{j \in J} \sim_j$ generated by $\{\sim_j : j \in J\}$. For $\omega \in \Omega$, write $[\omega]_J$ for the equivalence class of ω under $\sim_J$. Write $J' = I - J$ for the complement to J in I. Define X to be *nondegenerate* if for each $\omega \in \Omega$ and each $j \in I$,

$$\{\omega\} = \bigcap_{i \in I} [\omega]_{i'} \text{ and } [\omega]_j = \bigcap_{i \in j'} [\omega]_{i'}.$$

Example 7.5. Let V be an $r+1$-dimensional vector space and $I = \{1, \ldots, r\}$. Form the projective geometry $PG(V)$ of V. Thus $PG(V)$ is the simplicial complex whose vertices are the proper nonzero subspaces of V, with simplices the chains in the poset of such subspaces ordered by inclusion. The maximal simplices are the chains $(V_1 < \cdots < V_r)$ with $\dim(V_i) = i$. Form the chamber system $X = \mathbf{X}(PG(V))$ over I whose chambers are the maximal simplices, and with $(V_1 < \cdots < V_r) \sim_i (U_1 < \cdots < U_r)$ iff $U_j = V_j$ for all $i \neq j$. Then X is connected and nondegenerate.

Observe that, as defined in Exercise 7.4, $PG(V)$ is a geometric chamber system with type function $\tau(U) = \dim(U)$, and X is indeed the image of $PG(V)$ under the functor $\mathbf{X}$ defined in Exercise 7.4. Moreover by the same exercise, $\boldsymbol{\mathcal{C}}(X)$ is naturally isomorphic to $PG(V)$ as a geometric complex.

(7.6) *Let $P, Q \in \mathcal{P}$ and assume $H \leq N_S(P) \cap N_S(Q)$ is transitive on Ω. Then*

(1) P, Q, $P \vee Q$, and $P \wedge Q$ are regular partitions.

(2) H acts on $P \vee Q$ and $P \wedge Q$.

(3) For $B \in Q$, $N_H(B)$ is transitive on B, and if $Q \leq P$ then $N_H(B) \leq N_S(P_B)$, so P_B is a regular partition of B.

(4) If $P \vee Q = \infty$ then for $B \in P \wedge Q$, $\rho_B = (P_B, Q_B)$ is a CIR-chamber system on B, and $N_H(B) \leq N_S(\rho_B)$.

(5) If $P \vee Q = \infty$ and $P \wedge Q = 0$, then $\rho = (P, Q)$ is a CIR-chamber system on Ω and $H \leq N_S(\rho)$.

Proof. As S is a group of automorphisms of the lattice $\mathcal{P}$, and as H acts on P and Q, (2) holds.

As $H \leq N_S(Q)$, H permutes the blocks of Q. Then as H is transitive on Ω, H is transitive on the blocks of Q, and for $B \in Q$, $N_H(B)$ is transitive on B. As H is transitive on Q, Q is regular. Similarly P is regular, as are $P \vee Q$ and $P \wedge Q$ by (2). Thus (1) holds.

Suppose $Q \leq P$ and let $B \in Q$. Recall $P_B \in \mathcal{P}(B)$. As P and B are $N_H(B)$-invariant, so is P_B, so P_B is regular by (1). Thus (3) holds.

Suppose $P \vee Q = \infty$ and let $B \in P \wedge Q$. By (3), P_B and Q_B are regular. As $B \in P \wedge Q$, we have $P_B \wedge Q_B = 0$, and then as $P \vee Q = \infty$, also $P_B \vee Q_B = \infty$. Thus (4) follows, and then (4) implies (5).

We now begin to investigate the question of when the stabilizer in $T \in \{S, A\}$ of a nontrivial partition $P \in \mathcal{P}$ is maximal in T. Trivially a necessary condition is that either $|P| = 2$, or P is regular. It will turn out that these conditions are usually also sufficient. When P is regular, $\kappa(P)$, S_P, and A_P are normal subgroups of $N_S(P)$, and we also go a long way toward determining the maximal overgroups of these subgroups in S and A. Recall from the introduction that results of that sort are very useful in investigating the lattice of subgroups of S and A.

(7.7) *Let $\Delta \subseteq \Omega$ and assume either*

(i) $|\Delta| \geq 2$ and set $K = S_{\Omega - \Delta}$, or

(ii) $|\Delta| > 2$ and set $K = A_{\Omega - \Delta}$.

Let $G \in \mathcal{O}_S(K)$. Then

(1) If G is primitive on Ω then $A \leq G$.

(2) If $Q \in \mathcal{P}(G)$ then $K \leq G_Q$ and Δ is contained in some block B of Q.

(3) If $\Delta \subseteq B \subseteq \Omega$ and $N_G(B)$ is primitive on B then $\kappa_B \leq G$.

Proof. Observe that in (ii), $|\Delta| > 2$ and K acts faithfully as $Alt(\Delta)$ on Δ, so K is generated by 3-cycles t. Similarly in (i), K is generated by transpositions t. Now $t \in K \leq G$, so (1) follows from 6.13.

Assume $Q \in \mathcal{P}(G)$. Let $B \in Q$. Then either t acts on Q or the orbit of $\langle t \rangle$ on Q containing B is of order $|t|$. In the latter case, $|t| = |Mov(t)| \geq |t||B| > |t|$, a contradiction. Therefore t acts on B, and then as $\langle t \rangle$ is transitive on $Mov(t)$, B is contained in $Fix(t)$ or $Mov(t) \subseteq B$. Now K is generated by 3-cycles or transpositions, which all act on each B in Q, so $K \leq N_S(B)$. Indeed there exists $B \in Q$ such that t moves a point of B, so $Mov(t) \subseteq B$. Then as K acts on B and is transitive on Δ, $\Delta = Mov(t)K \subseteq B$. This establishes (2).

Finally assume the hypothesis of (3). Applying (1) to $N_G(B)^B$ in the role of G, we conclude that $N_G(B)^B$ contains $Alt(B)$. Thus $\kappa_B \leq \langle t^{N_G(B)} \rangle \leq G$, so (3) holds.

(7.8) *Let $P \in \mathcal{P}'$ be regular. Assume either*

(i) $|P| < n/2$ and set $K = \kappa(P)$, or

(ii) K is the kernel S_P of the action of $N_S(P)$ on P.

Let $G \in \mathcal{O}_S(K)$. Then

(1) If G is primitive on Ω then $A \leq G$.

(2) P is the greatest member of $\mathcal{P}(K)$, and $K \leq G_Q$ for each $Q \in \mathcal{P}(K)$.

(3) If G is transitive on Ω and $A \not\leq G$, then $\mathcal{P}(G)$ has a greatest member Q. Moreover $P = Q$ if $|Q| = n/2$ while $\kappa(Q) \leq G$ if $|Q| < n/2$.

(4) Suppose G is transitive on P, let $\Gamma(G)$ be the set of G-invariant partitions of P, and for $\gamma \in \Gamma(G)$ define $P(\gamma)$ to be the partition of Ω with blocks B_σ, $\sigma \in \gamma$, where

$$B_\sigma = \bigcup_{\Delta \in \sigma} \Delta.$$

Then the map $\gamma \mapsto P(\gamma)$ is a bijection of $\Gamma(G)$ with $\mathcal{P}(G)$ and $\{N_{AG}(Q) : Q \in \mathcal{P}(G)\}$ is the set of maximal overgroups of G in AG.

Proof. We apply 7.7 to a block Δ of P. Then (1) is immediate from 7.7.1, while if $Q \in \mathcal{P}(K)$ then by 7.7.2, each block Δ of P is contained in some block of Q. Thus $Q \leq P$, establishing (2). Similarly under the hypothesis of (3), $\mathcal{P}(G) \neq \varnothing$ by (1), and for each $Q \in \mathcal{P}(G)$ and $B \in Q$, B contains some block Δ of P, so by 7.7.3, $\kappa_B \leq G$. Thus $\kappa(Q) \leq G$. Further $Q \leq P$ by (2), so if $|Q| = n/2$, then $P = Q$ and hence P is the greatest member of $\mathcal{P}(G)$. On the other hand if $|Q| < n/2$ then as $\kappa(Q) \leq G$, we conclude from (2) that Q is the greatest member of $\mathcal{P}(G)$. This completes the proof of (3).

Finally assume the hypothesis and notation of (4). By (2), the map $\gamma \mapsto P(\gamma)$ is a bijection of $\Gamma(G)$ with $\mathcal{P}(G)$, while by (1), $\{N_{AG}(Q) : Q \in \mathcal{P}(G)\}$ is the set of maximal overgroups of G in AG.

(7.9) *Assume $n > 2$ and let $P \in \mathcal{P}$ be regular with blocks of size 2. Then*

(1) P is the greatest member of $\mathcal{P}(A_P)$.

(2) If $M \in \mathcal{O}_S(A_P)$ acts primitively on P then $\mathcal{P}(M) = \{P\}$.

Proof. Let $K = A_P$ and $Q \in \mathcal{P}(K)$. We first prove (1), so we must show $Q \leq P$. This holds if $K \leq A_Q$ as P is the set of orbits of K on Ω. Thus we may assume $K \not\leq A_Q$. Let $\mathcal{T} = \{t \in K : |Mov(t)| = 4\}$. Then $K = \langle \mathcal{T} \rangle$, so there is $t \in \mathcal{T} - M_Q$. We conclude from the proof of 7.7 that $|Q| = n/2$ and $t^Q = (\alpha, \beta)$ is a transposition with $Mov(t) = \alpha \cup \beta$. As $n > 4$ there is $s \in \mathcal{T}$ with $\gamma = Mov(t) \cap Fix(s)$ of order 2. Then $\gamma \subseteq \alpha \cup \beta$, so we may take $\alpha \cap \gamma \neq \varnothing$, so s acts on α and then $\alpha = \gamma$. But now as $[s, t] = 1$, t acts on $Fix_{Mov(t)}(s) = \alpha$, a contradiction. This establishes (1).

Now assume the hypothesis of (2), and let $Q \in \mathcal{P}(M)$. By (1), $Q \leq P$. Then from 7.2, $Q/P = \{P_B : B \in Q\}$ is an M-invariant partition of P, so as M is primitive on P, Q/P is trivial, and hence $P = Q$.

Theorem 7.10. *Let $P \in \mathcal{P}'$ be regular and let T be S or A. Then either*

(1) $N_T(P)$ is maximal in T, or

(2) $n = 8$, $|P| = 4$, $T = A$, $N_A(P)$ is a minimal parabolic subgroup of $A \cong L_4(2)$, and $\mathcal{O}_A(N_A(P)) = \{N_A(P), M_1, M_2, A\}$, where M_1 and M_2 are maximal parabolic subgroups of A and stabilizers of affine structures on Ω.

Proof. First observe that $N_S(P)$ contains a transposition t with $Mov(t) \subseteq B \in P$, so

(a) $N_S(P) \not\leq A$.

Next as P is nontrivial, $n \geq 4$, and when $n = 4$ we have $|P| = 2$ and (1) holds trivially. Thus we may assume:

(b) $n > 4$.

Let $M = N_T(P)$. If $|P| < n/2$ or $T = S$, then as M is primitive on P, applying 7.8.4 to M in the role of "G", we conclude that M is maximal in AM. But $T = AM$ by (a), so that (1) holds. Therefore we may assume:

(c) $T = A$ and $|P| = n/2$.

Suppose $n = 8$. Then $A \cong L_4(2)$ is of Lie type over $\mathbf{F}_2$, and we appeal to the theory of such groups; cf. [GLS3] or [FGT] sections 43 and 47. As $|M|_2 = 64 = |A|_2$, M contains a Sylow 2-subgroup U of A. As $A \cong L_4(2)$ is of Lie type over $\mathbf{F}_2$, U is a Borel subgroup of A, so if follows from the theory of groups of Lie type (cf. 43.7.2 in [FGT]) that M is a parabolic subgroup of A. Indeed as $|M : U| = 3$, M is a minimal parabolic, so as A is of Lie rank 3 there are exactly two maximal parabolics M_1 and M_2 over M (cf. 4.3.7.2 in [FGT]). Indeed one can check that each M_i is a split extension of $O_2(M_i) \cong E_8$ by $L_3(2)$, and that $O_2(M_i)$ is regular on Ω, so from 8.4, M_i is the stabilizer of an affine structure on Ω. We have shown that conclusion (2) of Theorem 7.9 holds in this case, so we may assume:

(d) $n \neq 8$.

It remains to show that M is maximal in A, so we may assume G is a maximal overgroup of M in A and $M < G$. As $M < G$, $P \notin \mathcal{P}(G)$, so as M is primitive on P we conclude from 7.9.2 that:

(e) G is primitive on Ω.

If $T = S$, then $M \not\leq A$ by (a), so in any event as G is proper in T, we conclude that

(f) $A \not\leq G$.

We may assume $\Omega = \{1, \dots, n\}$ and $P = \{\{1,2\}, \{3,4\}, \dots\}$. Then (using (b) and (c)), $t = (1,2)(3,4)$ and $s = (1,2)(5,6)$ are in $K = A_P$, and $r = (1,3)(2,4) \in M$. Thus $E_4 \cong R = \langle t, r\rangle \leq M$ with $Mov(R) = \theta = \{1,2,3,4\}$. Applying Jordon's Theorem 6.12 to $\Delta = \Omega - \theta$, we conclude that G is 2-transitive on Ω. Indeed $M_{1,2}$ is transitive on $\Gamma = \Omega - \{1,2\}$, so

(g) G is 3-transitive on Ω.

(h) $G_{1,2} = M_{1,2}$.

For t^Γ is a transposition, so by 6.13.1, either $G^\Gamma = Sym(\Gamma)$ or G^Γ is imprimitive. But M^Γ is the stabilizer of the partition $P' = P - \{1,2\}$ of Γ, so as we've proved 7.10 in the case where $T = S$, it follows that $G^\Gamma = M^\Gamma$ when G^Γ is imprimitive, so that (h) holds in that case. However if $G^\Gamma = Sym(\Gamma)$, then $G_{1,2}$ acts faithfully as $Alt(\Gamma)$ on Γ, so by (b), G contains a 3-cycle, contrary to (e), (f), and 6.13.2. This completes the proof of (h).

It follows from (h) that $Fix(G_{1,2,3}) = \theta$, so by (g) and Exercise 6.1, G^θ is 3-transitive. Then as s^θ is a transposition, $G^\theta = Sym(\theta)$. In particular there is $g \in N_G(\theta)$ such that $u = s^g = (1,3)(\alpha,\beta)$ for some $\alpha, \beta \in \Delta$. Then $v = su = (1,2,3) \cdot w$, where $w = (5,6)$, $(5,6)(\alpha,\beta)$, or $(5,6,\beta)$, when $(\alpha,\beta) = (5,6)$, $\alpha,\beta > 6$, or $\alpha = 5$ and $\beta > 6$, respectively. In the first two cases, G contains a 3-cycle, contrary to (e), (f), and 6.13.2. In the third case, we may take $\beta = 7$, so $v^\Delta = (5,6,7) \notin M^\Delta$. But then as M^Δ is maximal in $Sym(\Delta)$ (as we've proved 7.10 when $T = S$), we have $G^\Delta = Sym(\Delta)$. Further $n \geq 10$ by (d), so $N_G(\theta)^\infty \leq G_\theta$ contains a 3-cycle, for our usual contradiction.

Theorem 7.11. *Let Γ be a proper nonempty subset of Ω, let $1 \neq T$ be S or A, and let $M = N_T(\Gamma)$. Then either*

(1) M is maximal in T, or

(2) $|\Gamma| = n/2$, and $N_T(P)$ is the unique maximal overgroup of M in T, where $P = \{\Gamma, \Omega - \Gamma\} \in \mathcal{P}$.

Proof. The proof is similar to that of 7.10, only much easier, and is left as Exercise 7.1.

(7.12) *Let $P = \{\Delta_1, \dots, \Delta_r\}$ be a regular (m, r)-partition and set $M = N_S(P)$, $K = S_P$, and for $1 \leq i \leq r$, set $K_i = S_{\Omega - \Delta_i}$. Then*

(1) K_i acts faithfully on Δ_i as $Sym(\Delta_i)$.

(2) $K = K_1 \times \cdots \times K_r$.

(3) There is a complement T to K in M, and a faithful representation of T on $I = \{1, \dots, r\}$, so that T acts on P and $\{K_1, \dots, K_r\}$ via $K_i^t = K_{it}$ and $\Delta_i^t = \Delta_{it}$. Moreover T_i centralizes K_i and Δ_i.

Proof. Exercise 7.2.

Exercises for Section 7

1. Prove Theorem 7.11.

2. Prove Lemma 7.12.

3. Let G be a group, H a subgroup of G, $I = \{1, \dots, m\}$ a finite set, and $\mathcal{P} = (P_i : i \in I)$ a family of overgroups of H in G. For $J \subseteq I$, set $P_J = \langle P_j : j \in J \rangle$, with $P_\varnothing = H$. Define a relation $\sim_J$ on G/H by $Hu \sim_J Hv$ iff $vu^{-1} \in P_J$. Define $X = X(G, H, \mathcal{P})$ to be the set G/H together with the relations $\sim_i$, $i \in I$, regarded (after part (1) of this problem) as a chamber system on G/H over I. Prove:

(1) For each $J \subseteq I$, $\sim_J$ is an equivalence relation on G/H, so X is a chamber system on G/H over I.

(2) Under the representation π of G on G/H by right multiplication, G preserves $\sim_J$ for each $J \subseteq I$, so $G\pi \leq Aut(X)$.

(3) For $J \subseteq I$ and $\alpha \in G/H$, write $[\alpha]_J$ for the equivalence class of $\sim_J$ containing α. Then $[H]_J$ is the orbit of H under P_J, and P_J is the stabilizer in G of $[H]_J \in \sim_J$.

(4) For each $\varnothing \neq J \subseteq I$, $\sim_J$ is the equivalence relation generated by $\{\sim_j : j \in J\}$.

(5) X is injective iff $\bigcap_{i\in I} P_i = H$, and X is nondegenerate iff for each $j \in I$,

$$\bigcap_{i\in I} P_{i'} = \{H\} \text{ and } \bigcap_{i\in j'} P_{i'} = P_j.$$

(6) X is connected iff $G = \langle \mathcal{P} \rangle$.

(7) Assume $Y = \{\simeq_i : i \in I\}$ is a chamber system on Ω over I, and $\bar{G}$ is a group of automorphisms of Y transitive on Ω. Pick $\omega \in \Omega$, set $\bar{H} = \bar{G}_\omega$, and for $i \in I$, set $\bar{P}_i = N_{\bar{G}}([\omega]_i)$. Set $\bar{\mathcal{P}} = \{\bar{P}_i : i \in I\}$ and $\bar{X} = X(\bar{G}, \bar{\mathcal{P}})$. Define $\varphi : \Omega \to \bar{G}/\bar{H}$ by $\varphi : \omega g \mapsto \bar{H} g$ for $g \in \bar{G}$. Then $\varphi : Y \to \bar{X}$ is a $\bar{G}$-equivariant isomorphism of chamber systems.

4. Let $I = \{1, \ldots, m\}$ be a finite set. A *geometric complex* over I is a simplicial complex $\mathcal{C} = (V, \Sigma)$, together with a surjective function $\tau : V \to I$ on the vertex set V of $\mathcal{C}$, such that for each σ in the set Σ of simplices of $\mathcal{C}$, we have $\tau : \sigma \to I$ is an injection, and each simplex is contained in a simplex of order m. Define $\tau(\sigma)$ to be the *type* of σ. The *chambers* of $\mathcal{C}$ are the simplices of type I; write $\Omega(\mathcal{C})$ for the set of chambers of $\mathcal{C}$. A *morphism* $\alpha : \mathcal{C} \to \mathcal{C}'$ of complexes over I is a simplicial map $\alpha : V \to V'$ which preserves type.

Define $\mathbf{X}(\mathcal{C})$ to be the set $\Omega(\mathcal{C})$ together with the relations $(\sim_i : i \in I)$ on $\Omega(\mathcal{C})$ defined by $\omega \sim_i \lambda$ iff the subsimplices of ω and λ of type i' are the same. Define $\mathbf{X}(\alpha) : \mathbf{X}(\mathcal{C}) \to \mathbf{X}(\mathcal{C}')$ by $\omega \mathbf{X}(\alpha) = \omega\alpha$.

Let $X = (\simeq_i : i \in I)$ be a chamber system over I on a set Ω. Define $\boldsymbol{\mathcal{C}}(X) = (V(X), \Sigma(X)$ to be the geometric complex over I with

$$V(X) = \coprod_{i\in I} \Omega_{i'},$$

where $\Omega_{i'}$ is the set of equivalence classes of $\simeq_{i'}$, and $\sigma \subseteq V(X)$ is a simplex iff $\bigcap_{v\in\sigma} v \neq \varnothing$. Define $\tau(X)(v) = i$, where $v \in \Omega_{i'}$. Given a morphism $\beta : X \to X'$ of chamber systems over I, define $\boldsymbol{\mathcal{C}}(\beta) : \boldsymbol{\mathcal{C}}(X) \to \boldsymbol{\mathcal{C}}(X')$ by $\boldsymbol{\mathcal{C}}(\beta) : [\omega]_{i'} \mapsto [\omega\beta]_{i'}$.

Prove:

(1) $\mathbf{X}$ is a functor from the category $\mathfrak{C}$ of geometric complexes over I to the category $\mathfrak{X}$ of chamber systems over I.

(2) $\boldsymbol{\mathcal{C}}$ is a functor from $\mathfrak{X}$ to $\mathfrak{C}$.

(3) For $J \subseteq I$ and $\sigma \in \Sigma(\mathcal{C})$ of type J', define $\sigma\varphi = \{\omega \in \Omega(\mathcal{C}) : \sigma \subseteq \omega\}$. Then $\sigma\varphi \in \sim_J = \{\sim_j : j \in J\}$ and $\sim_J = \{\sigma\varphi : \sigma \in \Sigma(\mathcal{C}) \text{ and } \tau(\sigma) = J'\}$.

(4) The map $\varphi : V \to V(\mathbf{X}(\mathcal{C}))$ defined in (3) is an isimorphism $\varphi : \mathcal{C} \to \boldsymbol{\mathcal{C}}(\mathbf{X}(\mathcal{C}))$ of geometric complexes over I.

(5) For $\omega \in \Omega$ set $\omega\psi = \{[\omega]_{i'} : i \in I\}$. Then $\psi : X \to \mathbf{X}(\boldsymbol{\mathcal{C}}(X))$ is a surjective morphism of chamber systems over I, and ψ is an isomorphism iff X is nondegenerate.

(6) $\mathbf{X}(\mathcal{C})$ is nondegenerate, so that $\mathbf{X}(\mathcal{C}) \cong \mathbf{X}(\boldsymbol{\mathcal{C}}(\mathbf{X}(\mathcal{C})))$.

(7) Let $T = \mathbf{X} \circ \boldsymbol{\mathcal{C}}$ regarded as a functor from the category $\mathfrak{X}$ to $\mathfrak{X}$. Then $\psi : 1 \to T$ is a natural transformation, where 1 is the identity functor on $\mathfrak{X}$; that is for $X, Y \in \mathfrak{X}$ and $\beta \in Mor(X, Y)$, $\beta\psi = \psi T(\beta)$.

(8) Let $\tilde{T} = \boldsymbol{\mathcal{C}} \circ \mathbf{X}$ regarded as a functor from $\mathfrak{C}$ to $\mathfrak{C}$. Then $\varphi : 1 \to \tilde{T}$ is a natural equivalence.

(9) Let $\mathfrak{Y}$ be the category of nondegenerate chamber systems over I. Then $\mathbf{X} : \mathfrak{C} \to \mathfrak{Y}$ and $\boldsymbol{\mathcal{C}} : \mathfrak{Y} \to \mathfrak{C}$ define equivalences of categories.

Section 8. Affine structures

In this section p is a prime, e is a positive integer, and Ω is a finite set of order p^e. Set $S = Sym(\Omega)$. We discuss affine structures on Ω. Recall the discussion of relations and their stabilizers in Definition 7.1.

Definition 8.1. An *affine structure* on Ω is a 4-ary relation $\mathcal{A} = \mathcal{A}(\Omega, +)$ on Ω of the form

$$\mathcal{A} = \{(a, b, c, b + c - a) : a, b, c \in \Omega\},$$

defined by some e-dimensional vector space structure $(\Omega, +)$ on Ω over the field $\mathbf{F}_p$ of order p. The *stabilizer* in S of the affine space structure $\mathcal{A}$ is the subgroup $N_S(\mathcal{A})$ of S defined in 7.1.

(8.2) *Let $V = (\Omega, +)$ be an e-dimensional vector space structure over $\mathbf{F}_p$, $\mathcal{A} = \mathcal{A}(V)$ the corresponding affine structure on Ω, and $M = N_S(\mathcal{A})$ the stabilizer of $\mathcal{A}$. Then*

(1) For each $a \in \Omega$, the translation $\tau_a : b \mapsto a + b$ is in M.

(2) $D = D(\mathcal{A}) = \{\tau_a : a \in \Omega\}$ is a subgroup of M isomorphic to E_{p^e}, and D is regular on Ω.

Proof. We first prove (1). Let $d \in \Omega$ and $\tau = \tau_d$. Then

$$\begin{aligned}(a, b, c, b + c - a)\tau &= (a + d, b + d, c + d, b + c - a + d)\\ &= (a + d, b + d, c + d, (b + d) + (c + d) - (a + d)) \in \mathcal{A},\end{aligned}$$

so $\tau \in M$, establishing (1).

Visibly $\tau_a\tau_b = \tau_{a+b}$, so $D = \{\tau_d : d \in \Omega\}$ is a subgroup of M, and the map $a \mapsto \tau_a$ is an isomorphim of D with the group V. Therefore $D \cong E_{p^e}$. For $a, b \in \Omega$, $a\tau_{b-a} = b$, so D is transitive on Ω, and visibly $D_0 = 1$, so D is regular on Ω, establishing (2).

Definition 8.3. Given an $\mathbf{F}_p$-space structure $(\Omega, +)$ on Ω, define the group $D(\mathcal{A}(\Omega, +))$ of 8.2.2 to be the *group of translations* of $\mathcal{A}(\Omega, +)$. Given $E_{p^e} \cong D \leq S$ with D regular on S, and given a base point $\omega \in \Omega$, define a relation $+ = +_{D,\omega}$ on Ω by $\omega a + \omega b = \omega ab$ for $a, b \in D$. This is well defined as D is regular on Ω, so for each $\alpha \in \Omega$, there is a unique $d \in D$ with $\alpha = \omega d$.

(8.4) *Let $E_{p^e} \cong D \leq S$ be regular on Ω and pick $\omega \in \Omega$. Then*

(1) $(\Omega, +_{D,\omega})$ is an $\mathbf{F}_p$-space with zero vector ω.

(2) $\mathcal{A}(D) = \mathcal{A}(\Omega, +_{D,\omega})$ is an affine structure on Ω independent of ω.

(3) $D = D(\mathcal{A})$ is the group of translations of $\mathcal{A}$.

(4) $N_S(\mathcal{A}(D)) = N_S(D)$.

(5) $N_S(D)_\omega$ is a complement to D in $N_S(D)$, and $N_S(D)_\omega$ acts faithfully on D as $GL(D)$.

Proof. Write D additively, and set $+ = +_{D,\omega}$. By construction the map $\varphi : d \mapsto \omega d$ is an isomorphism of the group D with $(\Omega+)$ in the category of sets with binary operation, so (1) holds. Thus $\mathcal{A} = \mathcal{A}(D)$ is an affine structure on Ω. If δ is a second base point, then $\delta = \omega d$ for some $d \in D$ and for $a, b, c \in D$,

$$(\delta a, \delta b, \delta c, \delta(b+c-a)) = (\omega(a+d), \omega(b+d), \omega(c+d), \omega((b+d)+(c+d)-(a+b))) \in \mathcal{A},$$

so $\mathcal{A}$ is independent of the choice of ω, establishing (2).

Part (3) is immediate from the definitions.

Set $G = N_S(D)$ and $J = G_\omega$. As D is abelian and regular on Ω, $D = C_S(D)$ by Exercise 6.2. Then by 1.13.2, $G/D \cong GL(D)$. As D is regular on Ω, J is a complement to D in G, so $J \cong GL(D)$, and as $D = C_S(D)$, J acts faithfully on D. This completes the proof of (5).

Let $T = N_S(\mathcal{A})$; it remains to show that $G = T$. Identifying Ω with D via the bijection φ, we may assume $\Omega = D$ and $\omega = 0$ is the identity of D. Subject to this identification, J acts on Ω via conjugation by 6.6.2. As D is regular on Ω, $T = T_0 D$, where T_0 is the stabilizer in T of 0. Thus it remains to show $T_0 = J$. First for $j \in J$,

$$(a, b, c, b+c-a)j = (a^j, b^j, c^j, (b+c-a)^j) = (a^j, b^j, c^j, b^j + c^j - a^j)$$

as j preserves addition on D. Thus $J \leq T_0$. Second, for $t \in T_0$,

$$(0, at, bt, (a+b)t) = (0, a, b, a+b)t \in \mathcal{A}$$

for all $a, b \in D$, as T preserves $\mathcal{A}$, so $(a+b)t = at + bt$, and hence $t \in J$. This completes the proof of (4).

(8.5) *Let $V = (\Omega, +)$ be an e-dimensional vector space structure over $\mathbf{F}_p$, $\mathcal{A} = \mathcal{A}(V)$ the corresponding affine structure on Ω, $M = N_S(\mathcal{A})$ the stabilizer of $\mathcal{A}$, $0 \in \Omega$ the zero vector in V, and D the group of translations of $\mathcal{A}$. Then*

(1) $F^*(M) = D \cong E_{p^e}$, and D is regular on Ω.

(2) The stabilizer M_0 of 0 in M is a complement to D in M, $M_0 \cong GL(V)$, and the map $a \mapsto \tau_a$ is an equivalence of the representation of M_0 on Ω with the representation of M_0 on D via conjugation.

(3) $\mathcal{A} = \mathcal{A}(D)$.

(4) The map $D \mapsto \mathcal{A}(D)$ is an S-equivariant bijection between the set of elementary abelian regular subgroups of S and the set of affine structures on Ω.

(5) S is transitive on its affine space structures.

Proof. By 8.2.2, $D \cong E_{p^e}$ is regular on Ω, so by 8.4, $\mathcal{B} = \mathcal{A}(D)$ is an affine structure on Ω. Consider the $\mathbf{F}_p$ space $U = (\Omega, \boxplus)$, where $\boxplus = +_{D,0}$, and $0 \in \Omega$ is the zero vector of V. Observe for $a, b \in \Omega$,

$$a + b = 0\tau_a + 0\tau_b = 0(\tau_a\tau_b) = 0\tau_a \boxplus 0\tau_b = a \boxplus b,$$

so $U = V$, and hence $\mathcal{B} = \mathcal{A}$, establishing (3). Then by (3) and 8.4.3, the map $\psi : D \to \mathcal{A}(D)$ of (4) has inverse $\mathcal{A} \mapsto D(\mathcal{A})$, so ψ is a bijection. Visibly ψ is S-equivariant, so (4) holds.

By (4), $M = N_S(D)$. By 8.4.3, $M/D \cong GL(D)$, so in particular $O_p(M/D) = 1$. Thus $D = O_p(M)$. We saw during the proof of 8.4 that $D = C_S(D)$. Therefore $D = F^*(M)$ by 3.7 and 3.9, completing the proof of (1).

As $M = N_S(D)$, (2) follows from 8.4.5.

The representation of D on Ω is the regular representation, which is determined up to quasiequivalence, so S is transitive on such subgroups D by 1.13.1. Then (4) implies (5).

Section 9. Diagonal structures

In this section we assume the following hypothesis:

Hypothesis 9.1. L is a nonabelian finite simple group, $r > 1$ is an integer, and Ω is a finite set of order $|L|^{r-1}$. Set $S = Sym(\Omega)$.

Definition 9.2. Define $\mathrm{Diag}(L, \Omega)$ to be the set of transitive subgroups D of S such that D is the direct product of r copies of L, and for $\omega \in \Omega$, D_ω is a full diagonal subgroup of D. (See Definition 4.8 for the definition of a full diagonal subgroup of the direct product of isomorphic groups.)

(9.3) *S acts transitively via conjugation on* $\mathrm{Diag}(L, \Omega)$.

Proof. Let $D \in \mathrm{Diag}(L, \Omega)$. By 4.10.2, $Aut(D)$ is transitive on the set of full diagonal subgroups of D. Thus by 1.11, for each $D' \in \mathrm{Diag}(L, \Omega)$, the inclusion maps $\iota : D \to S$ and $\iota' : D' \to S$ are quasiequivalent. Then the lemma follows from 1.13.1.

(9.4) *Let $D \in \mathrm{Diag}(L, \Omega)$, $\omega \in \Omega$, $F = D_\omega$, and $M = N_S(D)$. Let $\mathcal{L} = \{L_1, \dots, L_r\}$ be the set of components of D. Then*

(1) $D = L_1 \times \cdots \times L_r$ with $L_i \cong L$ for each i.

(2) There exists a family $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r)$ of isomorphisms $\alpha_i : L_1 \to L_i$ with $\alpha_1 = 1$ the identity map on L_1, such that

$$F = \mathrm{diag}(\boldsymbol{\alpha}) = \{\prod_i a\alpha_i : a \in L_1\}.$$

(3) $M = M_\omega D$ with $M_\omega \cong N_{Aut(D)}(F)$ and $D = F^*(M)$.

(4) $M_\omega = N_M(F) = T \times K$, where T acts faithfully on $\mathcal{L}$ as $Sym(\mathcal{L})$ with $N_T(L_1) = C_T(L_1)$, K is the kernel of the action of M_ω on $\mathcal{L}$, and K acts faithfully on L_1 via conjugation as $Aut(L_1)$ with $F = F^*(K)$.

Proof. By 9.2, (1) holds and F is a full diagonal subgroup of D. Then (2) follows from 4.9.1. By 9.2, D is transitive on Ω, so $M = M_\omega D$.

Let $X = L_1 \cdots L_{r-1}$. As F is a full diagonal subgroup of D, F is a complement to X in D, so X is regular on Ω. Thus X is a regular normal subgroup of D, so by 6.6.2, the map $x \mapsto \omega x$ is an equivalence of the representation of F on X via conjugation with the representation of F on Ω. In particular as $C_X(F) = 1$, $Fix(F) = \{\omega\}$. Thus $C_S(D)$ fixes ω, so as D is transitive on Ω we conclude that $C_S(D) = 1$. Then by 1.13.2 and 1.11, $M_\omega \cong N_{Aut(D)}(F)$. As $D \trianglelefteq M$ and $C_S(D) = 1$, $D = F^*(M)$ by 3.5.2 and 3.9. This completes the proof of (3).

Next $F = D_\omega \trianglelefteq M_\omega$ as $D \trianglelefteq M$. But by 4.10.3, $F = N_D(F)$, so as $M = M_\omega D$, it follows that $M_\omega = N_M(F)$. Now 4.10.3 completes the proof of (4).

Remark 9.5. We wish to define a notion of a "diagonal structure" on Ω determined by L in such a way that there exists an S-equivariant bijection $D \mapsto \mathbf{d}(D)$ of $\mathrm{Diag}(L,\Omega)$ with the set $\mathrm{diag}(L,\Omega)$ of diagonal structures on Ω. The easiest way to accomplish this goal is to define $\mathrm{diag}(L,\Omega)$ to be $\mathrm{Diag}(L,\Omega)$. The reader is free to adopt this definition and ignore the remainder of this section. On the other hand, the classes of "structures" we have considered so far are all defined to be families of relations on Ω, so it would be nice to give a definition of $\mathrm{diag}(L,\Omega)$ in that language. In the remainder of the section, we work toward such a definition. It should be noted however that it is not clear the point of view we introduce below is all that useful, although it is used in our proof of a theorem of Burnside in 11.6, and in the proof of 14.6.4.

Definition 9.6. Let Δ be a finite set. An *orbital structure* on Δ is a partition $\mathcal{O} = \{O_i : i \in I\}$ of $\Delta \times \Delta$. Thus $\mathcal{O}$ is a collection of relations O_i on Δ. Alternatively on can regard $\mathcal{O}$ as a coloring of the complete directed graph on Δ with the set I of colors. The *stabilizer* $N_{Sym(\Delta)}(\mathcal{O})$ of $\mathcal{O}$ in $Sym(\Delta)$ is the subgroup

$$N_{Sym(\Delta)}(\mathcal{O}) = \bigcap_{i\in I} N_{Sym(\Delta)}(O_i)$$

of $Sym(\Delta)$ preserving each of the relations in $\mathcal{O}$.

Example 9.7. Let Δ be a finite set and G a transitive subgroup of $T = Sym(\Delta)$. Define the *orbital structure* $\mathcal{O}(G)$ of G to be the set of orbits of G on $\Delta \times \Delta$. Visibly $\mathcal{O}(G)$ *is* an orbital structure on Δ, and $G \le N_T(\mathcal{O}(G))$.

Observe that if G is 2-transitive on Δ, then $\mathcal{O}(G) = \{O_1, O_2\}$, where $O_1 = \{(\delta,\delta) : \delta \in \Delta\}$ is the diagonal orbital, and $O_2 = \Delta\times\Delta - O_1$. Hence in this case, $T = N_T(\mathcal{O}(G))$. But if G is *not* 2-transitive, then $N_T(\mathcal{O}(G))$ is a proper subgroup of T. In particular if G is maximal in T, then $G = N_T(\mathcal{O}(G))$.

Indeed the various graphs $\mathcal{G}$ on Ω preserved by G can be retrieved from $\mathcal{O}(G)$ by defining $\mathcal{G} = (\Omega, E)$, where the set E of edges of $\mathcal{G}$ is the union of some suitable subset of the relations in $\mathcal{O}(G)$. By construction, $G \le Aut(\mathcal{G})$, and if G is maximal in S then $G = Aut(\mathcal{G})$ for each such graph $\mathcal{G}$.

Definition 9.8. Define $\mathrm{diag}(L, \Omega)$ to be the set of orbital structures on Ω isomorphic to $\mathcal{O}(N_S(D))$ for some $D \in \mathrm{Diag}(L, \Omega)$. In particular by Example 9.7, $N_S(D) \leq N_S(\mathcal{O}(N_S(D)))$, and if $N_S(D)$ is maximal in S (which it is) then $N_S(D) = N_S(\mathcal{O}(N_S(D)))$. $\blacksquare$ Hence the map $D \mapsto \mathcal{O}(N_S(D))$ is a S-equivariant bijection of $\mathrm{Diag}(L, \Omega)$ with $\mathrm{diag}(L, \Omega)$.

Example 9.9. Here is a nice description of the members of $\mathrm{diag}(L, \Omega)$ when $r = 2$. There is a similar, but less attractive, description for larger r. See Exercise 9.1 for this description.

Assume $r = 2$. Then $|\Omega| = |L|$, so we take $\Omega = L$. Let

$$D = \{(a, b) : a, b \in L\}$$

be the direct product of two copies of L, and let

$$F = \{(a, a) : a \in L\}$$

be the standard full diagonal subgroup of D. Define $t \in Aut(D)$ by $(a, b)^t = (b, a)$, and for $\alpha \in Aut(L)$, define $k_\alpha \in Aut(D)$ by $(a, b)^{k_\alpha} = (a^\alpha, b^\alpha)$. Then the map $k : \alpha \mapsto k_\alpha$ is an embedding of $Aut(L)$ in $Aut(D)$; write K for the image of this embedding. Observe t centralizes F, and, identifying D with $Inn(D)$ via $d \mapsto c_d$ where $c_d \in Aut(D)$ is conjugation by $d \in D$, we have $Inn(L)k = F$. Thus $K \cap D = F$. By 4.10, $B = K\langle t\rangle = N_{Aut(D)}(F)$. Set $G = DB \leq Aut(D)$.

Represent D on $\Omega = L$ via $c \cdot (a, b) = a^{-1}cb$ for $a, b, c \in L$. Represent B on L via $c \cdot k_\alpha = c\alpha$ and $c \cdot t = c^{-1}$. These representations embed G in S as a transitive subgroup with the stabilizer G_1 of $1 \in L$ in G equal to B. Therefore $D \in \mathrm{Diag}(L, \Omega)$ and $G = N_S(D)$ by 4.10.

We next define an orbital structure $\mathcal{O}_L$ on Ω. Let $\mathcal{C}$ be the set of orbits of B on L, and for $C \in \mathcal{C}$, define

$$O_C = \{(a, b) \in L \times L : ba^{-1} \in C\}.$$

Finally set $\mathcal{O}_L = \{O_C : C \in \mathcal{C}\}$. The next lemma says that $\mathcal{O}_L = \mathcal{O}(G)$, so that $\mathcal{O}_L \in \mathrm{diag}(L, \Omega)$.

(9.10) *Assume $r = 2$ and take $\Omega = L$ and define D, G, and $\mathcal{O}_L$ as in Example 9.9. Then*

(1) $\mathcal{O}_L = \mathcal{O}(G)$.

(2) $\mathcal{O}_L \in \mathrm{diag}(L, \Omega)$.

Proof. Exercise 9.1.

Exercises for Section 9

1. Assume Hypothesis 9.1, let $D \in \mathrm{Diag}(L, \Omega)$, and adopt the notation of 9.4. Set $I = \{1, \dots, r\}$, $X = L_2 \cdots L_r$ and regard D as L^I and $F = \{\iota(a) : a \in L\}$, where $\iota(a)(i) = a$ for all $i \in I$. Prove:

(1) X is regular on Ω, so we can take $X = \Omega$ via the identification $x \mapsto \omega x$ for $x \in X$.

(2) For $a \in L$, define $\xi(a) \in X$ by $\xi(a)(i) = a$ for each $1 < i \in I$. Let $\pi_X : D \to X$ be the projection map of $D = L_1 \times X$ onto X. Then, subject to the identification of Ω with X in (1), D acts on Ω via $y \cdot d = \xi(d(1))^{-1} y(d\pi_X)$, for $d \in D$ and $y \in X$.

(3) The kernel K of the action of M_ω on $\mathcal{L}$ can be regarded as $Aut(L)$. Further K acts on D by conjugation via $(d^k)(i) = d(i)^k$ for $k \in K$ and $d \in D$, and, subject to the identification of Ω with X in (1), K acts on Ω via conjugation on X.

(4) We can choose $T = Sym(I)$ to act on D by conjugation via $d^t(i) = d(i^{t^{-1}})$ for $d \in D$ and $t \in T$.

(5) For $1 < i \in I$ and $a \in L$, define $\zeta_i(a) \in X$ by $\zeta_i(a)(j) = 1$ for $j \neq i$ and $\zeta_i(a)(i) = a$. Define $\sigma_i \in Sym(X)$ by $x^{\sigma_i} = \xi(x(i))^{-1} x \zeta_i(x(i))^{-1}$. Then T_1 acts on Ω via restriction to X from D, while $(1, i) \in T$ acts as σ_i.

(6) Let $\mathcal{C}$ be the set of orbits of $K \times T$ on X, and for $C \in \mathcal{C}$, set $\mathcal{O}_C = \{(x, y) \in X \times X : yx^{-1} \in C\}$. Then the map $C \mapsto \mathcal{O}_C$ is a bijection of $\mathcal{C}$ with $\mathcal{O}(M)$.

(7) For $x \in X$, the orbit of x under $K \times T$ is $\{x^{kt}, x^{kt\sigma_i} : k \in K,\ t \in T_1,\ 1 < i \in I\}$.

(8) For each $a \in L$, $C_a = \{\xi(b), \zeta_i(b)^{-1} : 1 < i \in I,\ b \in a^K\} \in \mathcal{C}$.

(9) Let $m = |\{|a| : a \in L\}|$. Then $|\mathcal{C}| \geq m > 3$, so $|\mathcal{O}(M)| \geq m > 3$.

Section 10. Regular product structures

In this section Ω is a finite set of order n and $S = Sym(\Omega)$. We record results about regular product structures on Ω. We begin by recalling some definitions from Section 2.

Definition 10.1. Let m, k be integers with $m \geq 5$ and $k > 1$. We recall the notion of a *regular (m,k)-product structure* on Ω. From Example 2.19, such a structure is a (similarity class of) bijection(s) $f : \Omega \to \Gamma^I$, where $I = \{1, \dots, k\}$ and Γ is an m-set. The function f may be thought of as a family of functions $(f_i : \Omega \to \Gamma : i \in I)$ via $f(\omega) = (f_1(\omega), \dots, f_k(\omega))$ for $\omega \in \Omega$.

Observe that if Ω admits a regular (m, k)-product structure, then $n = m^k$.

While the definition given in 10.1 is often useful, there is a second equivalent definition in terms of chamber systems which is usually more useful. When we wish to distinguish between the two types of objects, we refer to the functions f in Definition 10.1 as *informal regular product structures*, and call the chamber systems defined in the next definition as *formal regular product structures*.

Definition 10.2. Again let m, k be integers with $m \geq 5$ and $k > 1$. Formally a product structure is a family $\mathcal{F} = (\Omega_i : i \in I)$ of partitions Ω_i of Ω into m blocks of size m^{k-1}, such that $\mathcal{F}$ is *injective*: For each $\omega \in \Omega$,

$$\bigcap_{i \in I} [\omega]_i = \{\omega\},$$

where $[\omega]_i$ is the block in Ω_i containing ω.

Observe that if $\simeq_i = \sim_{\Omega_i}$ is the equivalence relation on Ω whose set of equivalence classes is Ω_i, then (cf. Definition 7.4) we can regard $\mathcal{F}$ as an injective chamber system on Ω over I.

Remark 10.3. The class of chamber systems defined in Definition 10.2 are also considered in [BPS1], where such objects are called *Cartesian decompositions* of Ω. Various properties of Cartesian decompositions are derived in [BPS1] and [BPS2].

Let us now see how to pass between our two definitions, and in particular convince ourselves that they are indeed equivalent.

Definition 10.4. Let $f : \Omega \to \Gamma^I$ be an informal (m, k)-product structure. We define a family $\mathcal{F} = \mathcal{F}(f)$ of partitions of Ω. The ith partition of the family is $\Omega_i = \{f_i^{-1}(\gamma) : \gamma \in \Gamma\}$, the set of fibers of f_i. By construction, Ω_i has m blocks of size m^{k-1}. As the function f is an injection, the chamber system $\mathcal{F}(f)$ is injective.

Definition 10.5. Next assume $\mathcal{F} = (\Omega_i : i \in I)$ is a formal (m, k)-product structure. An *indexing* of $\mathcal{F}$ is an indexing $\Omega_i = \{\Omega_{i,\gamma} : \gamma \in \Gamma\}$ of the blocks of the various partitions Ω_i by our m-set Γ. The function f in Definition 10.4 defines the indexing $\Omega_{i,\gamma} = f_i^{-1}(\gamma)$ of $\mathcal{F}(f)$. In the other direction, an indexing $\mathcal{I}$ of $\mathcal{F}$ defines a function $f = f_{\mathcal{F},\mathcal{I}} : \Omega \to \Gamma^I$ via $\omega \in \Omega_{i,f_i(\omega)}$. As $\mathcal{F}$ is injective, the function f defined by the indexing is injective, so as $|\Omega| = |\Gamma^I|$, $f : \Omega \to \Gamma^I$ is a bijection, and hence defines a informal product structure on Ω.

In short, the formal definition is a "coordinate free" definition of product structure, in that the various indexings of $\mathcal{F}$ define various (similar) informal product structure functions, corresponding to a choice of "coordinate system" for the description of Ω as a set product.

Write $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}(\Omega)$ for the set of all formal regular product structures on Ω. Observe that S is represented on $\boldsymbol{\mathcal{F}}$ via $s : \mathcal{F} \mapsto \mathcal{F}s = \{\Omega_i s : i \in I\}$ for $s \in S$. Recall from Definition 7.4 that the *stabilizer* $N_S(\mathcal{F})$ in S of $\mathcal{F}$ is the subgroup consisting of those $s \in S$ such that $\mathcal{F}s = \mathcal{F}$; that is $N_S(\mathcal{F})$ is the stabilizer of $\mathcal{F}$ in the representation of S on $\boldsymbol{\mathcal{F}}$.

(10.6) *(1) The map $\varphi : \langle f \rangle \mapsto \mathcal{F}(f)$ is a bijection between the set of similarity classes of informal regular (m,k)-product structure on Ω, and the set $\boldsymbol{\mathcal{F}}(\Omega)$ of formal regular (m,k)-product structures on Ω.*

(2) The inverse of φ is $\psi : \mathcal{F} \to \langle f_{\mathcal{F},\mathcal{I}} \rangle$, for any indexing $\mathcal{I}$ of $\mathcal{F}$.

(3) Represent S on the set $\tilde{P}$ of similarity classes of informal product structures via $s : \langle f \rangle \mapsto \langle s^{-1} f \rangle$. Then φ is S-equivariant.

(4) S is transitive on $\boldsymbol{\mathcal{F}}(\Omega)$.

Proof. Let P be the set of informal regular product structures on Ω, and set $\tilde{P} = \{\langle f \rangle : f \in P\}$. Let $f, g \in P$ and $\pi_i : \Gamma^I \to \Gamma$ the ith projection. Then $f_i = f\pi_i$ and $g_i = g\pi_i$, $i \in I$, are the corresponding projection maps from Ω to Γ. From Definition 2.10, $\langle f \rangle = \langle g \rangle$ iff

$$\Omega_i^f = \{f_i^{-1}(\gamma) : \gamma \in \Gamma\} = \{g_i^{-1}(\gamma) : \gamma \in \Gamma\} = \Omega_i^g.$$

Thus $\langle f \rangle = \langle g \rangle$ iff $\mathcal{F}(f) = \mathcal{F}(g)$, so the map $\varphi : \tilde{P} \to \boldsymbol{\mathcal{F}}$ is a well defined injection.

Similarly if $\mathcal{I}$, $\mathcal{J}$ are indexings of $\mathcal{F} \in \boldsymbol{\mathcal{F}}$, and $f = f_{\mathcal{F},\mathcal{I}}$ and $g = f_{\mathcal{F},\mathcal{J}}$, then from the definition of f, g in 10.4, for each $i \in I$,

$$\{f_i^{-1}(\gamma) : \gamma \in \Gamma\} = \Omega_i = \{g_i^{-1}(\gamma) : \gamma \in \Gamma\},$$

so $\langle f \rangle = \langle g \rangle$ by the previous paragraph. Hence $\psi : \boldsymbol{\mathcal{F}} \to \tilde{P}$ is well defined. Moreover this says that $\mathcal{F}(f) = \{\Omega_i : i \in I\}$, so $\mathcal{F}(f) = \mathcal{F}$, and hence $\psi\varphi = 1$ is the identity map. Therefore φ is a surjection, while from the previous paragraph, φ is an injection, so φ is a bijection. Then as $\psi\varphi = 1$, $\psi = \varphi^{-1}$, completing the proof of (1) and (2).

Let $h \in P$. Then $fh^{-1} = \alpha \in S$ with $f = \alpha h$, so for $i \in I$, $f_i = f\pi_i = \alpha h \pi_i = \alpha h_i$, and hence

$$\Omega_i^f = \{f_i^{-1}(\gamma) : \gamma \in \Gamma\} = \{h_i^{-1}(\gamma)\alpha^{-1} : \gamma \in \Gamma\} = \Omega_i^h \alpha^{-1},$$

so that $\Omega_i^f \alpha = \Omega_i^h$. Therefore $\mathcal{F}(f)\alpha = \mathcal{F}(h)$, so the representation of S on $\tilde{P}$ in (3) is well defined, and φ is S-equivariant. That is (3) holds. Further as $f = \alpha h$, $\langle f \rangle \alpha^{-1} = \langle h \rangle$, proving (4).

We next establish a criterion for a group D to preserve a regular product structure.

(10.7) *Assume $|\Omega| = m^k$ and let $\omega \in \Omega$ and $I = \{1, \dots, k\}$. Assume D is a transitive subgroup of S which is the direct product of a set $\mathcal{D} = \{D_i : i \in I\}$ of subgroups such that*

$$D_\omega = \prod_{i \in I} D_{i,\omega},$$

with $|D_i : D_{i,\omega}| = m$ for each $i \in I$. Define $\mathcal{F} = \mathcal{F}(\mathcal{D}) = (\Omega_i : i \in I)$ to be the chamber system on Ω such that Ω_i is the set of orbits of $D_{i'} = \langle D_j : j \in I - \{i\} \rangle$ on Ω. Then

(1) $\mathcal{F} \in \boldsymbol{\mathcal{F}}(\Omega)$.

(2) D is contained in the kernel of the action of $N_G(\mathcal{F})$ on $\mathcal{F}$.

(3) $D_{i'}$ is the kernel of the action of D on Ω_i, and D_i acts faithfully and transitively on Ω_i.

(4) If G is a subgroup of S permutating $\mathcal{D}$ via conjugation then $G \leq N_S(\mathcal{F})$.

Proof. By hypothesis, $D_\omega = \prod_{i \in I} D_{i,\omega}$. Hence as $|D_j : D_{j,\omega}| = m$ for each $j \in I$, $|D_{i'} : D_{i',\omega}| = m^{k-1}$, so the orbit $\omega D_{i'}$ is of order m^{k-1}.

As $D_{i'} \trianglelefteq D$ and D is transitive on Ω, the set Ω_i of orbits of $D_{i'}$ on Ω is a partition of Ω into m blocks of size m^{r-1}. Thus $\mathcal{F}$ is indeed a chamber system on Ω, and to complete the proof of (1) it remains to show $\mathcal{F}$ is injective. Suppose $\alpha \in \omega D_{i'}$ for each $i \in I$. Then for each $i \in I$, there is $g_i \in D_{i'}$ with $\omega g_i = \alpha$. Now $g_i = d_{i,1} \cdots d_{i,r}$ with $d_{i,l} \in D_l$. Further for $j \neq i$, $g_i g_j^{-1} \in D_\omega$, so as $D_\omega = \prod_i D_{i,\omega}$, $x_{i,l} x_{j,l}^{-1} \in D_{l,\omega}$ for each l. However $x_{j,j} = 1$, so $x_{i,j} \in D_{j,\omega}$ for each $j \in I$, and hence $g_i \in D_\omega$, so that $\alpha = \omega g_i = \omega$. This completes the proof of (1).

Part (2) follows by construction of $\mathcal{F}$, as does the fact that $D_{i'}$ is contained in the kernel K of the action of D on Ω_i. Thus $K = D_{i'} X$, where $X = D_i \cap K$. But $X \leq D_{j'}$ for each $j \neq i$, so X fixes all blocks in each partition Ω_j. But then as $\mathcal{F}$ is injective, for each $\alpha \in \Omega$, X acts on the intersection $\{\alpha\}$ of the blocks containing α, so $X \leq S_\Omega = 1$. Thus D_i is faithful on Ω_i, and as $D = D_{i'} D_i$ is transitive on Ω_i, so is D_i. This completes the proof of (3).

Let $G = N_S(\mathcal{D})$. Then $D \trianglelefteq G$, and G permutes $\mathcal{D}$ via conjugation, so G permutes $\{D_{i'} : i \in I\}$ and hence the set $\mathcal{F}$ of orbits of those groups. This establishes (4).

(10.8) *Assume the setup of 10.7, and assume for each $i \in I$ that $E_i \leq D_i$ with $D_i = D_{i,\omega}E_i$. Set $\mathcal{E} = \{E_i : i \in I\}$. Then $\mathcal{F}(\mathcal{D}) = \mathcal{F}(\mathcal{E})$.*

Proof. As $D_i = D_{i,\omega}E_i$ for each $i \in I$, $\omega D_{i'} = \omega E_{i'}$ and Ω_i is the set of orbits of $E_{i'}$.

Set Up 10.9. Assume $\Omega = \Gamma^I$ for some set Γ of order m and $I = \{1, \ldots, k\}$. Thus the members of Ω are functions from I into Γ. Let $\pi_i : \Omega \to \Gamma$ be the ith projection $\pi_i(u) = u(i)$.

Let $\mathrm{id} : \Omega \to \Omega$ be the identity map and $\mathcal{F} = \mathcal{F}(\mathrm{id}) \in \boldsymbol{\mathcal{F}}(\Omega)$. Hence $\mathcal{F} = \{\Omega_i : i \in I\}$, where $\Omega_i = \{\pi_i^{-1}(\gamma) : \gamma \in \Gamma\}$. Let $T = Sym(I)$ and represent T on Ω via $(u \cdot t)(i) = u(i^{t^{-1}})$ for $u \in \Omega$, $i \in I$, and $t \in T$. Identifying T with its image in S under this representation, we may regard T as a subgroup of S.

For $\sigma \in Sym(\Gamma)$ and $i \in I$, define $k_i(\sigma) \in S$ by $(u \cdot k_i(\sigma))(j) = u(j)$ for $j \neq i$, and $(u \cdot k_i(\sigma))(i) = u(i)\sigma$. Set $K_i = \{k_i(\sigma) : \sigma \in Sym(\Gamma)\}$, so that the map $\varphi : \sigma \mapsto k_i(\sigma)$ is an isomorphism of K_i with $Sym(\Gamma)$. Set $K = \langle K_i : i \in I\rangle \leq S$.

Let $\Delta = \coprod_{i \in I} \Omega_i$. Observe each K_i acts on each Ω_j, for $j \neq i$, K_i is trivial on Ω_j, and defining $\psi : \gamma \mapsto \pi_i^{-1}(\gamma)$, the pair ψ, φ defines a quasiequivalence of the representation of $Sym(\Gamma)$ on Γ, with the representation of K_i on Ω_i. That is K_i acts faithfully as $Sym(\Omega_i)$ on Ω_i.

It follows that $K = K_1 \times \cdots \times K_k$ is the direct product of k copies of S_m. Further $\langle K, T\rangle \leq Sym(\Delta)$ preserves the regular partition $\mathcal{F}$ on Δ. By 7.11, the stabilizer $\bar{M}$ in $Sym(\Delta)$ of this partition is isomorphic to the wreath product of S_m by S_k, with $\bar{M}_{\mathcal{F}} = \bar{K}$ the direct product of k copies of S_m and $\bar{M}^{\mathcal{F}} = Sym(\mathcal{F})$. As $K \leq \bar{K}$ it follows that $K = \bar{K}$, and then as $T \cong T^{\mathcal{F}} = Sym(\mathcal{F})$, it follows that $\bar{M} = KT$ with T a complement to K in $\bar{M}$. Observe by construction, $N_T(K_1) = C_T(K_1)$.

Finally let $M = N_S(\mathcal{F})$. Then M also permutes Δ and by an argument in the proof of 10.7.3, M is faithful on Δ. Thus $M \leq \bar{M}$, so $M = \bar{M}$.

(10.10) *Let $\mathcal{F} = (\Omega_i : i \in I) \in \boldsymbol{\mathcal{F}}$, pick $\omega \in \Omega$, and set $M = N_S(\mathcal{F})$. Let K be the kernel of the action of M on $\mathcal{F}$. Then*

(1) $K = K_1 \times \cdots \times K_k$, where K_i is the subgroup of K trivial on Ω_j for each $j \in I - \{i\}$. Moreover K_i acts faithfully as $Sym(\Omega_i)$ on Ω_i.

(2) K has a complement T in M, such that T acts faithfully on $\mathcal{F}$ as $Sym(\mathcal{F})$ and $N_T(K_1) = C_T(K_1)$.

(3) M is isomorphic to the wreath product of S_m by S_k.

(4) $F^*(M) = D = D_1 \times \cdots \times D_k$, *where for* $i \in I$, $D_i = F^*(K_i)$ *acts faithfully on* Ω_i *as the alternating group, and* $D_{i'} = \prod_{j \neq i}$ *acts transitively on each block in* Ω_i.

(5) $\mathcal{F} = \mathcal{F}(\mathcal{D})$, *where* $\mathcal{D} = \{D_i : i \in I\}$.

Proof. Pick a set Γ of order n and an isomorphism $f : \Omega \to \Gamma^I$ with $\mathcal{F}(f) = \mathcal{F}$. Identifying Ω with Γ^I via f, we may assume $\Omega = \Gamma^I$ and f is the identity map id on Ω. Thus we are in Set Up 10.9. Then (1)-(3) follows from the discussion in Set Up 10.9. Next (1)-(3) imply (4), while (4) implies (5).

Exercises for Section 10

1. Let $m, k > 1$ be integers, $I = \{1, \dots, k\}$, Γ an m-set, $f : \Omega \to \Gamma^I$ an informal product structure on Ω, and $\mathcal{F} = \mathcal{F}(f) = (\Omega_i : i \in I) \in \boldsymbol{\mathcal{F}}$. For $i \in I$, let $\simeq_i$ be the equivalence relation on Ω with set of equivalence classes Ω_i, and regard $\mathcal{F} = (\simeq_i : i \in I)$ as a chamber system on Ω over I. If $k > 2$ then $\mathcal{F}$ is not nondegenerate. In this exercise we form the *dual chamber system* $\mathcal{F}^*$ of $\mathcal{F}$, and show that $\mathcal{F}^*$ is nondegenerate. Then, using Exercise 7.4, we can naturally associate to $\mathcal{F}$ the simplicial complex $\boldsymbol{\mathcal{C}}(\mathcal{F}^*)$, which we investigate in later exercise. For $J \subseteq I$, set

$$\sim_J = \vee_{i \in J'} \simeq_i,$$

let $\pi_J : \Gamma^I \to \Gamma^J$ be the projection map $\pi_J(u) = u_{|J}$ for $u \in \Gamma^I$ and $j \in J$, and set $f_J = \pi_J \circ f : \Omega \to \Gamma^J$. Define the *dual* of $\mathcal{F}$ to be the chamber system $\mathcal{F}^* = (\sim_i : i \in I)$ on Ω over I. Prove:

(1) For each $i \in I$, $\simeq_i = \sim_{i'} = \{f_i^{-1}(u) : u \in \Gamma^i\}$.

(2) For each $J \subseteq I$, $\sim_J$ has equivalence classes $\{f_{J'}^{-1}(u) : u \in \Gamma^{J'}\}$.

(3) For each $J \subseteq I$,

$$\sim_J = \wedge_{j \in J} \sim_j,$$

is the equivalence relation generated by $\{\sim_j : j \in J\}$.

(4) For each $J \subseteq I$ and $\omega \in \Omega$,

$$\vee_{j \in J} \sim_{j'} = \sim_{J'} \quad \text{and} \quad \bigcap_{j \in J} [\omega]_{j'} = [\omega]_{J'},$$

where $[\omega]_J$ is the equivalence class of $\sim_J$ containing ω. Moreover $[\omega]_{I'} = \{\omega\}$.

(5) $\mathcal{F}^*$ is a nondegenerate chamber system.

2. Let G be a group, $I = \{1, \dots, m\}$ a finite set, and $\mathcal{F} = (G_i : i \in I)$ a family of subgroups of G. Define a simplicial complex $\mathcal{C}(G, \mathcal{F}) = (V(G, \mathcal{F}), \Sigma(G, \mathcal{F}))$ by $V = V(G, \mathcal{F}) = \coprod_{i \in I} G/G_i$ and $\Sigma = \Sigma(G, \mathcal{F}) = \{\sigma_{J,x} : \varnothing \neq J \subseteq I,\ x \in G\}$, where $\sigma_{J,x} = \{G_j x : j \in J\}$. Define $\tau : V \to I$ by $\tau(G_i x) = i$.

Let $\bar{\mathcal{C}} = (\bar{V}, \bar{\Sigma})$ be a geometric complex over I. (cf. Exercise 7.4) For $\sigma \in \bar{\Sigma}$ of type $J \subset I$, the *residue* or *link* of σ is the geometric complex $\mathrm{lk}(\sigma)$ over J' with simplices $\lambda - \sigma$ for $\sigma \subset \lambda \in \bar{\Sigma}$. We say $\bar{\mathcal{C}}$ is *residually connected* if for all $J \subseteq I$ with $|J'| > 1$, and for all $\sigma \in \bar{\Sigma}$ of type J, $\mathrm{lk}(\sigma)$ is connected.

Prove:

(1) $\mathcal{C} = \mathcal{C}(G, \mathcal{F})$ is a geometric complex over I, and the representation of G on V by right multiplication maps G to a flag transitive group of automorphisms of $\mathcal{C}$; that is for each $J \subseteq I$, G is transitive on simplices of type J.

(2) For each $J \subseteq I$, set $G_J = \bigcap_{j \in J} G_j$, with $G_\varnothing = G$. Then G_J is the stabilizer in G of $\sigma_J = \sigma_{J,1}$.

(3) Assume $\bar{G}$ is a flag transitive group of automorphisms of $\bar{\mathcal{C}}$, and pick a chamber $\sigma = \{v_i : i \in I\}$ of $\bar{\mathcal{C}}$ with $v_i \in \bar{V}$ of type i. Set $\bar{G}_i = \bar{G}_{v_i}$ and $\bar{\mathcal{F}} = (\bar{G}_i : i \in I)$. Define $\xi : \bar{V} \to V(\bar{G}, \bar{\mathcal{F}})$ by $\xi : v_i g \mapsto \bar{G}_i g$. Then ξ is a $\bar{G}$-equivariant isomorphism of $\bar{\mathcal{C}}$ with $\mathcal{C}(\bar{G}, \bar{\mathcal{F}})$.

(4) For $i \in I$ set $P_i = G_{i'}$ and $\mathcal{P} = (P_i : i \in I)$. Set $H = G_I$ and form the chamber system $X = X(G, H, \mathcal{P})$ of Exercise 7.3, and the chamber system $\mathbf{X} = \mathbf{X}(\mathcal{C})$ of Exercise 7.4. Define $\zeta : G/H \to \Omega = \{\sigma_{I,x} : x \in G\}$ by $\zeta : Hx \mapsto \sigma_{I,x}$. Then $\zeta : X \to \mathbf{X}$ is a G-equivariant isomorphism of chamber systems.

(5) $\mathcal{C} \cong \boldsymbol{\mathcal{C}}(X)$, where $\boldsymbol{\mathcal{C}}(X)$ is defined in Exercise 7.4.

(6) $\mathcal{C}$ is connected iff $G = \langle \mathcal{F} \rangle$.

(7) For $J \subseteq I$, set $\mathcal{F}_J = (G_{J \cup \{i\}} : i \in J')$. Show G_J is flag transitive on $\mathrm{lk}(\sigma_J)$, and $\mathrm{lk}(\sigma_J) \cong \mathcal{C}(G_J, \mathcal{F}_J)$.

(8) $\mathcal{C}$ is residually connected iff for all $J \subseteq I$, $G_J = P_{J'}$.

3. Let $\mathcal{C}_r = (V_r, \Sigma_r, \tau_r)$ be geometric complexes over I_r for $r = 1, 2$. Set $V = V_1 \coprod V_2$, $I = I_1 \coprod I_2$, and define $\tau : V \to I$ by $\tau_{|V_r} = \tau_r$. Let Σ consist of the nonempty sets $\sigma_1 \coprod \sigma_2$ such that $\sigma_r \in \Sigma_r \cup \{\varnothing\}$ for $r = 1, 2$. Set $\mathcal{C}_1 \oplus \mathcal{C}_2 = (V, \Sigma, \tau)$. Prove:

(1) $\mathcal{C}_1 \oplus \mathcal{C}_2$ is a geometric complex over I.

(2) For each $\sigma = \sigma_1 \coprod \sigma_2 \in \Sigma$ of type $J_1 \coprod J_2$, $\mathrm{lk}(\sigma) = \mathrm{lk}(\sigma_1) \oplus \mathrm{lk}(\sigma_2)$.

(3) Given positive integers $n_1, \dots, n_k$, let $\Gamma(n_i)$ be the 0-dimensional complex with n_i vertices, and set $\mathcal{C}(n_1, \dots, c_k) = \Gamma(n_1) \oplus \cdots \oplus \Gamma(n_k)$. Set $\mathcal{C}(n^k) = \mathcal{C}(n_1) \oplus \cdots \oplus \mathcal{C}(n_k)$

where $n_i = n$ for all i. Let $\bar{\mathcal{C}} = (\bar{V}, \bar{\Sigma})$ be a geometric complex over $I = \{1, \dots, k\}$, write $\bar{V}_i$ for the set of vertices of $\bar{\mathcal{C}}$ of type i, and set $n_i = |\bar{V}_i|$. Prove $\bar{\mathcal{C}} = \mathcal{C}(n_1, \dots, n_k)$ iff for each $v_i \in \bar{V}_i$, $1 \leq i \leq k$, we have $\{v_1, \dots, v_k\} \in \bar{\Sigma}$.

(4) Let $\bar{\mathcal{C}} = \mathcal{C}(n_1, \dots, n_k)$. Show for each $\varnothing \neq J \subseteq I$ and simplex σ of type J, $\mathrm{lk}(\sigma) \cong \mathcal{C}(n_i : i \in J')$.

(5) Assume $\mathcal{F} \in \boldsymbol{\mathcal{F}}$ is a regular (m, k)-product structure on a set Ω, regard $\mathcal{F}$ as a chamber system, and let $\mathcal{F}^*$ be the dual of $\mathcal{F}$ defined in Exercise 10.1. Let $\boldsymbol{\mathcal{C}}(\mathcal{F}^*)$ be the geometric complex constructed in Example 7.4. Prove $\boldsymbol{\mathcal{C}}(\mathcal{F}^*) \cong \mathcal{C}(m^k)$.

(6) Prove $\bar{\mathcal{C}} = \mathcal{C}(n_1, \dots, n_k)$ iff $\bar{\mathcal{C}}$ is residually connected, and for all $J \subseteq I$ with $|J'| = 2$, and for all simplices σ of type J, $\mathrm{lk}(\sigma) \cong \mathcal{C}(n_\sigma, m_\sigma)$ for some n_σ, m_σ.

Section 11. The structure of primitive permutation groups.

In this section we investigate the structure of a finite primitive permutation group. We will see that the structure of such a group is highly restricted. This is one reason why one should usually seek to reduce questions about permutation groups to the primitive case.

Throughout this section Ω is a finite set of order n, $S = Sym(\Omega)$, and G a primitive group of permutations on Ω; that is $G \leq S$ and G is primitive on Ω.

In this section we state the Structure Theorem for Primitive Permutation Groups and explore what it is saying. Then in the next section we prove the theorem.

In the literature the Structure Theorem is often called the O'Nan-Scott Theorem. Actually (as I understand it) O'Nan neither stated nor proved the theorem. Rather he and (independently) Len Scott were (I believe) the first to prove and state Theorem 13.1, which describes the subgroup structure of finite symmetric groups, and can be derived from the Structure Theorem. In [S], Scott states a version of the Structure Theorem, and gives a sketch of a proof; this seems to be the first appearance of the Structure Theorem. However, probably because he didn't write out all the details of a proof, Scott's statement is not quite correct. The first correct statement and proof of the Structure Theorem appears in [AS].

Theorem 11.1. *(Structure Theorem for Primitive Permutation Groups)*

Let G be a primitive permutation group on a finite set Ω of order n, $\omega \in \Omega$, $H = G_\omega$, and $D = F^*(G)$. Then $G = HD$ and one of the following holds:

(A) $D \cong E_{p^e}$ for some prime p and integer e, and D is regular on Ω, so $n = p^e$ and H is a complement to D in G, acting irreducible on D regarded as an $\mathbf{F}_p H$-module.

(B) G has exactly two minimal normal subgroups D_L and D_K, and $D = D_L \times D_K$ is the direct product of the r conjugates of $L \times K$ where L and K are isomorphic nonabelian simple groups with $L \leq D_L$ and $K \leq D_K$. Further $H = N_G(H \cap D)$ and $H \cap D$ is the direct product of the H conjugates of the full diagonal subgroup $H \cap LK$ of $L \times K$. In particular $n = |L|^r$.

(C) D is the direct product of the set $\mathcal{L}$ of r components of G, H is transitive on $\mathcal{L}$ via conjugation, each $L \in \mathcal{L}$ is a nonabelian simple subgroup, and one of the following holds:

(1) D is regular on Ω, H is a complement to D in G, and $Inn(L) \leq Aut_H(L)$ so $r \geq 6$, and $n = |L|^r$.

(2) There exists a maximal (in the partial order of Definition 7.2) H-invariant partition $\Sigma(G)$ of $\mathcal{L}$ of order $t < r$ such that $H = N_G(H \cap D)$ with $H \cap D$ the direct product of the full diagonal subgroups $F_\sigma = H \cap D_\sigma$ of $D_\sigma = \langle \sigma \rangle$, for $\sigma \in \Sigma(G)$. In particular $n = |L|^{n-t}$.

(3) $H = N_G(H \cap D)$ with $H \cap D$ the direct product of the H conjugates of $H \cap L$, and $Aut_H(L)$ is maximal in $Aut_G(L)$. In particular $n = |L : L \cap H|^r$.

The proof of Theorem 11.1 appears in the next section. Let us examine the theorem to get a better idea of what it is telling us. Our point of view will be that each case corresponds to a class of structures on Ω preserved by groups in the class. In case A the group G preserves an affine space structure on Ω, while in the remaining cases either G preserves a diagonal structure or a regular product structure, or G is almost simple. From the discussion in previous sections, the structures are determined up to conjugacy under S, and we have seen what the stabilizer of each structure looks like.

First consider Case A. Here as $D \cong E_{p^e}$, we may view D as an e-dimensional vector space over the field $\mathbf{F}_p$ of integers modulo p, and $Aut(D) = GL(D)$ is the general linear group on this vector space. It is convenient to write the group additively, so that the addition on the vector space is the addition in D, and the scalar multiple of a vector $d \in D$ by a congruence class $(p) + m$ in $\mathbf{F}_p$ is just the mth power md of d in D. From 8.4, D defines an affine space structure $\mathcal{A} = \mathcal{A}(D)$ on Ω, D is the group of translation for this affine space structure, and $N_S(D) = N_S(\mathcal{A})$ is the stabilizer in S of $\mathcal{A}$. In particular G stabilizes $\mathcal{A}$.

We call the groups in Case A the *affine primitive groups.*

Next consider Case B. We call the groups in this case *doubled primitive groups* as such groups have a pair D_L, D_K of isomorphic minimal normal subgroups, while in each of the other cases, G has a unique minimal normal subgroup.

In Case B set $\mathcal{D} = (LK)^G$, and observe that $\mathcal{D}$ is an orbit of length r under the action of G via conjugation, with D the direct product of the members of $\mathcal{D}$. Further for each $X = L \times K \in \mathcal{D}$, $X_\omega = H \cap LK$ is a full diagonal subgroup of X, so $|X : X_\omega| = |L|$. Finally $D_\omega = H \cap D$ is the direct product of the groups X_ω, $X \in \mathcal{D}$. Therefore if $r > 1$, then by 10.7, $N_S(D)$ preserves a regular $(|L|, r)$-product structure $\mathcal{F} = \mathcal{F}(\mathcal{D})$ on Ω. In particular $G \leq N_S(\mathcal{F})$.

On the other hand suppose $r = 1$. Then (cf. Definition 9.1) $D \in \mathrm{Diag}(L, \Omega)$ and by Remark 9.5, $N_S(D)$ preserves the diagonal structure on Ω determined by D. In particular G preserves this diagonal structure.

This leaves Case C. Recall in Case C that $\mathcal{L}$ is the set of components of G, $|\mathcal{L}| = r$, and D is the direct product of the set $\mathcal{L}$ of components.

We first consider Case C1. We call the groups in this case *complemented primitive groups* as $H = G_\omega$ is a complement to the regular normal subgroup D of G. As D is regular on Ω, $1 = D_\omega$ is the product of the subgroups $L_\omega = 1$, $L \in \mathcal{L}$. Thus once again by 10.7, $N_S(D)$ preserves the regular $(|L|, r)$-product structure $\mathcal{F} = \mathcal{F}(\mathcal{L})$, so in particular G preserves this structure.

Next we consider Case C2. Recall $t = |\Sigma|$ is the order of the G-invariant partition $\Sigma = \Sigma(G)$ of $\mathcal{L}$, and for $\sigma \in \Sigma$, $D_\sigma = \langle \sigma \rangle$ is the direct product of the components in σ, and $H \cap D_\sigma = F_\sigma$ is a full diagonal subgroup of D_σ. We call the groups in Case C2, *diagonal primitive groups* as D_ω is the direct product of the full diagonal subgroups F_σ of the subgroups D_σ.

Suppose first that $t > 1$. In this case set $\mathcal{D} = \{D_\sigma : \sigma \in \Sigma\}$. Then D is the direct product of the groups D_σ, which are permuted by G as Σ is G-invariant. Further $|D_\sigma : D_{\sigma,\omega}| = |D_\sigma : F_\sigma| = |L|^{(r-t)/t}$. Therefore by 10.7, $N_S(D)$ preserves the regular $(|L|^{(r-t)/t}, t)$-product structure $\mathcal{F} = \mathcal{F}(\mathcal{D})$, so in particular G preserves this structures.

Now assume $t = 1$. We call the groups in this subcase *strongly diagonal*, since as we will soon see, they preserve a diagonal structure on Ω. Indeed $\Sigma = \{\mathcal{L}\}$, so $F = H \cap D = D_\omega$ is a full diagonal subgroup of D, and hence (again see 9.2) $D \in \mathrm{Diag}(L, \Omega)$. Then by Remark 9.6, $N_S(D)$ is the stabilizer of the diagonal structure determined by D, and in particular G preserves this structure.

Finally we consider Case C3. Observe G is almost simple iff $r = 1$. In that event we of course call G an *almost simple primitive group.*

So assume that $r > 1$. Now in this case D_ω is the direct product of the groups L_ω, $L \in \mathcal{L}$, so as usual by 10.7, $N_S(D)$ preserves the regular $(|L : L_\omega|, r)$-product structure $\mathcal{F} = \mathcal{F}(\mathcal{L})$. In particular G preserves $\mathcal{F}$. Observe that, as L is simple, L is faithful on L/L_ω, so $Sym(L/L_\omega)$ is not solvable, and hence $|L : L_\omega| \geq 5$.

We call the groups in Case C3 *semisimple primitive groups*, since, at least when $r > 1$, such groups preserve the quite natural product structure $\mathcal{F}$ defined by D, much as a semisimple linear group D defines a product structure in the category of vector spaces: the direct sum decomposition of the space as irreducibles for D. It is convenient to regard almost simple groups as semisimple as they are also in Case C3.

In summary we have shown that:

Definition 11.2. There are six types of primitive subgroups G of S:

(1) The affine primitive groups from Case A, which each preserve the affine structure $\mathcal{A}(D)$ defined by $D = F^*(G)$.

(2) The doubled primitive groups from Case B, which preserve a regular (m, r)-product structure with $m \geq 60$ if $r > 1$, and a diagonal structure if $r = 1$.

(3) The complemented primitive groups from Case C1, which preserve a regular (m, r)-product structure on Ω with $m \geq 60$.

(4) The diagonal primitive groups from Case C2, which preserve a regular (m, t)-product structure with $m \geq 60$ if $t = |\Sigma(G)| > 1$, and preserve a diagonal structure if $t = 1$.

(5) The semisimple groups from Case C3 with $r > 1$ components, which preserve a regular (m, r)-product structure with $m \geq 5$.

(6) The almost simple groups in Case C3 with $r = 1$.

Remark 11.3. The Australian school of permutation group theory uses somewhat different terminolgy. See for example Cheryl Praeger's paper [P], where affine groups are said to be of type HA, doubled groups of type HC, complemented groups are of type TW, diagonal groups are of type CD, strongly diagonal groups are of type SD, and semisimple but not almost simple groups are of type PA.

(11.4) *Let G be a primitive subgroup of S. Then one of the following holds.*

(1) G preserves a regular (m,k)-product structure on Ω with $m \geq 5$.

(2) G preserves an affine space structure on Ω.

(3) G preserves a diagonal structure on Ω.

(4) G is almost simple.

Proof. This is immediate from 11.2.

Remark 11.5. In many problems, primitive groups preserving a regular product structure or a diagonal structure are not too difficult to deal with. Thus already we begin to see from Lemma 11.4 that the primitive groups which are of most interest are the affine groups and the almost simple groups. Lemma 11.6 supplies another illustration of this point. Thus it is important to describe as completely as possible the irreducible subgroups of the general linear group, and the primitive permutation representations of the almost simple groups. By Theorem 1.10 and 6.9.1, the latter problem is equivalent to a description of the maximal subgroups of the almost simple groups.

(11.6) *(Burnside) Assume G is doubly transitive on Ω. Then either*

(1) $n = p^e$ is a power of some prime p, G is the semidirect product of a regular normal subgroup $D \cong E_{p^e}$ by $H = G_\omega \leq GL(D)$, and H is transitive on $D^\#$ by conjugation.

(2) G is almost simple.

Proof. From 6.11, 2-transitive groups are primitive. Thus we can apply 11.4. As G is 2-transitive, $H = G_\omega$ is transitive on $\Omega - \{\omega\}$. But if G is affine, the map $d \mapsto xd$ is an equivalence of the actions of H on D via conjugation, and H on Ω by 6.6.2, so H is transitive on $D^\#$ and (1) holds.

Suppose G preserves a product structure; say $\Omega = Y^I$ with $I = \{1, \ldots, r\}$, and each $g \in G$ is of the form $\prod_i g_i \sigma(g)$ for some $g_i \in Sym(Y)$ and $\sigma(g) \in Sym(I)$. As G is transitive on Ω we may take the function $\omega : I \to Y$ to satisfy $\omega(i) = y_0$ for all i and some fixed $y_0 \in Y$. Then H consists of those $g \in G$ with $y_0 g_i = y_0$ for all i. In particular H permutes the set of elements $\alpha \in \Omega$ such that $\alpha(j) = y_0$ for all but one j. But then H is not transitive on $\Omega - \{\omega\}$.

Therefore by 11.4, we may assume G preserves a diagonal structure determine by some $D \in \mathrm{Diag}(L, \Omega)$, where $D = L_1 \times \cdots \times L_r$ with $r > 1$, $L_i \cong L$ for each i, and $D_\omega = F$ is a full diagonal subgroup of D. As $G \leq N_S(D)$ is 2-transitive, so is $N_S(D)$, so we may take $G = N_S(D)$. By Exercise 9.1, the permutation rank of G is greater than 2, whereas G is 2-transitive, so its permutation rank is 2, a contradiction. This completes the proof.

Exercises for Section 11

1. Let $I = \{1, \ldots, 6\}$, $K = Alt(I)$, $\Omega = K/K_{1,2}$, and embed K in $S = Sym(\Omega)$ via the representation of K by right multiplication. Prove:

(1) K preserves a regular $(5,6)$-partition $P = \{B_1, \ldots, B_6\}$ of Ω.

(2) Let $D = \kappa(P)$, and for $i \in I$ set $L_i = \kappa_{B_i}$. (cf. Definition 7.2) Then $D = L_1 \times \cdots \times L_6$ and $L_i \cong A_5$, so $\mathcal{L} = \{L_1, \ldots, L_6\}$ is the set of components of D. Further K is a complement to D in $G = KD \leq S$, K acts faithfully and transitively on $\mathcal{L}$ as $Alt(\mathcal{L})$, and $D = F^*(G)$.

(3) K is maximal in G.

(4) G is primitive of type (C1) on G/K.

Section 12. The proof of the structure theorem for primitive groups.

In this section we prove Theorem 11.1. Thus we assume G is a primitive permutation group on a set Ω of finite order n, $\omega \in \Omega$, $H = G_\omega$, and $D = F^*(G)$.

(12.1) *(1) H is maximal in G.*

(2) If $1 \neq X \trianglelefteq G$ then $X \not\leq H$ and $G = HX$.

Proof. As G is primitive on Ω, (1) follows from 6.9.1, and (2) follows from 6.10.2.

(12.2) *If G has a nontrivial solvable normal subgroup then case A of Theorem 8.1 holds.*

Proof. If G has a nontrivial solvable normal subgroup, then G has a solvable minimal normal subgroup V, and by 9.4 in [FGT], $V \cong E_{p^e}$ for some prime p and integer e. By 12.1.2, $G = HV$. As $V \trianglelefteq G$, $H \cap V \trianglelefteq H$ and as V is abelian, $H \cap V \trianglelefteq V$. Thus $H \cap V \trianglelefteq HV = G$, so $H \cap V = 1$ by 12.1.2. Therefore V is regular on Ω, so $C_G(V) = V$ by Exercise 6.2. Therefore $F^*(G) = V$ and case A of Theorem 11.1 holds.

Because of 12.2, we may assume in the remainder of this section that G has no nontrivial solvable normal subgroup. Therefore $D = E(G)$, so G has a component L, and L is a nonabelian simple group. Let $D_L = \langle L^G \rangle$. By 12.1.2, $G = HD_L$.

(12.3) *If $D \neq D_L$ then case B of Theorem 8.1 holds.*

Proof. If $D \neq D_L$ then there is $1 \neq X \trianglelefteq G$ with $X \leq C_G(D_L)$. Set $B = XD_L \cap H$ and $X_H = \langle (X \cap H)^X \rangle$. Now

$$XD_L = XD_L \cap G = XD_L \cap HD_L = BD_L$$

so as $X_H \leq X \leq C_G(D_L)$

$$X_H = X_H^{D_L} = \langle (X \cap H)^{XD_L} \rangle = \langle (X \cap H)^{BD_L} \rangle = \langle (X \cap H)^B \rangle \leq H$$

Also X_H is invariant under X and H, while by 12.1.2, $G = HX$, so $X_H \trianglelefteq G$. Therefore $X_H = 1$ by 12.1.2, so $X \cap H = 1$.

Interchanging the roles of X and D_L in the argument of the previous paragraph, we conclude $XD_L = XB$ and $H \cap D_L = 1$. As $X \cap D_L \leq Z(D_L) = 1$, $XD_L = X \times D_L$. As $B \leq H$ and $H \cap D_L = 1$, B is also a complement to D_L in $XD_L = BD_L$, so $X \cong B$. Then by symmetry between X and D_L, $B \cong D_L$, so $X \cong B \cong D_L$. As this is true for both $X = C_G(D_L)$ and a minimal normal subgroup X of G contained in $C_G(D_L)$, we conclude $X = C_G(D_L)$ *is* minimal normal in G. Therefore $D = D_L \times X$ and $B = D_L X \cap H = H \cap D$.

Indeed we have shown that if $\pi_X : B \to X$ and $\pi_D : B \to D_L$ are the projection maps from B onto the factors of the direct sum decomposition $D = D_L \times X$, then π_X and π_D are isomorphisms. That is (cf. Definition 4.8) we have shown B is a full diagonal subgroup of $X \times D_L$.

Let $\beta = \pi_D^{-1}\pi_X : D_L \to X$. As H acts on X, D_L, and $B = H \cap D$, π_X and π_D are H-equivariant isomorphisms, and hence so is β. Let $K = L\beta$. As $D_L = \langle L^H \rangle$,

$$X = D_L\beta = \langle L\beta^H \rangle = \langle K^H \rangle,$$

so $D = \langle (LK)^H \rangle$ and $H \cap D = B$ is the direct product of the H-conjugates of the full diagonal subgroup $\{l \cdot l\beta : l \in L\}$ of $L \times K$. Hence Case B of Theorem 11.1 holds.

Because of 12.3, we may assume in the remainder of this section that $D = D_L$ is the direct product of the H-conjugates of L.

(12.4) *If $D = L$ then $H \cap L \neq 1$ and $H = N_G(H \cap L)$.*

Proof. As $L = D \trianglelefteq G$, $H \cap L \neq L$ by 12.1.2, while $H \cap L \trianglelefteq H$, so if $H \cap L \neq 1$ then $H = N_G(H \cap L)$ as H is maximal in G. Thus we may assume $H \cap L = 1$ and it remains to derive a contradiction. To do so we need to know that the Schreier Property holds. This follows from the classification of the finite simple groups.

Schreier Property. *For each nonabelian finite simple group L, $Out(L)$ is solvable.*

We return to the proof of Lemma 12.4. As $H \cap L = 1$,

$$H \cong H/(H \cap L) \cong HL/L \leq Aut(L)/Inn(L) = Out(L)$$

so by the Schreier Property, H is solvable. As 1 is not a maximal subgroup of L, $H \neq 1$, so $X = O_p(H) \neq 1$ for some prime p. By maximality of H, $H = N_G(X)$, so

$$C_L(X) \leq N_G(X) \cap L = H \cap L = 1$$

Therefore by Exercise 3.2, L is a p'-group, so by 18.7.2 in [FGT], for each prime divisor q of $|L|$, there is a *unique* X-invariant Sylow q-subgroup Q of L. But now $H \leq N_G(Q)$, so QH is a proper subgroup of G and hence $Q \leq H$ by maximality of H. Then $Q \leq H \cap L = 1$, a contradiction.

By 12.4, if $D = L$ then Case C3 of Theorem 11.1 holds with G almost simple. Therefore we may assume during the remainder of this section that:

(12.5) $r = |L^H| > 1$.

Let $X = N_G(L)$ and $X^* = X/C_X(L) = Aut_G(L)$. Observe $D \leq X$, so as $G = HD$, $X = N_H(L)D$ and then as $D = L \times C_D(L)$, also $X^* = N_H(L)^*D^* = N_H(L)^*L^*$ with $N_H(L)^* = Aut_H(L)$. We record this as:

(12.6) $X^* = Aut_G(L)$, $L^* = Inn(L)$, $N_H(L)^* = Aut_H(L)$, *and* $X^* = N_H(L)^*L^*$.

(12.7) *Suppose* Y^* *is a proper nontrivial* $N_H(L)^*$*-invariant subgroup of* L^**, let* Y *be the preimage of* Y^* *in* L*, and set* $J = \langle Y^H \rangle$*. Then* $Y = J \cap L \leq H$ *and* $J = \prod_{K \in \mathcal{L}} (J \cap L)$.

Proof. As Y^* is $N_H(L)^*$-invariant, Y is $N_H(L)$-invariant. Let $\{h_K : K \in \mathcal{L}\}$ be a set of coset representatives for $N_H(L)$ in H with $L^{g_K} = K$. Then $Y^H = \{Y^{g_K} : K \in \mathcal{L}\}$ with $Y^{g_K} \leq K$, so $J = \langle Y^H \rangle$ is the direct product of the groups Y^{g_K}, $K \in \mathcal{L}$. and J is H-invariant. Further $Y < L$, so J is not normal in D, and hence $HJ \neq G$, so $J \leq H$ by 12.1.1, completing the proof of the lemma.

(12.8) *If* $L^* \not\leq N_H(L)^*$ *then* $H \cap L \neq 1$*,* $N_H(L)^*$ *is maximal in* X^**, and* $H \cap L$ *is the preimage in* L *of* $N_H(L)^* \cap L^*$.

Proof. As $L^* \not\leq N_H(L)^*$, $N_H(L)^*$ is contained in some maximal subgroup M of X^* which does not contain L^*. Then applying 12.4 to M, X^*, L^* in the roles of H, G, L, we conclude that $M \cap L^*$ is a proper nontrivial $N_H(L)^*$-invariant subgroup of L^* and $M = N_{X^*}(M \cap L^*)$. Now by 12.7, the preimage Y of $M \cap L^*$ in L is contained in H, so in particular $H \cap L \neq 1$. As $H \cap L \leq N_H(L)$ and $N_H(L)^* \leq M^*$, $H \cap L \leq Y$, so $Y = H \cap L$. Further as $X^* = L^* N_H(L)^*$ and $N_H(L)^* \leq M$, $M = M \cap L^* N_H(L)^* = (M \cap L^*) N_H(L)^* = Y^* N_H(L)^* = N_H(L)^*$, completing the proof of the lemma.

(12.9) *We may assume that $H \cap D \neq 1$.*

Proof. Suppose $H \cap D = 1$. Then D is regular on Ω, so if $Inn(L) \leq Aut_H(L)$ then Case C1 of Theorem 11.1 holds. Thus we may assume $Inn(L) \not\leq Aut_H(L)$, so $L^* \not\leq N_H(L)^*$ by 12.6. But then $H \cap L \neq 1$ by 12.8, contrary to our assumption that $H \cap D = 1$.

(12.10) *If $L^* \not\leq N_H(L)^*$ then case C3 of Theorem 11.1 holds.*

Proof. By 12.8, $N_H(L)^*$ is a maximal subgroup of X^*, and $Y = H \cap L \neq 1$ is the preimage in L of $N_H(L)^* \cap L^*$. Set $J = \langle Y^H \rangle$. By 12.7, $J \leq H \cap D$ is the direct product of the subgroups $J \cap K$, for $K \in \mathcal{L}$. Let Y_0 be the projection of $H \cap D$ on L. Then Y_0^* is $N_H(L)^*$-invariant, so by maximality of $N_H(L)^*$ in X^*, $Y_0^* \leq N_H(L)^* \cap L^* = Y^*$, so $Y_0 = Y$. Therefore $H \cap D = J$. By 12.1, $N_G(J) = H$, while $Aut_H(L) = N_H(L)^*$ and $Aut_G(L) = X^*$ by 12.6. Thus Case C3 if Theorem 11.1 holds.

By 12.10, we may assume $L^* \leq N_H(L)^*$. For $K \in \mathcal{L}$, let $\pi_K : H \cap D \to K$ be the projection of $H \cap D$ on K with respect to the direct sum decomposition $D = \prod_{I \in \mathcal{L}} I$.

(12.11) *$\pi_K : H \cap D \to K$ is a surjection for each $K \in \mathcal{L}$.*

Proof. As H is transitive on $\mathcal{L}$, it suffices to show π_L is surjective. As $H \cap D \neq 1$ by 12.9, the transitivity says $(H \cap D)\pi_L \neq 1$, so $1 \neq ((H \cap D)\pi_L)^* = (H \cap D)^* \cap L^*$. But $(H \cap D)^* \trianglelefteq N_H(L)^*$, so as $L^* \leq N_H(L)^*$ and L^* is simple, we conclude $L^* = ((H \cap D)\pi_L)^*$ and hence $L = (H \cap D)\pi_L$.

By 12.11 and 4.10, $H \cap D = \prod_{\sigma \in \Sigma} (H \cap D)\pi_\sigma$ for some partition Σ of $\mathcal{L}$ with $F_\sigma = (H \cap D)\pi_\sigma$ a full diagonal subgroup of $D\pi_\sigma$, where for $\sigma \in \Sigma$, $D_\sigma = \langle \sigma \rangle$ and $\pi_\sigma : D \to D_\sigma$ is the projection map with respect to the direct sum decomposition $D = \prod_{\sigma \in \Sigma} D_\sigma$. As H acts on $H \cap D$, Σ is an H-invariant partition of $\mathcal{L}$ of order $t = r/d$, where $d = |\sigma| > 1$. As $d > 1$, $t < r$. Hence if Σ is a maximal H-invariant partition of $\mathcal{L}$, then Case C2 of Theorem 11.1 holds, so the proof of Theorem 11.1 is complete.

Suppose Σ is not maximal. Then there exists a nontrivial H-invariant partition Δ of $\mathcal{L}$ with $\Sigma < \Delta$. Pick $\alpha \in \Delta$ with $\alpha \subset \sigma$, let $\pi : D \to D_\alpha$ be the projection map with respect to the direct sum decomposition $D = \prod_{\delta \in \Delta} D_\delta$, and set $F_\alpha = F_\sigma \pi$. As F_σ is a full diagonal subgroup of D_σ, for each $K \in \alpha$, $\pi_K : F_\sigma \to K$ is an isomorphism. Then as $\pi_K = \pi \cdot \pi_{K|D_\alpha}$, $\pi_K : F_\alpha \to K$ is a surjection, so F_α is a full diagonal subgroup of D_α. Moreover $H \cap D < P = \prod_{\delta \in \Delta} F_\delta$. Finally as Δ is H-invariant, also P is H-invariant, so as P is not normal in D, maximality of H implies $P \leq H$. Thus $P \leq H \cap D < P$, a contradiction. Hence the proof of Theorem 11.1 is finally complete.

Section 13. The Structure Theorem for the Symmetric Groups.

In this section Ω is a set of order $n \geq 5$, $S = Sym(\Omega) \cong S_n$, and $A = Alt(\Omega) \cong A_n$. Thus S is almost simple with $F^*(S) = A$. We first prove the Structure Theorem for S on Ω.

Structure Theorem for S_n. *(O'Nan-Scott) If $G \leq S$ then either G is almost simple and irreducible on Ω (ie. G is primitive on Ω) or G stabilizes one of the following structures:*

(1) (Substructure, coproduct structure) A proper nonempty subset of Ω.

(2) (Regular coproduct structure, admissible relation) A nontrivial regular partition of Ω.

(3) A regular (m,k)-product structure on Ω with $m \geq 5$.

(4) An affine space structure on Ω.

(5) A diagonal structure on Ω.

Proof. We may assume G is transitive, or else it acts on some proper nonempty subset of Ω. Similarly we may assume G is primitive, or else it preserves some nontrivial partition. Now 11.4 completes the proof.

(13.1) *The stabilizers in S of our structures are as follows:*

(1) If $\Delta \subseteq \Omega$ with $|\Delta| = m$ then $N_S(\Delta) = S_\Delta \times S_{\Omega-\Delta} \cong S_m \times S_{n-m}$.

(2) If Δ is a regular (m,k)-partition of Ω then $n = mk$ and $N_S(\Delta) \cong S_m$ wr S_k.

(3) If Δ is a regular (m,k)-product structure then $n = m^k$ and $N_S(\Delta) \cong S_m$ wr S_k.

(4) If Δ is an affine space structure on Ω then $n = p^e$ is a prime power and $N_S(\Delta) \cong E_{p^e} \cdot GL_e(\mathbf{F}_p)$.

(5) If Δ is a diagonal structure determined by $D \in \mathrm{Diag}(L,\Omega)$ for some nonabelian finite simple group L, then $n = |L|^{r-1}$ for some $r > 1$, $F^*(N_S(\Delta)) = D$ is the direct

product of r copies of L, $N_S(\Delta) = N_S(D)$ is transitive on the components of D, and $N_S(\Delta)/D \cong Out(L) \times S_r$.

Proof. Part (1) is easy, or see 2.6. Part (2) follows from 7.11, part (3) follow from 10.10.3, part (4) is 8.4, and part (5) is 9.4.

(13.2) *S is transitive on structures of each isomorphism type.*

Proof. Of course the symmetric group is transitive on subsets Δ of Ω of order m for each integer m, and on regular (m, k)-partitions of Δ for each divisor m of n. By 8.5.5, S is transitive on affine structures. By 10.6.4, S is transitive on regular (m, k)-product structures for any suitable m, and by 9.3, S is transitive on its diagonal structures $\mathrm{Diag}(L, \Omega)$ for each nonabelian simple group L.

Remark 13.3. The Structure Theorem tells us that each maximal subgroup of S or A is either a stabilizer of a structure, or an almost simple primitive group. To determine which of these subgroups is actually maximal, it remains to enumerate the inclusions among them. This is done in Theorem 13.5.

Remark 13.4. Suppose G is an almost simple primitive subgroup of S or A, and let $L = F^*(G)$. Then $G \leq N_S(L)$ which is also almost simple and primitive, so if we are interested in maximal subgroups of S we may as well take $G = N_S(L)$. Recall that the conjugation map $c : G \to Aut(L)$ identifies G with a subgroup of $Aut(L)$ containing $Inn(L)$, so modulo such identifications, we may take $L = Inn(L) \leq G \leq Aut(L)$. As $G = N_S(L)$ and $C_S(L) = 1$, $G = N_S(L)/C_S(L) = Aut_S(L)$.

Let $\iota : L \to S$ be the inclusion map and regard ι as a permutation representation of L. Then by 1.13.2,

$$G = Aut_S(L) = Aut_S(L\iota) = Aut(L)_{[\iota]}.$$

What is $Aut(L)_{[\iota]}$? By Theorem 1.10, the equivalence class $[\iota]$ of ι is determined by the conjugacy class of L_ω in L, for $\omega \in \Omega$. Hence by 1.11, the stabilizer in $Aut(L)$ of $[\iota]$ is the subgroup of $Aut(L)$ permutating L_ω^L, which is just $LN_{Aut(L)}(L_\omega)$. From Theorem 1.10, the permutation representation of G on Ω is equivalent to its representation on $G/G_\omega = G/N_G(L_\omega) = G/N_{Aut(L)}(L_\omega)$ by right multiplication.

Theorem 13.5. *(Liebeck-Prager-Saxl) Modulo an explicit list of exceptions, the stabilizers of our structures and the groups $N_S(F^*(G))$, where G is almost simple and primitive* *on Ω, are maximal in S. In particular almost always if $\iota : G \to S$ is a primitive permutation representation of an almost simple group G, then $N_S(L) \cong Aut(L)_{[\iota]}$ is maximal in S, where $L = F^*(G)$.*

Proof. See [LPS2]. The idea of the proof is as follows. Theorem 7.10 says that the stabilizer of proper nonempty subset Γ of Ω is maximal iff $|\Gamma| \neq n/2$. Thus it remains to consider the case when X and Y are transitive on Ω and $X < Y$. Then as X is transitive on Ω, $Y = Y_\omega X$ for $\omega \in \Omega$, so we have a factorization of Y. The most difficult case occurs when Y is almost simple, so it suffices to have good information about the factorizations of almost simple groups. This appears in [LPS1]. The key observation in analyzing such factorizations is that at least one of the subgroups $C = X$ or Y_ω must be large; ie. $|C|^2 \geq |Y|$.

Remark 13.6. When combined, the various results of this section give a classification of the maximal subgoups of S_n and A_n, in a weak sense. To completely enumerate the maximal subgroups, we would have to enumerate all primitive permutation representations of all almost simple groups. For example to enumerate the maximal subgroups of S_n depends upon previously enumerating the primitive representations of S_m of degree n for $m < n$, and hence enumerating the maximal subgroups of S_m.

However in addition to being impractical, such an enumeration is usually not even necessary. For purposes of applications, what is usually desirable is an enumeration of the "large" maximal subgroups. For example a subgroup G of S might be "large" if G contains certain types of elements or subgroups of S, or if $|G|$ is large relative to $|S|$; eg. we have just seen that it would be desirable to know maximal subgroups G of S such that $|G|^2 \geq |S|$.

Section 14. Some problems

In this section Ω is a finite set of order $n \geq 5$, $S = Sym(\Omega)$, and $A = Alt(\Omega)$. Take G to be S or A, so in particular G is almost simple with $F^*(G) = A$.

In this section we call attention to various problems suggested by the discussion in earlier sections. In some cases the problems are only stated for the almost simple group $G = A$ or S. In other cases the problems are stated for arbitrary almost simple groups. Sometimes we supply answers or partial answers to the problems in the case $G = A$ or S, and sometimes we don't.

Notation 14.1. Give a finite group X and a proper subgroup Y of X, write $\mathcal{M}(Y) = \mathcal{M}_X(Y)$ for the maximal members of the poset $\mathcal{O}_X(Y) - \{X\}$. Set $\mathcal{M} = \mathcal{M}_X = \mathcal{M}_X(1)$. Thus $\mathcal{M}(Y)$ is the set of maximal overgroups of Y in X, and $\mathcal{M}$ the set of maximal subgroups of X.

From Remark 13.3, the Structure Theorem for S_n supplies us with a collection of subgroups containing all maximal subgroups of G: the stabilizers in G of our structures, and the almost simple subgroups of G acting primitively on Ω. In Theorems 7.10 and 7.11 we showed that the stabilizers in G of proper nonempty subsets of Ω and nontrivial regular product structures on Ω are almost always maximal in G, and we determined those few cases where such a stabilizer is not maximal. The proofs of those two theorems (and indeed almost all the proofs in these notes) were reasonably elementary. However to go much further, and in particular to decide when our remaining candidates for maximal subgroups of G are actually maximal, it is often necessary to appeal to the classification of the finite simple groups, together with other deep results about the finite simple groups.

Recall also that the Liebeck-Praeger-Saxl Theorem 13.5 tells us which members of our collection are maximal. One way to view this result is that it (essentially) describes $\mathcal{O}_G(M)$ for M in our collection. Indeed Theorem 13.5 says that almost always, $\mathcal{O}_G(M) = \{M, G\}$. While the Liebeck-Praeger-Saxl Theorem does not actual pin down $\mathcal{O}_G(M)$ in the remaining cases, the proof does give information about the set of overgroups of M, and in particular information about $\mathcal{M}(M)$.

This leads us to the following problem:

Problem 14.2. Give a description of $\mathcal{O}_G(H)$ for each primitive subgroup H of G, in terms of the generalized Fitting subgroups of primitive subgroups X of G, and the structures preserved by X.

Remark 14.3. Observe that if H is primitive, then so is each member of $\mathcal{O}_G(H)$. In particular by 11.2, each $X \in \mathcal{O}_G(H)$ is affine, doubled, complemented, diagonal, or semisimple.

In [P], Cheryl Praeger gives one description of $\mathcal{O}_G(H)$, but her description is in terms of what she calls "blow-ups", rather than in terms of structures on Ω. References [A3] and [A4] begin to put in place a theory describing $\mathcal{O}_G(H)$ in terms of generalized Fitting subgroups and structures. This theory involves results which supply answers to questions of the following sort:

Problem 14.4. Let X be an almost simple finite group and M a maximal subgroup of X. Describe $\mathcal{O}_X(F^*(M))$, or perhaps $\mathcal{O}_X(Y)$ for suitable $Y \in \mathcal{O}_M(F^*(M))$.

Remark 14.5. Observe that Lemma 7.8 supplies a partial answer to Problem 14.4 in the case where M is the stabilizer in G of a regular partition P on Ω. For example 7.8.4 says that, for almost all P and any $Y \in \mathcal{O}_M(F^*(N_G(P))$ such that Y is transitive on Ω, $\mathcal{M}(Y) = \{N_G(P(\gamma)) : \gamma \in \Gamma(Y)\}$, where $\Gamma(Y)$ is the set of Y-invariant partitions of P, and $P(\gamma) \in \mathcal{P}(Y)$ is described in 7.8.4.

Suppose Δ is one of our structures and $G = S$. If $N_S(\Delta) \leq A$ then $N_S(\Delta)$ is not maximal in S as $N_S(\Delta) < A < S$. Usually however in this case $N_S(\Delta)$ *is* maximal in A. In any event this shows that it is of interest to determine just when $N_S(\Delta)$ is contained in A.

(14.6) *Let Δ be one of the structures appearing on the list in the Structure Theorem for S_n, and set $M = N_S(\Delta)$. Then*

(1) If Δ is a nontrivial proper subset of Ω, or a regular partition of Ω, then $M \nleq A$.

(2) If Δ is an affine structure on Ω then $M \leq A$ iff n is a power of 2.

(3) If Δ is a regular (m, k)-product structure on Ω with $m \geq 5$, then $M \leq A$ iff m is even and either $k > 2$ or $m \equiv 0 \mod 4$.

(4) Suppose $\Delta = D \in \mathrm{Diag}(L, \Omega)$ for some finite simple group L, with D the direct product of r copies of L. Let $K = Aut(L)$. Then $M \leq A$ iff $r > 2$ or $r = 2$ and the following hold:

(i) All 2-elements in $K - Inn(L)$ are even permutations of L, and

(ii) $|L| \equiv |J| \mod 4$, where $J = \{j \in L : j^2 = 1\}$.

Proof. This is Exercise 14.1.

Recall from Definition 5.12 that a subgroup Y of a group X is of *depth 2* in X if 2 is the maximal length of a chain in $\mathcal{O}_X(Y)$. From Section 5, subgroups of depth 2 in almost simple groups are of interest for a variety of reasons. Observe:

(14.7) *Let X be a group and Y a proper subgroup of X.*

(1) The following are equivalent:

(i) Y is of depth 2 in X.

(ii) $\mathcal{O}_X(Y) \cong \mathbf{M}_m$ for some positive integer m.

(iii) Y is maximal in each member of $\mathcal{M}(Y)$.

(2) If Y is of depth 2 in X then for each pair of distinct $M, N \in \mathcal{M}(Y)$, $M \cap N = Y$.

Recall Question 5.13:

Question 5.13. What are the subgroups of depth 2 in the almost simple groups, and what are the possible M-lattices that can occur as overgroup lattices in almost simple groups? In particular describe the subgroups of G of depth 2 and their overgroup lattices.

Once again a special case occurs when X is almost simple with $L = F^*(X)$ proper in X and Y maximal in L.

(14.8) *Assume X is an almost simple finite group such that $L = F^*(X)$ is of prime index in X, and Y is a maximal subgroup of L such that $N_X(Y) \not\leq L$. Then*
(1) $\mathcal{M}_X(Y) = \{L, N_X(Y)\}$, and
(2) $\mathcal{O}_X(Y) = \{Y, L, N_X(Y), X\}$, so that Y is of depth 2 in X and $\mathcal{O}_X(Y) \cong \mathbf{M}_2$.

Proof. As $|X : L|$ is prime, $L \in \mathcal{M}$, and then as $Y \leq L$, we have $L \in \mathcal{M}(Y)$. As L is maximal in X and $K = N_X(Y) \not\leq L$, $X = LK$. Let $J \in \mathcal{O}_X(Y) - \{X, L\}$. Then as Y is maximal in L, it follows that $Y = J \cap L \trianglelefteq J$, so $J \leq K$. In particular $Y = K \cap L$, so $|K : Y|$ is prime and hence $J = Y$ or K. The lemma follows.

Remark 14.9. Observe that if $H = N_A(\Delta)$ is the stabilizer of one of our structures such that H is a maximal subgroup of A and $N_S(\Delta) \not\leq A$, then Lemma 14.8 says that $N_S(\Delta)$ is maximal in S, and H is of depth 2 in S with $\mathcal{O}_S(H) \cong \mathbf{M}_2$. Moreover inspecting Lemma 14.6, we can decide for which Δ we have $N_S(\Delta) \not\leq A$. As an example, if either

(i) $H = N_A(\Delta)$ for some proper nonempty subset Δ of Ω with $|\Delta| \neq n/2$, or

(ii) $H = N_A(\Delta)$ for some nontrivial regular (m, k)-partition Δ of Ω with $(m, k) \neq (2, 4)$,

then H is maximal in A by Theorems 7.10 and 7.11, while by 14.6.1, $N_S(\Delta) \not\leq A$. Therefore by 14.8, H is of depth 2 in S with $\mathcal{O}_S(H) \cong \mathbf{M}_2$.

In looking for subgroups H of depth 2 in G, we can partition the problem into three cases: (i) H intransitive; (ii) H transitive but imprimitive; (iii) H primitive. Case (i) is the easiest case. Exercises 14.2 and 14.3 partition case (i) into two subcases. Exercise 14.2 gives a complete answer when H has at least three orbits on Ω. Exercise 14.3 gives a partial answer when H has two orbits on Ω, reducing the problem to certain problems involving doubly transitive groups. Using the classification of the almost simple doubly transitive groups, it it then possible to determine the examples which arise in the second subcase. In particular:

Remark 14.10. The examples which arise in case (4) of Exercise 14.3 are:

(1) $n = p+1$ for some prime $p \geq 5$, H is the stabilizer in G_α of an affine structure on Γ, and $\mathcal{Y} = \{Y\}$, where either $G = S$ and $Y \cong PGL_2(p)$, or $G = A$, $p \neq 7$, 11, 17, or 23, and $Y \cong L_2(p)$.

(2) $n = 7$, $G = A$, $H \cong S_4$ is the stabilizer in G_α of a regular $(2,3)$-partition on Γ, and $\mathcal{Y} = \{Y_1, Y_2\}$ with $Y_i \cong L_3(2)$.

(3) $n = 2^e$ for some $e \geq 3$, $G = A$, $H \cong L_e(2)$, and $\mathcal{Y} = \{Y\}$, where Y is the stabilizer in G of an affine structure on Ω.

(4) $n = 2^{m-1}(2^m + \epsilon)$ for some $m \geq 3$ and $\epsilon = \pm 1$, $H \cong O^\epsilon_{2m}(2)$, and $\mathcal{Y} = \{Y\}$ with $Y \cong Sp_{2m}(2)$.

(5) $n = 11$, 12, 22, 23, or 24, $F^*(Y)$ is isomorphic to a Mathieu group M_n for each $Y \in \mathcal{Y}$, and $\mathcal{O}_G(H) \cong \mathbf{M}_2$ or $\mathbf{M}_3$.

(6) $n = 176$, H is an extension of $U_3(5)$ by $\mathbf{Z}_2$, and $\mathcal{Y} = \{Y\}$, where $Y \cong HS$.

(7) $n = 276$, $H \cong Aut(Mc)$, and $\mathcal{Y} = \{Y\}$ with $Y \cong Co_3$.

Remark 14.11. In case (5) of Exercise 14.3, $|\mathcal{Z}| = 2$ so $\mathcal{O}_G(H) \cong \mathbf{M}_3$, and for $Z \in \mathcal{Z}$, one of the following holds:

(1) $n = 6$, $H \cong D_8$, and $Z \cong PGL_2(5)$.

(2) $n = 12$, $H \cong P\Gamma L_2(9)$, and $Z \cong M_{12}$.

(3) $n = 24$, $H \cong Aut(M_{22})$, and $Z \cong M_{24}$.

Here is a partial result in case (ii); as far as I know there is no definitive result in case (ii).

Theorem 14.12. *(Aschbacher-Shareshian) Assume $H \leq G$ with $\mathcal{O}_G(H) \cong \mathbf{M}_2$, and H transitive but imprimitive on Ω. Let $\mathcal{M}(H) = \{M_1, M_2\}$. Then one of the following holds:*

(1) $\mathcal{P}(H) = \{P_1, P_2\}$, $M_i = N_G(P_i)$, and for some $i \in \{1,2\}$, $P_i \leq P_{3-i}$. Further $n \geq 8$ and if $n = 8$ then $G = S$.

(2) $G = A$, $n = 2^{a+1}$, and for some $i \in \{1,2\}$, $\mathcal{P}(H) = \{P\}$, $M_i = N_G(P)$, M_{3-i} is affine with $D = F^*(M_{3-i}) \leq H$, D_P is a hyperplane of D, P consists of the two orbits of D_P on Ω, and $H = N_G(D_P)$.

(3) $G = A$, $n \equiv 0 \mod 4$, $n > 8$, and $\mathcal{P}(H) = \{P_1, P_2\}$ with $M_i = N_G(P_i)$, and for some $j \in \{1,2\}$, $\rho = (P_j, P_{3-j})$ is a $[n/2, 2]$-product structure on Ω with $H = N_G(\rho) \cong \mathbf{Z}_2 \times S_{n/2}$.

For m a proper divisor of n, a $[m, n/m]$-*product structure* on Ω is a pair $\rho = (P, Q)$ such that P is a regular $(m, n/m)$-partition of Ω, Q is a regular $(n/m, m)$-partition, and ρ is (cf Definition 7.4) a CIR-chamber system. Informally, ρ is an identification of Ω with a set product $\Omega \cong \Delta \times \Gamma$, where Δ and Γ are sets of order m and n/m, respectively.

Finally here is a result in case (iii):

Theorem 14.13. *(Aschbacher) Assume n is not a prime and H is a primitive subgroup of G of depth 2 in G. Then either $|\mathcal{M}(H)| = 1$, so that $\mathcal{O}_G(H) \cong \mathbf{M}_1$, or $\mathcal{O}_G(H) \cong \mathbf{M}_2$ and, setting $\mathcal{M}(H) = \{M_1, M_2\}$, one of the following holds:*

(1) H is semisimple and there exist regular (m_i, k_i)-product structures $\mathcal{F}_i$ such that $M_i = N_G(\mathcal{F}_i)$, $\mathcal{F}_1 < \mathcal{F}_2$ so that $m_1 = m_2^s$ for some $s > 1$, and if m_2 is even then $G = A$ and either $s > 2$ or $m_2 \equiv 0 \mod 4$.

(2) $n = q^k$ for some odd prime power q, H and $M_1 = N_G(D)$ are affine, where $D = F^*(H)$, H preserves a nontrivial direct sum decomposition $D = D_1 \oplus \cdots \oplus D_k$ with $|D| = q$, and setting $\mathcal{D} = \{D_1, \ldots, D_k\}$, $M_2 = N_G(\mathcal{F}(\mathcal{D}))$.

(3) $n = 8$, $G = A$, $H \cong L_3(2)$, and $\mathcal{M}(H)$ consists of the stabilizers of the two affine structures preserved by H.

(4) $n = 8$, $G = S$, $H \cong L_3(2)$, and $\mathcal{M}(H) = \{N_S(H), A\}$.

(5) $G = S$, $N_G(H)$ is the stabilizer of an affine structure, regular product structure, or diagonal structure Δ, $H = N_A(\Delta)$, and $\mathcal{M}(H) = \{A, N_S(H)\}$.

The partial order on $\boldsymbol{\mathcal{F}}$ in (1) is defined by $\mathcal{F} = (\Omega_i : i \in I) \leq \tilde{\mathcal{F}} = (\tilde{\Omega}_j : j \in \tilde{I})$ iff $\mathcal{F}$ is an (m,k)-structure, $\tilde{\mathcal{F}}$ is an $(\tilde{m}, \tilde{k})$-structure, and there exists a positive integer s such that $\tilde{k} = ks$ and there exists a regular (s,k)-partition $\Sigma = (\sigma_i : i \in I)$ of $\tilde{I}$ such that for each $i \in I$ and $j \in \sigma_i$, $\Omega_j \leq \Omega_i$.

In the case where n is prime, there exist examples of primitive subgroups H with $\mathcal{O}_G(H) \cong \mathbf{M}_m$ for $m > 2$ (cf. [Pe]), but there are still a finite number of examples, which can be enumerated.

Exercises for Section 14

1. Prove Lemma 14.6.

2. Let Ω be a finite set of order $n \geq 5$ and H a subgroup of $G = Sym(\Omega)$ or $Alt(\Omega)$ such that H is of depth 2 in G and H has at least three orbits on Ω. Then H has exactly three orbits Ω_i, $i \in I = \{1, 2, 3\}$, and setting $n_i = |\Omega_i|$ for $i \in I$, we have:

(1) For $i \in I$, $n_i \neq n/2$.

(2) Either

(i) for distinct $i, j \in I$, $n_i \neq n_j$, or

(ii) up to a permutation of I, $n_2 = n_3 = 1$ and $n_1 = n - 2$.

(3) $\mathcal{M}(H) = \{N_G(\Omega_i) : i \in I\}$, $H = N_G(\Omega_1) \cap N_G(\Omega_2)$, and $\mathcal{O}_G(H) \cong \mathbf{M}_3$.

3. Let Ω be a finite set of order $n \geq 5$ and H a subgroup of $G \in \{S, A\}$, where $S = Sym(\Omega)$ and $A = Alt(\Omega)$. Assume H is of depth 2 in G and H has two orbits θ and $\Gamma = \Omega - \theta$ on Ω. Prove that, interchanging θ and Γ if necessary, one of the following holds:

(1) $\mathcal{O}_G(H) \cong \mathbf{M}_1$.

(2) $G = S$, $H = N_A(\theta)$, $\mathcal{M}(H) = \{A, N_S(\theta)\}$, and $\mathcal{O}_G(H) \cong \mathbf{M}_2$.

(3) There is a regular partition Q of Ω with $\theta \in Q$, such that $H = N_G(\theta) \cap N_G(Q)$, $\mathcal{M}(H) = \{N_G(\theta), N_G(Q)\}$, and $\mathcal{O}_G(H) \cong \mathbf{M}_2$.

(4) $\theta = \{\alpha\}$ is of order 1, $H \neq A_\alpha$ is maximal in G_α, and $\mathcal{M}(H) = \{G_\alpha\} \cup \mathcal{Y}$, where $\mathcal{Y}$ is the set of 2-transitive subgroups Y of G such that $H = Y_\alpha$.

(5) $|\theta| = 2$, $G = A$, H is maximal in $N_G(\theta)$, and $\mathcal{M}(H) = \{N_G(\theta)\} \cup \mathcal{Z}$, where $\mathcal{Z}$ is the set of 3-transitive subgroups Z of G such that $H = N_Z(\theta)$.

Hint: Set $X = N_G(\theta)$, assume H is a counter example, and prove the following lemmas:

(a) $|\theta| \neq n/2$.

(b) H is maximal in X.

(c) There exists $M \in \mathcal{M}(H) - \{X\}$, and for each such M, H is maximal in M and M is transitive on Ω.

(d) $A \not\leq M$.

(e) If $|\theta| > 2$ then $A_\Gamma \not\leq H$, and if $|\theta| = 2$ then $S_\Gamma \not\leq H$.

(f) Set $J = \ker_H(X)$ and $X^* = X/J$. Then X^* is faithful and primitive on X^*/H^*.

(g) $|\theta| \leq 2$ and $G = A$ if $|\theta| = 2$.

(h) $G = A$ and $|\theta| = 2$.

(i) Use Exercise 6.3 to show that either M is 3-transitive on Ω or $Q = \{\theta g : g \in M\} \in \mathcal{P}(M)$.

(j) At most one member of $\mathcal{M}(H) - \{X\}$ is not 3-transitive.

(k) Use the argument in the last paragraph of the proof of 7.10 to show that if M is not 3-transitive then $\mathcal{M}(H) = \{X, M\}$.

Section 15. Δ and $D\Delta$ pairs

Recall the definition of the lattice $\Delta(m)$ and the class of $D\Delta$-lattices from Definition 5.8.

Definition 15.1. Let m be a positive integer. A $\Delta(m)$-*pair* is a pair (G, H) such that G is a finite group, H is a subgroup of G, and $\mathcal{O}_G(H) \cong \Delta(m)$. A Δ-*pair* is a $\Delta(m)$-pair for some positive integer m. A Δ-pair (G, H) is *reduced* if for each $K \in \mathcal{O}_G(H)$, $K = N_G(K)$.

In this section we study Δ-pairs and the class of $D\Delta$-lattices. Recall these lattices appear in Conjecture 5.9, which, if verified, would show that the Palfy-Pudlak Question from Section 5 has a negative answer.

We begin with some examples of Δ-pairs.

Example 15.2. Let L be a group of Lie type of Lie rank l and B a Borel subgroup of L. Then (B, L) is a reduced $\Delta(l)$-pair.

Example 15.3. Let $G = DH$ be the split extension of D by H such that D is an elementary abelian p-group for some prime p, and $D = D_1 \oplus \cdots \oplus D_m$ with D_i an irreducible $\mathbf{F}_pH$-module for each i, and D_i not $\mathbf{F}_pH$-isomorphic to D_j for $i \neq j$. Then (G, H) is a $\Delta(m)$-pair, and the pair is reduced iff D_i is a nontrivial $\mathbf{F}_pH$-module for each i.

Example 15.4. Let $L = G(q)$ be a simply connected group of Lie type with $q = q_0^r$, where $r = \prod_{i \in I} r_i$ is a product of distinct primes r_i, $i \in I = \{1, \ldots, m\}$. For $J \subseteq I$, set $r_J = \prod_{j \in J} r_j$. Let σ be a field automorphism of order r, and for $J \subseteq I$, set $L_J = C_L(\sigma^{r_J})$, so that $L_J \cong G(q_0^{r_J})$. Set $H = L_I \cong G(q_0)$. Then (at least generically), (L, H) is a reduced $\Delta(m)$-pair and $\mathcal{O}_L(H) = \{L_J : J \subseteq I\}$.

Example 15.5. Let $G = HD$ with H a complement in G to a subgroup $D = A \times B$ of G, such that A and B are normal in G, and $H \cap D$ a full diagonal subgroup of $A \times B$. Assume $A = A_1 \times \cdots \times A_m$ with A_i a minimal normal subgroup of G, and A_i is not H-isomorphic to A_j for $i \neq j$ if A_i is abelian. Then (G, H) is a $\Delta(m)$-pair, and the pair is reduced if $[A_i, H] \neq 1$ for each i with A_i abelian.

Notation 15.6. Suppose (G, H) is a Δ-pair. Write $\tilde{\mathcal{O}}_G(H)$ for the minimal members of $\mathcal{O}_G(H)-\{H\}$, and $\mathcal{O}_G(H)^*$ for the maximal members of $\mathcal{O}_G(H)-\{G\}$. For $\alpha \subseteq \tilde{\mathcal{O}}_G(H)$, set $L_\alpha = \langle A : A \in \alpha\rangle$, with $L_\varnothing = H$. For $K \leq L \leq G$, write $d(K, L)$ for the depth of K in L (cf. Definition 5.12).

(15.7) *Assume $\rho = (G, H)$ is a $\Delta(m)$-pair, set $\mathcal{A} = \tilde{\mathcal{O}}_G(H)$, and write Λ for the lattice of subsets of $\mathcal{A}$ under inclusion. Then*

(1) $m = |\tilde{\mathcal{O}}_G(H)| = |\mathcal{O}_G(H)^*| = d(H, G)$.

(2) The map $\varphi : \Lambda \to \mathcal{O}_G(H)$ is an isomorphism of lattices, where $\varphi : \alpha \mapsto L_\alpha$.

(3) For subsets α and β of $\mathcal{A}$, $\langle L_\alpha, L_\beta\rangle = L_{\alpha\cup\beta}$, and $L_\alpha \cap L_\beta = L_{\alpha\cap\beta}$.

(4) For $\alpha \subseteq \beta \subseteq \mathcal{A}$, (L_α, L_β) is a $\Delta(d)$-pair, where $d = |\beta| - |\alpha|$. If ρ is reduced, then so is (L_α, L_β).

(5) If α and β are subsets of $\mathcal{A}$ such that $\mathcal{A} = \alpha \cup \beta$, then $G = \langle L_\alpha, L_\beta\rangle$.

(6) If α and β are subsets of $\mathcal{A}$ such that $\alpha \cap \beta = \varnothing$, then $H = L_\alpha \cap L_\beta$.

Proof. Parts (1) and (2) follow as $\mathcal{O}_G(H) \cong \Delta(m)$. Then (2) implies (3). Then (3) implies the first statement in (4). Further if ρ is reduced, then for $L_\alpha \leq K \leq L_\beta$, $N_{L_\beta}(K) = N_G(K) \cap L_\beta = K \cap L_\beta = K$, completing the proof of (4).

Assume the hypothesis of (5). Then by (2) and (3),

$$G = L_{\mathcal{A}} = L_{\alpha\cup\beta} = \langle L_\alpha, L_\beta\rangle,$$

establishing (5). The dual proof establishes (6).

(15.8) *Assume $\rho = (G, H)$ is a $\Delta(m)$-pair, and $X \leq H$ with $X \trianglelefteq G$. Set $G^* = G/X$ and $\rho^* = (G^*, H^*)$. Then ρ^* is a $\Delta(m)$-pair, and ρ is reduced iff ρ^* is reduced.*

Proof. The map $K \mapsto K^*$ is an isomorphism of $\mathcal{O}_G(H)$ with $\mathcal{O}_{G^*}(H^*)$ with $N_{G^*}(K^*) = N_G(K)^*$.

(15.9) *Let G be a finite group and Λ a sublattice of $\mathcal{O}_G(1)$ containing 1 and G. Assume:*

(i) $\Lambda \cong \Delta(m)$ for some positive integer m, and

(ii) if $m \leq 2$ then $\Lambda = \mathcal{O}_G(1)$, and

(iii) for each $X \in \Lambda - \{G\}$, we have $\mathcal{O}_X(1) \subseteq \Lambda$.

Then

(1) There exists m distinct primes $p_1, \dots, p_m$ such that G is cyclic of order $p_1 \cdots p_m$.

(2) $\Lambda = \mathcal{O}_G(1)$.

Proof. Let $\mathcal{A}$ be the set of minmal members of $\Lambda-\{1\}$ and $I=\{1,\dots,m\}$. As $\Lambda\cong\Delta(m)$, $\mathcal{A}=\{A_i : i\in I\}$ is of order m. Observe that for each $i\in I$, $\mathcal{O}_{A_i}(1)\subseteq\Lambda$. This follows from (iii) if $A_i\neq G$, while if $A_i=G$ then $m=1$, so the remark is a consequence of (ii).

As A_i is minimal in $\Lambda-\{1\}$ and $\mathcal{O}_{A_i}(1)\subseteq\Lambda$, it follows that $|A_i|=p_i$ is prime. In particular if $m=1$ then the lemma holds, so we may assume $m>1$. Hence there exists $j\in I-\{i\}$. Set $A=\langle A_i,A_j\rangle$. As Λ is a sublattice of $\mathcal{O}_G(1)$, $A\in\Lambda$.

Observe that $\mathcal{O}_A(1)\subseteq\Lambda$. Again if $A\neq G$ this follows from (iii), while if $A=G$ then $m=2$, so the remark follows from (ii). As $\mathcal{O}_A(1)\subseteq\Lambda\cong\Delta(m)$, we conclude that

(a) A_i and A_j are the only proper nontrivial subgroups of A.

Then we conclude from (a) and Cauchy's Theorem that:

(b) $|A|=p_i^{e_i}p_j^{e_j}$ for some positive integers e_i and e_j.

We next claim that:

(c) $p_i\neq p_j$.

For if $p_i=p_j=p$ then A is a p-group by (b), so A has a subgroup Z of order p in its center. Then $Z\in\{A_i,A_j\}$ by (a), say $Z=A_i$. Then $Z<ZA_j\leq A$, so $ZA_j=A$ by (a). Therefore $A\cong E_{p^2}$, so A has $p+1>2$ nontrivial proper subgroups, contrary to (a). This completes the proof of (c).

By (b) and (c), for each $p\in\{p_i,p_j\}$, a Sylow p-subgroup of A is a proper nontrivial subgroup of A. We conclude from (a) that for $k\in\{i,j\}$, A_k is the unique Sylow p_k-subgroup of A. Hence:

(d) A is cyclic of order p_ip_j.

By (c) and (d), for all distinct $i,j\in I$, $p_i\neq p_j$ and $[A_i,A_j]=1$. It follows that $H=\langle A_i : i\in I\rangle$ is cyclic of order $p_1\cdots p_r$. Therefore $\mathcal{O}_H(1)\cong\Delta(m)$. Finally as $\Lambda\cong\Delta(m)$ is a sublattice of $\mathcal{O}_G(1)$ containing G, we conclude that $H=G$. Then

$$\mathcal{O}_G(1)=\mathcal{O}_H(1)\cong\Delta(m)\cong\Lambda\subseteq\mathcal{O}_G(1),$$

so $\Lambda=\mathcal{O}_G(1)$, completing the proof of the lemma.

(15.10) *Assume $(G,1)$ is a $\Delta(m)$-pair. Then there exist distinct primes $p_1,\dots,p_m$ such that G is cyclic of order $p_1\cdots p_m$.*

Proof. This is immediate from 15.9 applied to $\Lambda=\mathcal{O}_G(1)$.

(15.11) *Assume (G,H) is a $\Delta(m)$-pair and $H\trianglelefteq G$. Then there exist distinct primes $p_1,\dots,p_m$ such that G/H is cyclic of order $p_1\cdots p_m$.*

Proof. Passing to G/H and appealing to 15.8, we may assume $H=1$. Then the lemma follows from 15.10.

(15.12) *Assume $\rho = (G, H)$ is a $\Delta(m)$-pair with $H \neq N_G(H)$, and set $\mathcal{A} = \tilde{\mathcal{O}}_G(H)$. Then*

(1) $N_G(H) = L_\alpha$ for some $\alpha \subseteq \mathcal{A}$.

(2) Set $|\alpha| = r$. Then there exist distinct primes $p_1, \dots, p_r$ such that $N_G(H)/H$ is cyclic of order $p_1 \cdots p_r$.

(3) For each $K \in \mathcal{O}_G(H)$, $N_G(H) \leq N_G(K)$.

(4) Set $\mathcal{B} = \mathcal{A} - \alpha$ and $B = L_{\mathcal{B}}$. Then each $K \in \mathcal{O}_G(B)$ is normal in G.

(5) (B, H) is a reduced $\Delta(m-r)$-pair, $B \trianglelefteq G$, $G = BN_G(H)$, and $G/B \cong N_G(H)/H$ is cyclic.

Proof. Part (1) follows from 15.7.2. By 15.7.4, $(N_G(H), H)$ is a $\Delta(r)$-pair. Hence (2) follows by applying 15.11 to this pair.

Let $U = N_G(H)$, $A \in \mathcal{A}$, and $\beta = \alpha \cup \{A\}$. If $A \in \alpha$, then U acts on A by (2), so suppose $A \notin \alpha$. As $H \trianglelefteq U$, U permutes $\mathcal{O}_G(H)$ and then also $\mathcal{A}$. By 15.7.2, $\beta = \mathcal{A} \cap L_\beta$. Then as $U \leq L_\beta$, U acts on the unique member A of β not contained in α, so $U \leq N_G(A)$. Hence (3) follows from 15.7.2.

Let $K \in \mathcal{O}_G(B)$; then by 15.7.2, $K = L_\beta$ for some $\beta \subseteq \mathcal{A}$ containing $\mathcal{B}$. Then by 15.7.5, $G = \langle L_\beta, L_\alpha \rangle = \langle K, U \rangle$, so $G \leq N_G(K)$ by (3), establishing (4).

Next by 15.7.6, $U \cap B = L_\alpha \cap L_{\mathcal{B}} = H$, so $N_B(H) = B \cap U = H$. Then B is reduced by the next lemma, whose proof does not depend upon (5). By (4), $B \trianglelefteq G$, and we saw in the previous paragraph that $G = \langle B, U \rangle$, so $G = BU$. As $B \cap U = H$, $G/B = BU/B \cong U/(B \cap U) = U/H$, and then (2) completes the proof of (5).

(15.13) *Assume $\rho = (G, H)$ is a Δ-pair. Then ρ is reduced iff $N_G(H) = H$.*

Proof. Trivially if ρ is reduced then $H = N_G(H)$. Conversely assume $N_G(H) = H$ and let $K \in \mathcal{O}_G(H)$ and $J = N_G(K)$. Suppose $K \neq J$. Then by 15.7 there exists $A \in \tilde{\mathcal{O}}_G(H)$ with $A \not\leq K$. By 15.7.6, $A \cap K = H$, so as A acts on A and K, $A \leq N_G(H)$ and hence $H \neq N_G(H)$. Thus if ρ is not reduced then $H \neq N_G(H)$, so if $H = N_G(H)$ then ρ is reduced.

Remark 15.14. Let $\rho = (G, H)$ be a Δ-pair. Observe that by 15.12.5 and 15.13, if ρ is not reduced then there exists a normal subgroup B of G containing H such that $\rho' = (B, H)$ is a reduced Δ-pair, $G = BN_G(H)$, and G/B is cyclic. Thus, replacing (G, H) by (B, H), we can often reduce to the case where our Δ-pair is reduced, so there is little loss of generality in assuming a Δ-pair is reduced.

(15.15) *Assume G is a finite group and H is a subgroup of G such that $\mathcal{O}_G(H)$ is disconnected and $\mathcal{C}$ is a connected component of $\mathcal{O}_G(H)'$ isomorphic to $\Delta(m)'$ for some $m \geq 3$. Then for each $K \in \mathcal{C}$,*

(1) H is not normal in G and $N_K(H) = H$, and

(2) $N_G(K) = K$.

Proof. First suppose that $H \trianglelefteq G$, and set $G^* = G/H$. As in the proof of 15.8, the map $K \mapsto K^*$ is an isomorphism of $\mathcal{O}_G(H)$ with $\mathcal{O}_{G^*}(1)$. Now applying 15.9 to the sublattice $\Lambda = \mathcal{C}^* \cup \{1, G^*\}$ of $\mathcal{O}_{G^*}(1)$, we conclude that $\mathcal{O}_{G^*}(1) = \Lambda$. But then $\mathcal{O}_G(H) = \mathcal{C} \cup \{G, H\}$, contrary to the assumption that $\mathcal{O}_G(H)$ is disconnected.

Therefore H is not normal in G. Next suppose $K \in \mathcal{C}$ with $K \neq N_G(K)$. Observe (G, K) is a Δ-pair, so by 15.12.4, there exists a proper subgroup B of G in $\mathcal{O}_G(K)$ normal in G. As $\mathcal{O}_G(H)$ is disconnected there is a maximal subgroup $M \notin \mathcal{C}$. Then $G = BM$, so $B \cap M \trianglelefteq X$. But as $X \notin \mathcal{C}$, $B \cap M = H$, so $H \trianglelefteq M$. Next let A be minimal in $\mathcal{C}$ with $A \leq N_G(K)$ but $A \not\leq K$. Then A acts on $A \cap K = H$, so $G = \langle A, M \rangle \leq N_G(H)$, contrary to an earlier reduction. This completes the proof of (2).

Now assume $N_K(H) \neq H$. Without loss of generality, K is maximal in $\mathcal{C}$. Then (K, H) is a $\Delta(m-1)$-pair, so it follows from 15.12.4 that there is a maximal member of J of $\mathcal{O}_K(H) - \{K\}$ normal in K. Further as $m \geq 3$, $J \in \mathcal{C}$, so (2) supplies a contradiction which establishes (1).

(15.16) *(Shareshian) Assume G is a finite group and H is a subgroup of G such that $\mathcal{O}_G(H)$ is a $D\Delta$-lattice. Then*

(1) for each $K \in \mathcal{O}_G(H)$, $K = N_G(K)$, and

(2) for each $H \leq K \leq L \leq G$ such that $(K, L) \neq (H, G)$, (L, K) is a reduced Δ-pair.

Proof. As $\mathcal{O}_G(H)$ is a $D\Delta$-lattice, for each $K \in \mathcal{O}_G(H) - \{G, H\}$, the connected component $\mathcal{C}(K)$ of $\mathcal{O}_G(H)'$ containing K is isomorphic to $\Delta(m_K)$ for some $m_K \geq 3$. Therefore $K = N_G(K)$ by 15.15.2. Of course $G = N_G(G)$. Finally set $J = N_G(H)$. By 15.15.1, $J \neq G$, so if $H \neq J$ then $\mathcal{C}(J) \cong \Delta(m_J)$ and 15.15.1 supplies a contradiction. This establishes (1).

Assume $(K, L) \neq (H, G)$ is as in (2). Then $X = L$ or K is in $\mathcal{O}_G(H)'$, so $\mathcal{C}(X) \cong \Delta(m_X)'$, and (X, H) or (G, X) is a Δ-pair, respectively. Then applying 15.7.4 to this pair, (L, K) is also a Δ-pair, and the pair is reduced by (1).

Remark 15.17. Let $\rho = (G, H)$ be a $\Delta(m)$-pair. In Exercise 15.1, we see how to associate to ρ a residually connected geometric simplicial complex $\mathcal{C}(\rho)$ over $I = \{1, \dots, m\}$

on which G acts as a flag transitive group of automorphisms, (cf. Exercise 7.4 for the definitions) and a nondegenerate chamber system $X(\rho)$ over I, on which G acts as a chamber transitive group of automorphisms. Indeed the class of complexes $\mathcal{C} = (V, \Sigma)$ and flag transitive groups G of automorphisms of $\mathcal{C}$, obtained from Δ-pairs via this construction, is characterized by the property that for each chamber ω of $\mathcal{C}$, and for each proper subset σ of ω, the maximal overgroups of G_ω in G_σ are the stabilizers $G_{\sigma,v}$, $v \in \omega - \sigma$.

The complex $\mathcal{C}(\rho)$ supplies a geometric tool which is potentially useful for studying Δ-pairs, and which makes Δ-pairs more interesting.

Exercises for Chapter 15

1. Let $\rho = (G, H)$ be a $\Delta(m)$-pair, $I = \{1, \dots, m\}$, and $\mathcal{F} = \mathcal{O}^*_G(H) = \{G_i : i \in I\}$. For $J \subseteq I$ and $i \in I$, set

$$G_J = \bigcap_{j \in J} G_j,\ P_i = G_{i'},\ \text{and}\ P_J = \langle P_j : j \in J\rangle.$$

Set $\mathcal{C} = \mathcal{C}(\rho) = \mathcal{C}(G, \mathcal{F})$. (cf Exercise 10.2) Prove:

(1) For each $J \subseteq I$, $G_J = P_{J'}$.

(2) $\mathcal{C}$ is a residually connected geometric complex over I, and G acts as a flag transitive group of automorphism on $\mathcal{C}$.

(3) $\mathcal{P} = \{\mathcal{P}_i : i \in I\} = \tilde{\mathcal{O}}_G(H)$ and $X = X(H, H, \mathcal{P})$ (cf. Exercise 7.3) is a nondegenerate chamber system on I on which G acts as a chamber transitive group of automorphisms.

(4) $X \cong \mathbf{X}(\mathcal{C})$ and $\mathcal{C} \cong \boldsymbol{\mathcal{C}}(X)$.

(5) G is residually primitive on $\mathcal{C}$; that is for each $J \subset I$ and each simplex σ of $\mathcal{C}$ of type J, G_σ acts primitively on the vertices of $\mathrm{lk}(\sigma)$ of type i for each $i \in J'$.

Section 16. The structure of groups in a Δ-pair

In this section we begin to generate constraints on the structure of a finite group G and the embedding of H in G, when (G, H) is a Δ-pair.

Definition 16.1. Given a finite group G, a subgroup D of G, and $H \leq N_G(D)$, let $\mathcal{I}_D(H)$ be the set of H-invariant subgroups of D, and $\mathcal{V}_D(H) = \{I \in \mathcal{I}_D(H) : H \cap D \leq I\}$.

(16.2) *Assume $\rho = (G, H)$ is a $\Delta(m)$-pair and $1 \neq D \trianglelefteq G$. Set $G^* = G/D$ and $\mathcal{A} = \tilde{\mathcal{O}}_G(H)$. Then*

(1) $\mu = (HD, H)$ is a $\Delta(m')$-pair, where $m' = |\tilde{\mathcal{O}}_{HD}(H)|$, and if ρ is reduced then so is μ.

(2) $\eta = (G, HD)$ is a $\Delta(m - m')$-pair, and if ρ is reduced then so is η.

(3) $\rho^* = (G^*, H^*)$ is a $\Delta(m - m')$-pair, and if ρ is reduced then so is ρ^*.

(4) If ρ is reduced then $D = [H, D](H \cap D)$.

(5) Let $\mathcal{B} = \mathcal{A} - \tilde{\mathcal{O}}_{HD}(H)$ and $B = \langle \mathcal{B} \rangle$. Then $\rho' = (B, H)$ is a $\Delta(m - m')$-pair, $G = BD$, and $H \cap D = B \cap D \trianglelefteq B$. If ρ is reduced then so is ρ'.

(6) The map $\psi : \mathcal{V}_D(H) \to \mathcal{O}_{HD}(H)$ defined by $V \mapsto HV$ is an isomorphism of lattices with inverse $K \mapsto K \cap D$.

(7) B acts on each member of $\mathcal{V}_D(H)$.

(8) $\Phi(D) \leq H$.

Proof. Adopt Notation 15.6. As $D \trianglelefteq L$, $HD \in \mathcal{O}_G(H)$. Then by 15.7.2, $HD = L_\alpha$ for some $\alpha \subseteq \mathcal{A}$, and by 15.7.4, μ is a $\Delta(m')$-pair, where $m' = d(H, HD) = |\alpha|$. Moreover if ρ is reduced, so is μ. That is (1) holds. Similarly (2) follows from 15.7.4. As $\rho^* = ((HD)^*, H^*)$, (3) follows from (2) and 15.8 applied to D, HD in the role of X, H.

Assume for the moment that ρ is reduced. Now $K = H[H, D] \trianglelefteq HD$, and as ρ is reduced, we have $N_G(K) = K$, so $K = HD$. Hence $D = D \cap K = [H, D](H \cap D)$, establishing (4).

Adopt the notation of (5). Then $B = L_{\mathcal{B}}$ and $\mathcal{B} = \mathcal{A} - \alpha$. Thus by 15.7.4, ρ' is a $\Delta(m - m')$-pair, and ρ' is reduced if ρ is reduced. Further by 15.7.6, $B \cap HD = H$, so $B \cap D = B \cap HD \cap D = B \cap H \cap D = H \cap D$. As $B \cap D \trianglelefteq B$, we have $H \cap D \trianglelefteq B$. Similarly by 15.7.5, $G = \langle HD, B \rangle = DB$, completing the proof of (5).

Visibly the map ψ in (6) is a function from $\mathcal{V}_D(H)$ into $\mathcal{O}_{HD}(H)$ and the map $\varphi : K \mapsto K \cap D$ is a function from $\mathcal{O}_{HD}(H)$ to $\mathcal{V}_D(H)$. Check that $\varphi = \psi^{-1}$ to complete the proof of (6).

Let $V \in \mathcal{V}_D(H)$ and $U = \langle V^B \rangle$. By (6), $VH = L_\gamma$ for some $\gamma \subseteq \alpha$, so by 15.7.3,

$$UB = \langle HV, B \rangle = \langle L_\gamma, L_\beta \rangle = L_{\gamma \cup \beta}.$$

Further $U \in \mathcal{V}_D(H)$, so $UH = L_\delta$ for some $\gamma \subseteq \delta \subseteq \alpha$. Then $\delta \subseteq \alpha \cap (\gamma \cup \beta) = \alpha \cap \gamma = \gamma$, so $\delta = \gamma$ and hence $U = V$ by (6). This establishes (7).

Let $\mathcal{C}$ be the set of maximal proper subsets of α, and for $\gamma \in \mathcal{C}$, set $D_\gamma = L_\gamma \cap D$, let

$\mathcal{M}_\gamma$ be the set of maximal subgroups of D containing D_γ, and set

$$V_\gamma = \bigcap_{M \in \mathcal{M}_\gamma} M.$$

Then H permutes $\mathcal{M}_\gamma$, so $D_\gamma \leq V_\gamma \in \mathcal{V}_D(K)$. Thus by (6), $V_\gamma = L_\delta \cap D$ for some $\delta \subseteq \alpha$, so $\gamma = \delta$ by maximality of γ, and hence $V_\gamma = D_\gamma$. Therefore writing $\mathcal{M}$ for the set of maximal subgroups of D, and observing that

$$\bigcap_{\gamma \in \mathcal{C}} \gamma = \varnothing,$$

it follows fromm 15.7.3 that

$$\Phi(D) = \bigcap_{M \in \mathcal{M}} M \leq \bigcap_{\gamma \in \mathcal{C}, M \in \mathcal{M}_\gamma} M = \bigcap_{\gamma \in \mathcal{C}} V_\gamma = D \cap \bigcap_{\gamma \in \mathcal{C}} L_\gamma = D \cap L_\varnothing = D \cap H,$$

establishing (8).

(16.3) *Assume $\rho = (G, H)$ is a $\Delta(m)$-pair, $1 \neq D \trianglelefteq G$ is an abelian p-group for some prime p, and* $\ker_{H \cap D}(G) = 1$. *Set $\mathcal{A} = \tilde{\mathcal{O}}_G(H)$, $\mathcal{B} = \tilde{\mathcal{O}}_{HD}(H)$, and $B = L_{\mathcal{B}}$. Then*

(1) $H \cap D = 1$.

(2) B is a complement to D in G.

(3) $\Phi(D) = 1$.

(4) $D = D_1 \oplus \cdots \oplus D_r$, where D_i is an irreducible $\mathbf{F}_pH$-module, and for $i \neq j$, D_i is not $\mathbf{F}_pH$-isomorphic to D_j.

(5) $\{D_1, \ldots, D_r\}$ is the set of irreducible $\mathbf{F}_pH$-submodules of D.

(6) (HD, H) is a $\Delta(r)$-pair.

Proof. By 16.2.5, $G = BD$ and $H \cap D = B \cap D \trianglelefteq B$. As D is abelian, $H \cap D \trianglelefteq D$. Therefore $H \cap D \trianglelefteq G$ by 16.2.5, so (1) holds as $\ker_{H \cap D}(G) = 1$. Then as $G = BD$ and $B \cap D = H \cap D$, (2) holds. Similarly by 16.2.8, $\Phi(D) \leq H$, so (3) holds.

Adopt Notation 15.6. By 15.7.1, $HD = L_\alpha$, where $\alpha = \tilde{\mathcal{O}}_{HD}(H)$. Let $r = |\alpha|$, so that $\mu = (HD, H)$ is a $\Delta(r)$-pair by 16.2.1. Let $\alpha = \{X_1, \ldots, X_r\}$ and $D_i = X_i \cap D_i$. By 16.2.6, $\mathcal{D} = \{D_1, \ldots, D_r\}$ is the set of minimal H-invariant subgroups of D. Thus $\mathcal{D}$ is the set of irreducible $\mathbf{F}_pH$-submodules of D. By 15.7.2, $D = \langle \mathcal{D} \rangle$, so D is a semisimple $\mathbf{F}_pH$-module. Further for $1 \leq j \leq r$, $Y = \langle D_j : j \neq i \rangle = L_\beta \cap D$, where $\beta = \alpha - \{X_i\}$. Then by 15.7.6,

$$Y \cap D_i = L_\beta \cap X_i \cap D = H \cap D = 1,$$

so $D = D_1 \oplus \cdots \oplus D_r$. Finally as $\mathcal{D} = \{D_1, \ldots, D_r\}$, for $i \neq j$, $\{D_i, D_j\}$ are the set of irreducibles in $D_i \oplus D_j$, so D_i is not $\mathbf{F}_pH$-isomorphic to D_j, completing the proof of the lemma.

(16.4) *Assume (G,H) is a Δ-pair such that $\ker_H(G) = 1$. Then*

(1) $H \cap F(G) = 1$.

(2) $\Phi(O_p(G)) = 1$ for each prime p.

(3) All components of G are simple.

(4) There is a complement B to $F(G)$ in G containing $HE(G)$.

Proof. Part (2) follows from 16.2.8. Then (1) follows from 16.3.1. Indeed $\Phi(E(G)) = Z(E(G))$, so also $Z(E(G)) = 1$ by (1) and 16.2.8. Hence (3) holds.

To prove (4), we show that if N is an abelian normal subgroup of G with $\ker_{N\cap H}(G) = 1$ then here is a complement to N in G containing $HE(G)$. Assume otherwise and pick a counter example with N minimal. If $N = 1$ the claim is trivial, so we may assume p is a prime with $D = O_p(N) \neq 1$. By hypothesis, D is abelian and $\ker_{H\cap D}(G) = \ker_{H\cap N}(G) \cap D = 1$, so the hypotheses of 16.3 are satisfied. Hence by 16.3.2, there is a complement G_0 to D in G containing $O^p(N)E(G)H$ such that (G_0, H) is a Δ-pair. Then by minimality of G, there is a complement B to $O^p(N)$ in G_0 containing $HE(G)$, completing the proof.

(16.5) *Assume (G,H) is a $\Delta(m)$-pair and $D = D_1 \times D_2$ is a subgroup of G with $1 \neq D_i \trianglelefteq G$. Let V_i be the projection of $H \cap D$ on D_i, $H_i = H \cap D_i$, and $V = V_1V_2$. Then*

(1) $H_i \trianglelefteq V$.

(2) Set $(VH)^* = VH/H_1H_2$. Then $(H \cap D)^*$ is a full diagonal subgroup of $V_1^* \times V_2^*$.

(3) $V_1^* = X_1^* \times \cdots \times X_r^*$, where $\{X_1^*, \ldots, X_r^*\}$ are the minimal normal subgroups of $(VH)^*$ contained in V_1^*.

(4) (HV, V) is a $\Delta(r)$-pair.

(5) $\tilde{\mathcal{O}}_{DH}(H) = \tilde{\mathcal{O}}_{D_1H}(H) \cup \tilde{\mathcal{O}}_{D_2H}(H)$.

(6) V acts on each member of $\mathcal{V}_{D_1}(H)$.

(7) Let $\tilde{\mathcal{E}}_1$ be the set of minimal members of $\mathcal{V}_{D_1}(H) - \{H_1\}$ which are not contained in V_1, and $E_1 = \langle \tilde{\mathcal{E}}_1 \rangle$. Then $E_1 \trianglelefteq G$.

(8) If D_1 is a minimal normal subgroup of G then either

(a) $V_1 = H_1$, or

(b) $H_1 = 1$ and $D_1 = V_1$.

Proof. Let $J = H \cap D$ and $X = HV$. Then $V = V_2J$, and V_2 and J act on H_1, so $H_1 \trianglelefteq V$, establishing (1). By construction, $J^* \cap V_1^* = (J \cap V_1)^* = H_1^* = 1$, so (2) follows.

Let $\mathcal{Y}$ be the minimal members of $\mathcal{V}_{V_1}(H) - \{H_1\}$. As $V = V_2J$, and V_2 and J act on each $Y \in \mathcal{Y}$, $Y \trianglelefteq V_1$. Then $\mathcal{Y}^*$ is the set of minimal normal subgroups of X^* contained in V_1^*. Then arguing as usual, (3) and (4) hold.

Part (5) follows from 15.7.3. As $V = V_2J$, and V_2 and J acts on each member U of $\mathcal{V}_{D_1}(H)$, V acts on U, proving (6).

By 16.2.5 we have a factorization $G = BD_1$ and by 16.2.7, B acts on each member of $\tilde{\mathcal{E}}_1$. Thus B acts on E_1. By (6), V acts on E_1, and by 16.2.6, $D_1 = E_1V_1$. Thus D_1 acts on E_1, so $L = BD_1$ acts on E_1, establishing (7).

Assume D_1 is a minimal normal subgroup of L. Then by (7), either $D_1 = E_1$ or $E_1 = 1$. In the first case $V_1 \leq E_1$, while by 16.2.6, for A a minimal member of $\mathcal{V}_{D_1}(H)$ not contained in $\tilde{\mathcal{E}}_1$, $A \not\leq E_1$. Therefore $V_1 = H_1$, so (8a) holds in this case.

So assume $E_1 = 1$. Then $D_1 = V_1$ by 16.2.6. Recall we have a factorization $G = BD_1$, and by 15.17.7, $H_1 \trianglelefteq B$. By (1), $H_1 \trianglelefteq V_1 = D_1$, so $H_1 \trianglelefteq G$. Then by minimality of D_1, $H_1 = 1$ or D_1, completing the proof of (8).

(16.6) *Assume (G, H) is a reduced $\Delta(m)$-pair and let $X \trianglelefteq G$. Then*

(1) $G = \langle H^G \rangle$.

(2) If $H \leq X$ then $X = G$.

(3) If G/X is nilpotent then $G = XH$.

(4) Suppose $X \leq Y \trianglelefteq G$ with G/Y nilpotent and Y/X cyclic. Set $Y_H = X(Y \cap H)$. Then $G = HY$, and Y/Y_H is cyclic of order $r_1 \cdots r_s$ with the r_i distinct primes and $m = d(H, HX) + s$.

Proof. Let $Y = \langle H^G \rangle$. Then $Y \in \mathcal{O}_G(H)$ and $G = N_G(Y)$. But as (G, H) is reduced, $Y = N_G(Y)$, so (1) holds. Trivially (1) implies (2).

It remains to prove (3) and (4). By 16.2.2, (G, HX) is a reduced Δ-pair, and by 15.8, $(G/X, HX/X)$ is a reduced Δ-pair. By 15.7.4, $m = d(H, HX) + d(HX, G)$. Thus replacing (G, H) by $(G/X, HX/X)$, we may assume $X = 1$ and either G is nilpotent or G/Y is nilpotent and Y is cyclic.

If G is nilpotent, it remains to show that $G = H$. But by (1), $G = \langle H^G \rangle$, so as G is nilpotent, $G = H$, completing the proof of (3).

So assume $Y \trianglelefteq G$ with Y cyclic and G/Y nilpotent. By (3), $G = HY$. Observe that $Y_H = Y \cap H$, so as Y is cyclic, $Y_H \trianglelefteq G$. By 16.2.8, $\Phi(Y) \leq Y_H$. Therefore Y/Y_H is cyclic of order $r_1 \cdots r_s$ for distinct primes $r_1, \ldots, r_s$. In particular $d(H, Y) = s$. This completes the proof.

Exercises for Section 16

1. Assume G is a finite solvable group and H is a subgroup of G such that H has a normal complement X in G, and (G,H) is a $\Delta(m)$-pair. Let Π be the set of prime divisors of $|X|$. Prove:

(1) The map $K \mapsto K \cap X$ is an isomorphism of lattices from $\mathcal{O}_G(H)$ to $\mathcal{I}_X(H)$, with inverse $Y \mapsto YH$.

(2) Each minimal normal subgroup of G contained in X is in the set $\tilde{\mathcal{I}}_X(H)$ of minimal nonidentity members of $\mathcal{I}_X(H)$.

(3) For each $Y \in \mathcal{I}_X(H)$, there is a unique H-invariant complement to Y in X.

(4) For each $\pi \subseteq \Pi$, there is a unique H-invariant Hall π-subgroup X_π of X, and each H-invariant π-subgroup of X is contained in X_π.

(5) All Sylow groups of X are elementary abelian.

(6) $F(X)$ is abelian.

(7) Let $1 = F_0(X) < \cdots < F_l(X) = X$ be the Fitting series for X, and $l = l(X)$ the Fitting length of X. (cf. Exercise 3.4) Prove that for $1 \leq i \leq l$, there exists $A_i \in \tilde{I}_{F_i(X)}(H)$ with $A_i \not\leq F_{i-1}(X)$, such that A_i is a p_i-group for distinct primes $p_1, \ldots, p_l$, and $Y = A_1 \cdots A_l \in \mathcal{I}_X(H)$ with $F_i(Y) = A_1 \cdots A_i$ for each $1 \leq i \leq l$. Hence $l(X) \leq |\Pi|$.

(8) Is it true that if (G,H) is reduced, then $X/F(X)$ is abelian?

2. Let G be a finite group, $I = \{1, \ldots, m\}$ a finite set, and $\mathcal{F} = (G_i : i \in I)$ a family of subgroups of G. Define the geometric complex $\mathcal{C} = \mathcal{C}(G, \mathcal{F})$ as in Exercise 10.2, and adopt the notation of that exercise. For $i \in I$, set $n_i = |G : G_i|$. Prove:

(1) If $m = 2$ then $\mathcal{C} \cong \mathcal{C}(n_1, n_2)$ (cf. Exercise 10.3) iff $G = G_1 G_2$.

(2) $\mathcal{C} \cong \mathcal{C}(n_i : i \in I)$ iff $\mathcal{C}$ is residually connected and for all i, j, $P_{i,j} = P_i P_j$.

(3) Let $H = G_I$ and assume (G,H) is a $\Delta(m)$-pair such that G is solvable and H has a nontrivial normal complement X in G. Prove $\mathcal{C} \cong \mathcal{C}(n_i : i \in I)$.

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