"Wave-style" Geometry of Interaction Models in Rel are Graph-like Lambda-models*

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Abstract. We study the connections between *graph models* and *"wave-style" Geometry of Interaction* (GoI) λ-models. The latters arise when Abramsky's GoI axiomatization, which generalizes Girard's original GoI, is applied to a traced monoidal category with the *categorical product* as tensor, using a countable power as the traced strong monoidal functor !. Abramsky hinted that the category *Rel* of sets and relations is the basic setting for traditional denotational "static semantics". However, the category *Rel* together with the cartesian product apparently escapes original Abramsky's axiomatization. Here we show that, by moving to the category *Rel** of *pointed sets* and relations preserving the distinguished point, and by sligthly relaxing Abramsky's GoI axiomatization, we can recover a large class of graph-like models as wave models. Furthermore, we show that the class of untyped λ-theories induced by wave-style GoI models is richer than that induced by game models.
Keywords: (linear) graph model, traced monoidal category, weak linear category, categorical geometry of interaction.

Introduction

Geometry of Interaction and game models have been the most relevant novelties in the last decade in the field of semantic analysis of proof theory and functional languages.

In [1], Abramsky provides a categorical axiomatization/generalization of Girard's *Geometry of Interaction* (GoI) [18], embracing previous axiomatic approaches, such as that based on *dynamic algebras* [16, 17] and the one in [4]. This generalization is based on traced monoidal categories, [23], and it consists in building a compact closed category $\mathcal{G}(\mathcal{C})$ (*GoI category*) from a traced symmetric monoidal category $\mathcal{C}$. In [2, 3], the construction is extended to exponentials, which, in a general categorical setting, are captured by a strong monoidal functor ! on the traced category $\mathcal{C}$, together with some additional structure. Under these conditions on $\mathcal{C}$, the GoI category $\mathcal{G}(\mathcal{C})$ is a *weak linear category* (WLC), i.e. a weakening of a linear category (see [13]). Moreover, every reflexive object in a WLC gives rise to a *linear combinatory algebra* (LCA).

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Following [1, 2], there are two main instantiations of the GoI axiomatization. In the "particle-style" GoI, the tensor on the underlying category is a coproduct and the strong monoidal functor is a *countable* copower. Girard's GoI is an instance of this. Composition in the GoI category can be intuitively understood by simulating the flow of a particle around a network. Dually, in the "wave-style" GoI, the tensor is a product and the strong monoidal functor amounts to a *countable* power. Composition in the GoI category is defined now statically and globally. In [1], the category $(Rel, +)$ was suggested as the "basic setting" for particle/*dynamic semantics*, while $(Rel, \times)$ as the "basic setting" for wave/*static semantics*. This is clearly the case of the former, wich underlies many game categories in the style of [5], and contains as subalgebras those fruitfully used in [6, 8, 7]. On the other hand, the thesis that $(Rel, \times)$ is the basic setting for static semantics is less immediate, also because $(Rel, \times)$ itself apparently escapes the original GoI axiomatization of [3].

The connections between traditional (static) semantics and wave GoI have not received much attention in the literature, apart from the investigations of some special wave-style models in [10, 9]. In the present paper and in [20], this connection has been taken seriously, and categories of relations in the wave-style have been explored, *vis-à-vis* graph models.

In this paper, we show that, by moving to the category Rel^* of pointed sets and relations preserving the distinguished point, and by slightly relaxing the GoI axiomatization of [3], we can recover many familiar graph models of λ-calculus. In particular, we show that GoI algebras on $(Rel^*, \times^*)$ are essentially *graph-like models* in the sense of Scott-Plotkin-Engeler, [26, 24]. Moreover, we show that the λ-theories modeled in this setting do go beyond sensible and semi-sensible theories. This should be contrasted with the fact that, in [14], it has been shown that game models, i.e. particle-style GoI models, capture only a very limited number of λ-theories, related to Böhm trees and Levy-Longo trees.

The paper consists of three parts.

In the first part (Section 1), we study standard *graph models* (GMs) and *linear graph models* (LGMs) from a purely set-theoretical viewpoint. The latter have been introduced by Abramsky, and they are special cases of LCAs, providing combinatory models for the linear λ-calculus. A natural, but somewhat not immediate fact that we prove is that every GM can be recovered from a LGM via standardisation. An important consequence is that all λ-theories induced by GM's can be captured by LGMs. In the literature many variants of the original notion of GM have been studied (see e.g. [26, 24, 15]). Here we provide purely set-theoretical presentations of some variants of the original notions of (L)GM, which, as we will see, arise as wave GoI models.

In the second part of the paper (Section 2), we study LGMs from a categorical point of view. In particular, we show that many (*pointed*) LGMs can be captured as special WLCs in the wave-style. Somewhat surprisingly, the original Scott-Plotkin-Engeler graph model escapes this categorical description, because of the behaviour of the empty relation.

In the third part of the paper (Sections 3 and 4), we study wave GoI algebras. In particular, we introduce a weaker axiomatization of the *GoI situation* of [3], which still ensures that a GoI category is a WLC, and hence it allows to build a GoI algebra. This weaker axiomatization allows us to capture the case of $(Rel^*, \times^*)$. We show that the class of GoI algebras induced by $(Rel^*, \times^*)$ are *pointed* LGMs. Finally, we show that there are wave GoI models realizing non-sensible λ-theories.

Notation. Let $\mathcal{P}$, $\mathcal{P}_f$, $\mathcal{P}_{fne}$ denote the (finite, finite non-empty) powerset. Let $\subseteq$, $\subseteq_f$, $\subseteq_{fne}$ denote (finite, finite non-empty) subset inclusion. Let U, V be objects in a category $\mathcal{C}$. We denote by $\tau : U \lhd V : \tau'$ a retraction of U in V, i.e. $\tau' \circ \tau = id_U$. Let Pfn be the category of sets and *partial* functions. Let Rel be the category of sets and relations. Relations $f \subseteq A \times B$ will be denoted by $f : A \multimap\!\!\!\!\!\rightarrow B$. Let Rel^* be the category of pointed sets $(A, *_A)$ and relations which preserve the distinguished point, i.e. $f : (A, *_A) \multimap\!\!\!\!\!\rightarrow (B, *_B)$ iff $(*_A, *_B) \in f$. The sets of finite streams and the set of infinite streams on a set A will be denoted by $A^{<\omega}$ and A^ω, respectively. Streams in A^ω will be denoted by $\boldsymbol{a}, \boldsymbol{b}, \ldots$ The i-th component of a (finite) stream $\boldsymbol{a}$ is denoted by a_i; a^ω denotes the infinite stream whose components are all equal to a.

1 Linear Combinatory Algebras and Graph Models

In this section, first we recall basic facts concerning *linear combinatory algebras* (LCAs) and we introduce the new notion of *linear combinatory λ-model*. Then we focus on a special class of (linear) combinatory λ-models, i.e. *(linear) graph models* ((L)GMs), and we show that every graph model is induced by a linear one. Finally, we present some variants of the original notion of (L)GM, which will be of interest in the sequel, namely *Rel* (L)GMs, *pointed* (L)GMs, *stream-based* (L)GMs, together with their generalizations w.r.t. cardinality.

1.1 Linear Combinatory Algebras, λ-models and Graph Models

The notion of linear combinatory algebra refines the notion of combinatory algebra, in that it has an extra unary operation ! and a set of combinators, refining Curry's original set of combinators:

Definition 1 (Linear Combinatory Algebra). *A* linear combinatory algebra (LCA) $\mathcal{A} = (A, \cdot, !)$ *is an applicative structure* $(A, \cdot)$ *with a unary (injective) operation !, and distinguished elements (combinators)* $B, C, I, K, W, D, \delta, F$ *satisfying the following equations (we associate* $\cdot$ *to the left and we assume ! to have order of precedence greater than* $\cdot$*): for all* $x, y, z \in A$*,*

$$\begin{array}{ll} Bxyz = x(yz) & Wx!y = x!y!y \\ Cxyz = (xz)y & D!x = x \\ Ix = x & \delta!x = !!x \\ Kx!y = x & F!x!y = !(xy) \end{array}$$

The notion of LCA corresponds to a Hilbert style axiomatization of the $\{!, \multimap\}$-fragment of linear logic. Every LCA induces a standard combinatory algebra (CA), by the combinatory version of Girard's translation of Intuitionistic Linear Logic into Intuitionistic Logic (see [3] for more details), i.e.:

Proposition 1 (Standardisation). *Let $(A, \cdot, !)$ be a LCA. Then $(A, \cdot_s)$, where $x \cdot_s y \triangleq x \cdot !y$, is a CA with combinators B_s, C_s, I_s, K_s, W_s defined by:*

$$\begin{array}{ll} C_s \triangleq D' \cdot C & I_s \triangleq D' \cdot I \\ B_s \triangleq C \cdot (B \cdot (B \cdot B \cdot B) \cdot (D' \cdot I)) \cdot (C \cdot ((B \cdot B) \cdot F) \cdot \delta) & \\ K_s \triangleq D' \cdot K & W_s \triangleq D' \cdot W \ , \end{array}$$

where $D' \triangleq C(BBI)(BDI)$ is such that, for all x, y, $D'x!y = xy$.

It is well known that λ-models à la Hindley-Longo can be characterized as *combinatory λ-models* (see e.g. [11]):

Definition 2 (Combinatory λ-model). *A CA $\mathcal{A} = (A, \cdot)$ is a* combinatory λ-model *if there exists an extra* selector *combinator ϵ such that, for all $x, y \in A$,*

$$\epsilon xy = xy \quad \textit{and} \quad (\forall z.\ xz = yz) \Rightarrow \epsilon x = \epsilon y \ .$$

Here we introduce the linear version of the notion of combinatory λ-model:

Definition 3 (Combinatory Linear λ-model). *A LCA $\mathcal{A} = (A, \cdot, !)$ is a* combinatory linear λ-model *if there exist* linear selector *ϵ and* selector *combinator ϵ_s such that, for all $x, y \in A$,*

$$\begin{array}{c} \epsilon xy = xy \quad \textit{and} \quad (\forall z.\ xz = yz) \Rightarrow \epsilon x = \epsilon y \\ \epsilon_s !x!y = x!y \quad \textit{and} \quad (\forall z.\ x!z = y!z) \Rightarrow \epsilon_s !x = \epsilon_s !y \ . \end{array}$$

Then we have:

Proposition 2. *Every combinatory linear λ-model gives rise by standardisation to a combinatory λ-model.*

Graph models (GMs) à la Scott-Plotkin-Engeler and Abramsky's *linear graph models* (LGMs) are examples of combinatory (linear) λ-models:

Definition 4 ((Linear) Graph Model). *A* graph model (GM) *$\mathcal{U}$ is an applicative structure $(\mathcal{P}(U), \cdot_\tau)$, where U is a (infinite) set with a retraction in* Pfn[1] *$\tau : \mathcal{P}_f(U) \times U \lhd U$, and the application $\cdot_\tau$ is defined by: for all $x, y \in \mathcal{P}(U)$,*

$$x \cdot_\tau y \triangleq \{a \mid \exists w \subseteq_f y.\ \tau(w, a) \in x\} \ .$$

A linear graph model (LGM) *is a structure $\mathcal{U} = (\mathcal{P}(U), \cdot_{\tau_1}, !_{\tau_2})$, where U is a (infinite) set with retractions in* Pfn *$\tau_1 : U \times U \lhd U$ and $\tau_2 : \mathcal{P}_f(U) \lhd U$, and linear application $\cdot_{\tau_1}$ and $!_{\tau_2}$ are defined by: for all $x, y \in \mathcal{P}(U)$,*

$$x \cdot_{\tau_1} y \triangleq \{a \mid \exists b \in y.\ \tau_1(b, a) \in x\} \ ,$$

$$!_{\tau_2} x \triangleq \{\tau_2(w) \mid w \subseteq_f x\} \ .$$

One can define combinators on (linear) graph models in such a way that:

Proposition 3. *Every (L)GM is a combinatory (linear) λ-model.*

As one expects, given a LGM, by standardisation, we get a GM:

Proposition 4. *Let $\mathcal{U} \triangleq (\mathcal{P}(U), \cdot_{\tau_1}, !_{\tau_2})$ be a LGM with retractions $\tau_1 : U \times U \lhd U$ and $\tau_2 : \mathcal{P}_f(U) \lhd U$. Then by standardisation we get a GM $\mathcal{S}(\mathcal{U}) \triangleq (\mathcal{P}(U), \cdot_\tau)$, where the retraction $\tau : \mathcal{P}_f(U) \times U \lhd U$ is defined by $\tau_1 \circ (\tau_2 \times id_U)$.*

[1] I.e. an injection.

An interesting fact, which has a simple but non trivial proof, is that there exists a (non trivial) dual construction for building a LGM from a GM, for which standardisation is an inverse:

Proposition 5. *For any GM $\mathcal{U}$, there is a LGM $\mathcal{L}(\mathcal{U})$ such that $\mathcal{S}(\mathcal{L}(\mathcal{U})) = \mathcal{U}$.*

Proof. Let $\mathcal{U} \triangleq (\mathcal{P}(U), \cdot_\tau)$ be a GM. For any bijection $\xi : U \to \mathcal{P}_f(U)$ (which exists by cardinality reasons), the retractions $\tau_1 : U \times U \lhd U$, $\tau_1 \triangleq \tau \circ (\xi \times id_U)$, and $\tau_2 : \mathcal{P}_f(U) \lhd U$, $\tau_2 \triangleq \xi^{-1}$ induce a LGM $(\mathcal{P}(U), \cdot_{\tau_1}, !_{\tau_2})$. We take such LGM as $\mathcal{L}(\mathcal{U})$. $\square$

Remarkably, the construction in the proof of Proposition 5 holds for any choice of the bijection ξ. From the point of view of λ-theories, we have the following important consequence:

Corollary 1. *The class of λ-theories induced by GMs coincides with the class of λ-theories induced by GMs obtained from LGMs via standardisation.*

1.2 Rel (Linear) Graph Models

We introduce a class of generalized (L)GMs, where retractions are allowed to be *relations* instead of functions. The constructions in [26] and some weak variants of filter models [12], can be viewed as instances of *Rel* graph models.

Definition 5 (*Rel* (Linear) Graph Model). *A* *Rel* graph model *is an applicative structure $(\mathcal{P}(U), \cdot_\tau)$, where U is a (infinite) set with a retraction in* Rel *$\tau : \mathcal{P}_f(U) \times U \lhd U$, and the application $\cdot_\tau$ is defined by: for all $x, y \in \mathcal{P}(U)$,*

$$x \cdot_\tau y \triangleq \{a \mid \exists w \subseteq_f y. \exists c \in x.\ ((w, a), c) \in \tau\}\ .$$

A *Rel* linear graph model *is a structure $\mathcal{U} = (\mathcal{P}(U), \cdot_{\tau_1}, !_{\tau_2})$ where U is a (infinite) set with retractions in* Rel *$\tau_1 : U \times U \lhd U$ and $\tau_2 : \mathcal{P}_f(U) \lhd U$, and linear application $\cdot_{\tau_1}$ and $!_{\tau_2}$ are defined by: for all $x, y \in \mathcal{P}(U)$,*

$$x \cdot_{\tau_1} y \triangleq \{a \mid \exists b \in y. \exists c \in x.\ ((b, a), c) \in \tau_1\}$$
$$!_{\tau_2} x \triangleq \{a \mid \exists w \subseteq_f x.\ (w, a) \in \tau_2\}\ .$$

Proposition 6. Rel *(L)GMs are combinatory (linear) λ-models.*

Proof. We just show how to define selectors. Linear selector: $\epsilon \triangleq \{d \mid \exists a, b, c \in U.\ (((c, c), d) \in \tau_1\ \wedge\ ((a, b), c) \in \tau_1)\}$. Standard selector: $\epsilon_s \triangleq \{b \mid \exists w \subseteq v \subseteq_f U.\ \exists a, c, c' \in U.\ (((\{c\}, c'), b) \in \tau\ \wedge\ ((w, a), c) \in \tau\ \wedge\ ((v, a), c') \in \tau)\}$. $\square$

One can check that Propositions 4–5 extend to *Rel* (L)GMs.

1.3 Pointed Rel (Linear) Graph Models

Pointed *Rel* graph models arise when we carry out the graph model construction in *Rel**. Namely, we fix a special point $* \in U$, and we take as carrier the pointed powerset $(\mathcal{P}^* U, \{*\})$, i.e. the set of all subsets u of U such that $* \in U$, together with point-preserving codings.

Definition 6 (Pointed *Rel* (Linear) Graph Model). *Let U be a (infinite) set with a special point $* \in U$.*
A pointed *Rel* graph model *(pointed graph model, for short) is an applicative structure $(\mathcal{P}^*(U), \cdot_{\tau^*})$, where $\tau^* : \mathcal{P}^*_f(U) \times U \lhd U$ is a retraction in* Rel* *i.e. $((\{*\}, *), *) \in \tau^*$, $\mathcal{P}^*_f(U)$ denotes the set of all finite pointed subsets of U, and the application $\cdot_{\tau^*}$ is defined by: for all $x, y \in \mathcal{P}^*(U)$,*

$$x \cdot_{\tau^*} y \triangleq \{a \mid \exists w \subseteq_{fne} y.\ \exists c \in x.\ ((w, a), c) \in \tau^*\} \ .$$

A pointed *Rel* linear graph model *(pointed linear graph model, for short) is a structure $\mathcal{U} = (\mathcal{P}^*(U), \cdot_{\tau_1^*}, !_{\tau_2^*})$ where $\tau_1^* : U \times U \lhd U$, $\tau_2^* : \mathcal{P}^*_f(U) \lhd U$ are retractions in* Rel*, *i.e. $((*, *), *) \in \tau_1^*$ and $(\{*\}, *) \in \tau_2^*$ and linear application $\cdot_{\tau_1^*}$ and $!_{\tau_2^*}$ are defined by: for all $x, y \in \mathcal{P}^*(U)$,*

$$x \cdot_{\tau_1^*} y \triangleq \{a \mid \exists b \in y.\ \exists c \in x.\ ((b, a), c) \in \tau_1^*\} \ ,$$

$$!_{\tau_2^*} x \triangleq \{a \mid \exists w \subseteq_{fne} x.\ (w, a) \in \tau_2^*\} \ .$$

Proposition 7. *Every pointed (L)GM is a combinatory (linear) λ-model.*

Similar results to those in Propositions 4–5 hold then for pointed (L)GMs.

Remark 1. Notice that, in the spirit of [25], the above constructions of (L)GMs go through even if we consider $\mathcal{P}_{<\kappa}$, for any regular cardinal κ, in place of $\mathcal{P}_f$ (i.e. $\mathcal{P}_{<\omega}$). We call κ-(L)GMs the corresponding (L)GMs induced by such codings.

1.4 Stream-based (Linear) Graph Models

Interesting variants of graph models are obtained by considering codings on (possibly finite) streams, in place of codings on the powerset. As we will see, these can be actually viewed as special cases of graph models as defined in the previous sections. The interest of stream-based graph models lies in the fact that, as we will see, they arise in the categorical context of weak linear categories and of Geometry of Interaction. Both the standard, *Rel* and pointed graph models considered previously have corresponding stream based variants. Here we give the details only in the case of pointed graph models with finite streams. Similarly one can define (pointed) graph models with streams with finite codomain, or general streams.

Definition 7 (Finite-stream Graph Models). *Let U be a (infinite) set with a special point $* \in U$.*
A finite-stream graph model *is an applicative structure $(\mathcal{P}^*(U), \cdot_{\tau^*})$, where $\tau^* : U^{*<\omega} \times U \lhd U$, $U^{*<\omega}$ is the set of finite streams with at least one occurrence of $*$, and τ^* is a retraction in* Rel* *i.e. $((\langle * \rangle, *), *) \in \tau^*$, and the application $\cdot_{\tau^*}$ is defined by: for all $x, y \in \mathcal{P}^*(U)$,*

$$x \cdot_{\tau^*} y \triangleq \{a \mid \exists \boldsymbol{u}.\ \exists c \in x.\ (\forall i.\ u_i \in y \ \wedge \ ((\boldsymbol{u}, a), c) \in \tau^*)\} \ .$$

A finite-stream linear graph model *is a structure* $\mathcal{U} = (\mathcal{P}^*(U),\ \cdot_{\tau_1^*},\ !_{\tau_2^*})$ *where* $\tau_1^* : U \times U \lhd U$, $\tau_2^* : U^{*<\omega} \lhd U$ *are retractions in* Rel^*, *i.e.* $((*, *), *) \in \tau_1^*$ *and* $(\langle * \rangle, *) \in \tau_2^*$ *and linear application* $\cdot_{\tau_1^*}$ *and* $!_{\tau_2^*}$ *are defined by: for all* $x, y \in \mathcal{P}^*(U)$,

$$x \cdot_{\tau_1^*} y \triangleq \{a \mid \exists b \in y.\ \exists c \in x.\ ((b, a), c) \in \tau_1^*\}\ ,$$

$$!_{\tau_2^*} x \triangleq \{a \mid \exists \boldsymbol{u}.\ ((\forall i.\ u_i \in x\ \wedge\ (\boldsymbol{u}, a) \in \tau_2^*)\}\ .$$

Proposition 8. *Every finite-stream (L)GM is a combinatory (linear) λ-model.*

Similar results to those in Propositions 4–5 hold also for finite-stream (L)GMs.

The connection between stream-based (L)GMs and powerset-based (L)GMs is given by the following

Theorem 1. *Every finite-stream (L)GM is isomorphic to a pointed (L)GM.*

Proof. We give the proof for the linear case. Let $(\mathcal{P}^*(U), \cdot_{\tau_1^*}, !_{\tau_2^*})$ be a finite-stream linear graph model. Now we take $\xi : \mathcal{P}_f^* U \multimap\!\!\!\!\to U^{*<\omega}$ to be the injective relation defined by $(v, \boldsymbol{a}) \in \xi$ iff, for all i, $a_i \in v$ and for all $b \in v$ there exists i s.t. $a_i = b$. Then $\theta_2^* : \mathcal{P}_f^* U \multimap\!\!\!\!\to U$, $\theta_2^* \triangleq \tau_2^* \circ \xi$ is s.t. $!_{\theta_2^*} x = \{a \mid v \subseteq_{fne} x.\ (v, a) \in \theta_2^*\} = !_{\tau_2^*} x$, i.e. the finite-stream LGM $(\mathcal{P}^*(U), \cdot_{\tau_1^*}, !_{\theta_2^*})$ is a pointed LGM with codings τ_1^*, θ_2^*. Notice that the coding θ_2^* in the proof above is forced to be non functional. $\square$

However, notice that the converse of the above theorem fails.

2 Weak Linear Categories and Linear Graph Models

In this section, we discuss Abramsky's axiomatic construction of an LCA from a *weak linear category* (WLC). WLCs are the counterpart for linear combinatory algebras of the notion of linear category for linear λ-models (see [13]). In particular, we show that the category Rel^* with tensor the cartesian product, together with suitable stream-based functors turns out to be a WLC. Moreover, the LCAs arising from the WLC Rel^* capture exactly the classes of stream-based (L)GMs.

We start by recalling Abramsky's notion of WLC and the construction of an LCA from a WLC (for basic categorical definitions see Appendix A).

Definition 8 (Weak Linear Category). *A* weak linear category (WLC) $(\mathcal{C}, \otimes, !)$ *consists of:*

- *a symmetric monoidal closed category* $(\mathcal{C}, \otimes)$;
- *a symmetric monoidal functor* $! : \mathcal{C} \to \mathcal{C}$;
- *the following four monoidal pointwise natural transformations:*

  $der : ! \Rightarrow Id$ (dereliction)
  $\delta : ! \Rightarrow !!$ (comultiplication)
  $con : ! \Rightarrow ! \otimes !$ (contraction)
  $weak : ! \Rightarrow \mathcal{K}_I$ (weakening) ,

  where $\mathcal{K}_I$ *is the constant* I *functor.*

Definition 9. *A* reflexive *object in a WLC* $\mathcal{C}$ *is an object* V *in* $\mathcal{C}$ *with the following retracts:*

$$V \multimap V \lhd V \qquad !V \lhd V \qquad I \lhd V\ .$$

Theorem 2 ([3]). *Let $(\mathcal{C}, \otimes, !)$ be a WLC and V be a reflexive object in $\mathcal{C}$ with retracts $\theta_1 : V \multimap V \lhd V : \theta_1'$ and $\theta_2 : !V \lhd V : \theta_2'$. Then $(\mathcal{C}(I, V), \cdot, !)$ is a LCA, where $\cdot$ and $!$ are defined by: for $f, g \in \mathcal{C}(I, V)$,*

$$f \cdot g \triangleq ev \circ ((\theta_1' \circ f) \otimes g) \circ \phi_I' \qquad !f \triangleq \theta_2 \circ (!f) \circ \phi_I' \ ,$$

where $\phi_I' : I \to !I$ is the isomorphism associated to the strong monoidal functor $!$ (see Appendix A).

The category Rel^* is symmetric monoidal closed w.r.t. the product $\times^*$ inherited from Rel. Moreover, $(Rel^*, \times^*)$ together with any of the following symmetric monoidal functors based on streams is a WLC:

Definition 10. *i) Let $(\ )^{*<\omega} : Rel^* \to Rel^*$ be the functor defined by:*

- *for any pointed set $(A, *_A)$, let $(A, *_A)^{<\omega} \triangleq (A^{*<\omega}, \langle *_A \rangle)$, where $A^{*<\omega}$ is the set of finite streams with at least one occurrence of $*_A$;*
- *for any $f : (A, *_A) \twoheadrightarrow (B, *_B)$, let $f^{*<\omega} : (A^{*<\omega}, \langle *_A \rangle) \twoheadrightarrow (B^{*<\omega}, \langle *_B \rangle)$ be defined by $(\boldsymbol{a}, \boldsymbol{b}) \in f^{*<\omega}$ iff $|\boldsymbol{a}| = |\boldsymbol{b}|$ and $\forall i.\ (a_i, b_i) \in f$.*

ii) Let $(\ )^{*\omega} : Rel^* \to Rel^*$ be the functor defined by:

- *for any pointed set $(A, *_A)$, $(A, *_A)^{*\omega} \triangleq (A^{*\omega}, *_A^{\omega})$ where $A^{*\omega}$ is the set of (infinite) streams with at least one occurrence of $*_A$;*
- *for any $f : (A, *_A) \twoheadrightarrow (B, *_B)$, let $f^{*\omega} : (A^{*\omega}, \langle *_A \rangle) \twoheadrightarrow (B^{*\omega}, \langle *_B \rangle)$ be defined by $(\boldsymbol{a}, \boldsymbol{b}) \in f^{*\omega}$ iff $\forall i.\ (a_i, b_i) \in f$.*

iii) Let $(\ )_f^{*\omega} : Rel^* \to Rel^*$ be the functor defined by:

- *for any pointed set $(A, *_A)$, $(A, *_A)_f^{*\omega} \triangleq (A_f^{*\omega}, *_A^{\omega})$, where $A_f^{*\omega}$ is the restriction of $A^{*\omega}$ to the streams with finite codomain[2];*
- *the definition on arrows is similar to (ii).*

Proposition 9. *$(Rel^*, \times^*)$ with one of the functors in Definition 10 is a WLC.*

Proof. The proof of the fact that $(Rel^*, \times^*)$ is symmetric monoidal closed, with right adjoint of $\times^*$ the product $\times^*$ itself, and of the fact that the functors of Definition 10 are monoidal is routine. We just sketch the definition of some natural transformations for $!^* = (\ )^{*\omega}$. Let $con : (\ )^{*\omega} \Rightarrow (\ )^{*\omega} \times (\ )^{*\omega}$, $con_A : (A)^{*\omega} \twoheadrightarrow (A)^{*\omega} \times (A)^{*\omega}$ be defined by $con_A \triangleq \{(a, (a', a'')) \mid a, a', a'' \in A^{*\omega} \ \wedge\ \forall i \geq 0.\ a''_{2i+1} = a_i \ \wedge\ \forall i \geq 1.\ a''_{2i} = a'_i\}$. Let $weak_A : A^{*\omega} \to I$, $weak_A \triangleq \{(a^\omega, *) \mid a \in A\}$. □

Theorem 3. *The class of LCAs generated by reflexive objects in the WLC $(\mathrm{Rel}^*, \times^*, !^*)$, where $!^*$ is either $(\ )^{*<\omega}$ or $(\ )_f^{*\omega}$ or $(\ )^{*\omega}$, is isomorphic to the class of finite-stream, finite-codomain stream, and stream LGMs, respectively.*

Proof. We consider the case $!^* = (\ )^{*<\omega}$. The other cases can be dealt with similarly. Using Theorem 2, one can easily check that any set U with retracts $\theta_1^* : U \times^* U \lhd U$ and $\theta_2^* : U^{*<\omega} \lhd U$ gives rise to a LCA isomorphic, via $\lambda_U : I \times^* U \to U$, to a finite-stream graph model. And vice versa. □

WLC	UNIVERSAL OBJECT	LCA
$(Rel^*, \times^*, (\ )^{*<\omega})$	$\theta_1^* : U \times U \lhd U,\ \theta_2^* : U^{*<\omega} \lhd U \ \Rightarrow$	$\exists \xi : \mathcal{P}_f^* U \lhd U^{*<\omega}$ s.t. $(\mathcal{P}^* U, \cdot_{\theta_1^*}, !_{\theta_2^* \circ \xi})$ pointed LGM
$(Rel^*, \times^*, (\ )_f^{*\omega})$	$\theta_1^* : U \times U \lhd U,\ \theta_2^* : U_f^{*\omega} \lhd U \ \Rightarrow$	$\exists \xi : \mathcal{P}_f^* U \lhd U_f^{*\omega}$ s.t. $(\mathcal{P}^* U, \cdot_{\theta_1^*}, !_{\theta_2^* \circ \xi})$ pointed LGM
$(Rel^*, \times^*, (\ )^{*\omega})$	$\theta_1^* : U \times U \lhd U,\ \theta_2^* : U^{*\omega} \lhd U \ \Rightarrow$	$\exists \xi : \mathcal{P}_{<\omega_1}^* U \lhd U^{*\omega}$ s.t. $(\mathcal{P}^* U, \cdot_{\theta_1^*}, !_{\theta_2^* \circ \xi})$ pointed ω_1-LGM

Table 1. Wave graph models on Rel^*.

As a consequence, by Theorem 1, the LCAs generated by reflexive objects in the WLC $(Rel^*, \times^*, !^*)$ are pointed LGMs.

The situation is summarized in Table 1. Notice that, in the case of infinite streams (possibly with finite codomain), the cardinality of U is at least $2^{\aleph_0}$. Moreover, for $! = (\ )^{*\omega}$, we get a pointed ω_1-LGM.

Remark 2. Notice that the basic category *Rel* of sets and relations fails to be a WLC, since there is no notion of "empty" stream, and the definition of weakening $weak_A : A^\omega \multimap I$, $weak_A \triangleq \{(a^\omega, *) \mid a \in A,\ * \in I\}$ is *not* natural on the *empty* relation. This is the reason why one has to shift to pointed relations.
Moreover, the powerset functor $\mathcal{P}_f$ fails to induce a structure of WLC on *Rel* (and Rel^*). Namely, the "natural" definition of dereliction, i.e. $der_A : \mathcal{P}_f(A) \multimap A$, $der_A \triangleq \{(\{a\}, a) \mid a \in A\}$, is not pointwise natural.

3 The Geometry of Interaction Construction

In this section, we recall the categorical axiomatization of the *Geometry of Interaction* (GoI) developed in [1–3]. This is based on traced categories, see [23]. Any traced monoidal category gives rise, by the construction of [1], to a *GoI category.* If moreover the traced category we start from has a strong traced monoidal functor together with suitable retractions, then the GoI category is a WLC, and hence, by Theorem 2, it can generate a (GoI) LCA. This situation, called *GoI situation*, is axiomatized and studied in [3]. However, a GoI situation gives only sufficient conditions for a GoI category to be a WLC. In this section, we introduce the notion of *weak GoI situation* (wGoI situation), in which we give weaker, but still sufficient conditions on the retractions for the GoI category to be a WLC. In the next section, we will see that the notion of wGoI situation captures many graph models introduced in Section 1.

GoI categories arise from traced symmetric monoidal categories by the following construction:

Proposition 10 (GoI Construction, [1]). *Given a traced symmetric monoidal category $\mathcal{C}$, we define a* GoI category $\mathcal{G}(\mathcal{C})$ *by:*

- **Objects***: pairs of objects of $\mathcal{C}$, denoted by (A^+, A^-), where A^+ and A^- are objects of $\mathcal{C}$.*
- **Arrows***: an arrow $f : (A^+, A^-) \to (B^+, B^-)$ in $\mathcal{G}(\mathcal{C})$ is $f : A^+ \otimes B^- \to A^- \otimes B^+$ in $\mathcal{C}$.*

[2] I.e. the set $[\mathbf{N} \to_{fcod} X]$ of functions from $\mathbf{N}$ in X with *finite* codomain.

– **Identity**: $id_{(A^+,A^-)} = \sigma_{A^+,A^-}$.
– **Composition**: *it is given by* symmetric feedback. *Given* $f : (A^+, A^-) \to (B^+, B^-)$ *and* $g : (B^+, B^-) \to (C^+, C^-)$, $g \circ f : (A^+, A^-) \to (C^+, C^-)$ *is given by:* $g \circ f \triangleq Tr^{B^- \otimes B^+}_{A^+ \otimes C^-, A^- \otimes C^+}(\gamma' \circ (f \otimes g) \circ \gamma)$, *where* $\gamma \triangleq (id_{A^+} \otimes id_{B^-} \otimes \sigma_{C^-,B^+}) \circ (id_{A^+} \otimes \sigma_{C^-,B^-} \otimes id_{B^+})$ *and* $\gamma' \triangleq (id_{A^-} \otimes id_{C^+} \otimes \sigma_{B^+,B^-}) \circ (id_{A^-} \otimes \sigma_{B^+,C^+} \otimes id_{B^-}) \circ (id_{A^-} \otimes id_{B^+} \otimes \sigma_{B^-,C^+})$.
– **Tensor**: $(A^+, A^-) \otimes (B^+, B^-) \triangleq (A^+ \otimes B^+, A^- \otimes B^-)$, *and, for any* $f : (A^+, A^-) \to (B^+, B^-)$ *and* $g : (C^+, C^-) \to (D^+, D^-)$, $f \otimes g \triangleq (id_{A^-} \otimes \sigma_{B^+,C^-} \otimes id_{D^+}) \circ (f \otimes g) \circ (id_{A^+} \otimes \sigma_{C^+,B^-} \otimes id_{D^-})$.
– **Unit**: (I, I).

Then $\mathcal{G}(\mathcal{C})$ *is compact closed. Moreover,* $F : \mathcal{C} \to \mathcal{G}(\mathcal{C})$ *with* $F(A) = (A, I)$ *and* $F(f) = f$ *is a full and faithful embedding.*

In [3], sufficient conditions are given on the traced monoidal category $\mathcal{C}$ for $\mathcal{G}(\mathcal{C})$ to be a WLC, and hence, by Theorem 2, to give rise to a GoI LCA. We recall this construction:

Definition 11 (GoI Situation, [3]). *A* GoI situation *is a triple* $(\mathcal{C}, T, U)$, *where:*
– $\mathcal{C}$ *is a traced symmetric monoidal category;*
– $T : \mathcal{C} \to \mathcal{C}$ *is a traced strong symmetric monoidal functor with the following retractions (which are monoidal natural transformations):*
  1. $e : TT \lhd T : e'$ (Comultiplication)
  2. $d : Id \lhd T : d'$ (Dereliction)
  3. $c : T \otimes T \lhd T : c'$ (Contraction)
  4. $w : K_I \lhd T : w'$ (Weakening), *where* K_I *denotes the constant* I *functor.*
– U *is an object of* $\mathcal{C}$, *called a* GoI reflexive object, *with retractions*
  1. $\theta_1 : U \otimes U \lhd U : \theta_1'$
  2. $I \lhd U$
  3. $\theta_2 : TU \lhd U : \theta_2'$.

Theorem 4 ([3]). *Let* $(\mathcal{C}, T, U)$ *be a GoI situation. Then*
i) $(\mathcal{G}(\mathcal{C}), !)$ *is a WLC with* $! : \mathcal{G}(\mathcal{C}) \to \mathcal{G}(\mathcal{C})$ *defined as follows:* $!(A^+, A^-) \triangleq (TA^+, TA^-)$, *and, for* $f : (A^+, A^-) \to (B^+, B^-)$, $!f \triangleq TA^+ \otimes TB^- \xrightarrow{\simeq} T(A^+ \otimes B^-) \xrightarrow{Tf} T(A^- \otimes B^+) \xrightarrow{\simeq} TA^- \otimes TB^+$.
ii) $(\mathcal{C}(U, U), \cdot, !)$ *is a LCA, where for any* $f, g \in \mathcal{C}(U, U)$, $f \cdot g \triangleq Tr^U_{U,U}((id_U \otimes g) \circ (\theta_1' \circ f \circ \theta_1))$, *and* $!f \triangleq \theta_2 \circ Tf \circ \theta_2'$.

Definition 12. *We call* GoI LCA, *(or* GoI algebra*), a LCA which comes from a GoI situation.*

As pointed out in [3], particle GoI situations arise when the strong monoidal functor is given by a countable copower. Dually, in the wave case, GoI situations arise when the monoidal functor is given by a countable power. In what follows, we focus on the wave case. An example of wave GoI category is $(\omega\text{-}CPO, \times)$ together with the stream functor $(\ )^\omega$, [3]. However, the basic setting for wave

GoI, i.e. $(Rel^*, \times^*)$ (and $(Rel, \times)$) together with the stream functor fail to give rise to a GoI situation. The induced GoI category, however, is still a WLC. In the next definition we introduce the notion of *weak GoI situation*, which captures the basic case of $(Rel^*, \times^*)$. In a weak GoI situation, the naturality condition of the retractions is relaxed, by requiring only *naturality up-to retraction*:

Definition 13 (Weak GoI Situation). *A* weak GoI situation *(wGoI situation) is a triple $(\mathcal{C}, T, U)$ where $\mathcal{C}$ and U satisfy the conditions in Definition 11, and T is a traced strong symmetric monoidal functor with retractions which are natural only up-to retraction, i.e.:*

1. $\{e_A : TTA \lhd TA : e'_A\}_A$ (Comultiplication) *is a family of monoidal retractions s.t., for all $f : A \to B$, $e'_B \circ Tf \circ e_A = TTf$;*
2. $\{d_A : A \lhd TA : d'_A\}_A$ (Dereliction) *is a family of monoidal retractions s.t., for all $f : A \to B$, $d'_B \circ Tf \circ d_A = f$;*
3. $\{c_A : TA \otimes TA \lhd TA : c'_A\}_A$ (Contraction) *is a family of monoidal retractions s.t., for all $f : A \to B$, $c'_B \circ Tf \circ c_A = Tf \otimes Tf$;*
4. $\{w_A : I \lhd TA : w'_A\}_A$ (Weakening) *is a family of monoidal retractions s.t., for all $f : A \to B$, $w'_B \circ Tf \circ w_A = id_I$.*

It is immediate to see that a GoI situation is in particular a wGoI Situation. Moreover, a direct calculation shows that *der*, δ, *con*, *weak*, as defined in [3] in terms of the retractions on $\mathcal{C}$, are monoidal pointwise natural transformations on $\mathcal{G}(\mathcal{C})$. Therefore:

Theorem 5. *A wGoI situation gives rise to a GoI category which is a WLC.*

The situation is summarized in Figure 1.

$$\begin{tikzcd}
\text{GoI situation} \arrow[r, dotted, "is\ a"] \arrow[d, "GoI\ construction"'] & \text{weak GoI situation} \arrow[dl, "GoI\ construction"] \\
\text{WLC} \arrow[d, "universal\ object"'] & \\
\text{LCA} \arrow[d, "standardization"'] & \\
\text{CA} &
\end{tikzcd}$$

Fig. 1. (Weak) GoI construction.

4 Wave GoI Algebras and Linear Graph Models

In this section, first we show that some pointed linear graph models can be viewed as wave GoI algebras arising from the traced category $(Rel^*, \times^*)$, together with suitable stream functors as monoidal functors. Moreover we show that such wave models can induce λ-theories where not all unsolvables of order 0 are equated.

4.1 GoI algebras on $(\boldsymbol{Rel}^*, \times^*)$

The category $(Rel^*, \times^*)$ is traced with the trace operator $Tr^U_{A,B}(\ )$ defined by: for $f : A \times U \multimap B \times U$, $Tr^U_{A,B}(f) \triangleq \{(a,b) \mid \exists u.\ (a,u,b,u) \in f\}$.

Both the functor of streams $(\ )^{*\omega}$ and that of streams with finite codomain $(\ )^{*\omega}_f$ induce on $(Rel^*, \times^*)$ a wGoI situation. We focus on $(\ )^{*\omega}_f$.

Proposition 11. *For any GoI reflexive object U in Rel^*, $(Rel^*, \times^*, (\ )^{*\omega}_f, U)$ is a wGoI situation.*

Proof. The functor $(\ )^{*\omega}_f$ is traced strong monoidal with isomorphism $\phi'_I : I \to I^{*\omega}_f$, defined by $\phi'_I(*) = *^\omega$, and natural isomorphism $\phi : (\ )^{*\omega}_f \times (\ )^{*\omega}_f \to (\ \otimes\ )^{*\omega}_f$ with components $\phi_{A,B} : A^{*\omega}_f \times B^{*\omega}_f \to (A \times B)^{*\omega}_f$ defined by $\phi_{A,B}(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{c}$, where $c_i = (a_i, b_i)$. We only sketch the definitions of the monoidal retractions:
- $e_A : (A^{*\omega}_f)^{*\omega}_f \to A^{*\omega}_f$, $e_A \triangleq \xi \circ \chi_A$, where $\chi_A : [\mathbf{N} \to_{fcod} [\mathbf{N} \to_{fcod} A]] \lhd [\mathbf{N} \times \mathbf{N} \to_{fcod} A]$ is a component of the retraction natural in A induced by curryfication, and $\xi \triangleq \lambda f \in [\mathbf{N} \times \mathbf{N} \to_{fcod} A].\lambda n \in \mathbf{N}. f(\epsilon^{-1}(n))$, where $\epsilon : \mathbf{N} \times \mathbf{N} \simeq \mathbf{N}$ is any bijective coding of pairs;
- $c_A : A^{*\omega}_f \times A^{*\omega}_f \to A^{*\omega}_f$, $c_A(\boldsymbol{a}, \boldsymbol{a}') = \boldsymbol{a}''$, where, for all $i \geq 0$, $a''_{2i+1} = a_i$ and, for all $i \geq 1$, $a''_{2i} = a'_i$;
- $d_A : A \to A^{*\omega}_f$, $d_A(a) = a^\omega$, $d'_A : A^{*\omega}_f \to A$, $d'_A(a^\omega) = a$;
- $w_A : I \multimap A^{*\omega}_f$, $w_A \triangleq \{(*, a^\omega) \mid a \in A\}$, $w'_A : A^{*\omega}_f \to I$, $w'_A \triangleq \{(a^\omega, *) \mid a \in A\}$. □

Notice that d in the proof above is natural up-to retraction, but not in the full sense. This justifies the definition of weak GoI situation.

Remark 3. Notice that both the finite powerset functor $\mathcal{P}^*_f$ and the finite stream functor $(\ )^{*<\omega}$ fail to be strong monoidal, and hence they do not give a GoI situation on Rel^*. Moreover, the category $(Rel, \times)$ together with the (infinite) stream functor fails to give a wGoI situation, because weakening (as it is defined in the proof of Proposition 11) does not satisfy the condition of Definition 13 on the empty relation.

Proposition 12. *i) The GoI category $\mathcal{G}(Rel^*, \times^*, (\ )^{*\omega}_f)$ is a WLC.*
ii) Let U be a GoI reflexive object in Rel^*, with retractions $\theta^*_1 : U \times^* U \lhd U$, $\theta^*_2 : U^{*\omega}_f \lhd U$. Then $(\mathcal{P}^*(U \times U), \cdot_{\theta^*_1}, !_{\theta^*_2})$ is a LCA, where, $\cdot_{\theta^*_1}$ and $!_{\theta^*_2}$ are defined by (using "functional" notation): for all $x, y \in \mathcal{P}(U \times U)$
- $x \cdot_{\theta^*_1} y = \{(a,b) \mid \exists (c,d) \in y.\ (\theta^*_1(a,d), \theta^*_1(b,c)) \in x\}$;
- $!_{\theta^*_2} x = \{(\theta^*_2(\boldsymbol{a}), \theta^*_2(\boldsymbol{b})) \mid \boldsymbol{a}, \boldsymbol{b} \in U^{*\omega}_f \ \wedge\ \forall i.(a_i, b_i) \in x\}$.

A crucial fact for our purposes is that all the GoI algebras of Proposition 12 give rise, for a suitable choice of the coding relations, to finite-codomain stream LGMs, and hence, by suitable analogue of Theorem 1, to pointed LGMs.

Theorem 6. *Let $\mathcal{U} = (\mathcal{P}^*(U \times U), \cdot_{\theta^*_1}, !_{\theta^*_2})$ be a GoI algebra induced by $(Rel^*, \times^*, (\ )^{*\omega}_f)$, with $\theta^*_1 : U \times U \lhd U$, $\theta^*_2 : U^{*\omega}_f \lhd U$. Then $\mathcal{U}$ coincides with the pointed LGM $\mathcal{U}' = (\mathcal{P}^*(U'), \cdot_{\tau^*_1}, !_{\tau^*_2})$, where $U' \triangleq U \times U$, $\tau^*_1 : U' \times U' \lhd U'$ and $\tau^*_2 : \mathcal{P}^*_f(U') \lhd U'$ are defined by:*

$$\tau^*_1 \triangleq \langle \theta^*_1 \circ (\pi_1 \circ \pi_2 \times \pi_2 \circ \pi_1), \theta^*_1 \circ (\pi_2 \circ \pi_2 \times \pi_1 \circ \pi_1) \rangle \ ,$$
$$\tau^*_2 \triangleq (\theta^*_2 \times \theta^*_2) \circ \zeta \ ,$$

where $\zeta : \mathcal{P}^*_f(U \times U) \multimap U^{*\omega}_f \times U^{*\omega}_f$ is the injective relation defined by $(u, (\boldsymbol{a}, \boldsymbol{b})) \in \zeta$ iff for all i, $(a_i, b_i) \in u$ and for all $(c,d) \in u$ there exists i such that $a_i = c$ and $b_i = d$.

Proof. Let $x, y \in \mathcal{P}^*(U \times U)$. By Proposition 12, $x \cdot_{\theta_1^*} y = \{(a,b) \mid \exists (c,d) \in y.\ (\theta_1^*(a,d), \theta_1^*(b,c)) \in x\} = \{(a,b) \mid \exists (c,d) \in y.\ \tau_1^*((c,d),(a,b)) \in x\}$, by definition of τ_1^*; i.e. $\cdot_{\theta_1^*}$ is the application on the LGM $\mathcal{U}'$. Moreover, by Proposition 12, $!_{\theta_2^*} x = \{(\theta_2^*(\boldsymbol{a}), \theta_2^*(\boldsymbol{b})) \mid \boldsymbol{a}, \boldsymbol{b} \in U_f^{*\omega}\ \wedge\ \forall i.(a_i, b_i) \in x\} = \{(\theta_2^*(\boldsymbol{a}), \theta_2^*(\boldsymbol{b})) \mid \boldsymbol{a}, \boldsymbol{b} \in U_f^{*\omega}\ \wedge\ \exists u \subseteq_{fne} x.\ \forall i.(a_i, b_i) \in u \wedge \forall (c,d) \in u\ \exists i.(a_i = c \wedge b_i = d)\} = \{(\theta_2^* \times \theta_2^*)(\zeta(u)) \mid u \subseteq_{fne} x\} = \{\tau_2^*(u) \mid u \subseteq_{fne} x\}$, by definition of τ_2^*; i.e. $!_{\theta_2^*}$ is the ! operator on the pointed LGM $\mathcal{U}'$. $\square$

Notice that by considering the GoI construction over $(Rel^*, \times^*)$, we get a class of LGMs which is a subclass of the one obtained in Section 2, where $(Rel^*, \times^*)$ itself is viewed as a WLC.

The broad abstract pattern is given in Figure 2.

$$
\begin{tikzcd}
(Rel^*, \times^*, (\ )_f^{*\omega}) \text{ weak GoI situation} \arrow[d, "GoI\ construction"'] \\
\mathcal{G}(Rel^*, \times^*, (\ )_f^{*\omega}) \text{ WLC} \arrow[d, "universal\ object"'] \\
\text{finite-codomain stream LGM} \arrow[d, "is\ isomorphic\ to"'] \\
\text{pointed LGM} \arrow[d, "standardization"'] \\
\text{pointed GM}
\end{tikzcd}
\qquad
\begin{tikzcd}
(Rel^*, \times^*, (\ )^{*\omega}) \text{ weak GoI situation} \arrow[d, "GoI\ construction"'] \\
\mathcal{G}(Rel^*, \times^*, (\ )^{*\omega}) \text{ WLC} \arrow[d, "universal\ object"'] \\
\text{stream LGM} \arrow[d, "is\ isomorphic\ to"'] \\
\omega_1\text{-pointed LGM} \arrow[d, "standardization"'] \\
\omega_1\text{-pointed GM}
\end{tikzcd}
$$

Fig. 2. Wave GoI constructions on *Rel**.

4.2 Wave GoI λ-theories

The class of λ-theories induced by wave models is quite rich, or at least it goes beyond theories where all unsolvable of order 0 are equated in the bottom element, as in models based on games *à la* [5, 22] (see [14]). Namely:

Lemma 1. *For any $k \geq 0$, there exists a GoI algebra on $(Rel^*, \times^*, (\ )_f^{*\omega})$ with k* self-singletons *(up-to-$*$) different from $*$, i.e. elements $a \in U$ such that $\theta_1^*(a,a) = a$ and $\theta_2^*(\{a, *\}) = a$.*

One can easily check that any self-singleton (up-to-$*$) a belongs to the interpretation of a term iff it reduces to a closed λ-term (see [21]). Therefore:

Theorem 7. *There exists a wave model in which* $[\![\lambda x.\Delta\Delta]\!] \neq [\![\lambda x.\Delta\Delta x]\!]$.

5 Conclusions and Future Work

Building on [1, 3], we have investigated the connections between graph models and wave GoI models arising in the basic setting of $(Rel^*, \times^*)$. We have shown that such wave models are (pointed) graph models, which yield models of the λ-calculus capturing a rich class of λ-theories. The category $(Rel, \times)$ apparently fails to give a WLC and a wGoI situation, because weakening is not well-behaved on the empty relation.

However, in [20], in order to capture the case of $(Rel, \times)$, a *strict* variant of the GoI situation of [3] has been introduced, giving rise to a *strict* WLC, where only a restricted form of weakening holds. LCAs arising from strict WLCs are themselves strict, i.e. application is strict, and only a restricted form of K combinator is available. These are models for restricted λ-calculi, such as Church's λI-calculus and the $\lambda\beta_{KN}$-calculus of [19].

In summing up, we make a disclaimer. The objective of the paper is not that of characterizing λ-theories arising from graph models or linear graph models, or wave-style graph models. This problem is very difficult, but orthogonal to the one we have addressed. We have shown nonetheless that the class of λ-theories induced by GoI wave-style graph models is richer than that of game models. Our goal in this paper is rather that of showing that the basic intuitions underlying the applicative machinery of graph models can be subsumed in a more general setting, by a suitable relaxation of the categorical axiomatization of GoI provided by Abramsky. Actually, graph models arising from GoI constructions generalize in various intriguing ways the original graph model construction.

The present paper provides new insights in the theory of graph models, in that it opens up the possibility of applying Abramky's paradigm of "splitting the atom of computation" also to this class of models. In the opposite direction, the paper illustrates that the expressive power of the GoI axiomatization goes well-beyond game-like models and subsumes also graph-like models.

Finally, here is a list of intriguing open questions. Definitive answers or even well-motivated conjectures appear to be rather difficult.
- Many classes of graph-like models have been considered in the literature, see [15]. Do they all induce the same λ-theories as the class of standard GMs?
- Can original LGMs and the generalizations of graph models in [15] be captured as wave WLCs?
- Are all the theories of GMs induced by wave GoI algebras? Which theories escape GoI characterizations?
- Are there particle-style λ-algebras alternative to the ones based on game categories, which induce a richer class of λ-theories?
- Finally, an interesting issue to be investigated is that of giving a logical characterization via *intersection types* [12] to the graph models arising from wave GoI constructions. We feel that this will shed more light both on intersection types and on wave models.

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A Categorical Definitions

We collect some categorical definitions. For more details (in particular for the definition of *traced category*), we refer to [3, 23].

A *monoidal functor* between monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is a triple (F, ϕ, ϕ'_I), where $F : \mathcal{C} \to \mathcal{D}$ is a functor, ϕ is a natural transformation with components $\phi_{A,B} : FA \otimes FB \to F(A \otimes B)$ and $\phi'_I : I \to FI$ is a morphism in $\mathcal{D}$ such that the following diagrams commute

$$\begin{tikzcd}
FA \otimes (FB \otimes FC) \arrow[r, "\alpha"] \arrow[d, "id_{FA} \otimes \phi"'] & (FA \otimes FB) \otimes FC \arrow[d, "\phi \otimes id_{FC}"] \\
FA \otimes F(B \otimes C) \arrow[d, "\phi"'] & F(A \otimes B) \otimes FC \arrow[d, "\phi"] \\
F(A \otimes (B \otimes C)) \arrow[r, "F\alpha"] & F((A \otimes B) \otimes C)
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes FA \arrow[r, "\lambda"] \arrow[d, "\phi'_I \otimes id_{FA}"'] & FA \\
FI \otimes FA \arrow[r, "\phi"] & F(I \otimes A) \arrow[u, "F\lambda"']
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes I \arrow[r, "\rho"] \arrow[d, "id_{FA} \otimes \phi'_I"'] & FA \\
FA \otimes FI \arrow[r, "\phi"] & F(A \otimes I) \arrow[u, "F\rho"']
\end{tikzcd}$$

A monoidal functor is *strong* when ϕ is a natural isomorphism and ϕ' is an isomorphism.

A monoidal functor $F : \mathcal{C} \to \mathcal{D}$, with $\mathcal{C}$ and $\mathcal{D}$ symmetric monoidal categories, is *symmetric* if the following diagram commutes:

$$\begin{tikzcd}
FA \otimes FB \arrow[r, "\phi_{A,B}"] \arrow[d, "\sigma_{FA,FB}"'] & F(A \otimes B) \arrow[d, "F\sigma_{A,B}"] \\
FB \otimes FA \arrow[r, "\phi_{B,A}"] & F(B \otimes A)
\end{tikzcd}$$

A strong symmetric monoidal functor $F : \mathcal{C} \to \mathcal{D}$ between traced categories is *traced* if, for all $f : A \otimes U \to B \otimes U$, $\mathrm{Tr}^{FU}_{FA,FB}(\phi^{-1}_{B,U} \circ Ff \circ \phi_{A,U}) = F(\mathrm{Tr}^{U}_{A,B}(f))$.

A *monoidal* natural transformation m between monoidal functors (F, ϕ, ϕ'_I) and (G, ψ, ψ'_I) is a natural transformation $m : F \Rightarrow G$ s.t. the following diagrams commute:

$$\begin{tikzcd}
FA \otimes FB \arrow[r, "\phi_{A,B}"] \arrow[d, "m_A \otimes m_B"'] & F(A \otimes B) \arrow[d, "m_{A \otimes B}"] \\
GA \otimes GB \arrow[r, "\psi_{A,B}"] & G(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "\phi'_I"] \arrow[dr, "\psi'_I"'] & FI \arrow[d, "m_I"] \\
& GI
\end{tikzcd}$$

A monoidal *pointwise* natural transformation is a family of maps $m_A : FA \to GA$ s.t. the naturality diagram commutes for morphisms of the form $f : I \to A$, for all object A.