Semantical Analysis of Perpetual Strategies in λ-calculus *

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Perpetual strategies in λ-calculus are analyzed from a *semantical* perspective. This is achieved using suitable denotational models, *computationally adequate* with respect to the *observational* (*operational*) *equivalence*, $\approx_p$, induced by perpetual strategies. A necessary and sufficient condition is given for an ω-algebraic lattice, isomorphic to the space of its *strict continuous self-maps*, to be computationally adequate w.r.t $\approx_p$. While many such models exist, it is shown, however, that none is *fully abstract* w.r.t. $\approx_p$. The computationally adequate lattice model $\mathcal{D}^p$ is studied in detail. It is used to give a semantical proof of the Conservation Theorem for the λ-calculus, and to provide *coinductive* and *mixed inductive-coinductive* characterizations of $\approx_p$. The coinductive characterization allows to show that the term model of $\approx_p$ is a denotational model; this yields a new characterization of *perpetual* redexes in λ-calculus.

Key words: semantics, lambda calculus, observational equivalences, reduction strategies, normal forms

Introduction

In this paper we carry out a semantical investigation of *perpetual strategies* in λ-calculus. A *perpetual* strategy is a strategy which, for each term, computes an infinite reduction path, if it exists, and otherwise some finite path to normal form. Perpetual strategies have been extensively studied from a syntactical viewpoint, see e.g. [3,6,20,22]; but no study of perpetual strategies based on semantical tools has yet appeared in the literature.

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This paper can be seen as a further chapter in the general programme of investigating the denotational semantics of λ-calculi, some of whose earlier chapters are [3,8,9,13,10,11]. We feel that also in this case semantical investigations provide very fruitful insights.

More specifically, we use semantical techniques for deriving logical tools for reasoning on the *observational equivalence*, $\approx_p$, induced by any perpetual strategy $\rightarrow_p$. This is the equivalence obtained by taking terms to be equal if we cannot tell them apart by *observing* that, for a given *closing context*, the strategy halts successfully when one is used to fill the hole but does not halt when the other one is used.

In effect, the semantical investigations carried out in this paper are independent from the particular structure of the perpetual strategy $\rightarrow_p$. They depend only on the judgement $M \Downarrow_p N$, whose intended meaning is " the $\rightarrow_p$ strategy, starting from M, halts on N" or equivalently "M is strongly normalizing with normal form N". This is in line with the fact that we use standard *extensional* denotational semantics, and hence we do not describe the actual *dynamics* of the reduction strategy, but we account only for its *statics*.

We exploit extensively an S.O.S. axiomatization of $M \Downarrow_p N$.

Clearly, β-reduction cannot be correct w.r.t. $\approx_p$. Hence, in order to develop equational tools for reasoning on $\approx_p$, one has to consider restricted notions of β-reduction. The situation however is rather unsatisfactory. As we will see, the strongest restricted calculus, correct w.r.t. $\approx_p$, is as ineffective as $\approx_p$ itself. Moreover, in the case of $\rightarrow_p$ we do not even have the pleasant situation which arises for the *call-by-value* strategy, where at least there exists a correct calculus, i.e. Plotkin's λ_v-calculus (see [19]), whose *leftmost* strategy coincides with $\rightarrow_v$. So we shall not investigate any new restricted λ-calculus *per se*, but rather we will take a *semantical* approach and describe equational theories by means of suitable models. For the sake of completeness, however, we shall present an effective restricted λ-calculus, the $\lambda_{\beta_{KN}}$-calculus, and we shall briefly compare it with the λ_{δ_K}-calculus of Böhm and Intrigila (see [6]).

In order to develop semantical tools for establishing $\approx_p$, we need first to introduce a suitable notion of *denotational* model called *strict λ-model* and then to find useful examples of such models which are *computationally adequate* for $\approx_p$, i.e. models which induce semantical equivalences not coarser than $\approx_p$. The nature of restricted calculi essentially dictates that, in a lattice-theoretic setting, we have to consider *strict* function spaces. We focus on ω-algebraic lattices, which are homeomorphic to the space of their *continuous strict self-functions*. We call such structures *strict extensional λ-lattices*. The main result of our denotational investigation is a *necessary* and *sufficient* condition for a strict extensional λ-lattice to be *computationally adequate* for $\approx_p$. Such a condition allows to define easily a plethora of computationally adequate λ-models. The proof of sufficiency is an interesting application of the *computability technique*.

The class of strict extensional λ-lattices is rather large. Nevertheless, we show that no such model can capture precisely $\approx_p$, i.e. none is *fully abstract* for $\approx_p$. This is yet another example of the mismatch between the essentially *sequential* nature of λ-calculus and the essentially *parallel* nature of *Scott-continuous* functions.

A particular strict extensional λ-lattice, $\mathcal{D}^p$, is useful in two ways. It provides immediately a proof of the, so called, Conservation Theorem for λ-calculus. This is the theorem which asserts that $I\beta$-redexes are perpetual, i.e. if $C[(\lambda x.M)N]$ is not strongly normalizing and $x \in FV(M)$ then also $C[M[N/x]]$ is not strongly normalizing. Secondly, it is an essential tool in one of the proofs of the applicative characterization of $\approx_p$.

We give a *coinductive* (*applicative*) *characterization* of the observational equivalence $\approx_p$, by showing that, in testing for $\Downarrow_p$, we can restrict ourselves to applicative contexts. This allows us to derive a *co-induction principle* for establishing $\approx_p$ in the line of [11]. We discuss various proof methods for establishing this (see e.g. [15,16]). We give details only of a semantical proof based on *logical relations*, which generalizes Pitts' technique for lazy strategies. This proof makes use of a *mixed induction-coinduction* principle for establishing $\approx_p$, based on $\mathcal{D}^p$. As a by product, it provides also an alternative proof of the computational adequacy of $\mathcal{D}^p$.

The applicative characterization of $\approx_p$ is crucial in establishing that the term model of $\approx_p$ is a denotational λ-model. This latter fact is used to provide a new characterization of *perpetual* redexes in λ-calculus, equivalent to that of Bergstra and Klop (see [5]).

Some of the constructions in this paper are closely linked to those appearing in an earlier paper of ours, [10], where we analyze the semantics of λI-calculus, and to [8], where weakly normalizing terms are analyzed using *intersection type theory*.

The paper is organized as follows. In Section 1 we introduce perpetual strategies and observational equivalences. In Section 2 we introduce the notion of *strict* λ-model and discuss restricted λ-calculi such as the $\lambda_{\beta_{KN}}$-calculus. In Section 3 we present two examples of denotational strict λ-models: the term model of $\approx_p$ and the strict extensional λ-lattices. We derive also the new characterization of perpetual redexes. In Section 4 we give the necessary and sufficient condition for a strict extensional λ-lattice to be a computationally adequate model for $\approx_p$; we discuss examples, and we prove that fully abstract models of this kind do not exist. In Section 5 we prove coinduction and mixed induction-coinduction principles for establishing $\approx_p$. Final remarks and conjectures appear in Section 6.

In this paper we use standard λ-calculus notions, notations, and facts without explicitly introducing them. Full details can be found in [3].

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1 Perpetual Strategies

We start by recalling basic notions and introducing a bit of notation.

A *reduction strategy* is a procedure for determining, for each λ-term, a specific β-redex in it to contract. A (deterministic) strategy can be formalized as a partial function on λ-terms, $\to_\sigma \subseteq \Lambda \times \Lambda$, such that, if $M \to_\sigma N$, then N is the result of applying $\to_\sigma$ to M. Reduction paths are defined as usual, by repeatedly applying the reduction strategy (possibly 0 times). A particular set of terms which do not belong to the domain of $\to_\sigma$ are called *σ-values*, denoted by Val_σ. Given $\to_\sigma$, we can define the *evaluation relation* $\Downarrow_\sigma \subseteq \Lambda \times \Lambda$, such that $M \Downarrow_\sigma N$ holds if and only if there exists a reduction path leading from M to N and N is a σ-value. If there exists N such that $M \Downarrow_\sigma N$, then $\to_\sigma$ *halts successfully* (terminates) on M, otherwise it *diverges* on M ($M \Uparrow_\sigma$).

Given any perpetual strategy $\to_p$, the set Val_p is the set of λ-terms in normal form, and the judgement $M \Downarrow_p N$ has also the natural reading of "the term M is β-strongly normalizing and its normal form is N". Following [11], we can conveniently axiomatize $\Downarrow_p$ using the S.O.S. format as follows: [1]

Theorem 1 *Let $\to_p$ be a perpetual strategy. The evaluation relation $\Downarrow_p$ is the least binary relation over $\Lambda \times Val_p$ satisfying the following rules:*

$$(var)\quad \frac{M_1 \Downarrow_p M_1' \ \ \ldots \ \ M_n \Downarrow_p M_n'}{xM_1 \ldots M_n \Downarrow_p xM_1' \ldots M_n'} \quad n \geq 0$$

$$(abstr)\quad \frac{M \Downarrow_p N}{\lambda x.M \Downarrow_p \lambda x.N}$$

$$(I\beta)\quad \frac{(\lambda x.M)N \ \ I\beta\text{-}redex \quad M[N/x]M_1 \ldots M_n \Downarrow_p V}{(\lambda x.M)NM_1 \ldots M_n \Downarrow_p V} \quad n \geq 0$$

[1] Recall that $(\lambda x.M)N$ is an $I\beta$-redex if $x \in FV(M)$, otherwise it is a $K\beta$-redex.

$$(K\beta) \quad \frac{(\lambda x.M)N \ \mathit{K\beta\text{-}redex} \quad N \Downarrow_p P \quad MM_1 \ldots M_n \Downarrow_p V}{(\lambda x.M)NM_1 \ldots M_n \Downarrow_p V} \quad n \geq 0.$$

PROOF. In order to show that if M is β-strongly normalizing, with normal form N, then $M \Downarrow_p N$, we proceed by induction on the length of the longest derivation of M to normal form.
Conversely, in order to show that, if $M \Downarrow_p N$, then M is β-strongly normalizing, with normal form N, we proceed by induction on the structure of the derivation of $M \Downarrow_p N$. The only non-trivial cases are the rules $(I\beta)$ and $(K\beta)$. They are dealt with by double induction on the length of the longest derivation of $M[N/x]M_1 \ldots M_n$ to normal form, and on the length of the longest derivation of N to normal form. $\Box$

Notice that the evaluation relation $\Downarrow_p$ can be equivalently axiomatized using the following three rules:

$$(var') \quad \frac{M_1 \Downarrow_p M_1' \ \ldots \ M_n \Downarrow_p M_n'}{xM_1 \ldots M_n \Downarrow_p xM_1' \ldots M_n'} \quad n \geq 0$$

$$(abstr') \quad \frac{M \Downarrow_p N}{\lambda x.M \Downarrow_p \lambda x.N}$$

$$(\beta) \quad \frac{N \Downarrow_p \quad M[N/x]M_1 \ldots M_n \Downarrow_p V}{(\lambda x.M)NM_1 \ldots M_n \Downarrow_p V} \quad n \geq 0.$$

It is worth pointing out that the results in this paper concerning perpetual strategies will, in effect, depend only on the property of the evaluation relation $\Downarrow_p$ expressed by Theorem 1. This theorem can be viewed as a reformulation of well-known facts (see e.g. [14,20]). An alternative, more direct, proof of Theorem 1 can be obtained from the perpetuality of the strategy F_∞ introduced by Barendregt, Bergstra, Klop, and Volken ([3]). This strategy reduces the leftmost β-redex which is not contained in the operator of another redex, and which is either an $I\beta$-redex, or a $K\beta$-redex whose argument is a normal form. The relation $\rightarrow_{F_\infty} \subseteq \Lambda \times \Lambda$ is expressible in S.O.S. format as follows:

$$\frac{M_1 \ldots M_h \text{ normal forms} \quad M_{h+1} \rightarrow_{F_\infty} M_{h+1}'}{xM_1 \ldots M_n \rightarrow_{F_\infty} xM_1 \ldots M_h M_{h+1}' M_{h+2} \ldots M_n} \quad 0 \leq h < n$$

$$\frac{M \rightarrow_{F_\infty} N}{\lambda x.M \rightarrow_{F_\infty} \lambda x.N}$$

$$\frac{(\lambda x.M)N \ K\beta\text{-redex} \quad N \rightarrow_{F_\infty} N'}{(\lambda x.M)NM_1 \ldots M_n \rightarrow_{F_\infty} (\lambda x.M)N'M_1 \ldots M_n} \quad n \geq 0$$

$$\frac{(\lambda x.M)N \ I\beta\text{-redex}}{(\lambda x.M)NM_1 \ldots M_n \rightarrow_{F_\infty} M[N/x]M_1 \ldots M_n} \quad n \geq 0$$

$$\frac{(\lambda x.M)N \ K\beta\text{-redex} \qquad N \ \beta\text{-normal form}}{(\lambda x.M)NM_1 \ldots M_n \rightarrow_{F_\infty} MM_1 \ldots M_n} \quad n \geq 0.$$

Theorem 2 (Theorem 13.4.6 [3]) *$\to_{F_\infty}$ is an effective perpetual one-step strategy.*

PROOF. The fact that $M \to^*_{F_\infty} N \in Val_p \implies M \Downarrow_p N$ is proved, using Theorem 1, by induction on pairs (m, n), where m is the length of the $\to_{F_\infty}$-normalizing path and n is the complexity of M. The reverse implication is trivial, since $\to_{F_\infty}$ halts only on normal forms. $\square$

Other examples of effective perpetual strategies can be found in [21]. From now onwards $\to_p$ will denote a generic perpetual strategy.

1.1 Observational and applicative equivalences

The *observational pre-order* induced by a perpetual strategy $\to_p$ is defined as usual: syntax phrases are considered as *black boxes*, and a syntax phrase is said to approximate another if it is not possible to tell the first apart from the second by observing that, for a given (closing) program context, the machine halts successfully when the first is used as a subphrase, but does not halt when the other is used as a subphrase.

Definition 3 (p-observational preorder and equiv.) *Let $M, N \in \Lambda$.*

(i) The observational pre-order $\sqsubseteq_p$ *is defined by:*
$M \sqsubseteq_p N$ *iff* $\forall C[\ ].(C[M], C[N] \in \Lambda^0 \Rightarrow (C[M] \Downarrow_p \Rightarrow C[N] \Downarrow_p))$.
(ii) The observational equivalence $\approx_p$ *is defined by:*
$M \approx_p N$ *iff* $(M \sqsubseteq_p N$ *and* $N \sqsubseteq_p M)$.

Notice that $\approx_p$ is a congruence both w.r.t. application and w.r.t. abstraction.

Other important relations induced by a reduction strategy on λ-terms are the *applicative pre-order* and the corresponding *applicative equivalence*. These differ from the observational ones, in that we focus *only* on the behaviour of *closed* substitution instances of the terms in *applicative* (closed) contexts.

Definition 4 (p-applicative pre-order and equivalence) *Let $M, N \in \Lambda$, such that $FV(M) \cup FV(N) \subseteq \{x_1, \ldots, x_m\}$.*

(i) The applicative pre-order $\sqsubseteq_p^{app}$ *is defined by:* $M \sqsubseteq_p^{app} N$ *iff*
$\forall P_1 \ldots P_n \in \Lambda^0, \forall Q_1 \ldots Q_m \in (\Lambda^0 \cap Val_p)$.
$M[Q_1/x_1, \ldots, Q_m/x_m]P_1 \ldots P_n \Downarrow_p \Rightarrow\ N[Q_1/x_1, \ldots, Q_m/x_m]P_1 \ldots P_n \Downarrow_p$
.

(ii) *The* applicative equivalence $\approx_p^{app}$ *is defined by:*
$M \approx_p^{app} N$ *iff* $(M \sqsubseteq_p^{app} N$ *and* $N \sqsubseteq_p^{app} M)$.

The restriction of the applicative equivalence $\sqsubseteq_p^{app}$ to closed terms has the following standard coinductive characterization:

Lemma 5 $\sqsubseteq_p^{app} \cap(\Lambda^0 \times \Lambda^0)$ *can be viewed as the greatest fixed point of the monotone operator* $\Psi_p : \mathcal{P}(\Lambda^0 \times \Lambda^0) \to \mathcal{P}(\Lambda^0 \times \Lambda^0)$ *defined by:*
$\Psi_p(R) \equiv \{(M, N) \mid (M \Downarrow_p \implies N \Downarrow_p)\ \&\ \forall P \in \Lambda^0.\ (MP, NP) \in R\}$.

Using the above lemma we can derive immediately the validity of the following coinduction principle:

$$\frac{(M, N) \in R \quad R \text{ is a } \Psi_p\text{-bisimulation}}{M \sqsubseteq_p^{app} N} \ ,$$

where a Ψ_p-*bisimulation* is a relation $R \subseteq \Lambda^0 \times \Lambda^0$ s.t. $R \subseteq \Psi_p(R)$.

The coinduction principle above can be used to establish directly observational equivalences, since, as we will see in Section 5, $\approx_p = \approx_p^{app}$. A similar coinduction principle can be derived also for $(\approx_p^{app})_{|\Lambda^0 \times \Lambda^0}$, by suitably modifying the structure of the operator Ψ_p.

2 Restricted λ-calculi and strict λ-models

When faced with the problem of establishing $\approx_p$ we do not want to prove everything from first principles, using always just induction on the length of the reduction paths dictated by the $\to_p$ strategy. These inductions are rather complex, unperspicuous and ultimately error-prone. Semantically significant reasoning principles are called for in order to factor out such complex tasks.

When trying to find such *effective, logical* tools for establishing observational equivalences, it is natural to consider first *equational theories* derived from λ-calculus. We recall that an equational theory, $\sim$ on Λ (i.e. $\sim \subseteq \Lambda \times \Lambda$), is correct w.r.t. $\approx_p$ if

$$M \sim N \implies M \approx_p N \ .$$

Ordinary β-conversion, $=_\beta$, is clearly incorrect w.r.t. $\approx_p$, consider e.g. $\lambda x.x$ and its β-expansion $(\lambda z.(\lambda xy.y)(zz))(\lambda z.(\lambda xy.y)(zz))$. So, in order to find correct calculi for $\approx_p$ one has necessarily to look for suitable restrictions of $=_\beta$.

A similar situation arises in connection with the equivalence $\approx_v$ determined by Plotkin's standard *call-by-value* reduction $\to_v$, see [19,9]. A possible correct calculus, in this case, is Plotkin's call-by-value λ_v-calculus, where correct redexes are those whose operand is a variable or an abstraction. In a way, also $\to_p$ has a *call-by-value* flavour, in that any calculus correct for it has to be *at least* operand sensitive. Although Plotkin's λ_v-conversion is far from being the largest correct notion of conversion included in $\approx_v$, nevertheless it is interesting in that $M \Downarrow_v N$ if and only if M β_v-reduces to N following the *leftmost* β_v-strategy. However, one can easily see that there exists no restricted calculus which has such a property with respect to $\Downarrow_p$. Consider for instance $(\lambda yz.z)(xx)$. This should necessarily be a redex for any calculus, whose leftmost strategy be normalizing. But, there are many contexts where such a redex cannot be correctly contracted, e.g. $(\lambda x.[\ ])\lambda x.xx$.

In this paper, however, we do not want to study new *restricted* calculi *per se*, but only to the extent to which the equational theories that they determine shed light on $\approx_p$. Hence, we prefer to take a *semantical* approach and try to describe equational theories by means of suitable *denotational* models, called *strict λ-models*. Strict λ-models are very closely related to the *call-by-value* λ-models, in the style of Hindley and Longo, introduced in [9].

We will see that interesting *restricted calculi*, correct for $\approx_p$, are those valid in *strict λ-models*. Some noteworthy examples will be introduced below.

Definition 6 (strict λ-models) *A* strict λ-model *consists of a quadruple* $\mathcal{D} \equiv \langle D, \bullet^D, \perp_D, [\![\ ]\!]^D \rangle$ *where* $(D, \bullet^D)$ *is an* applicative structure, *i.e.* $\bullet^D : D \times D \to D$, $\perp_D \in D$ *is intended to represent divergence, and* $[\![\ ]\!]^D : D \times (Var \to (D \setminus \{\perp_D\})) \to D$ *is an* interpretation function *such that:*

i) $\bullet^D$ *is right strict, i.e., for all* $d \in D$, $d \bullet^D \perp_D =_D \perp_D$*;*
ii) $[\![x]\!]^D_\rho =_D \rho(x)$*, for all* $x \in Var$*;*
iii) $[\![MN]\!]^D_\rho =_D [\![M]\!]^D_\rho \bullet^D [\![N]\!]^D_\rho$*;*
iv) $[\![\lambda x.M]\!]^D_\rho \bullet^D d =_D [\![M]\!]^D_{\rho[d/x]}$*, if* $d \neq \perp_D$*;*
v) *if* $\forall d \in D \setminus \{\perp_D\}.\ [\![M]\!]^D_{\rho[d/x]} =_D [\![M']\!]^D_{\rho[d/x]}$ *, then* $[\![\lambda x.M]\!]^D_\rho =_D [\![\lambda x.M']\!]^D_\rho$*;*
vi) *if* $FV(M) \subseteq \{x_1, \ldots, x_n\}$*, and for all* $i = 1, \ldots, n.$ $\rho(x_i) =_D \rho'(x_i)$*, then* $[\![M]\!]^D_\rho =_D [\![M]\!]^D_{\rho'}$.

The following lemma can be easily proved by induction on the structure of M:

Lemma 7 *Let $\mathcal{D}$ be a strict λ-model. For all $M, N \in \Lambda$ such that* $[\![N]\!]^D_\rho \neq \perp_D$*,* $[\![M[N/x]]\!]^D_\rho =_D [\![M]\!]^D_{\rho[[\![N]\!]^D_\rho/x]}$.

Theorem 8 *Let $\mathcal{D}$ be a strict λ-model, then:* $\forall \rho.\ [\![(\lambda x.M)N]\!]^D_\rho =_D [\![M[N/x]]\!]^D_\rho$ *if and only if* $\forall \rho.\ ([\![N]\!]^D_\rho =_D \perp_D \Longrightarrow [\![M[N/x]]\!]^D_\rho =_D \perp_D)$.

PROOF. ($\Rightarrow$) This is immediately using items i) and iii) of Definition 6.
($\Leftarrow$) If $[\![N]\!]_{\rho}^{D} =_{D} \perp_{D}$, then thesis follows immediately from items i) and iii) of Definition 6. If $[\![N]\!]_{\rho}^{D} \neq \perp_{D}$, then, using items iii) and iv), $[\![(\lambda x.M)N]\!]_{\rho}^{D} =_{D} [\![M]\!]_{\rho[[\![N]\!]_{\rho}^{D}/x]}^{D}$. Then the thesis follows from Lemma 7. $\square$

Models determine equational theories in the following sense:

Definition 9 *Let $\mathcal{D}$ be a strict λ-model.*
i) The theory of $\mathcal{D}$, *$\mathcal{T}_D$, is the set of pairs*

$$\{(M, N) \in \Lambda \times \Lambda \mid \forall \rho \in (Var \to (D \setminus \{\perp_D\})).\ [\![M]\!]_{\rho} =_D [\![N]\!]_{\rho}\}.$$

ii) Let $\sim \subseteq \Lambda \times \Lambda$ be a congruence, $\mathcal{D}$ is a model *of $\sim$ if $\sim \subseteq \mathcal{T}_D$.*

Proposition 10 *Let $\mathcal{D}$ be a strict λ-model. Then $\mathcal{T}_D$ is a congruence.*

PROOF. Immediate.

2.1 Some examples of correct notions of reduction w.r.t $\approx_p$

A natural effective restriction of β-reduction, correct with respect to $\approx_p$, can be obtained by focusing only on the structure of the operand, in analogy to the λ_v-calculus. This amounts to allowing $(\lambda x.M)N$ to contract to $\to M[N/x]$ if and only if N is a *closed normal form*. But this is clearly too restrictive. A more significant restriction of β-reduction, which takes into account both the shape of the *operator* and the *operand*, is suggested by the class of strict λ-models of Section 3.2 and it was first introduced in [15].

Definition 11 (β_{KN}-reduction [15]) *The notion of β_{KN}-reduction, $\to_{\beta_{KN}}$, is the contextual closure of the relation generated by the axiom:*
$(\lambda x.M)N \to_{\beta_{KN}} M[N/x]$ *if $(\lambda x.M)N$ is either an $I\beta$-redex or a $K\beta$-redex whose argument is a* variable *or a* closed β-normal *form.*

Let $\to_{\beta_{KN}\eta}$ be the notion of reduction obtained by adding to $\to_{\beta_{KN}}$ the η-rule. One can easily see that $\to_{\beta_{KN}} \subset \to_{\beta}$, and that $\to_{\beta_{KN}} \subset \to_{\beta_{KN}\eta} \subset \to_{\beta\eta}$; moreover, all these inclusions are strict. Using standard techniques (see e.g. [3]), one can show also that all these notions of reduction are Church Rosser.

We shall refer to the equational theory induced by the reflexive, symmetric, transitive, contextual closure of $\to_{\beta_{KN}}$, $=_{\beta_{KN}}$, as the $\lambda_{\beta_{KN}}$*-calculus*. Similarly, we define the $\lambda_{\beta_{KN}\eta}$-calculus, $=_{\beta_{KN}\eta}$. In Section 4.1.1, we will prove that both $=_{\beta_{KN}}$ and $=_{\beta_{KN}\eta}$ are correct w.r.t $\approx_p$.

One can easily see that the $\rightarrow_{\beta_{KN}}$-conversion is not the largest $\approx_p$-correct conversion included in $\rightarrow_\beta$. For instance, one could safely allow the erasure of an operand, provided it coincide with the body of the operator. In the next section we will characterize precisely the notion of conversion $\approx_p \cap =_\beta$. Such a calculus however is rather ineffective.

3 Examples of strict λ-models

3.1 The term model of $\approx_p$

In this subsection we introduce a first, very important, example of a strict λ-model: the term model of $\approx_p$.

Theorem 12 *Let D^T be the set of equivalence classes of closed λ-terms modulo $\approx_p$, i.e. $\Lambda^0/\approx_p$. Put $[M]_{\approx_p} \bullet^{D^T} [N]_{\approx_p} =_{D^T} [MN]_{\approx_p}$. Take as distinguished element $\perp_{D^T}$ the equivalence class of non-strongly normalizing terms, e.g. $\perp_{D^T} \equiv [(\lambda x.xx)(\lambda x.xx)]_{\approx_p}$. And define $[\![M]\!]^{D^T}_{\lambda x \in Var.\ [\rho(x)]} =_{D^T} [M^\rho]_{\approx_p}$, where M^ρ denotes the term obtained from M by substituting any occurrence of a free variable x in M with $\rho(x)$.*
Then $\langle D^T, \bullet^{D^T}, \perp_{D^T}, [\![\ ]\!]^{D^T}\rangle$ is a strict λ-model.

PROOF. For ease of notation we shall omit the subscript $\approx_p$ in denoting equivalence classes.
First of all we need to show that $\bullet^{D^T}$ and $[\![\ ]\!]^{D^T}$ are well-defined. It is immediate to see that the former is well-defined since $\approx_p$ is a congruence w.r.t. application. In order to show that the latter is well-defined, assume that $M \approx_p N$ and that, for all $x \in Var$, $\rho(x) \approx_p \rho'(x)$. Then, by Theorem 55, $M^\rho \approx_p N^\rho$, and since $\approx_p$ is a congruence, $N^\rho \approx_p N^{\rho'}$, so we can conclude that $M^\rho \approx_p N^{\rho'}$, since $\approx_p$ is an equivalence.
Conditions i), ii), iii) and vi) are immediate by definition.
In order to prove iv), i.e., if $N \in \Lambda^0$ is strongly normalizing, then
$[\![\lambda x.\ M]\!]_{\lambda v \in Var.[\rho(v)]_{\approx_p}} \bullet^{D^T} [N]_{\approx_p} =_{D^T} [\![M]\!]_{\lambda v.[\rho[N/x](v)]_{\approx_p}}$,
by Theorem 56, we need to show only that $(\lambda x.\ M^\rho)N \approx_p^{app} M^\rho[N/x]$. This latter equivalence is immediate from the S.O.S. specification of the perpetual strategy $\rightarrow_p$ of Section 1.
Finally we have to prove v), i.e., if for all strongly normalizing closed λ-terms P, $M^\rho[P/x] \approx_p N^\rho[P/x]$, then $\lambda x.\ M^\rho \approx_p \lambda x.\ N^\rho$. But this is again an immediate consequence of Theorem 55. $\square$

A direct application of Theorem 8 to the term model $\mathcal{D}^T$ yields the following characterization of the *strongest* restricted calculus correct for $\approx_p$. This calculus is however almost as difficult to use as the equivalence $\approx_p$ itself.

Theorem 13 *Let Val_p^0 denote the set of closed normal forms i.e. $\Lambda^0 \cap Val_p$. Then the congruence $\approx_p \cap =_\beta$ is equal to:*
$\{\langle(\lambda x.M)N, M[N/x]\rangle \mid \forall \rho \in [Var \to Val_p^0].\ N^\rho \Uparrow_p \Longrightarrow (M[N/x])^\rho \Uparrow_p\}$.

PROOF. By Theorem 8 and Theorem 55.

The above Theorem can be viewed as a strengthening of Bergstra and Klop's characterization of *perpetual redexes* (see [5], and [3], Corollary 13.4.16), in that we consider only substitutions by *closed* values. We recall that a redex $(\lambda x.M)N$ is said to be *perpetual* if, for all (not necessarily closed) contexts $C[\,]$, $C[(\lambda x.M)N] \Uparrow_p \Longleftrightarrow C[M[N/x]] \Uparrow_p$. Clearly $(\lambda x.M)N$ is a perpetual redex if and only if $(\lambda x.M)N \approx_p M[N/x]$, since a term is strongly normalizing if and only if also its closure is strongly normalizing.

3.2 Strict λ-lattices

We recall first some standard definitions:

Definition 14 *Let $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$ be lattices.*
0) D is non-trivial *if it has at least two points.*
i) $c \in D$ is finite *if $\forall X \subseteq D.\ (X$ directed & $c \sqsubseteq_D \bigsqcup X) \Longrightarrow \exists x \in X.\ c \sqsubseteq_D x$;*
ii) $(D, \sqsubseteq_D)$ is ω-algebraic if the set of its finite elements K_D is countable, and $\forall d \in D.\ d =_D \bigsqcup\{c \in K_D \mid c \sqsubseteq_D d\}$;
iii) The set $[D \to_\perp D]$ is the ω-algebraic complete lattice consisting of the strict continuous self-functions *of D, i.e. of those continuous functions $f : D \to D$ such that $f(\perp_D) =_D \perp_D$;*
iv) A strict retraction *from D to E, $D \rhd E$, is a pair of continuous functions (Ψ^D, Φ^D) such that $\Phi^D \circ \Psi^D =_D id_E$ and $(\Psi^D \circ \Phi^D)(\perp_D) =_D \perp_D$.*

We introduce now the main tool of our semantical investigation of perpetual strategies: strict λ-lattices. We will see that they are models also of the $\lambda_{\beta_{KN}}$-calculus.

Definition 15 (strict λ-lattices) *A* (ω-algebraic complete) strict λ-lattice, *$\mathcal{D}$, is a quintuple $\langle D, \sqsubseteq_D, \bullet^D, \perp_D, [\![\ ]\!]^D\rangle$, where:*

- *$(D, \sqsubseteq_D)$ is a non-trivial (ω-algebraic complete) lattice such that $[D \to_\perp D]$ is a strict retraction of D ;*

- $\bullet^D : D \times D \to D$ *is the function defined by:* $d \bullet^D e =_D (\Phi^D(d))(e)$*;*
- $\perp_D$ *is the least element of* D*;*
- *the interpretation function* $[\![\ ]\!]^D : \Lambda \times (Var \to (D \setminus \{\perp_D\})) \to D$*, where* $\rho : Var \to (D \setminus \perp_D)$*, is defined* canonically *as follows:*
  - $[\![x]\!]^D_\rho =_D \rho(x)$
  - $[\![MN]\!]^D_\rho =_D [\![M]\!]^D_\rho \bullet^D [\![N]\!]^D_\rho$
  - $[\![\lambda x.M]\!]^D_\rho =_D \Psi^D(\ \lambda d \in D.\ \ \mathbf{if}\ d \neq \perp_D\ \ \mathbf{then}\ [\![M]\!]^D_{\rho[d/x]}\ \mathbf{else}\ \perp_D)$.

The structure of $[\![\ ]\!]^D$ defined above yields immediately:

Proposition 16 *Let $\mathcal{D}$ be a strict λ-lattice. Then $\mathcal{D}$ is a strict λ-model.*

Theorem 17 *Let $\mathcal{D}$ be a non-trivial strict λ-lattice. Then* $=_{\beta_{KN}} \subseteq \mathcal{T}^D$.

PROOF. We only need to show that $\forall \rho.\ [\![(\lambda x.M)N]\!]^D_\rho =_D [\![M[N/x]]\!]^D_\rho$, if $(\lambda x.M)N$ is either an $I\beta$-redex or a $K\beta$-redex whose argument is a *variable* or a *closed β-normal* form. In all the three cases we show that the condition of Theorem 8 applies. In the first case, which is essentially Proposition 2 ii) of [10], one can show directly by induction on the structure of M that, if $[\![N]\!]^D_\rho =_D \perp_D$ and $x \in FV(M)$, then $[\![M[N/x]]\!]^D_\rho =_D \perp_D$.
If $x \notin FV(M)$ then the condition of Theorem 8 applies trivially. In fact, if $N \equiv x$, by definition we have that, for all ρ, $\rho(x) \neq \perp_D$. If N is a closed normal form, one can see by induction on N that $[\![N]\!]_{\rho_{\top_D}} \neq \perp_D$, for $\rho_{\top_D} \equiv \lambda x \in Var.\ \top_D$. $\square$

Definition 18 *A* strict extensional λ-lattice *is a strict λ-lattice $\mathcal{D}$, where D is a non-trivial (ω-algebraic complete) lattice such that $[D \to_\perp D] \simeq D$.*

Theorem 19 *A strict extensional λ-lattice is a model of the $\lambda_{\beta_{KN}\eta}$-calculus.*

Theorem 20 *Let $\mathcal{D}$ be a strict extensional λ-lattice. $\mathcal{T}_D$ is non trivial if and only if the lattice has at least three points.*

PROOF. In any strict extensional λ-lattice D with at least three points, we have that $[\![\lambda xy.x]\!]^D \neq_D [\![\lambda yx.x]\!]^D$. This can be seen immediately by computing $[\![\lambda xy.x]\!]^D_\rho \bullet^D d_1 \bullet^D d_2 =_D d_1$ and $[\![\lambda yx.x]\!]^D_\rho \bullet^D d_1 \bullet^D d_2 =_D d_2$, where $d_1 \neq d_2$ are any two distinct elements of D different from $\perp_D$. On the other hand, if D has only two elements, then one can show by induction on the structure of $M \in \Lambda$ that $[\![M]\!]^D_{\rho_{\top_D}} =_D \top_D$, where $\top_D$ is the element of D different from $\perp_D$ and $\rho_{\top_D} \equiv \lambda x \in Var.\ \top_D$. $\square$

4 Denotational Semantics of $\approx_p$

In this section we discuss topological tools for establishing the observational equivalence $\approx_p$.

We begin by giving two crucial definitions:

Definition 21 *Let $\mathcal{D}$ be a strict extensional λ-model.*

(i) $\mathcal{D}$ is computationally adequate *w.r.t. $\approx_p$ if, for all $M, N \in \Lambda$,*

$$(\forall \rho \in Var \to (D \setminus \{\perp_D\}).\ [\![M]\!]^D_\rho \sqsubseteq_D [\![N]\!]^D_\rho)\ \to M \sqsubseteq_p N;$$

(ii) $\mathcal{D}$ is fully abstract *w.r.t. $\approx_p$ if, for all $M, N \in \Lambda$,*

$$(\forall \rho \in Var \to (D \setminus \{\perp_D\}).\ [\![M]\!]^D_\rho \sqsubseteq_D [\![N_\rho]\!]^D)\ \Longleftrightarrow M \sqsubseteq_p N.$$

The main results of this section can now be summarized as follows:
- a necessary and sufficient condition for a strict extensional λ-lattice to be computationally adequate for $\approx_p$ (Section 4.1);
- the proof that no strict extensional λ-lattice is fully abstract w.r.t. $\approx_p$ (Section 4.3).

4.1 Strict extensional λ-lattices computationally adequate for $\approx_p$

Since the seminal papers [4,7], it is well known that, given an ω-algebraic complete λ-lattice model D, such that $[D \to_\perp D] \simeq D$, we can give a *finitary logical presentation* of D in terms of *intersection types*. Namely, we can introduce an intersection type theory $\mathcal{T}_D$ which is the Stone dual of the model D in the sense of the classical Theorem 25 below:

Definition 22 *Let $\mathcal{D}$ be a strict extensional λ-lattice, and let $\mathcal{T}_D = (T_D, \leq_D)$ be the intersection type theory defined by*

$$(T_D \ni)\ \phi ::=\ c_k\ \mid\ \phi \wedge \phi\ \mid\ \phi \to \phi\ ,$$

where k is a compact element in $D \setminus \{\perp_D\}$.

The relation $\leq_D$ on types is the least relation closed under the following rules:

$c_k \leq_D c'_k \Longleftrightarrow k \sqsupseteq_D k'$

$c_k =_D \bigwedge_{i_1,i_2 \in I}(c_{k_{i_1}} \to c_{k_{i_2}}) \Longleftrightarrow \Phi^D(k) = \bigcup_{i\in I} f_{k_{i_1},k_{i_2}}$

$\phi \leq_D \phi$

$\phi_1 \wedge \phi_2 \leq_D \phi_1$

$\phi_1 \wedge \phi_2 \leq_D \phi_2$

$(\phi \to \phi_1) \wedge (\phi \to \phi_2) \leq_D \phi \to \phi_1 \wedge \phi_2$,

$\dfrac{\phi_1 \leq_D \phi_2 \quad \phi_2 \leq_D \phi_3}{\phi_1 \leq_D \phi_3}$

$\dfrac{\phi \leq_D \phi_1 \quad \phi \leq_D \phi_2}{\phi \leq_D \phi_1 \wedge \phi_2}$

$\dfrac{\phi_2 \leq_D \phi_1 \quad \psi_1 \leq_D \psi_2}{\phi_1 \to \psi_1 \leq_D \phi_2 \to \psi_2}$

where $\phi_1 =_D \phi_2$ *if and only if* $\phi_1 \leq_D \phi_2$ & $\phi_2 \leq_D \phi_1$, *and* $f_{k_{i_1},k_{i_2}}$ *is the step function defined by* $\lambda d \in D.$ **if** $d \sqsupseteq_D k_{i_1}$ **then** k_{i_2} **else** $\perp_D$.

A type environment *is a partial function* $\Gamma : Var \to T_D$. *Let* Env_Γ *denote the set of type environments. The type assignment system* S_D *consists of the following rules:*

$(var) \quad \Gamma[\phi/x] \vdash_D x : \phi$

$(\wedge I) \quad \dfrac{\Gamma \vdash_D M : \phi \quad \Gamma \vdash_D M : \psi}{\Gamma \vdash_D M : \phi \wedge \psi}$

$(\leq) \quad \dfrac{\Gamma \leq_D \Delta \quad \Delta \vdash_D M : \phi \quad \phi \leq_D \psi}{\Gamma \vdash_D M : \psi}$

$(\to I) \quad \dfrac{\Gamma[\phi/x] \vdash_D M : \psi}{\Gamma \vdash_D \lambda x.M : \phi \to \psi}$

$(\to E) \quad \dfrac{\Gamma \vdash_D M : \phi \to \psi \quad \Gamma \vdash_D N : \phi}{\Gamma \vdash_D MN : \psi}$,

where $\Gamma \leq_D \Delta$ *if and only if* $dom(\Gamma) \supseteq dom(\Delta)$ *and* $\Gamma(x) \leq_D \Delta(x)$, *for all* $x \in dom(\Delta)$.

Definition 23 *Let* K^D *be the set of compact elements of* $D \setminus \{\perp_D\}$. *The function* $* : T_D \to K^D$, *usually written postfix, is inductively defined as follows:*
- $c_k^* =_D k$
- $(\sigma \to \tau)^* =_D f_{\sigma^*,\tau^*}$
- $(\sigma \wedge \tau)^* =_D \sigma^* \sqcup \tau^*$.

Lemma 24 *Let* $\sigma, \tau \in \mathcal{T}_D$. *Then* $\sigma \leq_D \tau$ *iff* $\sigma^* \sqsupseteq_D \tau^*$.

Theorem 25 *i) If* $\Gamma \vdash_D M : \sigma$, *then* $\sigma^* \sqsubseteq_D [\![M]\!]^D_\rho$, *if for all* $x \in dom(\Gamma)$, $\Gamma(x)^* \sqsubseteq_D \rho(x)$.
ii) If $\sigma^* \sqsubseteq_D [\![M]\!]^D_\rho$, *then there exists* Γ *such that* $\forall x \in dom(\Gamma).\ \Gamma(x)^* \sqsubseteq_D \rho(x)$ *and* $\Gamma \vdash_D M : \sigma$.

It is well known that in case $(D, \sqsubseteq_D)$ is the inverse limit solution in the category $CPO_\perp$ of the domain equation $D \simeq [D \to_\perp D]$, obtained starting from the initial lattice $(D_0, \sqsubseteq_{D_0})$, one can restrict the set of type constants in $\mathcal{T}_D$ to be the set consisting just of the representatives of the compact elements in $D_0 \setminus \{\perp_{D_0}\}$.

We proceed now to give the necessary and sufficient condition for a strict extensional λ-lattice to be computationally adequate for $\approx_p$. To this end we shall distil the essence of the generalization of Krivine's *computability technique*, as presented by the authors in [10]. The computability argument was used in [10] for showing that, for the λI-calculus, the terms which receive an interpretation different from $\perp_D$ are exactly the strongly normalizing terms.

We start by defining the set Sat of saturated sets of $\Downarrow_p$-convergent λ-terms.

Definition 26 *A set X of $\Downarrow_p$-convergent λ-terms is saturated if:*
i) if $N \Downarrow_p$ and $(M_1[N/x])M_2 \ldots M_n \in X$, then $(\lambda x.M_1)NM_2 \ldots M_n \in X$;
ii) if $M_1, \ldots, M_n$ are $\Downarrow_p$-convergent λ-terms, then $xM_1M_2 \ldots M_n \in X$.

Proposition 27 *The set Sat of saturated sets of $\Downarrow_p$-convergent λ-terms is a non empty complete lattice closed under the operation $\Rightarrow$ defined by:*

$$(B \Rightarrow C) = \{M \in \Lambda \mid \forall N.(N \in B \Rightarrow MN \in C)\} \ .$$

PROOF. The set of strongly normalizing terms can be seen to be saturated using an argument similar to that used in Theorem 1. Saturated sets are closed under set-theoretic intersection. The rest is routine. □

Definition 28 *An intersection type theory $\mathcal{T}_D$ is* interpretable *in Sat if there exists a function $Comp : \mathcal{T}_D \to Sat$ such that, for all $\phi, \psi \in \mathcal{T}_D$:*
i) if $\phi \leq_D \psi$ then $Comp(\phi) \subseteq Comp(\psi)$;
ii) $Comp(\phi \wedge \psi) = Comp(\phi) \cap Comp(\psi)$;
iii $Comp(\phi \to \psi) = Comp(\phi) \Rightarrow Comp(\psi)$;

Theorem 29 *A strict extensional λ-lattice $(D, \sqsubseteq_D)$ is computationally adequate w.r.t. $\approx_p$ iff*
$(\star)$ the theory $\mathcal{T}_D$ is interpretable in Sat.

4.1.1 Proof of Theorem 29: Sufficiency

Assume that the type theory $\mathcal{T}_D$ is interpretable in Sat. In order to show that the strict extensional λ-lattice D is computationally adequate with respect to $\approx_p$, we only need to show that, for all $M \in \Lambda$,

$$M \Downarrow_p \iff \exists \Gamma\ \exists k \in K^D.\ \Gamma \vdash_D M : c_k \ .$$

Then, computational adequacy follows immediately using Theorem 25 and the continuity of the operation of "filling a hole" in a context.

First of all we show that

$$M \Downarrow_p \Longrightarrow \exists \Gamma\ \exists k \in K^D.\ \Gamma \vdash_D M : c_k\ .$$

Theorem 30 *Let $M \in \Lambda$. If $\mathcal{T}_D$ is interpretable in Sat, then*

$$M \Downarrow_p \Longrightarrow \exists k \in K^D.\ \Gamma_{\top_D} \vdash_D M : c_k\ ,$$

where $\forall x.\ \Gamma_{\top_D}(x) =_D c_{\top_D}$, $\top_D$ being the maximal element of D.

PROOF. The proof is by induction on the structure of the S.O.S. derivation of $M \Downarrow_p$. In the (var) case one uses the fact that $\forall k.\ c_{\top_D} \leq_D c_k \to c_{\top_D}$. The $(abstr)$ case is immediate. Cases $(I\beta)$ and $(K\beta)$ are proved using Theorems 8 and 25. $\square$

Next we show that

$$\exists \Gamma\ \exists k \in K^D.\ \Gamma \vdash_D M : c_k \ \Longrightarrow\ M \Downarrow_p\ .$$

First we need a Lemma

Lemma 31 *Let $M, N_1, \ldots, N_n \in \Lambda$ and $\{x_1, \ldots, x_n\} \supseteq FV(M)$.*
If $\Gamma[\phi_1/x_1, \ldots, \phi_n/x_n] \vdash_D M : \phi$ is derivable in S_D and $\forall i.N_i \in Comp(\phi_i)$, then $M[N_i/x_i] \in Comp(\phi)$.

PROOF. By induction on the length of the proof $\Gamma[\phi_1/x_1, \ldots, \phi_n/x_n] \vdash_D M : \phi$, using the property i) of the type interpretation $Comp$. By way of example, we discuss the case of $\to I$. Suppose that $\Gamma[\phi_1/x_1, \ldots, \phi_n/x_n, \psi_0/x] \vdash_D M : \psi_1$. Then, by induction hypothesis we have that, if $\forall i.N_i \in Comp(\phi_i)$ and $N \in Comp(\psi_0)$, then $M[N_i/x_i, N/x] \in Comp(\psi_1)$. But now, since $N \in Comp(\psi_0)$ then $N \Downarrow_p$, hence, using the first clause of Definition 26, we have that $(\lambda x.M[N_i/x_i])N \in Comp(\psi_1)$. But this amounts precisely to the fact that $\lambda x.M[N_i/x_i] \in Comp(\psi_0 \to \psi_1)$. $\square$

Hence, we have:

Theorem 32 (computational adequacy) *Let $M \in \Lambda$. If $\mathcal{D}$ is a strict extensional λ-lattice, whose theory $\mathcal{T}_D$ is interpretable in Sat, then:*
$M \Downarrow_p \Longleftrightarrow \exists \Gamma\ \exists k \in K^D.\ \Gamma \vdash_D M : c_k$.

Using Theorem 25, we immediately have

Corollary 33 $M \Uparrow_p \iff \forall \rho. [\![M]\!]^D_\rho =_D \perp_D$.

4.1.2 Proof of Theorem 29: Necessity

We will now show that condition $(\star)$ of Theorem 29 is necessary for a strict extensional λ-model $\mathcal{D}$ to be computationally adequate for $\approx_p$. Suppose $\mathcal{D}$ is computationally adequate. $\rightarrow_p$-divergent terms have to be interpreted in $\perp_D$. Otherwise, if there existed a term M such that $M \Uparrow_p$ and $[\![M]\!]_\rho \neq \perp_D$ for some ρ, the equality $[\![(\lambda x.I)M]\!]_\rho =_D [\![I]\!]_\rho$ would be valid in D contradicting computational adequacy. Hence $\forall M \in \Lambda.\ ([\![M]\!]_{\rho_\top} \neq \perp_D \iff M \Downarrow_p)$. Hence, using the properties of extensional strict λ-lattices, one can easily see that for all $k \in (D \setminus \{\perp_D\})$ the set

$$Comp(k) \equiv \{M \in \Lambda \mid [\![M]\!]_{\rho_\top} \sqsupseteq_D k\}.$$

is in Sat. It is immediate to check that $Comp$ satisfies the properties of a type interpretation.

4.2 Examples of computationally adequate models

Using Theorem 19, one has immediately that $\beta_{KN}\eta$-reduction is correct w.r.t. $\approx_p$ if there exists at least one computationally adequate strict extensional λ-lattice.

The simplest example of a strict extensional λ-lattice computationally adequate for $\approx_p$ is the model originally introduced in [10], in the context of the semantics of the λI-calculus, which is derived from the model in [8].

Definition 34 *Let $(D^p, \sqsubseteq_p)$ be the solution in the category $CPO_\perp$ of the domain equation $D \simeq [D \rightarrow_\perp D]$, obtained as the inverse limit starting from the three point lattice $D_0 = \{\perp, \nu, \top\}$, with $\perp$ as bottom element and $\top$ as top element, and initial projection i_0 defined as $i_0(\perp) =_{D^p} f_{\perp,\perp}$, $i_0(\nu) =_{D^p} f_{\top,\nu}$, $i_0(\top) =_{D^p} f_{\nu,\top}$, where $f_{a,b}$ denotes the function $\lambda x.$ **if** $x \sqsupseteq_p a$ **then** b **else** $\perp$.*

It is immediate to see that $(D^p, \sqsubseteq_p)$ can be extended to a strict extensional λ-lattice D^p. Using Theorem 29, one can show that D^p is computationally adequate with respect to $\approx_p$. In fact, using a fixed point construction, an interpretation $Comp^{D^p}$ of the theory $\mathcal{T}_{D^p}$ in Sat, satisfying the conditions of Defintion 28, can be easily defined. Namely:

Definition 35 *The sets $Comp^{D^p}(\phi)$ are inductively defined as follows:*

- $Comp^{D^p}(c_\nu) = A_0$
- $Comp^{D^p}(c_\top) = B_0$
- $Comp^{D^p}(\phi \to \psi) = Comp^{D^p}(\phi) \Rightarrow Comp^{D^p}(\psi)$
- $Comp^{D^p}(\phi \wedge \psi) = Comp^{D^p}(\phi) \cap Comp^{D^p}(\psi)$,

where (A_0, B_0) is the least fixed point of the monotone operator $R : (Sat \times Sat, (\supseteq, \subseteq)) \to (Sat \times Sat, (\supseteq, \subseteq))$ defined by

$$R(A, B) = ((B \Rightarrow A), (A \Rightarrow B)) \; .$$

Let SN denote the set of strongly normalizing λ-terms, it is interesting to point out that, using Corollary 33 and suitably reformulating the argument in Theorem 4.3. of [8], we have:

$$Comp^{D^p}(c_\top) = \{M \mid [\![M]\!]_{\rho_\top} =_{D^p} \top\} =$$

$$\{M \mid \forall N_1, \ldots N_k.\ N_i \in SN \Longrightarrow MN_1 \ldots N_k \in SN\},$$

$$Comp^{D^p}(c_\nu) = SN.$$

Building on the idea of a monotone operator, a plethora of models satisfying $(\star)$, not isomorphic to $\mathcal{D}^p$, can be defined. For instance, consider a system of recursive type equations of the form $\{\phi_i = C_i\}_{i \in I}$ in the set of indeterminates $\{\phi_i \mid i \in I\}$, where the C_i's are type expressions built from the indeterminates using only the $\to$ type constructor. Moreover, assume that the system satisfies the syntactical constraint of Mendler, [17], i.e. each ϕ_i occurs only positively in all C_j judged equal to C_i. Then we can define immediately an inverse limit strict λ-model satisfying $(\star)$, by taking as D_0 a flat lattice whose elements different from $\perp_D$ and $\top_D$ are into one-to-one correspondence with the indeterminates and whose functional behaviour is dictated by the system of equations. A simple concrete example is the 4-point lattice D^{bool} where the two incomparable elements, say $true$ and $false$, satisfy $i_0(true) =_{D^{bool}} f_{false,true}$ and $i_0(false) =_{D^{bool}} f_{true,false}$, and $i_0(true \sqcup false)$ is the supremum of $i_0(true)$ and $i_0(false)$.

Suitably modifying the results in [17], one can see that Mendler's condition is necessary and sufficient if restricted to type theories which are generated by a set of incomparable basic constants whose applicative behaviour is that of step functions. Mendler's syntactical condition is clearly not necessary when also the $\wedge$ operator is considered. Consider for instance again D^{bool}, and take i_0 to be the same as above except on $true$, which now is $i_0(true) =_{D^{bool}} f_{false,true} \sqcup f_{true,true}$. One can always take $Comp(true)$ to be the least saturated set, and hence define an interpretation of $\mathcal{T}_{D^{bool}}$ in Sat. More investigation is necessary to determine whether there exists a condition in the style of Mendler for the general case.

Using the existence of a non-trivial strict extensional λ-lattice, which is computationally adequate for $\approx_p$, one obtains immediately a semantical proof of the Conservation Theorem for λ-calculus (see [3], Theorem 13.4.12), and also of the Conservation Theorem for λI-calculus (see [3], Theorem 11.3.4). In fact, such a model is a model of the β_{KN}-calculus where all and only $\rightarrow_p$-divergent terms are equated to $\perp$; and, by Theorem 1, $\rightarrow_p$-divergent terms are precisely the λ-terms which are not strongly normalizable.

Theorem 36 (conservation) *Let $M, N \in \Lambda$. If $C[(\lambda x.M)N] \Uparrow_p$ and $x \in FV(M)$ then $C[M[N/x]] \Uparrow_p$.*

Theorem 37 (conservation for λI-calculus) *Let M be a term of the λI-calculus. If M is weakly normalizing, then M is strongly normalizing.*

4.3 Yet another topological incompleteness phenomenon

Reasoning in the line of [13], one can show that there exists no fully abstract model for $\approx_p$ in the class of strict extensional λ-lattices. Strict extensional λ-lattices are, in fact, continuously complete, i.e. *all* strict continuous self-functions are *representable* in them. But then, a mismatch between the purely *sequential* nature of $\rightarrow_p$ reduction and the *unstable* nature of many continuous functions arises. Similar phenomena were described in [13] for ordinary λ-calculus, and in [9] for *call-by-value* reduction.

Let $O \equiv \lambda xy.y$ and $O^- \equiv \lambda x.O(x\lambda y.y)$ and consider the terms

$$P \equiv \lambda z.(\lambda xyw.w)(zOO^-)(zO^-O)$$

$$Q \equiv \lambda z.(\lambda xy.y)(zO^-O^-)$$

Theorem 38 *The terms P and Q are observationally equivalent w.r.t. $\rightarrow_p$, i.e. $P \approx_p Q$.*

PROOF. We use the fact that $\approx_p$ coincides with $\approx_p^{app}$ (see Section 5). The inclusion $Q \sqsubseteq_p^{app} P$ is immediate by the computational adequacy of $\mathcal{D}^p$, since $[\![Q]\!]^{D^p} \sqsubseteq_{D^p} [\![P]\!]^{D^p}$. This follows from the fact that, for all $d \neq_{D^p} \perp$, $[\![O]\!]^{D^p} \bullet^{D^p} d =_{D^p} [\![I]\!]^{D^p}$, while $[\![O^-]\!]^{D^p} \bullet^{D^p} d =_{D^p} [\![I]\!]^{D^p}$ if $d \bullet^{D^p} [\![I]\!]^{D^p} \neq_{D^p} \perp$. The reverse inclusion, i.e. $P \sqsubseteq_p^{app} Q$, is a complex argument by induction on the number of computation steps in the perpetual strategy F_∞. Suppose the converse, then there exists a closed λ-term, say M, such that $PM \Downarrow_p$ but $QM \Uparrow_p$. We will reach a contradiction by showing that there is no such M for which the number of F_∞-reduction steps such that $PM \Downarrow_p$ is minimal. The structure of P and Q implies that $MOO^- \Downarrow_p$ and $MO^-O \Downarrow_p$ but $MO^-O^- \Uparrow_p$. Then, at least one of the descendants of the arguments of M will have to act as an operator in the F_∞-reduction path, as can be seen by induction on the structure of M. Assume without loss of generality that it is the first. We have then:

$$C[OD[O,O^-]E_1[O,O^-]\dots E_m[O,O^-],O,O^-]\Downarrow_p$$

$$C[O^-D[O^-,O]E_1[O^-,O]\dots E_m[O^-,O],O^-,O]\Downarrow_p$$

$$C[O^-D[O^-,O^-]E_1[O^-,O^-]\dots E_m[O^-,O^-],O^-,O^-]\Uparrow_p$$

for suitable contexts $C[\ ,\ ]$, $D[\ ,\ ]$, $E_1[\ ,\ ]\dots E_m[\ ,\ ]$. If $O^-D[O^-,O^-]\Downarrow_p$, we can take as argument for P and Q, with a shorter reduction path, the term $\lambda uv.C[E_1[u,v]\dots E_m[u,v],u,v]$. Otherwise we have that $O^-D[O^-,O^-]\Uparrow_p$. If $D[O^-,O^-]\Downarrow_p$, by induction on the structure of $D[\ ]$, one can see that there exists a normal form D^* such that $D[O^-,O^-]\Downarrow_p D^*$, $D[O^-,O]\Downarrow_p D^*$, $D[O,O^-]\Downarrow_p D^*$. But then we reach immediately a contradiction, because necessarily we have $O^-D[O^-,O]\Downarrow_p$, since $PM\Downarrow_p$, and so, by definition of F_∞, also $O^-D[O^-,O^-]\Downarrow_p$. Thence, necessarily we have that $D[O^-,O^-]\Uparrow_p$. But then, again using the fact that $PM\Downarrow_p$, we have $D[O^-,O]\Downarrow_p$, $D[O,O^-]\Downarrow_p$. So we can take as argument for P and Q, with a shorter reduction path, the term $\lambda uv.C[D[u,v],u,v]$. $\Box$

Notice that the above proof is the only proof in this paper, which strictly depends on the specific nature of the effective perpetual strategy F_∞.

On the other hand we have:

Theorem 39 *Let $\mathcal{D}$ be a strict extensional λ-lattice computationally adequate for $\approx_p$. Then $[\![P]\!]^D \neq_D [\![Q]\!]^D$.*

PROOF. The interpretation of $\rightarrow_p$-divergent terms in strict extensional computationally adequate λ-lattices is necessarily $\bot_D$. Hence $[\![O^-]\!]^D \sqsubseteq_D [\![O]\!]^D$ but $[\![O^-]\!]^D \neq_D [\![O]\!]^D$. The function:

$$f \equiv \lambda de.\ \textbf{if}\ (d \not\sqsubseteq_D [\![O^-]\!]^D \vee e \not\sqsubseteq_D [\![O^-]\!]^D)\ \ \textbf{then}\ e\ \textbf{else}\ \ \bot_D$$

is such that $[\![P]\!]^D \bullet_D f =_D \Psi_D(\ \lambda d.d)$ while $[\![Q]\!]^D \bullet_D f =_D \bot_D$. $\Box$

In order to overcome this topological incompleteness, one could try to modify or to extend the class of strict extensional λ-lattices in various ways. Considering other classes of Scott-continuous functions with a common fixed point should not make much difference, while forcing us to have rather awkward interpretations. The same kind of counterexamples should work *mutatis mutandis* in these settings. One should rather try to consider models where not *all* Scott continuous functions are representable. But working with spaces of stable functions should be of no use. We conjecture that, as was the case for call-by-value λ_v-calculus in [10], we would have that $[\![P]\!] = [\![Q]\!]$, but new *exotic* stable functions would then arise, which would allow to discriminate terms equated in strict extensional λ-lattices.

One can see that the theories of strict extensional λ-lattices are incomplete already w.r.t. pure redexes:

Theorem 40 *There exists no strict extensional λ-lattice $\mathcal{D}$, such that $(\mathcal{T}^D \cap =_\beta) = (\approx_p \cap =_\beta)$.*

PROOF. Consider the terms Pz and Qz, where P, Q are the terms introduced above. Since $Pz \approx_p Qz$, then $(\lambda x.Pz)(Qz) \approx_p Pz$, but reasoning as in Theorem 39 one can see that in any strict extensional model there are ρ's such that $[\![(\lambda x.Pz)(Qz)]\!]^D_\rho =_D \bot_D$ but $[\![Pz]\!]^D_\rho \sqsupset \bot_D$. $\square$

5 Coinductive Characterization of the Observational Equivalence

In this section we show that $\approx_p^{app} = \approx_p$. The inclusion $\approx_p \subseteq \approx_p^{app}$ is immediate. On the other hand, the reverse inclusion is rather difficult to show and requires very subtle techniques. In particular we need to show that (see Theorem 41):

- $\approx_p^{app}$ is a congruence w.r.t. application, i.e. for all $M, N, P, Q \in \Lambda$,
  $M \approx_p^{app} N \;\&\; P \approx_p^{app} Q \implies MP \approx_p^{app} NQ$;
- $\approx_p^{app}$ is a congruence w.r.t. λ-abstraction, i.e., for all $M, N \in \Lambda$
  $M \approx_p^{app} N \implies \lambda x.M \approx_p^{app} \lambda x.N$.

It is immediate to see that, when discussing congruence with respect to application, we can restrict ourselves to the case that M, N, P, Q are all closed.

The congruence of $\approx_p^{app}$ w.r.t. λ-abstraction is not problematic. The congruence w.r.t. application is the real challenge.

In two papers by the second author, i.e. [15,16], various general techniques for showing that $\approx_\sigma^{app}$ is a congruence w.r.t. application, for various notions of reduction σ, are presented. In particular, in [15], two methods are presented which are semantical, in that they make use of suitable computationally adequate CPO λ-models. The first is the generalization of the *domain logic method* used by Abramsky and Ong in [1] for the lazy strategy; while the second is the generalization of Pitts' *logical relation method* originally used in [18] also for the lazy strategy. In [15], both methods are shown to apply to many strategies, including perpetual strategies. In [16], a general and uniform version of a purely syntactical method is presented, which generalizes to non-lazy or nondeterministic strategies Howe's *congruence candidate method* ([12]). In [16] it is shown how this method applies to many strategies, including F_∞.

In this paper, we carry out the details of the proof that $\approx_p^{app}$ is a congruence w.r.t. application, using the logical relation method presented in [15]. It is based on the model $\mathcal{D}^p$ introduced in Definition 34. This method is also interesting in itself for two reasons. It allows us to show a *mixed induction-coinduction* principle for establishing $\approx_p$. And moreover, it provides as a by-product an alternative proof of computational adequacy for $\mathcal{D}^p$. The role of the structure $\mathcal{D}^p$ can be played also by other computationally adequate structures, thus yielding new mixed induction-coinduction principles. We use $\mathcal{D}^p$ because it simplifies matters.

Theorem 41 *Suppose that $\approx_p^{app}$ is a congruence w.r.t. λ-abstraction and application. Then $\approx_p^{app} \subseteq \approx_p$.*

PROOF. We prove by induction on the context $C[\,]$ that:
$M \approx_p^{app} N \implies \forall C[\,].(FV(C[M], C[N]) \subseteq \{x_1, \ldots, x_n\} \implies$
$\forall P_1 \ldots P_n \in \Lambda^0 \cap Val_p.\ C[M][P_i/x_i] \approx_p^{app} C[N][P_i/x_i])$. □

5.1 $\approx_p^{app}$ is a congruence w.r.t. λ-abstraction

Theorem 42 *$\approx_p^{app}$ is a congruence w.r.t. λ-abstraction.*

PROOF. It is sufficient to show that, for all $M, N \in \Lambda$, if, for all $P \in \Lambda^0$, $M[P/x] \approx_p^{app} N[P/x]$, then $\lambda x.M \approx_p^{app} \lambda x.N$. This follows from the fact that: $\forall M \in \Lambda.\ (FV(M) \subseteq \{x\}) \implies (\exists P \in \Lambda^0.\ M[P/x] \Downarrow_p \iff M \Downarrow_p)$.
The implication ($\Rightarrow$) follows immediately from the fact that, for all M, $M \Downarrow_p$ if and only if M is strongly normalizing. The reverse implication can be shown by taking as P a "permutator", *i.e.* a term of the shape $\lambda z_1 \ldots z_k y.y z_1 \ldots z_k$, with k larger than the number of arguments of any occurrence of the variable x in any term occurring in a reduction path starting from M. Since M is strongly normalizing, this number is clearly finite. One can then show that $M \Downarrow_p \implies M[P/x] \Downarrow_p$ by induction on the height of the derivation of $M \Downarrow_p$. All cases are straightforward, in the (var) case, one uses the condition on the number of abstractions of the permutator P. □

We prove that $\approx_p^{app}$ is a congruence w.r.t. application using the logical relation method presented in [15]. This method uses a computationally adequate CPO λ-model, $\mathcal{D}$, obtained as inverse limit. The method in [15] extends the one originally introduced by Pitts (see e.g. [18]). Pitts' method applies only to initial models, i.e. models for which a "minimal invariance property" à la Plotkin holds. A related technique appears in [9] for Plotkin's call-by-value strategy.

The core of the logical relation method consists in defining a relation $\lhd \subseteq D \times \Lambda^0(C)$, where C is a set of constants standing for suitable compact elements of D_0. This definition usually requires a deep analysis of the inverse limit model. Then, one proceeds to show that:
1. $\lhd$ is a congruence w.r.t. application;
2. $M \sqsubseteq_p^{app} N \iff [\![M]\!]^D \lhd N$.

The essential ingredient for proving 2 is a *mixed induction-coinduction principle* for the relation $\lhd$. In order to derive this principle, we give an explicit inductive characterization of $\lhd$ in terms of relations $\lhd^n \subseteq D_n \times \Lambda^0(C)$, where D_n denotes the n-th projection of D in the inverse limit construction.

Here we apply the logical relation method to the model $\mathcal{D}^p$ of Definition 34. As a by-product, we get an alternative proof of the computational adequacy of $\mathcal{D}^p$.

For ease of reading, in the rest of this section, we shall omit to mention explicitly $\bullet^{D^p}$ in applying two elements of D^p, and we shall simply write de instead of $d \bullet^{D^p} e$. Finally, for convenience, we will denote by $\mathcal{E}$ the union $\bigcup_{n\in\mathbf{N}}(D^p_n \times \Lambda)$.

The relation $\lhd^p$ is defined inductively in terms of the relations $\lhd^p_n \subseteq D^p_n \times \Lambda^0$ as follows:

Definition 43 – *The relations $\trianglelefteq^\sigma_n \subseteq D^\sigma_n \times \Lambda^0$ are defined inductively as follows:*

$$\trianglelefteq^\sigma_0 = \{(d,P) \mid P \in \Lambda^0 \ \wedge \ d \sqsubseteq_{D^p} ([\![P]\!]^{D^p})_0\}$$
$$\begin{aligned}\trianglelefteq^p_{n+1} = \{(d,P) \in D^p_{n+1} \times \Lambda^0 \mid \\ &(d_0 =_{D^p} \bot \ \wedge \ \forall (d',P') \in \ \trianglelefteq^p_n.\ (dd',PP') \in \ \trianglelefteq^p_n) \ \vee \\ &(d_0 \neq_{D^p} \bot \ \wedge \ P \Downarrow_p \ \wedge \ \forall (d',P') \in \ \trianglelefteq^p_n.\ (dd',PP') \in \ \trianglelefteq^p_n)\}.\end{aligned}$$

– *Let $\trianglelefteq^p \subseteq \mathcal{E}$ be defined by $\trianglelefteq^p = \bigcup_{n\in\mathbf{N}} \trianglelefteq^p_n$.*
– *Let $\lhd^p_n = \{(d,P) \in D^p \times \Lambda^0 \mid d_n \trianglelefteq^p_n P\}$.*
– *Finally, let $\lhd^p = \bigcap_{n\in\mathbf{N}} \lhd^p_n$.*

Definition 44 *A relation $\mathcal{R} \subseteq D^p \times \Lambda^0$ is called* limit-closed *if, whenever for all $n \in \mathbf{N}$ $(d_n,P) \in \mathcal{R}$, then also $(d,P) \in \mathcal{R}$.*

Given a relation $\mathcal{R}\subseteq D^p\times\Lambda^0$, we will denote by $\overline{\mathcal{R}}$ the least limit-closed relation including $\mathcal{R}$.

The following lemma is instrumental:

Lemma 45 i) $\forall d \in D^p_n,\ \forall P \in \Lambda^0,\ d \trianglelefteq^p_n P \implies \forall m.\ d_m \trianglelefteq^p_m P.$
ii) $\triangleleft^p = \overline{\trianglelefteq^p}$.

PROOF. i) One can easily check that $d \trianglelefteq^p_0 P \;\Rightarrow\; d \trianglelefteq^p_1 P$. Moreover, one can prove, by induction on n, that, for all $n \geq 0$,

$$d \trianglelefteq^p_n P \implies (d_{n-1} \trianglelefteq^p_{n-1} P \;\wedge\; d \trianglelefteq^p_{n+1} P) .$$

The non trivial part in proving this fact is the base case, i.e. $d \trianglelefteq^p_1 P \implies (d_0 \trianglelefteq^p_0 P \wedge d \trianglelefteq^p_2 P)$. In particular, $d \trianglelefteq^p_1 P \implies d \trianglelefteq^p_2 P$ amounts to showing that, for all $d, d' \in D^p_1$, $P, P' \in \Lambda^0(C)$, $d \trianglelefteq^p_1 P \;\wedge\; d' \trianglelefteq^p_1 P' \implies dd' \trianglelefteq^p_1 PP'$. But this follows from the following two conditions which hold in the model D^p: $d \trianglelefteq^p_0 P \;\Rightarrow\; d \trianglelefteq^p_1 P$ and $d \trianglelefteq^p_1 P \;\Rightarrow\; d_0 \trianglelefteq^p_0 P$. In fact, from $d' \trianglelefteq^p_1 P'$, by the second condition, $d'_0 \trianglelefteq^p_0 P'$, hence $dd' =_{D^p} dd'_0 \trianglelefteq^p_0 PP'$, and, by the first condition, $dd' \trianglelefteq^p_1 PP'$.
ii) The thesis follows from item i).

The crucial point of the logical relations method is the definition of an operator T_p on a complete lattice $\mathcal{R}_p$, depending on the domain D^p, such that its least fixed point is $(\trianglelefteq^p, \trianglelefteq^p)$:

Proposition 46 *i) Let $\mathcal{R}_p$ be defined as follows:*
$\mathcal{R}_p = \{((\mathcal{R}^-, \mathcal{R}^+), (\supseteq, \subseteq)) \mid \mathcal{R}^-, \mathcal{R}^+ \subseteq \mathcal{E} \;\wedge\; \trianglelefteq^p_0 \subseteq \mathcal{R}^-, \mathcal{R}^+ \;\wedge$
$A \cap \mathcal{R}^- = \emptyset \;\wedge\; A \cap \mathcal{R}^+ = \emptyset\}$,
where $A = \{(d, P) \in D^p_0 \times \Lambda^0 \mid d \not\trianglelefteq^p_0 P\}$.
Then $\mathcal{R}_p$ is a complete lattice.
ii) The following operator $T_p : \mathcal{R}_p \to \mathcal{R}_p$ is well defined:
$T_p(\mathcal{R}^-, \mathcal{R}^+) =$
$(\{(d, P) \mid d \trianglelefteq^p_0 P \;\vee\; (d_0 =_{D^p} \perp \;\wedge \forall (d', P') \in \mathcal{R}^+ .\ (dd', PP') \in \mathcal{R}^-) \;\vee$
$(d_0 \neq_{D^p} \perp \;\wedge\; P \Downarrow_p \;\wedge \forall (d', P') \in \mathcal{R}^+ .\ (dd', PP') \in \mathcal{R}^-)\} \cap \mathcal{E},$
$\{(d, P) \mid d \trianglelefteq^p_0 P \;\vee\; (d_0 =_{D^p} \perp \;\wedge \forall (d', P') \in \mathcal{R}^+ .\ (dd', PP') \in \mathcal{R}^-) \;\vee$
$(d_0 \neq_{D^p} \perp \;\wedge\; P \Downarrow_p \;\wedge \forall (d', P') \in \mathcal{R}^+ .\ (dd', PP') \in \mathcal{R}^-)\} \cap \mathcal{E})$.

Theorem 47 i) $(\trianglelefteq^p, \trianglelefteq^p)$ *is the least fixed point of T_p.*
ii) *The following induction-coinduction principle holds:*

$$\frac{\mathcal{R}^- \subseteq \pi_1(T_p(\mathcal{R}^-, \mathcal{R}^+)) \qquad \mathcal{R}^+ \supseteq \pi_2(T_p(\mathcal{R}^-, \mathcal{R}^+))}{\overline{\mathcal{R}^-} \subseteq \triangleleft^p \subseteq \overline{\mathcal{R}^+}} .$$

PROOF. i) Using Lemma 45i), one can prove that $(\trianglelefteq^{p}, \trianglelefteq^{p})$ is a fixed point of T_p. Moreover, one can check that, for all $n > 0$,
$T_p^{n+1}(\mathcal{E} \setminus A, \trianglelefteq^{p}_{0}) = (\triangleleft^{p}_{n} \cap \mathcal{E}, \trianglelefteq^{p}_{n})$.
This follows from the fact that, for $n > 0$, $T(\triangleleft^{p}_{n} \cap \mathcal{E}, \trianglelefteq^{p}_{n}) = (\triangleleft^{p}_{n+1} \cap \mathcal{E}, \trianglelefteq^{p}_{n+1})$.
Hence the least fixed point of T_p is
$(\bigcap_{n\in\mathbf{N}} \pi_1(T_p^n(\mathcal{E} \setminus A, \trianglelefteq^{p}_{0})), \bigcup_{n\in\mathbf{N}} \pi_2(T_p^n(\mathcal{E} \setminus A, \trianglelefteq^{p}_{0}))) = (\triangleleft^{p} \cap \mathcal{E}, \trianglelefteq^{p}) = (\trianglelefteq^{p}, \trianglelefteq^{p})$.
ii) The proof follows from Lemma 45ii) and item i).

The relation $\triangleleft^{p}$ is a congruence with respect to application:

Theorem 48 $d \triangleleft^{p} P \iff \forall e \triangleleft^{p} Q.\ de \triangleleft^{p} PQ$.

PROOF. ($\Rightarrow$) The proof follows from the fact that application in the model D^p satisfies the following property: for all $d, e \in D^p$, $de =_{D^p} \bigsqcup_{n\in\mathbf{N}} d_{n+1}e_n$. Suppose that $d \triangleleft^{p} P$ and $e \triangleleft^{p} Q$. We will prove that, for all n, $(de)_n \triangleleft^{p} PQ$. From the definition of $\triangleleft^{p}$, using the property of application in the model $\mathcal{D}^p$, we immediately get, for all n, k, $((\bigsqcup_{m\in\mathbf{N}} d_{m+1}e_m)_n)_k \trianglelefteq^{p}_{k} PQ$, i.e., from the definition of $\triangleleft^{p}$, we have, for all n, $(\bigsqcup_{m\in\mathbf{N}} d_{m+1}e_m)_n \triangleleft^{p} PQ$.
($\Leftarrow$) From the Definition of $\triangleleft^{p}$.

The proof of the following lemma makes use of the induction-coinduction principle of Theorem 47:

Lemma 49 *For all $d, d' \in D^p$, and for all $P, P' \in \Lambda^0$,*

$$d \sqsubseteq_{D^p} d' \triangleleft^{p} P' \sqsubseteq_p^{app} P \implies d \triangleleft^{p} P .$$

PROOF. Apply the induction-coinduction principle to $\mathcal{R}^+ = \trianglelefteq^{p}$, $\mathcal{R}^- = \{(d, P) \mid \exists d' \in D^p\ \exists P' \in \Lambda^0.\ d \sqsubseteq_{D^p} d' \triangleleft^{p} P' \sqsubseteq_p^{app} P\} \cap \mathcal{E}$.

The following theorem is crucial:

Theorem 50 *Let $M \in \Lambda$, with $FV(M) \subseteq \{x_1, \ldots, x_n\}$. Then*

$$\forall i \leq n.\ (d_i \in D^p \setminus \{\bot_{D^p}\}\ \wedge\ d_i \triangleleft^{p} P_i) \implies [\![M]\!]^{D^p}_{\rho[d_i/x_i]} \triangleleft^{p} M[P_i/x_i].$$

PROOF. The proof is by induction on M, using Theorem 48.

Corollary 51 *For all* $P \in \Lambda^0$*:* $[\![P]\!]^{D^p} \lhd {}^p P$.

Lemma 52 $[\![P']\!]^{D^p} \lhd {}^p P \iff P' \sqsubseteq_p^{app} P$.

PROOF. The implication ($\Rightarrow$) can be proved by coinduction. In fact it is easy to check, using Corollary 51 and Theorem 48 ($\Rightarrow$), that $\{(P', P) \mid [\![P']\!]^{D^p} \lhd {}^p P\}$ is a Ψ_p-bisimulation (see Section 1). The proof of the implication ($\Leftarrow$) follows from Lemma 49, and from Corollary 51.

Finally, we are in the position of stating the main theorem:

Theorem 53 i) $\sqsubseteq_p^{app}$ *is a congruence w.r.t. application.*
ii) *If* $\sqsubseteq_p^{app}$ *is a congruence w.r.t.* λ*-abstraction, then* D^p *is computationally adequate.*

PROOF. i) From Theorem 48 and Lemma 52 it follows that $\sqsubseteq_p^{app}$ is a congruence w.r.t. application.
ii) Using Corollary 51 and Lemma 49, one can show that, if $[\![M]\!]^{D^p} \sqsubseteq_{D^p} [\![N]\!]^{D^p}$, then $[\![M]\!]^{D^p} \lhd {}^p N$, hence by Lemma 52 $M \sqsubseteq_p^{app} N$. Then, by i) and Theorem 41, we have $M \sqsubseteq_p N$.

Theorem 54 $(\approx_p \cap (\Lambda^0 \times \Lambda^0)) = (\approx_p^{app} \cap (\Lambda^0 \times \Lambda^0))$.

PROOF. Use Theorems 41, 42 and 53.

Theorem 55 *Let* $M, N \in \Lambda$ *be such that* $FV(M) \cup FV(N) \subseteq \{x_1, \ldots, x_n\}$*. Then*

$$M \approx_p N \iff \forall P_1 \ldots P_n \in (\Lambda^0 \cap Val_p).\ M[P_i/x_i] \approx_p\ N[P_i/x_i].$$

PROOF. Assume that $FV(M) \cup FV(N) \subseteq \{x_1, \ldots, x_n\}$ and that $P_1, \ldots, P_n$ are closed and strongly normalizing. We have the following chain of implications:
$\forall P_1 \ldots P_n \in (\Lambda^0 \cap Val_p).\ M[P_i/x_i] \approx_p\ N[P_i/x_i]$
$\iff \forall P_1 \ldots P_n \in (\Lambda^0 \cap Val_p).\ M[P_i/x_i] \approx_p^{app}\ N[P_i/x_i]$,
by Theorem 54
$\iff \lambda x_1 \ldots x_n.\ M \approx_p^{app}\ \lambda x_1 \ldots x_n.\ N$,
by Theorem 42 and the S.O.S. definition of $\Downarrow_p$
$\iff \lambda x_1 \ldots x_n.\ M \approx_p\ \lambda x_1 \ldots x_n.\ N$,
by Theorem 54
$\Longrightarrow (\lambda x_1 \ldots x_n.\ M) x_1 \ldots x_n \approx_p\ (\lambda x_1 \ldots x_n.\ N) x_1 \ldots x_n$,

since $\approx_p$ is a congruence

$\Longleftrightarrow M \approx_p N$,

using the fact that $\approx_p$ is a congruence and by Theorem 32

$\Longrightarrow \lambda x_1 \ldots x_n.\ M \approx_p\ \lambda x_1 \ldots x_n.\ N$,

since $\approx_p$ is a congruence. $\square$

Theorem 56 $\approx_p = \approx_p^{app}$.

PROOF. By Theorem 55, and Theorem 54.

6 Final Remarks

Syntactical approaches to perpetuality.

In the literature there are various, very successful, purely syntactical investigations of perpetual strategies, see e.g [5,3,20,22,21]. It would be very interesting to compare in detail these approaches to the one presented in this paper. This would contribute to the more general problem of understanding the rôle and the relevance of semantical tools in the study of λ-calculus.

Böhm and Intrigila's λ_{δ_K}-calculus.

In [6], Böhm and Intrigila raise the problem of characterizing calculi, and terms within particular calculi, which have the SN-normalization property, i.e. for which weak normalization implies strong normalization.

In this context they introduce the λ_{δ_K}-calculus, which is obtained by extending the set of λI-terms with a constant δ_K, representing the combinator $\mathbf{K}$, whose δ-reduction rule is:

$$\delta_K M N \to M \ , \qquad \text{if } N \text{ is a } \textit{closed term in } \delta_K\textit{-normal form.}$$

The side condition, in the above reduction, is clearly recursive, and one needs to spell out inductively, on the structure of terms, what is to be intended as a δ_K-normal form. There is a natural compositional translation from Λ to terms of the λ_{δ_K}-calculus and viceversa obtained by replacing inductively subterms of the form $\delta_K M$ with $\lambda x.M$, for $x \notin M$. Such a translation yields immediately a new notion of conversion $\to_{\delta_K}$ on Λ. One can easily see that $\to_{\beta_{KN}}$ and $\to_{\delta_K}$ are incomparable. On one hand $\to_{\beta_{KN}}$ can erase free variables.

On the other hand $\to_{\delta_K}$ can erase non strongly normalizing terms, consider e.g. the non-strongly normalizing λ-term:

$$M \equiv (\lambda xy.y)(\lambda x.(\lambda xy.y)x(\lambda x.xx)(\lambda x.xx)),$$

which $\to_{\delta_K}$-reduces to $\lambda x.x$. Hence, the λ_{δ_K}-calculus is incorrect w.r.t. $\approx_p$.

It is rather problematic to model adequately the λ_{δ_K}-calculus using semantical tools, since the δ_K-reduction discriminates between open and closed terms. The behaviour of the term M, introduced above, clearly indicates that standard interpretations of δ_K in strict extensional λ-lattices are not satisfactory. In order to overcome these difficulties, one has two alternatives. One could either consider non-standard interpretations of δ_K, sensitive to the nature of the second argument, within models where it never occurs that for all environments the interpretation of an open term is less than that of a closed term. Or else, one could consider spaces of *doubly strict* functions, i.e. functions which map the bottom element to the bottom element and the top element to the top element, provided they are not the least function. To give an idea of the intricacies of modeling the λ_{δ_K}-calculus, we just point out that, if $\to_{p_{\delta_K}}$ is any perpetual strategy for it, we have that $\approx_{p_{\delta_K}} \neq \approx^{app}_{p_{\delta_K}}$. Consider again the term M, introduced above, we have $M \approx^{app}_{p_{\delta_K}} \lambda x.x$ but $M \not\approx_{p_{\delta_K}} \lambda x.x$.

The $\lambda_{\beta_{KN}}$-calculus.

The congruence $\approx_p$ is a maximal (non-trivial) theory of the $\lambda_{\beta_{KN}}$-calculus. I.e. it is a maximal (non-trivial) congruence closed under $\to_{\beta_{KN}}$-conversion. Assume, in fact, that there is a proper extension $\mathbf{T}$ of $\approx_p$. Then, since $\approx_p = \approx^{app}_p$, there exist terms M and N such that $(M, N) \in \mathbf{T}$ and $M \Downarrow_p$ but $N \Uparrow_p$. Hence, given any P and Q we have

$$P =_{\mathbf{T}} (\lambda x.P)M =_{\mathbf{T}} (\lambda x.P)N =_{\mathbf{T}} (\lambda x.Q)N =_{\mathbf{T}} (\lambda x.Q)M =_{\mathbf{T}} Q.$$

Suitably modifying the technique used in [6], Theorem 2.4.3, one can show that the $\lambda_{\beta_{KN}}$-calculus satisfies the *SN property* in the sense of Böhm and Intrigila. And suitably modifying the technique used in [6], Theorem 2.4.4, one can show also that Böhm's Theorem can be extended to the $\lambda_{\beta_{KN}}$-calculus, in the sense that $\beta\eta$-normal forms can be separated by terms of the $\lambda_{\beta_{KN}}$-calculus. Full separability for $\approx_p$ is problematic since there are syntactically different normal forms under $\lambda_{\beta_{KN}}$-reduction which are nonetheless equal in $\approx_p$. Consider e.g.: $\lambda x.\ (\lambda yzw.w)(x(\lambda x.x))(x(\lambda x.x))$ and $\lambda x.\ (\lambda yz.z)(x(\lambda x.x))$.

Exact conditions for computational adequacy in general.

Necessary and sufficient conditions, in the style of those in Theorem 29, can be given also for other notions of reduction, e.g. those in [11]. We do not elaborate on these here for lack of space.

Open questions and conjectures.

We conclude this paper with a short list of conjectures:

(i) Fully abstract models for $\approx_p$ can be found in appropriate categories of games.

(ii) (Using the notation of Section 5) The subset of the model $\mathcal{D}^p$, consisting of the interpretation of $\Lambda^0(\{\top\})$ can be quotiented into a fully abstract model of $\approx_p$ proceeding as follows. Define, for $d, e \in \Lambda^0(\{\top\})$, $d \cong_\infty e$ if and only if $\forall n. d_n \cong_n e_n$, where Λ^n is the set of the interpretations of the n^{th} projections of terms in $\Lambda^0(\{\top\})$ and the equivalences $\cong_n \subseteq \Lambda^n \times \Lambda^n$ are defined inductively as follows:
$d \cong_0 e$ iff $d =_{D_p} e$ and $d \cong_{n+1} e$ iff $\forall f, g.\ f \cong_n g \Longrightarrow d \bullet^{D^p} f \cong_n e \bullet^{D^p} g$.

(iii) (Using the notation of Section 4.2): $\sigma \leq \tau \Longleftrightarrow Comp(\sigma) \subseteq Comp(\tau)$.

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