The subgroup structure of finite alternating and symmetric groups

Permutation representations of groups and subgroup structure are essentially equivalent notions. In the case of finite groups, the modern approach to the study of these topics proceeds by reduction to the almost simple case. By now there is a fairly satisfactory theory giving a qualitative description of primitive permutation groups and maximal subgroups of almost simple groups. Thus the time seems right to begin a deeper study of the lattice of subgroups of finite groups, and, in particular, of almost simple groups.

The focus of this course is on the most accessible class of finite almost simple groups: the alternating and symmetric groups. We review the by now standard description of primitive groups, the O'Nan-Scott Theorem giving a qualitative description of the set $\mathcal{M}$ of maximal subgroups of a finite symmetric group S, and the work of Liebeck-Praeger-Saxl which gives (in a weak sense) a complete description of $\mathcal{M}$. Then we go on to discuss the lattice $\mathcal{O}_S(H)$ of overgroups of a primitive subgroup H of S, and more generally begin to put in place a qualitative theory of how maximal subgroups of S intersect.

To supply motivation, we discuss a program to show the following well known question has a negative answer:

Question. Is every nonempty finite lattice isomorphic to an interval in the lattice of subgroups of some finite group?