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Coinductive characterizations of applicative structures†

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dedicated to Roger Hindley

We discuss *new* ways of characterizing, as *maximal fixed points* of monotone operators, observational congruences on λ-terms and, more generally, equivalences on applicative structures. These characterizations naturally induce *new* forms of *coinduction principles*, for reasoning on program equivalences, which are not based on Abramsky's *applicative bisimulation*. We discuss in particular, what we call, the *cartesian* coinduction principle, which arises when we exploit the elementary observation that functional behaviours can be expressed as *cartesian graphs*. Using the paradigm of final semantics, the soundness of this principle over an applicative structure can be expressed easily by saying that the applicative structure can be construed as a *strongly extensional coalgebra* for the functor $(\mathcal{P}(\_\times\_))\oplus(\mathcal{P}(\_\times\_))$. In this paper, we present two general methods for showing the soundenss of this principle. The first applies to *approximable applicative structures*. Many CPO λ-models in the literature, and the corresponding quotient models of indexed terms turn out to be approximable applicative structures. The second method is based on Howe's congruence candidates, and it applies to many observational equivalences. Structures satisfying cartesian coinduction are precisely those applicative structures which can be modeled using the standard *set-theoretic* application in non-wellfounded theories of sets, where the Foundation Axiom is replaced by the Axiom X_1 of Forti and Honsell.

1. Introduction

Brute force induction on the number of computation steps is a rather complex and opaque way of reasoning on the operational behaviour of programs. As it is well known, proof principles which allow to factor out, or modularize, such inductive arguments are therefore extremely valuable. Recently, much attention has been devoted to the possibility of characterizing observational equivalences of programs as maximal fixed points of suitable operators, thus obtaining *coinduction proof principles* for reasoning on program equivalence, see e.g. (Abramsky and Ong 1993; Egidi et al. 1992; Fiore 1996; Gordon 1995;

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Honsell and Lenisa 1995; Howe 1996; Lenisa 1996; Mason et al.; Pitts 1996a; Rutten and de Vink 1997; Lenisa 1997; Lenisa 1997a; Lenisa 1998). As far as functional languages, or λ-calculi, are concerned, however, almost all coinduction principles considered so far in the literature have always had the same *applicative* pattern, based on Abramsky's notion of *applicative bisimulation.* All principles, apart from (Honsell and Lenisa 1995, Section 3), arise from monotone operators of the same shape, in the line of (Abramsky and Ong 1993; Egidi et al. 1992), and exploit the fact that the observational behaviour of functional programs needs to be tested essentially only in *applicative contexts* (Lenisa 1998).

In (Honsell and Lenisa 1995, Section 3), the authors considered for the first time another kind of monotone operator, thereby introducing a new form of *coinduction principle* for establishing observational equivalences between terms of λ-calculi, which they proved to be sound for the *call-by-value* λ-calculus. For reasons which will become clear in a few paragraphs, in this paper, we shall call this principle *cartesian* coinduction, and we shall call *cartesian* both those applicative structures and those λ-theories for which it is sound. In (Lenisa 1998), cartesian coinduction was proved sound also for the *lazy* λ-calculus.

A first objective of this paper is to show that the "unorthodox" move of (Honsell and Lenisa 1995) can be widely generalized. We present, in fact, plenty of *new* meaningful monotone operators, besides the traditional "applicative" one, to be used for manifacturing coinduction principles for applicative structures.

The main purpose of this paper, however, is to investigate the general status and theory of the operator introduced in (Honsell and Lenisa 1995), which has remained largely unexplored since then, and to answer some of the many open problems raised there. In particular, we will show that there are rich classes of cartesian applicative structures, and plenty of cartesian λ-theories.

Using concepts from *final semantics* (Aczel 1988; Rutten and Turi 1993; Rutten 1996; Turi 1996; Lenisa 1998), one can easily express the difference between the traditional *applicative* coinduction and *cartesian* coinduction. Both kinds of coinduction correspond to the *categorical coinduction principles* which arise when the set of closed λ-terms, Λ^0, (or more in general an applicative structure), is endowed with a *coalgebra* structure for a suitable functor. Applicative coinduction principles for λ-theories are obtained when the functor $F(\_) = (\Lambda^0 \to \_) \oplus (\Lambda^0 \to \_)$ is used.[†] This approach is investigated in full generality in (Honsell and Lenisa 1995). Cartesian coinduction, on the contrary, arises when the functor $G(\_) = \mathcal{P}(\_\times\_) \oplus \mathcal{P}(\_\times\_)$ is used. In both cases the direct sum of two copies of the same structure is taken so as to distinguish between *observable values* and *non-values.* Notice how the functor F enforces the view by which elements are functions defined over the set of closed λ-terms. Also the functor G supports the view of objects as functions, but this time functions are represented as their *cartesian graphs*, whence the name. The nature of the functor G is more general, it does not depend explicitly on any applicative structure specified in advance as F does, namely Λ^0. This allows for a more uniform treatment.

[†] $A \oplus B$ denotes the *disjoint union* of A and B, i.e. the set $\{a \cup \{u\} \mid a \in A\} \cup \{b \cup \{v\} \mid b \in B\}$, for suitable elements $u, v \notin \bigcup A \cup \bigcup B$.

We shall discuss two techniques for showing cartesianity of structures. The first is semantical and it applies to ordered λ-models, such as CPO's or quotients of interiors of CPO-models. The second technique is syntactical and uses the generalizations of Howe's technique (Howe 1989) of "congruence candidates" as carried out in (Lenisa 1998). This latter technique applies to term models of λ-theories determined by observing termination under various reduction strategies.

The existence of cartesian applicative structures is closely related to the existence of *set-theoretical* applicative structures in non-wellfounded theories of sets, where the Foundation Axiom is replaced by the Antifoundation Axiom X_1 of (Forti and Honsell 1983). Set-theoretical applicative structures are applicative structures whose points contain the set-theoretic description of their functional behaviour, so that application can be rendered by the usual set-theoretic application, i.e. for any given points d, d', e in the model, $d \bullet d' = e$ if and only if $(d', e) \in d$. The results in this paper allow to show that there are plenty of "well-behaved" set-theoretical applicative structures.

Finally, we would like to point out that this paper can be viewed also as, yet another, chapter in the general programme of investigating the denotational semantics of λ-calculi, some of whose earlier chapters are (Barendregt 1984; Coppo et al. 1987; Egidi et al. 1992; Honsell and Ronchi 1992; Abramsky and Ong 1993; Honsell and Lenisa 1993; Honsell and Lenisa 1995; Honsell and Lenisa 1998).

The paper is organized as follows. In Section 2 we discuss in general the problem of characterizing coinductively congruences over applicative structures. In particular, we present a number of monotone operators which can be utilized in coinduction principles. We conclude the section by introducing special classes of "enriched" applicative structures. In Section 3 we introduce the basic ideas of final semantics and we give the categorical accounts of the coinduction principles introduced in the previous section. In Section 4 we discuss λ-congruences and λ-models. In particular, we present the six λ-theories which have been most extensively studied in the literature, and which we shall deal with explicitly. The main theorems of this paper concerning the existence of cartesian applicative structures appear in Section 5. In Section 6 we describe a syntactical technique based on the notion of congruence candidate for establishing cartesianity of λ-theories. In Section 7 we discuss set-theoretical applicative structures in non-wellfounded Set Theories, and their connection with cartesian structures. Concluding remarks and open problems are presented in Section 8. In Appendix A we give informations concerning non-wellfounded sets. In Appendix B we recall some facts about final coalgebras.

We assume the reader familiar with basic concepts and results in λ-calculus, final semantics, and non-wellfounded Set Theory. The reader may consult (Barendregt 1984), (Rutten and Turi 1993; Lenisa 1998), and (Forti and Honsell 1983) respectively, for more details.

2. The "coinductive characterization" problem

The basic notions we shall be concerned with in this paper are those of *applicative structure*, and of *congruence* over an applicative structure:

Definition 2.1. (applicative structure) *An* applicative structure $\mathcal{D}$ *is a pair* $(D, \bullet_D)$, *where* $\bullet_D : D \times D \to D$; $\bullet_D$ *is called* application, *and it is often written infix.*

Definition 2.2. *Let* $\mathcal{D}$ *be an applicative structure. An equivalence relation* $\approx \subseteq D \times D$ *is a* congruence *if*

$$\forall d_1, d_2, e_1, e_2 \in D.\ (d_1 \approx d_2 \ \wedge \ e_1 \approx e_2 \ \Rightarrow \ d_1 \bullet_D e_1 \approx d_2 \bullet_D e_2).$$

Notation Let $\mathcal{D}$ be an applicative structure. For $d, e \in D$, we shall often write de, instead of $d \bullet_D e$. We shall denote a, possibly empty, sequence of elements $d_1 \ldots d_n \in D^n$, for $n \geq 0$, by $\vec{d}$. Moreover, we shall abbreviate $(\ldots(d \bullet_D d_1)\ldots) \bullet_D d_n$, for $n \geq 0$, by $d\vec{d}$.

The most obvious, and finest, congruence over an applicative structure $\mathcal{D}$ is *equality*, $=_D$. Many interesting applicative structures arising in Computer Science, however, are (observational) quotients of syntactical objects. In this case it is convenient to focus on the congruence inducing the quotient.

Given a congruence $\approx_D$ over an applicative structure $\mathcal{D}$, it is natural to ask for logical characterizations of it. Such characterizations will be the more useful when they induce proof principles for proving term congruence. By far, one of the most important example of such characterizations is the one which describes $\approx_D$ as the *maximal fixed point* of a *monotone operator* on relations. As it is well known, this induces immediately a *coinduction principle* for establishing that two points are $\approx_D$-congruent.

Definition 2.3. (Ξ-coinduction principle) *Let* $\mathcal{D}$ *be an applicative structure, and let* $\Xi : \mathcal{P}(D \times D) \longrightarrow \mathcal{P}(D \times D)$ *be a monotone operator w.r.t.* $\subseteq$. *Then*
i) $\mathcal{R} \in \mathcal{P}(D \times D)$ *is a* Ξ-bisimulation *if* $\mathcal{R} \subseteq \Xi(\mathcal{R})$;
ii) The Ξ-coinduction principle *is the following rule:*

$$\frac{(d,e) \in \mathcal{R} \quad \mathcal{R} \subseteq \Xi(\mathcal{R})}{d \approx^{\Xi} e} \ ,$$

where $\approx^{\Xi}$ *denotes the* maximal fixed point *of* Ξ.

The "coinductive characterization problem" for a congruence $\approx_D$, over an applicative structure $\mathcal{D}$, is the problem of finding appropriate operators whose maximal fixed point is $\approx_D$. The solution to this problem is not immediate. For instance, the natural operator for which congruences are fixed points is $\Theta_D : \mathcal{P}(D \times D) \to \mathcal{P}(D \times D)$, where $\Theta_D(\mathcal{R}) = \{(d,e) \mid \forall f, g.\ f\mathcal{R}g \Rightarrow df\mathcal{R}eg\}$. But Θ_D is not monotone, thus, in order to reason coinductively on applicative structures, alternative operators to it have to be looked for. Following the seminal work of Abramsky (Abramsky 1989; Abramsky and Ong 1993), all investigations on applicative structures (but for (Honsell and Lenisa 1995; Lenisa 1998)) have focused mainly on characterizing congruences as maximal fixed points of the following class of monotone operators:

Definition 2.4. *Let* $\mathcal{D}$ *be an applicative structure and let* $\mathsf{Eq}_D \subseteq D \times D$ *be an equivalence relation. The operator* $\Phi_{\mathsf{Eq}_D} : \mathcal{P}(D \times D) \longrightarrow \mathcal{P}(D \times D)$ *is defined as follows:*

$$\Phi_{\mathsf{Eq}_D}(\mathcal{R}) = \{(d,e) \mid d \,\mathsf{Eq}_D\, e \ \wedge \ \forall f \in D.\ df \ \mathcal{R} \ ef\} \ .$$

It is easy to show that the maximal fixed point of the operator Φ_{Eq_D}, namely $\approx^{\Phi_{\mathsf{Eq}_D}}$, is the applicative relation $\approx^{app}_{\mathsf{Eq}_D}$ defined as follows:

Definition 2.5. (applicative relation) *Let $\mathcal{D}$ be an applicative structure and let $\mathsf{Eq}_D \subseteq D \times D$ be an equivalence relation. The* applicative relation $\approx^{app}_{\mathsf{Eq}_D} \subseteq D \times D$ *is defined by:*

$$d \approx^{app}_{\mathsf{Eq}_D} e \iff \forall \vec{f}.\ d\vec{f}\,\mathsf{Eq}_D\, e\vec{f}.$$

Many examples of coinductive characterizations of congruences, in terms of the operators Φ_{Eq_D} above, over both syntactical and semantical structures have been extensively discussed in (Egidi et al. 1992; Abramsky and Ong 1993; Howe 1996; Lenisa 1997; Lenisa 1997a; Pitts 1996a).

Besides the Φ_{Eq_D}'s, one can consider many other operators, which, in our opinion, yield interesting characterizations and are worthy of being investigated. The most intriguing example is that of Ψ_{Eq_D}, the operator introduced in (Honsell and Lenisa 1995). This operator arises when we capitalize on the fact that the functional behaviour of elements in applicative structures can be described by a *cartesian graph*, *i.e.* a subcollection of the cartesian product $D \times D$.

Definition 2.6. *Let $\mathcal{D}$ be an applicative structure and let $\mathsf{Eq}_D \subseteq D \times D$ be an equivalence relation. The operator $\Psi_{\mathsf{Eq}_D} : \mathcal{P}(D \times D) \longrightarrow \mathcal{P}(D \times D)$ is defined as follows:*

$$\Psi_{\mathsf{Eq}_D}(\mathcal{R}) = \{(d,e) \mid d\ \mathsf{Eq}_D\ e\ \wedge\ \left(\begin{array}{l} \forall f \in D\ \exists g \in D.\ f\,\mathcal{R}\,g \quad\wedge\quad df\,\mathcal{R}\,eg \\ \wedge \\ \forall f \in D\ \exists g \in D.\ f\,\mathcal{R}\,g \quad\wedge\quad ef\,\mathcal{R}\,dg \end{array}\right)\}\ .$$

We denote by $\approx^{\Psi_{\mathsf{Eq}_D}}$ the greatest Ψ_{Eq_D}-bisimulation.

Variants of the above operators can be obtained when we consider that, in view of currying, points of applicative structures exhibit also the behaviour of n-ary functions:

Definition 2.7. *Let $\mathcal{D}$ be an applicative structure, let $\mathsf{Eq}_D \subseteq D \times D$ be an equivalence relation, and let $n > 0$.*
i) Let $\Phi^n_{\mathsf{Eq}_D} : \mathcal{P}(D \times D) \longrightarrow \mathcal{P}(D \times D)$ be the operator defined as follows:

$$\Phi^n_{\mathsf{Eq}_D}(\mathcal{R}) = \{(d,e) \mid d\,\mathsf{Eq}_D\,e\ \wedge\ \forall \vec{f} \in D^n.\ d\vec{f}\ \mathcal{R}\ e\vec{f}\}\ .$$

ii) Let $\Psi^n_{\mathsf{Eq}_D} : \mathcal{P}(D \times D) \longrightarrow \mathcal{P}(D \times D)$ be the operator defined as follows:

$$\Psi^n_{\mathsf{Eq}_D}(\mathcal{R}) = \{(d,e) \mid d\ \mathsf{Eq}_D\ e\ \wedge\ \left(\begin{array}{l} \forall \vec{f} \in D^n\ \exists \vec{g} \in D^n.\ \forall i. f_i\,\mathcal{R}\,g_i\ \wedge\ d\vec{f}\,\mathcal{R}\,e\vec{g} \\ \wedge \\ \forall \vec{f} \in D^n\ \exists \vec{g} \in D^n.\ \forall i. f_i\,\mathcal{R}\,g_i\ \wedge\ e\vec{f}\,\mathcal{R}\,d\vec{g} \end{array}\right)\}\ .$$

More operators arise if we take the existential quantification over n in the above definition.

We can combine also Φ's and Ψ's, as follows

Definition 2.8. *Let $\mathcal{D}$ be an applicative structure and let $\mathsf{Eq}_D \subseteq D \times D$ be an equivalence relation.*
Let $Z_{\mathsf{Eq}_D} : \mathcal{P}(D \times D) \longrightarrow \mathcal{P}(D \times D)$ be the operator defined as follows:

$$Z_{\mathsf{Eq}_D}(\mathcal{R}) = \{(d,e) \mid d \ \mathsf{Eq}_D \ e \ \wedge \ \begin{pmatrix} \forall f \in D \ \exists g \in D. \ f \,\mathcal{R}\, g \ \wedge \ df \approx^{app}_{\mathsf{Eq}_D} eg \\ \wedge \\ \forall f \in D \ \exists g \in D. \ f \,\mathcal{R}\, g \ \wedge \ ef \approx^{app}_{\mathsf{Eq}_D} dg \end{pmatrix} \} \ .$$

We shall not present other operators, but we simply point out that using the above templates one can easily device yet further combined operators.

Notice that all the operators introduced above are monotone, hence each of them induces a coinduction principle.

For simplicity, we shall refer to Φ_{Eq_D}-coinduction as Eq_D-*applicative coinduction.* For the reasons we mentioned earlier, and which will become clearer in Section 3, we shall refer to Ψ_{Eq_D}-coinduction as Eq_D-*cartesian coinduction.*

The above coinduction principles can be used for reasoning on the following kinds of congruences on applicative structures

Definition 2.9. (Ξ_{Eq_D}-coinductive congruence) *Let $\approx_D$ be a congruence on an applicative structure $\mathcal{D}$, and let $\mathsf{Eq}_D \subseteq D \times D$ be an equivalence relation. Then $\approx_D$ is a* Ξ_{Eq_D}-coinductive congruence, *or equivalently, $\approx_D$* satisfies Ξ_{Eq_D}-coinduction, *if $\approx_D$ coincides with $\approx^{\Xi_{\mathsf{Eq}_D}}$.*

Again, for simplicitly, we shall refer to Ψ_{Eq_D}-coinductive congruences as Eq_D-*cartesian congruences.* Rather than saying that $=_D$ is Ξ_{Eq_D}-coinductive, we shall simply say that the applicative structure $\mathcal{D}$ is Ξ_{Eq_D}-coinductive.

An elementary result concerning the Ξ_{Eq_D}-coinduction principles for $\Xi = \Phi, \Psi, \Phi^n, \Psi^n, Z$ is:

Proposition 2.10. *Let $(D, \bullet_D)$ be an applicative structure, $\mathsf{Eq}_D \subseteq D \times D$, and let $\mathcal{R} \subseteq D \times D$ be a reflexive relation:*

— *if $\mathcal{R}$ is a Φ_{Eq_D}-bisimulation then it is also a Z_{Eq_D}-bisimulation;*
— *if $\mathcal{R}$ is a Φ_{Eq_D}-bisimulation then it is also a Ψ_{Eq_D}-bisimulation;*
— *if $\mathcal{R}$ is a Z_{Eq_D}-bisimulation then $(R \cup \approx^{\Psi_{\mathsf{Eq}_D}})$ is also a Ψ_{Eq_D}-bisimulation;*
— *if n divides m and $\mathcal{R}$ is a $\Phi^n_{\mathsf{Eq}_D}$-bisimulation then it is also a $\Phi^m_{\mathsf{Eq}_D}$-bisimulation;*
— *if n divides m and $\mathcal{R}$ is a $\Psi^n_{\mathsf{Eq}_D}$-bisimulation then it is also a $\Psi^m_{\mathsf{Eq}_D}$-bisimulation.*

It is surprisingly difficult to extend results for one of the above operators to the others. This paper is devoted essentially to showing that large classes of applicative structures satisfying the Eq_D-applicative coinduction principle, satisfy also the Eq_D-cartesian coinduction principle.

2.1. *Enriched applicative structures*

In this paper we shall be mainly concerned with applicative structures which have some "extra" structure. In particular, we shall consider "order enriched" and "approximable" applicative structures. First we introduce

Definition 2.11. (ordered applicative structure) *An* ordered applicative structure, *$\mathcal{D} = (D, \sqsubseteq_D, \bullet_D)$, is a triple such that*

1 *$(D, \sqsubseteq_D)$ is a non-trivial partial order;*

2 $\bullet_D : D \times D \to D$ *is continuous, i.e., for all* $\{d_n\}_{n\in\omega}$, $\{e_n\}_{n\in\omega}$ ω*-chains, if there exist* $\bigsqcup_n d_n$ *and* $\bigsqcup_n e_n$, *then* $\bigsqcup_n d_n \bullet_D \bigsqcup_n e_n =_D \bigsqcup_n d_n \bullet_D e_n$.

In dealing with ordered applicative structures, it is natural to ask for coinductive characterizations of the order relation itself, rather than just $=_D$. Here is an important definition:

Definition 2.12. (Φ_{pEq_D}-coinductive applicative structure) *Let* $\mathcal{D} = (D, \sqsubseteq_D, \bullet_D)$ *be an ordered applicative structure, and let* $\mathsf{pEq}_D \subseteq D \times D$ *be a preequivalence relation. The structure* $\mathcal{D}$ *is* pEq_D-coinductive *if the order relation* $\sqsubseteq_D$ *satisfies the following* pEq_D*-applicative coinduction principle:*

$$d \sqsubseteq_D e \iff \exists \mathcal{R} \subseteq D \times D.\ (\mathcal{R} \subseteq \Phi_{\mathsf{pEq}_D}(\mathcal{R})\ \wedge\ d\,\mathcal{R}\,e)\ ,$$

where
$\Phi_{\mathsf{pEq}_D} : \mathcal{P}(D \times D) \to \mathcal{P}(D \times D)$
$\Phi_{\mathsf{pEq}_D}(\mathcal{R}) = \{(d,e) \mid d\,\mathsf{pEq}_D\,e\ \wedge\ \forall f \in D.\ df\ \mathcal{R}\ ef\}$.

Coinductive characterizations of inequality appear to be useful especially in the case of the operator Φ_{Eq_D}, since we have that $\approx^{app}_{\mathsf{Eq}_D} = \sqsubseteq^{app}_{\mathsf{pEq}_D} \cap (\sqsubseteq^{app}_{\mathsf{pEq}_D})^{-1}$, where $\mathsf{Eq}_D = \mathsf{pEq}_D \cap (\mathsf{pEq}_D)^{-1}$ and $\sqsubseteq^{app}_{\mathsf{pEq}_D}$ is the maximal fixed point of Φ_{pEq_D}. For many other operators, and notably for Ψ_{Eq_D}, such a neat correspondence between preorder and equality does not arise. In fact, it is not the case in general, that $\approx^{\Psi_{\mathsf{Eq}_D}} = \sqsubseteq^{\Psi_{\mathsf{pEq}_D}} \cap (\sqsubseteq^{\Psi_{\mathsf{pEq}_D}})^{-1}$, where $\sqsubseteq^{\Psi_{\mathsf{pEq}_D}}$ denotes the maximal fixpoint of the operator $\Psi_{\mathsf{pEq}_D} : \mathcal{P}(D \times D) \to \mathcal{P}(D \times D)$ defined by

$$\Psi_{\mathsf{pEq}_D}(\mathcal{R}) = \{(d,e) \mid d\,\mathsf{pEq}_D\,e\ \wedge\ \forall f \in D. \exists g \in D.\ (f\mathcal{R}g\ \wedge\ df\mathcal{R}eg)\}\ .$$

We conclude this subsection by introducing another important class of enriched applicative structures: the *approximable applicative structures* (aas). These are ordered applicative structures $(D, \sqsubseteq_D, \bullet_D)$, together with a countable system of elements of D, $\{p_n\}_{n\in\omega}$, representing projection functions, which approximate the elements of D. Clearly all "inverse limit" λ-models are aas's:

Definition 2.13. (approximable applicative structure) *An* approximable applicative structure (aas) *is a structure* $\mathcal{D} = (D, \sqsubseteq_D, \bullet_D, \{p_n\}_{n\in\omega})$ *such that*

1 $(D, \sqsubseteq_D, \bullet_D)$ *is an ordered applicative structure;*
2 $\{p_n\}_{n\in\omega}$ *is a countable set of elements of* D *whose applicative behaviour is that of a complete system of* projection *functions* $\{\pi_n : D \to D\}_{n\in\omega}$, *i.e.* $\forall n \geq 0.\ \forall d \in D.\ p_n d =_D \pi_n(d)$, *such that*
   — $\forall n, m \geq 0.\ p_m(p_n d) =_D p_{min\{m,n\}} d$;
   — $\forall n > 0.\ \forall d, e \in D.\ (p_n d)e =_D p_{n-1}(d(p_{n-1}e))$;
   — $\forall d \in D.\ d =_D \bigsqcup_n p_n d$;
   — $\exists n, m \geq 0.\ \exists d, e \in D.\ n \neq m\ \wedge\ (p_n d)e \neq (p_m d)e$.

The latter condition in the above definition may seem odd. If we drop it, then trivially all applicative structures can be turned into aas's, by taking as projections any set of elements $\{p_n\}_{n\in\omega}$ satisfying $\forall d, e.\ (p_n d)e =_D de$. In fact, one can see that: either all projections satisfy this latter condition, or the last condition in the definition above is met.

Notation When clear from the context, for $d \in D$, we will use d_n to denote $p_n \bullet_D d$, and D_n to denote the set $\{p_n \bullet_D d \mid d \in D\}$.

Correspondingly, we introduce also the notion of :

Definition 2.14. (Φ_{pEq_D}-coinductive approximable applicative structure) *A Φ_{pEq_D}-coinductive approximable applicative structure (pEq_D-caas) is an approximable applicative structure $\mathcal{D} = (D, \sqsubseteq_D, \bullet_D, \{p_n\}_{n\in\omega})$ such that the ordered applicative structure $(D, \sqsubseteq_D, \bullet_D)$ is a Φ_{pEq_D}-coinductive applicative structure.*

3. The final perspective

Following the Final Semantics Paradigm, introduced in (Aczel 1988), and further developed in (Aczel and Mendler 1989; Rutten and Turi 1993; Rutten 1996; Turi 1996; Rutten 1996; Lenisa 1998), coinduction principles can receive a very neat *categorical explanation.* It is interesting to point out that many of the monotone operators described in Section 2 were actually suggested by this categorical analysis.

We work in Set^*, the category whose objects are sets of a universe of the *non-wellfounded* set theory $\mathsf{ZF}^\circ\mathsf{X}_1$, and whose morphisms are the set theoretic functions. Set-theoretic and categorical concepts are defined in Appendices A and B. The general pattern of the Final Semantics justification of a coinduction principle over an applicative structure $\mathcal{D}$, induced by the monotone operator Ξ, is the following. We endow $\mathcal{D}$ with the structure of a *H_Ξ-coalgebra*, (D, α_Ξ), for a suitable endofunctor H_Ξ (see Theorem 3.2 below), which preserves weak pullbacks and has *final* coalgebra. Then the *unique* mapping into the final coalgebra induces an equivalence on $\mathcal{D}$ which is the union of all *categorical H_Ξ-bisimulations.* Full definitions appear in Appendix B. If the functor H_Ξ and the H_Ξ-coalgebra structure have been given appropriately, we have the following crucial theorem:

Theorem 3.1. *$\mathcal{R}$ is a categorical H_Ξ-bisimulation on the coalgebra (D, α_Ξ) if and only if $\mathcal{R}$ is a Ξ-bisimulation.*

In (Honsell and Lenisa 1995), the authors succeeded in providing a final justification of the *applicative coinduction* principle (Φ_{Eq_D}-coinduction principle) for various λ-theories over the applicative structure consisting of the set of closed λ-terms.

Generalizing (Honsell and Lenisa 1995), we can establish the following correspondence between monotone operators and functors in Set^*.

Theorem 3.2. *Let $\mathcal{D} = (D, \bullet_D)$ be an applicative structure and let $\mathsf{Eq}_D \subseteq D \times D$ be an equivalence relation on D such that $|D/\mathsf{Eq}_D| \leq \kappa$, i.e. the cardinality of the set of equivalence classes of Eq_D is less than or equal to κ.*
Let Ξ be one of the operators $\{\Phi_{\mathsf{Eq}_D}, \Psi_{\mathsf{Eq}_D}, \Phi^n_{\mathsf{Eq}_D}, \Psi^n_{\mathsf{Eq}_D}, Z_{\mathsf{Eq}_D}\}$. Ξ-bisimulations are categorical H_Ξ-bisimulations on the H_Ξ-coalgebra (D, α_Ξ) for the endofunctor H^Ξ_κ in Set^* given by the following table

$$\begin{array}{lll}
\Xi \equiv \Phi_{\mathsf{Eq}_D} & : & H^{\Xi}_{\kappa}(X) = \bigoplus_{\kappa}(D \to X) \\
& & H^{\Xi}_{\kappa}(f) = \bigoplus_{\kappa}(id_D \to f) \\
& & \alpha_{\Xi}(d) = \{i_{[d]}\} \cup \lambda e \in D.\ de, \\
& & \textit{where } i_{[d]} \textit{ is the tag of the } \mathsf{Eq}_D\textit{-equivalence class of } d.
\end{array}$$

$$\begin{array}{lll}
\Xi \equiv \Psi_{\mathsf{Eq}_D} & : & H^{\Xi}_{\kappa}(X) = \bigoplus_{\kappa} \mathcal{P}_{<\aleph_1}(X \times X) \\
& & H^{\Xi}_{\kappa}(f) = \bigoplus_{\kappa}(f \times f)^{+} \\
& & \alpha_{\Xi}(d) = \{i_{[d]}\} \cup \lambda e \in D.\ de, \\
& & \textit{where } i_{[d]} \textit{ is the tag of the } \mathsf{Eq}_D\textit{-equivalence class of } d, \textit{ and,} \\
& & \textit{for } f : X \to Y,\ f^{+} : \mathcal{P}_{<\aleph_1}(X) \to \mathcal{P}_{<\aleph_1}(Y) \textit{ is defined by} \\
& & \forall x \in \mathcal{P}_{<\aleph_1}(X).\ f^{+}(x) = \{f(z) \mid z \in x\}.
\end{array}$$

$$\begin{array}{lll}
\Xi \equiv \Phi^{n}_{\mathsf{Eq}_D} & : & H^{\Xi}_{\kappa}(X) = \bigoplus_{k}(D^{n} \to X), \\
& & H^{\Xi}_{\kappa}(f) = \bigoplus_{k}(id^{n}_{D} \to f) \\
& & \alpha_{\Xi}(d) = \{i_{[d]}\} \cup \lambda e_1 \ldots e_n \in D.\ (\ldots(de_1)\ldots e_n), \\
& & \textit{where } i_{[d]} \textit{ is the tag of the } \mathsf{Eq}_D\textit{-equivalence class of } d.
\end{array}$$

$$\begin{array}{lll}
\Xi \equiv \Psi^{n}_{\mathsf{Eq}_D} & : & H^{\Xi}_{\kappa}(X) = \bigoplus_{\kappa} \mathcal{P}_{<\aleph_1}(X^{n} \times X) \\
& & H^{\Xi}_{\kappa}(f) = \bigoplus_{\kappa}(f^{n} \times f)^{+} \\
& & \alpha_{\Xi}(d) = \{i_{[d]}\} \cup \lambda e_1 \ldots e_n \in D.\ (\ldots(de_1)\ldots e_n), \\
& & \textit{where } i_{[d]} \textit{ is the tag of the } \mathsf{Eq}_D\textit{-equivalence class of } d.
\end{array}$$

$$\begin{array}{lll}
\Xi \equiv Z_{\mathsf{Eq}_D} & : & H^{\Xi}_{\kappa}(X) = \bigoplus_{\kappa} \mathcal{P}_{<\aleph_1}(X \times (D/\approx^{app}_{\mathsf{Eq}_D})) \\
& & H^{\Xi}_{\kappa}(f) = \bigoplus_{\kappa}(f \times id_{(D/\approx^{app}_{\mathsf{Eq}_D})})^{+} \\
& & \alpha_{\Xi}(d) = \{i_{[d]}\} \cup \lambda e \in D.\ [de]_{\approx^{app}_{\mathsf{Eq}_D}}, \\
& & \textit{where } i_{[d]} \textit{ is the tag of the } \mathsf{Eq}_D\textit{-equivalence class of } d \textit{ and} \\
& & [de]_{\approx^{app}_{\mathsf{Eq}_D}} \textit{ is the equivalence class of } de \textit{ modulo } \approx^{app}_{\mathsf{Eq}_D}.
\end{array}$$

For A, B sets, the notation $\bigoplus_{|A|} B$ stands for the disjoint sum of $|A|$ copies of B, i.e. $\{\{\alpha\} \cup b \mid \alpha \in |A| \ \wedge\ b \in B\}$, assuming $|A| \cap TC(B) = \emptyset$. The proof of the above theorem is standard, see (Aczel 1988; Rutten and Turi 1993; Lenisa 1998).

It is interesting to notice that, in each case, the structure of the functor corresponding to the monotone operator reflects the way in which we construe objects of an applicative structure: as a function on a constant set, as the graph of a self-function, as an n-ary function, as the graph of an n-ary self-function, or as the graph of a function which takes values over a suitable constant set. In each case, however, we need $|D/\mathsf{Eq}_D|$ copies of the same domain, one for each equivalence class of the equivalence Eq_D we use as start-up.

The categorical account allows to give also a clear characterization of applicative structures which satisfy Ξ-coinduction principles.
In particular, for $\Xi \in \{\Phi_{\mathsf{Eq}_D}, \Psi_{\mathsf{Eq}_D}, \Phi^{n}_{\mathsf{Eq}_D}, \Psi^{n}_{\mathsf{Eq}_D}, Z_{\mathsf{Eq}_D}\}$

Corollary 3.3. *Let $\mathcal{D}$ be an applicative structure, such that $|D/\mathsf{Eq}_D| \leq \kappa$. Then the H^{Ξ}_{κ}-coalgebra (D, α_{Ξ}) is strongly extensional if and only if $\mathcal{D}$ is a Ξ-coinductive applicative structure.*

4. Theories and models of the λ-calculus

In this section we discuss λ-theories, the very important class of congruence relations over the paradigm applicative structure consisting of closed λ-terms.

We start by giving basic definitions and general results on λ-theories and λ-models. In Subsection 4.1 we present the specific λ-theories we shall be mainly concerned with.

Let Λ^C denote the class of λ-terms built over the set of basic constants C, i.e.:

$$(\Lambda \ni)\ M \ ::= \ x \ \mid \ c \ \mid \ MM \ \mid \ \lambda x.M \ ,$$

where $x \in Var$, $c \in C$.
Let $(\Lambda^C)^0$ denote the class of *closed* λ-terms built over the set of basic constants C.

Definition 4.1.

— *A* λ-preequivalence *is a reflexive and transitive relation on* $\Lambda^C \times \Lambda^C$.
— *A* λ-equivalence *is a symmetric* λ*-preequivalence.*
— *A* λ-precongruence, $\leq$, *is a* λ*-preequivalence which is a congruence w.r.t. application and* λ*-abstraction, i.e., for all* $M, N, M', N' \in \Lambda^C$,

$$M \leq N \ \wedge \ M' \leq N' \ \Rightarrow \ MM' \leq NN' \ , \ \textit{and}$$

$$M \leq N \ \Rightarrow \ \lambda x.M \leq \lambda x.N \ .$$

— *A* λ-congruence *is a* λ*-equivalence which is a congruence.*
— *A* λ-theory *is the restriction of a* λ*-congruence to* $(\Lambda^C)^0$.

We focus on λ-(pre)congruences which arise from relations $\mathsf{Eq} \subseteq (\Lambda^C)^0 \times (\Lambda^C)^0$ in the following sense:

Definition 4.2. *Let* $\mathsf{Eq} \subseteq (\Lambda^C)^0 \times (\Lambda^C)^0$. *The* contextual relation $\approx^{\mathsf{Eq}} \subseteq \Lambda^C \times \Lambda^C$ *is defined as follows:*

$$M \approx^{\mathsf{Eq}} N \iff \forall C[\ ].\ (C[M], C[N] \in (\Lambda^C)^0 \Rightarrow C[M] \,\mathsf{Eq}\, C[N]) \ .$$

Notice that, if Eq is a (pre)equivalence, then the relation $\approx^{\mathsf{Eq}}$ is a (pre)congruence.

All λ-congruences are induced by an equivalence relation with just *two* equivalence classes, i.e.:

Proposition 4.3. (Honsell and Ronchi 1992) *Any* λ*-congruence* $\approx \subseteq \Lambda^C \times \Lambda^C$ *is induced by a suitable set* $\mathcal{V} \subseteq (\Lambda^C)^0$ *in the following sense:*

$$M \approx N \iff \forall C[\ ].\ (C[M], C[N] \in (\Lambda^C)^0 \Longrightarrow (C[M] \in \mathcal{V} \Longleftrightarrow C[N] \in \mathcal{V})) \ .$$

Proof. Just take $\mathcal{V} = \{\lambda x.xPQ \mid P \approx Q\}$. □

Question 4.4. Is there an analogue of Proposition 4.3 for λ-precongruences?

The relation between λ-theories and β-reduction is formalized and clarified by the following proposition, whose proof is straightforward.

Proposition 4.5. *Let* $\approx^{\mathsf{Eq}}$ *be the congruence induced by the equivalence relation* $\mathsf{Eq} \subseteq$

$(\Lambda^C)^0 \times (\Lambda^C)^0$. *Then the notion of* β*-reduction* $\rightarrow_{\beta_r}$ [‡] *is correct w.r.t.* $\approx^{\mathsf{Eq}}$, *i.e.* $=_{\beta_r} \subseteq \approx^{\mathsf{Eq}}$, *if and only if* Eq *is closed under* β_r*-conversion, i.e.* $(=_{\beta_r} \cap ((\Lambda^C)^0 \times (\Lambda^C)^0)) \subseteq \mathsf{Eq}$.

Finally we recall some useful semantical definitions:

Definition 4.6. (ordered λ-model) *An* ordered λ-model *is a quadruple* $\mathcal{D} = (D, \sqsubseteq_D, \bullet_D, [\![\ ]\!]^D)$, *where*

1 $(D, \sqsubseteq_D, \bullet_D)$ *is an ordered applicative structure;*
2 $[\![\ ]\!]^D : \Lambda \times Env \rightarrow D$, *for* $Env = [Var \rightarrow D_{var}]$,

$$D_{var} = \begin{cases} D \setminus \{\bot\} & \text{if } \bot \in D \text{ is the least element of } D \text{ and } \forall d \in D.\ d \bullet_D \bot = \bot, \\ D & \text{otherwise,} \end{cases}$$

is the interpretation function*;*
3 $[\![x]\!]^D_\rho = \rho(x)$*;*
4 $[\![c]\!]^D_\rho = d_c$, *for some* $d_c \in D$*;*
5 $[\![MN]\!]^D_\rho = [\![M]\!]^D_\rho \bullet_D [\![N]\!]^D_\rho$*;*
6 $\forall d \in D_{var}.\ [\![\lambda x.M]\!]^D_\rho \bullet_D d = [\![M]\!]^D_{\rho[d/x]}$*;*
7 $\forall d \in D_{var}.\ [\![M]\!]^D_{\rho[d/x]} = [\![N]\!]^D_{\rho[d/x]} \ \Rightarrow\ [\![\lambda x.M]\!]^D_\rho = [\![\lambda x.N]\!]^D_\rho$*;*
8 $\forall \rho, \rho' : Var \rightarrow D_{var}\ (\forall x \in FV(M).\ \rho(x) = \rho'(x) \ \Rightarrow\ [\![M]\!]^D_\rho = [\![M]\!]^D_{\rho'})$.

It is immediate to see that, given a λ-model $\mathcal{D} = (D, \sqsubseteq_D, \bullet_D, [\![\ ]\!]^D)$, the theory $\mathcal{T}^{\mathcal{D}}$ induced by it, i.e. the set of pairs of terms which have the same interpretation in a λ-model, is a λ-congruence. Correspondingly, the set of pairs of terms whose first component has an interpretation which is less than that of the second component, is a λ-precongruence. A λ-model $(D, \sqsubseteq_D, \bullet^D, [\![\ ]\!])$ is *computationally adequate* with respect to a λ-precongruence $\leq$ if

$$\forall M, N \in \Lambda^0.\ [\![M]\!]^D \sqsubseteq_D [\![N]\!]^D \implies M \leq N.$$

A λ-model is *fully abstract* if the above implication is a bi-implication.

In this paper we shall be concerned with two kinds of λ-models: *finitary λ-models* and *term models.*

As far as finitary λ-models we shall focus on:

Definition 4.7. (λ-aas, λ-caas)

— *An aas* $(D, \sqsubseteq_D, \bullet_D, \{p_n\}_{n\in\omega})$ *is a* λ-aas *if it is a λ-model.*
— *A structure* $(D, \sqsubseteq_D, \bullet_D, \{p_n\}_{n\in\omega})$ *is a* $\Phi_{\mathsf{pEq_D}}\lambda$-caas *if it is a* $\Phi_{\mathsf{pEq_D}}$*-caas which is a λ-model.*

Many CPO λ-models in the literature turn out to be λ-aas. Other interesting examples of λ-aas are models obtained by quotienting sets of indexed terms (see Section 5.1 for examples).

Term models are induced by λ-precongruences and they are defined as follows:

Definition 4.8. (term model) *Let* $\leq$ *be a* λ*-precongruence and let* T *be the quotient*

‡ A notion of β*-reduction*, $\rightarrow_{\beta_r}$, is the λ-precongruence generated by a set of pairs (redex, β-reduct). The λ-congruence generated by the symmetric closure of a β-reduction $\rightarrow_{\beta_r}$ is the $=_{\beta_r}$*-conversion.*

$\Lambda^0/(\leq \cap (\leq)^{-1})$, *i.e. the set of* $(\leq \cap (\leq)^{-1})$*-equivalence classes of closed terms. The term model* $\mathcal{T}_{\leq}$ *is the applicative structure* $(T, \bullet)$*, where, for all* $M, N \in (\Lambda^C)^0$, $[M] \bullet [N] = [MN]$*. The interpretation function* $[\![\ ]\!] : \Lambda \times Env \to T$*, for* $Env = [Var \to T_{var}]$ *and*

$$T_{var} = \begin{cases} T \setminus \perp & \text{if } \perp \in T \text{ is the least element of } T \text{ and } \forall M.\ [M] \bullet \perp = \perp \\ T & \text{otherwise,} \end{cases}$$

is defined as follows:

$$[\![M]\!]_\rho = [\rho(M)]\ ,$$

where $FV(M) \subseteq \{x_1, \ldots, x_n\}$ *and* $\rho(M) = M[\rho(x_1)/x_1, \ldots, \rho(x_n)/x_n]$.

The equivalence determined by the interpretation function into $\mathcal{T}_{\leq}$ is, of course, $\leq \cap (\leq)^{-1}$. Notice that, since we consider only *closed* terms, it is not always the case that a term model is an ordered λ-model, since clause 7 of Definition 4.6 may fail. In any case, this will hold for all the term models we shall deal with explicitly in this paper. Our notion of term model is sligthly non standard, so as to accomodate also "restricted calculi" such a Plotkin's call-by-value λ_v-calculus. In these cases $\perp$ is intended to denote *non-values.* All the precongruences which we shall explicitly mention in this paper satisfy this condition.

4.1. *Examples of λ-theories*

In this paper we are mainly concerned with λ-precongruences which arise from reduction strategies. Namely, we take a term to approximate another if we cannot *observe* that, for a given *closing context*, the strategy halts successfully when one is used to fill the hole, but does not halt when the other one is used. A *reduction strategy* is a procedure for determining, for each λ-term, a specific β-redex in it, to contract. A (possibly non-deterministic) strategy can be formalized as a relation $\to_\sigma \subseteq \Lambda \times \Lambda$ $(\Lambda^0 \times \Lambda^0)$ such that, if $(M, N) \in \to_\sigma$ (also written infix as $M \to_\sigma N$), then N is a possible result of applying $\to_\sigma$ to M. We denote by $\to_\sigma^*$ the reflexive and transitive closure of $\to_\sigma$. The set of terms which do not belong to the domain of $\to_\sigma$ are partitioned into two disjoint sets: the set of *σ-values*, denoted by Val_σ, and the set of *σ-deadlocks*. Given $\to_\sigma$, we can define the *evaluation relation* $\Downarrow_\sigma \subseteq \Lambda \times \Lambda$ $(\Lambda^0 \times \Lambda^0)$, such that $M \Downarrow_\sigma N$ holds if and only if there exists a (possible empty) reduction path from M to a σ-value N. If there exists N such that $M \Downarrow_\sigma N$, then $\to_\sigma$ *halts successfully* on M and M converges ($M \Downarrow_\sigma$), otherwise $\to_\sigma$ does not terminate on M, or reaches a *deadlock* from M, and M diverges ($M \not\Downarrow_\sigma$). Each reduction strategy induces an *operational semantics*, in that we can imagine a machine which evaluates terms by implementing the given strategy. The *observational preequivalence* arises when we consider programs as *black boxes* and only observe their "halting properties".

Definition 4.9. *Let* $\to_\sigma$ *be a reduction strategy and let* $M, N \in \Lambda$.

— *The* observational precongruence $\leq_\sigma$ *is defined by*
  $M \leq_\sigma N \iff \forall C[\ ].(C[M], C[N] \in \Lambda^0 \Rightarrow (C[M] \Downarrow_\sigma \Rightarrow C[N] \Downarrow_\sigma))$.
— *The* observational λ*-theory* $\approx_\sigma$ *is the congruence defined by*
  $M \approx_\sigma N \iff \forall C[\ ].(C[M], C[N] \in \Lambda^0 \Rightarrow (C[M] \Downarrow_\sigma \Leftrightarrow C[N] \Downarrow_\sigma))$.

It is immediate to check that the σ-observational precongruence, $\leq_\sigma$, is the λ-precongruence induced by the preequivalence relation $\mathsf{pEq}_\sigma = \{(M,N) \mid M \Downarrow_\sigma \Rightarrow N \Downarrow_\sigma\}$. And, similarly, the σ-observational congruence, $\approx_\sigma$, is the λ-theory induced by the equivalence relation $\mathsf{Eq}_\sigma = \{(M,N) \in \Lambda^0 \times \Lambda^0 \mid M \Downarrow_\sigma \Leftrightarrow N \Downarrow_\sigma\}$. Notice also that $\approx_\sigma = \leq_\sigma \cap (\leq_\sigma)^{-1}$.

There is no loss of generality in considering σ-observational congruences, rather than λ-theories. In fact, by Proposition 4.3, any λ-theory can be viewed as an observational congruence induced by a trivially empty strategy, whose values are the terms in the set $\mathcal{V}$ arising in the proof of Proposition 4.3.

As we remarked earlier, λ-theories can be viewed as congruences on the applicative structure consisting of closed λ-terms. Hence we shall say that a λ-theory $\approx_\sigma$ is Ξ_{Eq_σ}-coinductive if $\approx_\sigma$ coincides with $\approx^{\Xi_{\mathsf{Eq}_\sigma}}$, *i.e.* the maximal fixed-point of the operator Ξ_{Eq_σ}. Hence we shall call Eq_σ-*applicative*, or Eq_σ-*cartesian* respectively, those theories which are $\Phi_{\mathsf{Eq}_\sigma}$-coinductive, or $\Psi_{\mathsf{Eq}_\sigma}$-coinductive.

Now we present six examples of σ-observational precongruences, and corresponding *computationally adequate finitary denotational λ-models*, which have been extensively studied in the literature (Plotkin 1977; Barendregt 1984; Coppo et al. 1987; Honsell and Ronchi 1992; Egidi et al. 1992; Abramsky and Ong 1993; Honsell and Lenisa 1998).

4.1.1. $\leq_\mathbf{l}$. The precongruence $\leq_l$ is induced by the *lazy call-by-name* strategy $\rightarrow_l \subseteq \Lambda^0 \times \Lambda^0$, which reduces the *leftmost* β-redex not appearing in a λ-abstraction. $Val_l = \{\lambda x.M \mid M \in \Lambda\} \cap \Lambda^0$. The evaluation $\Downarrow_l$ is the least binary relation over $\Lambda^0 \times Val_l$ satisfying the rules:

$$\frac{}{\lambda x.M \Downarrow_l \lambda x.M} \qquad\qquad \frac{M \Downarrow_l \lambda x.P \quad P[N/x] \Downarrow_l Q}{MN \Downarrow_l Q}$$

Classical β-reduction is correct w.r.t. $\approx_l$ (Abramsky and Ong 1993).
This is the reduction strategy of lazy functional languages.

A computationally adequate ordered λ-model for $\leq_l$ is the model $\mathcal{D}^l$, studied in (Abramsky and Ong 1993). The model $\mathcal{D}^l$ is the quadruple $(D^l, \sqsubseteq^l, \bullet^l_D, [\![\ ]\!]^l)$, where D^l is the inverse limit initial solution of the equation $D \simeq [D \to D]_\perp$ in the category $CPO_\perp$, and $\bullet^l_D$ and $[\![\ ]\!]^l$ are defined using the canonical isomorphisms given by the inverse limit construction. Computational adequacy follows from

Theorem 4.10. (computational adequacy of $\mathcal{D}^l$)

$$M \Downarrow_l \iff \forall \rho.\ [\![M]\!]^l_\rho \sqsupseteq_l \boldsymbol{\lambda} d \in D^l.\ \perp \ .$$

4.1.2. $\leq_v$. The precongruence $\leq_v$ is determined by Plotkin's *lazy call-by-value* strategy $\rightarrow_v \subseteq \Lambda^0 \times \Lambda^0$, which reduces the *leftmost* β-redex, not appearing within a λ-abstraction, whose argument is a λ-abstraction. $Val_v = \{\lambda x.M \mid M \in \Lambda\} \cap \Lambda^0$. The evaluation $\Downarrow_v$ is the least binary relation over $\Lambda^0 \times Val_v$ satisfying the following rules:

$$\frac{}{\lambda x.M \Downarrow_v \lambda x.M} \qquad\qquad \frac{M \Downarrow_v \lambda x.P \quad N \Downarrow_v Q \quad P[Q/x] \Downarrow_v U}{MN \Downarrow_v U}$$

A notion of β-reduction which is correct w.r.t. $\approx_v$ is Plotkin's $\rightarrow_{\beta_v} \subseteq \Lambda \times \Lambda$, i.e.:

$(\lambda x.M)N \rightarrow_{\beta_v} M[N/x]$, if N is a variable or an abstraction.
Notice that the β_v-reduction is far from being the largest notion of β-reduction correct w.r.t. $\approx_v$. E.g., we can extend this notion by allowing the reduction whenever N β_v-reduces to a variable or an abstraction. The characterization of *call-by-value*-redexes, *i.e.* $\approx_v \cap =_\beta$, is given by :

$$\{\langle(\lambda x.M)N, M[N/x]\rangle \mid \forall \rho \in [Var \rightarrow Val_v].\ N^\rho \Uparrow_v \Longrightarrow (M[N/x])^\rho \Uparrow_v\},$$

where, for $M \in \Lambda$ such that $FV(M) \subseteq \{x_1, \ldots, x_n\}$, and $\rho \in [Var \rightarrow Val_v]$, M^ρ denotes the closed term $M[\rho(x_1)/x_1, \ldots, \rho(x_n)/x_n]$.
The reduction strategy $\rightarrow_v$ is the one implemented by the $SECD$ machine of Landin and used in ML.

A computationally adequate ordered λ-model for $\leq_v$ is the model $\mathcal{D}^v$, studied in (Egidi et al. 1992). The model $\mathcal{D}^v$ is the quadruple $(D^v, \sqsubseteq^v, \bullet^v_D, [\![\ ]\!]^v)$, where D^v is the inverse limit initial solution of the equation $D \simeq [D \rightarrow_\perp D]_\perp$ in the category $CPO_\perp$, and $\bullet^v_D$ and $[\![\ ]\!]^v$ are defined using the canonical isomorphisms, given by the inverse limit construction. We recall that $[D \rightarrow_\perp D]$ denotes the cpo of *strict* continuous functions. Computational adequacy follows from

Theorem 4.11. (computational adequacy of $\mathcal{D}^v$)

$$M \Downarrow_v \iff \forall \rho.\ [\![M]\!]^v_\rho \sqsupseteq_v \boldsymbol{\lambda} d \in D^v.\ \perp\ .$$

4.1.3. $\leq_h$. The precongruence $\leq_h$ is determined by the *head call-by-name* strategy $\rightarrow_h \subseteq \Lambda \times \Lambda$, which reduces the *leftmost* β-redex, if the term is not in head normal form. Val_h is the set of λ-terms in head normal form. The evaluation $\Downarrow_h$ is the least binary relation over $\Lambda \times Val_h$ satisfying the following rules, for $n \geq 0$:

$$\frac{}{xM_1 \ldots M_n \Downarrow_h xM_1 \ldots M_n} \qquad \frac{M \Downarrow_h N}{\lambda x.M \Downarrow_h \lambda x.N} \qquad \frac{M[N/x]M_1 \ldots M_n \Downarrow_h P}{(\lambda x.M)NM_1 \ldots M_n \Downarrow_h P}$$

β-reduction is correct w.r.t. $\approx_h$ (see e.g. (Barendregt 1984)).

In the next definition we introduce an alternative axiomatization of the notion of $\rightarrow_h$-convergence on closed terms, $\Downarrow^0_h$, which will be useful in Section 6. Notice the somewhat "lazy" flavour of the abstraction rule in this proof system.

Definition 4.12. *Let $\Downarrow^o_h$ be the least binary relation over $\Lambda^0 \times \Lambda^0$ satisfying the following rules, for $n \geq 0$:*

$$\frac{M \in Val_h}{M \Downarrow^o_h M} \qquad \frac{\lambda x.M \notin Val_h \quad M[P/x] \Downarrow^o_h N}{\lambda x.M \Downarrow^o_h \lambda x.M} \qquad \frac{M[N/x]M_1 \ldots M_n \Downarrow^o_h P}{(\lambda x.M)NM_1 \ldots M_n \Downarrow^o_h P}$$

The following theorem clarifies the meaning of the $\Downarrow^o_h$:

Theorem 4.13. *For all $M \in \Lambda^0$,*
i) $\exists N \in \Lambda^0.\ M \Downarrow^o_h N \ \Rightarrow\ M \Downarrow_h$;
ii) $\forall N \in \Lambda^0.\ M \Downarrow^o_h N \ \Rightarrow\ M \rightarrow^*_h N$;
iii) $M \Downarrow_h \Rightarrow M \Downarrow^o_h$.

Proof. i) and ii) are proved by induction on the length of the derivation of $M \Downarrow_h^o N$. The proof of iii) follows from the fact that, for all $M \in \Lambda$ such that $FV(M) \subseteq \{x_1, \ldots, x_n\}$, $M \Downarrow_h \Rightarrow \exists N, P_1, \ldots, P_n \in \Lambda^0.\ M[P_1/x_1, \ldots, P_n/x_n] \Downarrow_h^o N$. This latter fact is proved by induction on the length of the derivation of $M \Downarrow_h$. $\square$

A fully abstract ordered λ-model for $\leq_h$ is given the well-known Scott $\mathcal{D}_\infty$ model, which, by uniformity, we will call $\mathcal{D}^h$. The model $\mathcal{D}^h$ is given by the quadruple $(D^h, \sqsubseteq^h, \bullet_{D^h}, [\![\ ]\!]^h)$, where D^h is the inverse limit solution of the equation $D \simeq [D \to D]$ in the category CPO, starting from the initial domain $D_0^h = \{\bot, \top\}$ and using the initial projection $j_{\top,\bot}^h : D_1^h \to D_0^h$ defined by: $j_{\top,\bot}^h(d) = \bot$, if $d \neq \boldsymbol{\lambda} d \in D_0^h.\top$, $j_{\top,\bot}^h(\boldsymbol{\lambda} d \in D_0^h.\top) = \top$. The definition of application and intepretation are given using the canonical isomorphisms given by the inverse limit construction. Computational adequacy follows from

Theorem 4.14. (computational adequacy of $\mathcal{D}^h$)

$$M \Downarrow_h \iff \exists \rho.\ [\![M]\!]_\rho^h \neq \bot\ .$$

4.1.4. $\leq_o$. The precongruence $\leq_o$ is determined by the non-deterministic strategy $\to_o \subseteq (\Lambda^{\{\Omega\}})^0 \times (\Lambda^{\{\Omega\}})^0$ (Honsell and Ronchi 1992). This strategy rewrites λ-terms which contain occurrences of the constant Ω by reducing any β-redex. $Val_o = \Lambda^0$. Normal forms which are not in Val_o are the $\to_o$-deadlock terms. The evaluation relation $\Downarrow_o$ is the least binary relation over $(\Lambda^{\{\Omega\}})^0 \times Val_o$ satisfying the following rules:

$$\frac{M \in Val_o}{M \Downarrow_o M} \qquad \frac{C[(\lambda x.M)N] \notin Val_o \quad C[M[N/x]] \Downarrow_o P}{C[(\lambda x.M)N] \Downarrow_o P}$$

β-reduction is trivially correct w.r.t. $\approx_o$.

A computationally adequate ordered λ-model for $\leq_o$ is the model $\mathcal{D}^o$, introduced (Honsell and Ronchi 1992). The model $\mathcal{D}^o$ is the quadruple $(D^o, \sqsubseteq^o, \bullet_{D^o}, [\![\ ]\!]^o)$, where D^o is Park's inverse limit solution of the equation $D \simeq [D \to D]$ in the category CPO, obtained starting from the domain $D_0^o = \{\bot, \nu\}$ and the initial projection $j_{1,0}^o : D_1^o \to D_0^o$ defined by: $j_{1,0}^o(\bot) = \bot$, $j_{1,0}^o(d) = \nu$, for $d \neq_{D^o} \bot$. Application and abstraction in $\mathcal{D}^o$ are defined using the canonical isomorphism given by the inverse limit construction. Computational adequacy of $\mathcal{D}^o$ follows from

Theorem 4.15. (computational adequacy of $\mathcal{D}^o$)

$$M \Downarrow_o \iff \exists \rho.\ [\![M]\!]_\rho^o \neq \bot\ .$$

4.1.5. $\leq_n$. The precongruence $\leq_n$ is determined by the *normalizing* strategy $\to_n \subseteq \Lambda \times \Lambda$, which reduces the leftmost β-redex. Val_n is the set of λ-terms in normal form. The evaluation $\Downarrow_n$ is the least binary relation over $\Lambda \times Val_n$ satisfying the following rules, for $n \geq 0$:

$$\frac{M_1 \Downarrow_n M_1' \ \ldots\ M_n \Downarrow_n M_n'}{xM_1 \ldots M_n \Downarrow_n xM_1' \ldots M_n'} \qquad \frac{M \Downarrow_n N}{\lambda x.M \Downarrow_n \lambda x.N} \qquad \frac{M[N/x]M_1 \ldots M_n \Downarrow_n P}{(\lambda x.M)NM_1 \ldots M_n \Downarrow_n P}$$

β-reduction is correct w.r.t. $\approx_n$.

A computationally adequate ordered λ-model for $\leq_n$ is the model $\mathcal{D}^n$, studied in (Coppo et al. 1987). The model $\mathcal{D}^n$ is the quadruple $(D^n, \sqsubseteq^n, \bullet_{D^n}, [\![\ ]\!]^n)$, where D^n is the inverse limit solution of the equation $D \simeq [D \to D]$ in the category CPO, obtained starting with the initial domain $D_0^n = \{\perp, 0, 1\}$, with $0 \sqsubseteq_n 1$, and the initial projection $j_{1,0}^n : D_1^n \to D_0^n$ defined by: $j_{1,0}^n(\perp) = \perp$, $j_{1,0}^n(d) = 0$, if $d \neq \perp, \boldsymbol{\lambda} d \in D_0^n.1$, $j_{1,0}(\boldsymbol{\lambda} d \in D_0^n.1) = 1$. Application and interpretation in D^n are defined using the canonical isomorphism given by the invers limit construction. Computational adequacy of $\mathcal{D}^n$ follows from

Theorem 4.16. (computational adequacy of $\mathcal{D}^n$)

$$M \Downarrow_n \iff \exists \rho. [\![M]\!]_\rho^n \sqsupseteq^n 0.$$

4.1.6. $\leq_p$. The precongruence $\leq_p$ is determined by any *perpetual strategy*, such as Barendregt's *perpetual* strategy $\to_p \subseteq \Lambda \times \Lambda$ which reduces the leftmost β-redex not in the operator of a redex, which is either an $I\beta$-redex[§], or a $K\beta$-redex[¶] whose argument is a normal form. Val_p is the set of λ-terms in normal form. One can easily show that the evaluation $\Downarrow_p$ is the least binary relation over $\Lambda \times Val_p$ satisfying the following rules, for $n \geq 0$:

$$\frac{M_1 \Downarrow_p M_1' \ \ldots \ M_n \Downarrow_p M_n'}{xM_1 \ldots M_n \Downarrow_p xM_1' \ldots M_n'} \qquad \frac{M \Downarrow_p N}{\lambda x.M \Downarrow_p \lambda x.N}$$

$$\frac{N \Downarrow_p \quad M[N/x]M_1 \ldots M_n \Downarrow_p V}{(\lambda x.M)NM_1 \ldots M_n \Downarrow_p V}$$

The following notion of β-reduction, $\to_{\beta_{KN}}$, defined by

$$(\lambda x.M)N \to_{\beta_{KN}} M[N/x], \quad \begin{array}{l}\text{if } (\lambda x.M)N \text{ is either an } I\beta\text{-redex}\\ \text{or a } K\beta\text{-redex with } N \in \Lambda^0 \text{ and } N \Downarrow_p,\end{array}$$

is correct w.r.t. $\leq_p$. The characterization of *perpetual*-redexes, *i.e.* $\approx_p \cap =_\beta$, is given by:

$$\{\langle (\lambda x.M)N, M[N/x]\rangle \mid \forall \rho \in [Var \to Val_p].\ N^\rho \Uparrow_p \Longrightarrow (M[N/x])^\rho \Uparrow_p\}\ ,$$

where, for $M \in \Lambda$ such that $FV(M) \subseteq \{x_1, \ldots, x_n\}$, and $\rho \in [Var \to Val_p]$, M^ρ denotes the closed term $M[\rho(x_1)/x_1, \ldots, \rho(x_n)/x_n]$.
See (Honsell and Lenisa 1998) for more informations.

Notice that $\approx_p$ is not very well behaved, we do not have, for instance, that if $M \to_p N$ then $M \approx_p N$. Consider for example $(\lambda x.(\lambda xy.x)xx)$ and $\lambda x.x$.

As we did in the case of $\to_h$, we introduce an alternative "lazy" axiomatization of $\Downarrow_p$, to be used in Section 6.

Definition 4.17. *Let $\Downarrow_p^o$ be the least binary relation over $\Lambda^0 \times \Lambda^0$ satisfying the following rules, for $n \geq 0$:*

§ $(\lambda x.M)N$ is an $I\beta$-redex if $x \in FV(M)$.
¶ $(\lambda x.M)N$ is a $K\beta$-redex if $x \notin FV(M)$.

$$\frac{M \in Val_p}{M \Downarrow_p^o M} \qquad \frac{\lambda x.M \notin Val_p \quad M[P/x] \Downarrow_p^o N}{\lambda x.M \Downarrow_p^o \lambda x.M} \qquad \frac{N \Downarrow_p^o \quad M[N/x]M_1 \dots M_n \Downarrow_p^o V}{(\lambda x.M)NM_1 \dots M_n \Downarrow_p^o V}$$

The proof of the following theorem is similar to the proof of Theorem 4.13.

Theorem 4.18. *For all* $M \in \Lambda^0$,
i) $\exists N \in \Lambda^0.\ M \Downarrow_p^o N \ \Rightarrow\ M \Downarrow_p$;
ii) $\forall N \in \Lambda^0.\ M \Downarrow_p^o N \ \Rightarrow\ M \to_p^* N$;
iii) $M \Downarrow_p \Rightarrow M \Downarrow_p^o$.

A computationally adequate model for $\leq_p$ is the model $\mathcal{D}^p$, studied in (Honsell and Lenisa 1998). The model $\mathcal{D}^p$ is the quadruple $(D^p, \sqsubseteq^p, \bullet_{D^p}, [\![\ ]\!]^p)$, where D^p is the inverse limit solution of the equation $D \simeq [D \to_\perp D]$ in the category $CPO_\perp$, obtained starting form the initial domain $D_0^p = \{\perp, 0, 1\}$, with $0 \sqsubseteq_p 1$, and the initial projection $j_{1,0}^p : D_1^p \to D_0^p$ defined by: $j_{1,0}^p(\perp) = \perp$, $j_{1,0}^p(d) = 0$, if $d \neq_{D^p} \perp$, $d \neq_{D^p} \boldsymbol{\lambda} d \in D_0^p.$ if $d =_{D^p} \perp$ then $\perp$ else 1 and $j_{1,0}(\boldsymbol{\lambda} d \in D_0^p.$ if $d =_{D^p} \perp$ then $\perp$ else $1) = 1$. Application and interpretation are defined using the canonical isomorphism given by the inverse limit construction. Computational adequacy follows from

Theorem 4.19. (computational adequacy of $\mathcal{D}^p$)

$$M \Downarrow_p \iff \exists \rho.\ [\![M]\!]_\rho^p \neq \perp\ .\ [\![M]\!]^p \sqsupseteq^p 0\ .$$

5. Cartesian applicative structures

As we pointed out in the Introduction and in Section 2, most of the existing literature on coinduction principles for applicative structures has focussed on Abramsky's notion of *applicative bisimulation*, and hence, in our terminology, only on the operator Φ_{Eq_D}. The most important results which have been obtained in this direction are that all the six observational λ-theories presented in Section 4.1 can be characterized using the $\Phi_{\mathsf{Eq}_\sigma}$-coinduction principle, see (Egidi et al. 1992; Abramsky and Ong 1993; Howe 1996; Pitts 1996; Lenisa 1997; Lenisa 1997a; Lenisa 1998). We can rephrase these results by saying that the term models $\mathcal{T}_{\leq_\sigma}$, which are indeed λ-models (Lenisa 1998, Theorem 7.6.4), are $\Phi_{\mathsf{Eq}_\sigma}$-coinductive applicative structures, and hence, by Corollary 3.3, they can be viewed as strongly extensional $H_2^{\Phi_{\mathsf{Eq}_\sigma}}$-coalgebras.

The only exceptions to the hegemony of applicative coinduction principles in the literature appear in (Honsell and Lenisa 1995) and (Lenisa 1998). In the former, the authors introduced the coinduction operator Ψ_{Eq_D} and they proved, using a semantical technique, that $\mathcal{T}_{\leq_v}$ is an Eq_v-cartesian applicative structure. In (Lenisa 1998), it is proved, using a syntactical technique, that both $\mathcal{T}_{\leq_v}$ and $\mathcal{T}_{\leq_l}$ are cartesian.

However, general theorems which allow to extend the results for Φ_{Eq_D} to the other coinduction operators presented in Section 2 are far from obvious. In this section we show that large classes of applicative structures satisfying a Φ_{Eq_D}-coinduction principle, are also Eq_D-cartesian. The technique presented in this section is semantical, in Section 6 we shall discuss and extend the purely syntactical technique of (Lenisa 1998).

The semantical technique, called *Finitary Method*, applies to the *approximable applicative structures* introduced in Definition 2.13. The main result is that all $\Phi_{\mathsf{pEq}^0{}_D}$-

coinductive approximable applicative structures (see Defintion 2.14) are $\mathsf{Eq^0}_D$-cartesian. Where pEq^0_D and Eq^0_D are defined as follows:

Definition 5.1. *Let $\mathcal{D} = (D, \sqsubseteq_D, \bullet_D, \{p_n\}_{n\in\omega})$ be an aas.*

1 Let $\mathsf{pEq}^0_D \subseteq D \times D$ be defined by $\{(d,e) \in D \times D \mid d_0 \sqsubseteq_D e_0\}$;
2 Let $\mathsf{Eq}^0_D \subseteq D \times D$ be defined by $\{(d,e) \in D \times D \mid d_0 =_D e_0\}$.

In particular, the Finitary Method can be used to prove $\Phi_{\mathsf{Eq^0}_D}$-cartesianity of many λ-aas.

Notice that the equality relation $=_D$ on a pEq^0_D-caas satisfies immediately the $\Phi_{\mathsf{Eq}^0_D}$-applicative coinduction principle.

In the following proposition, we provide an alternative characterization of the partial order relation on a pEq^0_D-caas.

Proposition 5.2. *Let $(D, \sqsubseteq_D, \bullet_D, \{p_n\}_{n\in\omega})$ be a pEq^0_D-caas, then*

$$d \sqsubseteq_D e \iff \forall n \geq 0.\ \forall \vec{f} \in D^n.\ (d\vec{f})_0 \sqsubseteq_D (e\vec{f})_0 \ .$$

Proof. The implication ($\Leftarrow$) follows from the fact that $\{(d,e) \mid \forall \overrightarrow{f}.\ (d\overrightarrow{f})_0 \sqsubseteq_D (e\overrightarrow{f})_0\}$ is a $\Phi_{\mathsf{pEq}^0_D}$-bisimulation. The converse, i.e. ($\Rightarrow$), follows immediately from the fact that $d \sqsubseteq_D e \implies d\overrightarrow{f} \sqsubseteq_D e\overrightarrow{f}$. $\square$

Finally, we can give one of the crucial theorems of this paper:

Theorem 5.3. *Let $\mathcal{D} = (D, \sqsubseteq_D, \bullet_D, \{p_n\}_{n\in\omega})$ be a pEq^0_D-caas. Then $\mathcal{D}$ is a Eq^0_D-cartesian applicative structure,* i.e.

$$\approx^{\Psi_{\mathsf{Eq}^0_D}} = =_D \ .$$

Proof. The inclusion $=_D \subseteq \approx^{\Psi_{\mathsf{Eq}^0_D}}$ is immediate, since a $\Phi_{\mathsf{Eq}^0_D}$-bisimulation is a $\Psi_{\mathsf{Eq}^0_D}$-bisimulation. In order to show the reverse inclusion, i.e. $\approx^{\Psi_{\mathsf{Eq}^0_D}} \subseteq =_D$, we proceed as follows. Assume the contrary, i.e. $\exists d,e.\ d \approx^{\Psi_{\mathsf{Eq}^0_D}} e \ \wedge\ d \not\sqsubseteq_D e$. Suppose n is the least natural such that $d \approx^{\Psi_{\mathsf{Eq}^0_D}} e$ but $d_n \not\sqsubseteq_D e$. Then $n \geq 1$, by definition of $\approx^{\Psi_{\mathsf{Eq}^0_D}}$. Hence, by Proposition 5.2, there exist $f^1, \ldots, f^m \in D$, such that $(d_n f^1 \ldots f^m)_0 \not\sqsubseteq_D (ef^1 \ldots f^m)_0$. But, since $d \approx^{\Psi_{\mathsf{Eq}^0_D}} e$, there exist $g^1, \ldots, g^m$ such that $g^i \approx^{\Psi_{\mathsf{Eq}^0_D}} f^i$, for all i, and $dg^1 \ldots g^m \approx^{\Psi_{\mathsf{Eq}^0_D}} ef^1 \ldots f^m$; hence $(dg^1 \ldots g^m)_0 =_D (ef^1 \ldots f^m)_0$, and $(d_n f^1 \ldots f^m)_0 \not\sqsubseteq_D (dg^1 \ldots g^m)_0$. Assume for simplicity $m \leq n$ (the other case is dealt with similarly). By induction hypothesis, for all $i = 1, \ldots, m$, $(f^i)_{n-i} \sqsubseteq g^i$, hence we have $(d_n f^1 \ldots f^m)_0 =_D (d_n (f^1)_{n-1} \cdots (f^m)_{n-m})_0 \sqsubseteq_D (dg^1 \ldots g^m)_0$, which contradicts $(d_n f^1 \ldots f^m)_0 \not\sqsubseteq_D (dg^1 \ldots g^m)_0$. $\square$

Using the categorical framework of Final Semantics, the above theorem states that pEq^0_D-caas's can be construed as strongly extensional coalgebras for the functor $H^{\Psi_{\mathsf{Eq}^0_D}}_{|D_0|}$.

5.1. *Examples of cartesian approximable applicative structures*

In this section, we present many examples of $\mathsf{pEq}^0_D\lambda$-caas's.

5.1.1. *CPO λ-models.* Many "inverse limit" CPO λ-models studied in the literature, e.g. those mentioned in Section 4.1, are $\mathsf{pEq}^0_D\lambda$-caas. Therefore, by Theorem 5.3, they are Eq^0_D-cartesian applicative structures. Notice that, for $\sigma = \{l, v\}$, we have to take as D^σ_0 the 1-projection domain of D^σ, i.e. the two-element CPO $\{\bot, \lambda d.\ \bot\}$.

5.1.2. *Quotient λ-models.* Another interesting class of $\mathsf{pEq}^0_D\lambda$-caas arises from quotienting the *applicative and projective closure* of the *interior* of a suitable λ-aas. We recall that $\mathcal{I}^D$, the interior of a model $\mathcal{D}$, is the subset consisting of the denotations of $(\Lambda^C)^0$. The *applicative and projective closure* of the *interior* of a λ-model $\mathcal{D}$ can be easily defined using *indexed terms*, *i.e.* those terms generated by extending the set of constants with symbols to denote projections.

Definition 5.4. *The set of indexed λ-terms Λ^{C^+} is defined as follows:*

$$A \ ::= \ x \ \mid \ AA \ \mid \ \lambda x.A \ \mid \ c \ \mid \ c^{p_n} \ ,$$

where $x \in Var$, $c \in C$.
The set of closed indexed terms will be denoted by $(\Lambda^{C^+})^0$.

In the sequel we shall assume that $\mathcal{D} = (D, \sqsubseteq_D, \bullet_D, \{p_n\}_{n\in\omega}, [\![\ ]\!])$ is a λ-aas modeling the set of λ-terms Λ^C. We can easily extend it canonically to a λ-aas $\mathcal{D}^+$, which models Λ^{C^+}, by extending $[\![\ ]\!]^D$ on the new constants as follows: for all $n \in \omega$, $[\![c^{p_n}]\!]^{D^+}_\rho = p_n$. Then the applicative and projective closure of $\mathcal{I}^D$ is $\mathcal{I}^{D^+}$.

In the rest of this section, we will show how to quotient $\mathcal{I}^{D^+}$ by a suitable equivalence relation $\equiv^\omega$, and we will discuss conditions under which this quotient structure is a $\mathsf{pEq}^0\lambda$-caas. In effect, we start by introducing an appropriate relation, which will be shown to be a preequivalence relation in Proposition 5.9 below. Notice that, for all $n \geq 0$, we have that $\forall d.\ p_n d =_D ([\![\lambda x.x]\!]^D_\rho)_{n+1} d$.

Definition 5.5. *Let $\sqsubseteq^\omega_D \subseteq \mathcal{I}^{D^+} \times \mathcal{I}^{D^+}$ be the relation defined by*

$$a \sqsubseteq^\omega_D b \iff \forall n \geq 0.\ a_n \sqsubseteq^n_D b_n \ ,$$

where the relations $\sqsubseteq^n_D \subseteq (\mathcal{I}^{D^+} \cap D_n) \times (\mathcal{I}^{D^+} \cap D_n)$ are inductively defined as follows:

— $a \sqsubseteq^0_D b \iff a \sqsubseteq_D b$.
— $a \sqsubseteq^{n+1}_D b \iff (a_0 \sqsubseteq^0_D b_0 \ \wedge \ \forall c \sqsubseteq^n_D d.\ ac \sqsubseteq^n_D bd)$.

Let $\equiv^\omega_D$ be the relation $\sqsubseteq^\omega_D \cap (\sqsubseteq^\omega_D)^{-1}$.

The notion $\sqsubseteq^\omega_D$ above generalizes the notion of *call-by-value applicative bisimulation* introduced in (Egidi et al. 1992), and the corresponding notion used in (Abramsky and Ong 1993) for the lazy λ-calculus. In (Egidi et al. 1992), this relation was utilized for showing that the theory $\approx_v$ is Eq_v-applicative. Notice that it is not at all obvious that the relation $\sqsubseteq^\omega_D$ is a preequivalence and a congruence w.r.t. application. The proof of this (Proposition 5.9) and in particular of the fact that $\sqsubseteq^\omega_D$ is reflexive is rather involved. It generalizes the one carried out in (Egidi et al. 1992) for *call-by-value applicative bisimulation.* Once we have this, we can define a quotient λ-model by endowing $Q^+_D = (\mathcal{I}^{D^+} / \equiv^\omega_D)$ with the structure of a λ-aas (Corollary 5.10). Then, we will show that the relation $\sqsubseteq^\omega_D$ satisfies a $\Phi_{\mathsf{pEq}^0_{Q^+_D}}$-coinduction principle. Therefore the quotient structure Q^+_D is a $\mathsf{pEq}^0_{Q^+_D}$ λ-caas and hence, by Theorem 5.3, it is $\mathsf{pEq}^0_{Q^+_D}$-cartesian.

We start by isolating a crucial property of the relations $\sqsubseteq^n_D$.

Lemma 5.6. *$\forall n \geq 0,\ \forall a, b \in \mathcal{I}^{D^+}$, i) $a \sqsubseteq^{n+1}_D b \implies a_n \sqsubseteq^n_D b_n$; ii) $a \sqsubseteq^n_D b \implies a \sqsubseteq^{n+1}_D b$.*

Proof. The proof of i) and ii) is by mutual induction on n.
Base case: $a \sqsubseteq^1_D b \implies a_0 \sqsubseteq^0_D b_0$, by definition. $a \sqsubseteq^0_D b \implies a \sqsubseteq^1_D b$, by monotonicity of $\bullet_D$.
Induction step: Assume $a \sqsubseteq^{n+1}_D b$. For i), we have only to show that $a_n \sqsubseteq^n_D b_n$, i.e. $\forall c \sqsubseteq^{n-1}_D d.\ a_n c \sqsubseteq^{n-1}_D b_n d$. By induction hypothesis, $c \sqsubseteq^{n-1}_D d \implies c \sqsubseteq^n_D d$, and hence $ac \sqsubseteq^n_D bd$, by hypothesis. Again, by induction hypothesis, $ac \sqsubseteq^n_D bd \implies (ac)_{n-1} \sqsubseteq^{n-1}_D (bd)_{n-1}$, i.e. (by properties of projections) $a_n c \sqsubseteq^{n-1}_D b_n d$. To establish ii), we have only to show that $a \sqsubseteq^{n+2}_D b$, i.e. $\forall c \sqsubseteq^{n+1}_D d.\ ac \sqsubseteq^{n+1}_D bd$. If $c \sqsubseteq^{n+1}_D d$, then, by i), $c_n \sqsubseteq^n_D d_n$. Therefore, by hypothesis, $ac_n \sqsubseteq^n_D bd_n$. By induction hypothesis, $ac_n \sqsubseteq^{n+1}_D bd_n$, i.e., since $a, b \in \mathcal{I}^{D^+} \cap D_{n+1}$, $ac \sqsubseteq^{n+1}_D bd$. □

Lemma 5.7. *For all $a, b \in \mathcal{I}^{D^+}$,*

$$a \sqsubseteq^\omega_D b \iff a_0 \sqsubseteq^\omega_D b_0\ \wedge\ \forall c, d \in \mathcal{I}^{D^+}.\ (c \sqsubseteq^\omega_D d\ \Rightarrow\ ac \sqsubseteq^\omega_D bd)\ .$$

Proof. ($\Rightarrow$) This follows from the fact that in finitary models $ab =_D \bigsqcup_{n\in\mathbf{N}} a_n b$, and from the fact that $\sqsubseteq^\omega_D$ is inclusive, i.e., if $\forall n.\ a_n \sqsubseteq^\omega_D b_n$, then $a \sqsubseteq^\omega_D b$.
($\Leftarrow$) First of all, notice that, using Lemma 5.6, one can easily show that $\forall a, b \in \mathcal{I}^{D^+}_n.\ a \sqsubseteq^n_D b \iff a \sqsubseteq^\omega_D b$. Using this fact, one can easily show that $\forall n \geq 0.\ a_{n+1} \sqsubseteq^{n+1}_D b_{n+1}$. The thesis follows using the hypothesis $a_1 \sqsubseteq^1_D b_1 \implies a_0 \sqsubseteq^0_D b_0$. □

Lemma 5.8. *Let $D_{var} \cap \mathcal{I}^{D^+} \neq \emptyset$. Then, for all $M \in \Lambda^{C^+}$, for all $\rho, \rho' : Var \to (D_{var} \cap \mathcal{I}^{D^+})$,*

$$\rho \sqsubseteq^\omega_D \rho' \implies [\![M]\!]^D_\rho \sqsubseteq^\omega_D [\![M]\!]^D_{\rho'}\ ,$$

where $\rho \sqsubseteq^\omega_D \rho'$ denotes the fact that $\forall x \in Var.\ \rho(x) \sqsubseteq^\omega_D \rho'(x)$.

Proof. By structural induction. In particular, for the application case use Lemma 5.7($\Rightarrow$), for the abstraction case use Lemma 5.7($\Leftarrow$). □

Proposition 5.9. *Let $D_{var} \cap \mathcal{I}^{D^+} \neq \emptyset$. Then, the relation $\sqsubseteq^\omega_D$ is a preequivalence which is a congruence w.r.t. application.*

Proof. Reflexivity follows from Lemma 5.8, since the interpretation of closed terms in $(\Lambda^{C^+})^0$ does not depend on the environment, and, by assumption, there exist environments $\rho, \rho' : Var \to (D_{var} \cap \mathcal{I}^{D^+})$ such that $\rho \sqsubseteq^\omega_D \rho'$. Transitivity follows by showing, by induction on n, that, the relations $\sqsubseteq^n_D$ are all transitive. The fact that $\sqsubseteq^\omega_D$ is a congruence w.r.t. application follows from Lemma 5.7($\Rightarrow$). □

Finally we can define a λ-caas out of the quotient Q^+_D.

Corollary 5.10. *Let $\mathcal{D} = (D, \sqsubseteq_D, \bullet_D, \{p_n\}_{n\in\omega}, [\![\ ]\!])$ be a λ-aas. If $(\mathcal{I}^{D^+} / \equiv^\omega_D)$ is non-trivial, then the structure $(Q^+_D, \tilde{\sqsubseteq}^\omega_D, \bullet, \{[p_n]_{\equiv^\omega_D}\}_{n\in\omega}, [\![\ ]\!])$ is a λ-aas, where:*

— *$Q^+_D \equiv (\mathcal{I}^{D^+} / \equiv^\omega_D)$;*

— $\widetilde{\sqsubseteq}^{\omega}_D$ *is the partial order induced on the quotient by* $\sqsubseteq^{\omega}_D$*;*
— $[d]_{\equiv^{\omega}_D} \bullet [e]_{\equiv^{\omega}_D} = [de]_{\equiv^{\omega}_D}$ *, where* $[d]_{\equiv^{\omega}_D}$ *denotes the equivalence class of* d *modulo* $\equiv^{\omega}_D$*;*
— *for all* $\rho : Var \to (Q^+_D)_{var}$*,* $[\![M]\!]_\rho = [M[\tilde{\rho}(x_1)/x_1, \ldots, \tilde{\rho}(x_n)/x_n]]_{\equiv^{\omega}_D}$*, where* $FV(M) \subseteq \{x_1, \ldots, x_n\}$*, and* $\tilde{\rho}(x_i)$ *is a representative of the equivalence class* $\rho(x_i)$*.*

We can show that all non-trivial quotient structures are $\Phi_{pEq^0_{Q^+_D}}$-coinductive. To this end we need to give an alternative *applicative* characterization of the relation $\sqsubseteq^{\omega}_D$. But first we need the following lemma, which is easily proved by induction on n, using Lemma 5.6.

Lemma 5.11. *For all* $n \geq 0$*,*

$$a \sqsubseteq^n_D b \implies \forall \vec{a}.\ (a\vec{a})_0 \sqsubseteq_D (b\vec{a})_0\ .$$

Proposition 5.12.

$$a \sqsubseteq^{\omega}_D b \iff \forall \vec{a}.\ (a\vec{a})_0 \sqsubseteq_D (b\vec{a})_0\ .$$

Proof. ($\Rightarrow$) This follows by contradiction, using Lemma 5.11 and properties of application in an aas.
($\Leftarrow$) Suppose, by contradiction, that $\exists n.\ a_n \not\sqsubseteq^n_D b_n$. Then $n > 0$. By definition of $\sqsubseteq^n_D$, there exist $a^1_{n-1}, \ldots, a^n_0$, $b^1_{n-1}, \ldots, b^n_0$ such that $\forall i = 1, \ldots, n.\ a^i_{n-i} \sqsubseteq^{n-i}_D b^i_{n-i}\ \wedge$ $(a_n a^1_{n-1} \cdots a^n_0)_0 \not\sqsubseteq_D (b_n b^1_{n-1} \cdots b^n_0)_0$. Moreover, by Lemma 5.6, $a^i_{n-i} \sqsubseteq^{\omega}_D b^i_{n-i}$; then, since $\sqsubseteq^{\omega}_D$ is a congruence, we have $(b_n a^1_{n-1} \cdots a^n_0)_0 \sqsubseteq^{\omega}_D (b_n b^1_{n-1} \cdots b^n_0)_0$, and hence *a fortiori* $(a a^1_{n-1} \cdots a^n_0)_0 \not\sqsubseteq_D (b a^1_{n-1} \cdots a^n_0)_0$, which is a contradiction by Lemma 5.11. □

The proof of the following theorem can be easily derived from the proposition above.

Theorem 5.13. *Let* $(D, \sqsubseteq_D, \bullet_D, \{p_n\}_{n\in\omega}, [\![\ ]\!]^D)$ *be a* λ*-aas. If* Q^+_D *is non-trivial, then the structure* $(Q^+_D, \widetilde{\sqsubseteq}^{\omega}_D, \bullet, \{[p_n]_{\approx^{\omega}_D}\}_{n\in\omega}, [\![\ ]\!])$*, defined in Corollary 5.10, is a* $\mathsf{pEq}^0_{Q^+_D}$ λ*-caas.*

Using Theorem 5.3, we get immediately the following

Corollary 5.14. *Let* $\mathcal{D}$ *be a* λ*-aas. If* Q^+_D *is non-trivial, then the approximable applicative structure* $(Q^+_D, \widetilde{\sqsubseteq}^{\omega}_D, \bullet, \{[p_n]_{\equiv^{\omega}_D}\}_{n\in\omega})$ *is a* $\mathsf{pEq}^0_{Q^+_D}$*-cartesian applicative structure.*

Corollary 5.14 can be applied to the quotient λ-aas generated by all the models $\mathcal{D}^h$, $\mathcal{D}^l$, $\mathcal{D}^v$, $\mathcal{D}^o$, $\mathcal{D}^n$, and $\mathcal{D}^p$ of Section 4.1. Again, for $\sigma = \{l, v\}$, we have to take as D^{σ}_0 the 1-projection domain of D^{σ}, i.e. the two-element CPO $\{\perp, \lambda d.\ \perp\}$. Moreover, since projection functions are λ-definable in the models $\mathcal{D}^v$ and $\mathcal{D}^o$, we can use Corollary 5.14 also for showing that, for $\sigma = v, o$ the theories $\approx_\sigma$ are Eq_σ-cartesian. In fact, using the Computational Adequacy Theorems 4.11 and 4.15, one can check that $\mathsf{Eq}_\sigma = \mathsf{Eq}^0_{Q^+_{D^\sigma}}$.

6. The congruence candidate method

In this section, we present a purely syntactical technique for establishing Eq_σ-cartesianity of observational λ-theories $\approx_\sigma$. We recall that $\mathsf{Eq}_\sigma = \{(M, N) \in (\Lambda^C)^0 \times (\Lambda^C)^0 \mid M \Downarrow_\sigma \Leftrightarrow N \Downarrow_\sigma\}$. This method, which we call *congruence candidate method*, was introduced in (Lenisa 1998). It is inspired by the congruence candidate method used in (Howe 1989; Howe 1996), for showing that $\approx_\sigma$ for lazy strategies is $\Phi_{\mathsf{Eq}_\sigma}$-coinductive. A first extension of Howe's method is in (Lenisa 1997a) for showing $\Phi_{\mathsf{Eq}_\sigma}$-coinduction for strategies other than lazy (e.g. $\sigma = h, n, p$). The congruence candidate method will be used here for showing that the λ-theories $\approx_\sigma$, for $\sigma = h, l, v, p$, are Eq_σ-cartesian. In general, it can be applied to observational λ-theories, which satisfy the $\Phi_{\mathsf{Eq}_\sigma}$-coinduction and the technical condition $(*)$ of Theorem 6.5 below.

The congruence candidate method makes an essential use of the Eq_σ-cartesian coinduction principle itself. One starts out by defining a *candidate relation* on Λ^C, which is a congruence, and which includes the greatest $\Psi_{\mathsf{Eq}_\sigma}$-bisimulation $\approx^{\Psi_{\mathsf{Eq}_\sigma}}$. This candidate relation is then shown to be a $\Psi_{\mathsf{Eq}_\sigma}$-bisimulation. Hence the Eq_σ-cartesian coinduction principle guarantees that $\approx^{\Psi_{\mathsf{Eq}_\sigma}}$ itself is a congruence w.r.t. application and thence it coincides with $\approx^{\Phi_{\mathsf{Eq}_\sigma}}$. Since this method is very technical we outline the:

General pattern of the congruence candidate method:

— *Build a* candidate relation $\widehat{\approx}^{\Psi_{\mathsf{Eq}_\sigma}} \subseteq \Lambda^C \times \Lambda^C$ *such that*

1 $\widehat{\approx}^{\Psi_{\mathsf{Eq}_\sigma}} \supseteq \approx^{\Psi_{\mathsf{Eq}_\sigma}}$*;*

2 $\widehat{\approx}^{\Psi_{\mathsf{Eq}_\sigma}}$ *is a congruence w.r.t. application;*

3 $(\widehat{\approx}^{\Psi_{\mathsf{Eq}_\sigma}})_{|(\Lambda^C)^0}$ *is a* $\Psi_{\mathsf{Eq}_\sigma}$*-bisimulation.*

— *Use the* Eq_σ*-cartesian coinduction principle to deduce that* $\approx^{\Psi_{\mathsf{Eq}_\sigma}}$ *is a congruence w.r.t. application.*

We shall now present the congruence candidate method in general. The following Definitions and Lemmata build up to Theorem 6.5. In the next subsection we shall apply Theorem 6.5 to various strategies.

In order to discuss uniformly both call-by-value and eager strategies we introduce the following notation:

Notation Let $\rightarrow_\sigma$ be a strategy. We put:

$$\Delta_\sigma = \begin{cases} \{M \in (\Lambda^C)^0 \mid M \Downarrow_\sigma\} & \text{if } \forall M, N \in (\Lambda^C)^0.\ N \not\Downarrow_\sigma \Rightarrow MN \not\Downarrow_\sigma, \\ (\Lambda^C)^0 & \text{otherwise.} \end{cases}$$

First of all, we have to explain how to build the *candidate relation* $\widehat{\approx}^{\Psi_{\mathsf{Eq}_\sigma}}$. These are defined in terms of the extensions to open terms of $\Psi_{\mathsf{Eq}_\sigma}$-bisimulations:

Definition 6.1. *Let* $\mathcal{R}$ *be a* $\Psi_{\mathsf{Eq}_\sigma}$*-bisimulation. For all* $M, N \in \Lambda^C$ *such that* $FV(M, N) \subseteq \{x_1, \ldots, x_k\}$*, we define*

$M \, \mathcal{R}^{ext} \, N \iff$
$\forall P_1, \ldots, P_k \in \Delta_\sigma.\ \exists Q_1, \ldots, Q_k \in \Delta_\sigma.$
$(\forall i = 1, \ldots, k.\ P_i \, \mathcal{R} \, Q_i \ \wedge \ M[P_1/x_1 \ldots P_k/x_k] \, \mathcal{R} \ N[Q_1/x_1 \ldots Q_k/x_k]) \ \wedge$
$\forall Q_1, \ldots, Q_k \in \Delta_\sigma.\ \exists P_1, \ldots, P_k \in \Delta_\sigma.$
$(\forall i = 1, \ldots, k.\ P_i \, \mathcal{R} \, Q_i \ \wedge \ M[P_1/x_1 \ldots P_k/x_k] \, \mathcal{R} \ N[Q_1/x_1 \ldots Q_k/x_k])$.

In the sequel, by abuse of notation, we will simply denote $\mathcal{R}^{ext}$ by $\mathcal{R}$.

Definition 6.2. (candidate relation) *Let* $\mathcal{R} \subseteq \Lambda^C \times \Lambda^C$ *be a reflexive and transitive* $\Psi_{\mathsf{Eq}_\sigma}$*-bisimulation. Define the* candidate relation $\widehat{\mathcal{R}} \subseteq \Lambda^C \times \Lambda^C$ *by induction on* M *as follows:*

$$\frac{x\ \mathcal{R}\ N}{x\ \widehat{\mathcal{R}}\ N} \qquad \frac{M_1\ \widehat{\mathcal{R}}\ M_1' \quad M_2\ \widehat{\mathcal{R}}\ M_2' \quad M_1'M_2'\ \mathcal{R}\ N}{M_1M_2\ \widehat{\mathcal{R}}\ N}$$

$$\frac{c\ \mathcal{R}\ N}{c\ \widehat{\mathcal{R}}\ N} \qquad \frac{M\ \widehat{\mathcal{R}}\ M' \quad \lambda x.M'\ \mathcal{R}\ N}{\lambda x.M\ \widehat{\mathcal{R}}\ N}$$

Notice that the candidate relation is not simply the contextual closure of $\mathcal{R}$; this subtle definition of $\widehat{\mathcal{R}}$, originally due to Howe, is necessary to guarantee the crucial Substitutivity Lemma 6.4. The following lemma is an easy consequence of the definition of $\widehat{\mathcal{R}}$.

Lemma 6.3. *Let $\mathcal{R} \subseteq \Lambda^C \times \Lambda^C$ be a reflexive and transitive $\Psi_{\mathsf{Eq}_\sigma}$-bisimulation. Then:*
i) $\widehat{\mathcal{R}}$ is reflexive.
ii) $\mathcal{R} \subseteq \widehat{\mathcal{R}}$.
iii) $\widehat{\mathcal{R}}$ is a congruence w.r.t. application.
iv) $M\widehat{\mathcal{R}}M' \ \wedge\ M'\mathcal{R}N \implies M\widehat{\mathcal{R}}N$.

Lemma 6.4. (substitutivity) *Let $\mathcal{R} \subseteq \Lambda^C \times \Lambda^C$ be a reflexive and transitive $\Psi_{\mathsf{Eq}_\sigma}$-bisimulation. For all $M, M' \in \Lambda^C$, $N, N' \in \Delta_\sigma$,*

$$M\widehat{\mathcal{R}}M' \ \wedge\ N\widehat{\mathcal{R}}N' \implies M[N/x]\widehat{\mathcal{R}}M'[N'/x]\ .$$

Proof. By induction on the structure of M.

- $M \equiv x$: $\quad \dfrac{x\ \mathcal{R}\ M'}{x\ \widehat{\mathcal{R}}\ M'}$

$x\mathcal{R}M' \implies \exists P \in \Delta_\sigma.\ N'\mathcal{R}P \ \wedge\ P\mathcal{R}M'[N'/x]$, by definition of $\mathcal{R}$, and hence, by transitivity of $\mathcal{R}$, $N'\mathcal{R}M'[N'/x]$
$N\widehat{\mathcal{R}}N' \ \wedge\ N'\mathcal{R}M'[N'/x] \implies N\widehat{\mathcal{R}}M'[N'/x]$, from iv) of Lemma 6.3.

- $M \equiv c$: $\quad \dfrac{c\ \mathcal{R}\ M'}{c\ \widehat{\mathcal{R}}\ M'}$ $\quad$ this case is immediate.

- $M \equiv M_1M_2$: $\ \exists M_1', M_2'$ s.t. $\quad \dfrac{M_1\ \widehat{\mathcal{R}}\ M_1' \quad M_2\ \widehat{\mathcal{R}}\ M_2' \quad M_1'M_2'\ \mathcal{R}\ M'}{M_1M_2\ \widehat{\mathcal{R}}\ M'}$

By definition of $\mathcal{R}$, $\exists P \in \Delta_\sigma$ such that $N'\mathcal{R}P$ and $M_1'M_2'[P/x]\mathcal{R}M'[N'/x]$. In particular, from $N\widehat{\mathcal{R}}N'$ and $N'\mathcal{R}P$, we get $N\widehat{\mathcal{R}}P$. By induction hypothesis, $M_1[N/x]\widehat{\mathcal{R}}M_1'[P/x]$ and $M_2[N/x]\widehat{\mathcal{R}}M_2'[P/x]$. Hence:

$$\frac{M_1[N/x]\ \widehat{\mathcal{R}}\ M_1'[P/x] \quad M_2[N/x]\ \widehat{\mathcal{R}}\ M_2'[P/x] \quad M_1'M_2'[P/x]\ \mathcal{R}\ M'[N'/x]}{M_1M_2[N/x]\ \widehat{\mathcal{R}}\ M'[N'/x]}\ .$$

- $M \equiv \lambda y.M_1$: $\ \exists M_1'$ s.t. $\quad \dfrac{M_1\ \widehat{\mathcal{R}}\ M_1' \quad \lambda y.M_1'\ \mathcal{R}\ M'}{\lambda y.M_1\ \widehat{\mathcal{R}}\ M'}$

By definition of $\mathcal{R}$, $\exists P \in \Delta_\sigma$ such that $N'\mathcal{R}P$ and $(\lambda y.M_1')[P/x]\mathcal{R}M'[N'/x]$. In particular, from $N\widehat{\mathcal{R}}N'$ and $N'\mathcal{R}P$, we get $N\widehat{\mathcal{R}}P$. By induction hypothesis, $M_1[N/x]\widehat{\mathcal{R}}M_1'[P/x]$. Hence:

$$\frac{M_1[N/x]\ \widehat{\mathcal{R}}\ M_1'[P/x] \quad (\lambda y.M_1')[P/x]\ \mathcal{R}\ M'[N'/x]}{(\lambda y.M_1)[N/x]\ \widehat{\mathcal{R}}\ M'[N'/x]}\ .$$ $\square$

Thus, if we take $\mathcal{R}$ to be the equivalence $\approx^{\Psi_{\mathsf{Eq}_\sigma}}$, we get a relation $\widehat{\approx}^{\Psi_{\mathsf{Eq}_\sigma}}$, which, by ii) of Lemma 6.3, includes $\approx^{\Psi_{\mathsf{Eq}_\sigma}}$. Moreover, by iii) of the same lemma, it is a congruence w.r.t. application. In order to show that $\approx^{\Psi_{\mathsf{Eq}_\sigma}}$ is itself a congruence w.r.t. application, we prove that $(\widehat{\approx}^{\Psi_{\mathsf{Eq}_\sigma}})_{|(\Lambda^C)^0} = (\approx^{\Psi_{\mathsf{Eq}_\sigma}})_{|(\Lambda^C)^0}$. This is done using the $\Psi_{\mathsf{Eq}_\sigma}$-coinduction principle, by proving that $(\widehat{\approx}^{\Psi_{\mathsf{Eq}_\sigma}})_{|(\Lambda^C)^0}$ is a $\Psi_{\mathsf{Eq}_\sigma}$-bisimulation. In order to prove that $(\widehat{\approx}^{\Psi_{\mathsf{Eq}_\sigma}})_{|(\Lambda^C)^0}$ is a $\Psi_{\mathsf{Eq}_\sigma}$-bisimulation, it is sufficient to show that, for all $M, N \in (\Lambda^C)^0$,

$$M \widehat{\approx}^{\Psi_{\mathsf{Eq}_\sigma}} N \ \wedge\ M \Downarrow_\sigma \implies N \Downarrow_\sigma \ .$$

Hence we can state the following

Theorem 6.5. *If, for all $M, N \in (\Lambda^C)^0$,*

$$M \widehat{\approx}^{\Psi_{\mathsf{Eq}_\sigma}} N \ \wedge\ M \Downarrow_\sigma \implies N \Downarrow_\sigma \qquad (*)\ ,$$

then $\approx^{\Psi_{\mathsf{Eq}_\sigma}}$ is a congruence w.r.t. application.

6.1. *The congruence candidate method at work*

The validity of hypothesis $(*)$ of Theorem 6.5 depends on the particular strategy. In this section we show that hypothesis $(*)$ holds for all the observational congruences induced by the strategies $\to_l$, $\to_v$, $\to_h$, $\to_p$. First we need the following two lemmata. Notice that for the "non lazy" strategies h and p we refer to the alternative "lazy" axiomatizations $\Downarrow^0_\sigma$ introduced in Definitions 4.12 and 4.17. This is an essential ingredient for proving Theorem 6.9 below. Notice that, in (Lenisa 1997a; Lenisa 1998), the proof of the fact that the theory $\approx_\sigma$ is Φ_{Eq_D}-coinductive, for $\sigma \in \{h, n, p\}$, is based directly on $\Downarrow_\sigma$. But a straightforward application of a similar argument for cartesianity breaks down. This explains also why proving cartesianity for $\approx_n$ is problematic, since for this theory there is no immediate "lazy" axiomatization of convergence.

Lemma 6.6. *Let $\sigma \in \{l, v, h, p\}$. Then, for all $(\lambda x.M)N \in \Lambda^0$ and $N \in \Delta_\sigma$,*

$$(\lambda x.M)N \approx^{\Psi_{\mathsf{Eq}_\sigma}} M[N/x]\ .$$

Proof. It is easy to see that

$$(\lambda x.M)N \approx^{\Phi_{\mathsf{Eq}_\sigma}} M[N/x]\ .$$

The thesis follows from $\approx^{\Phi_{\mathsf{Eq}_\sigma}} \subseteq \approx^{\Psi_{\mathsf{Eq}_\sigma}}$. □

Lemma 6.7. *i) Let $\sigma \in \{h, p\}$. Then, for all $M, P \in \Lambda^0$*

$$M \Downarrow^0_\sigma P \ \Rightarrow\ M \approx^{\Psi_{\mathsf{Eq}_\sigma}} P\ .$$

ii) Let $\sigma \in \{l, v\}$. Then, for all $M \in \Lambda^0$

$$M \Downarrow_\sigma P \ \Rightarrow\ M \approx^{\Psi_{\mathsf{Eq}_\sigma}} P\ .$$

Proof. i) It is sufficient to prove that $M \approx^{\Phi_{\mathsf{Eq}_\sigma}} P$, and this follows from the correctness of $\to_\beta(\to_{\beta_{KN}})$ reduction.
ii) Similar to item i). □

First we prove that condition $(*)$ holds for the "lazy" strategies, namely:

Theorem 6.8. *Let $M, N \in \Lambda^0$, and let $\sigma \in \{l, v\}$. Then*

$$M \widehat{\approx}^{\Psi_{\mathsf{Eq}_\sigma}} N \ \wedge \ M \Downarrow_\sigma \lambda x.P \implies \exists Q.(N \Downarrow_\sigma \lambda x.Q \ \wedge \ P \widehat{\approx}^{\Psi_{\mathsf{Eq}_\sigma}} Q) \ .$$

Proof. The proof proceeds by induction of the derivation of $M \Downarrow_\sigma \lambda x.P$.

- $M \equiv \lambda x.P : \ \exists N'$ s.t. $\qquad \dfrac{P \widehat{\approx}^{\Psi_{\mathsf{Eq}_\sigma}} N' \quad \lambda x.N' \approx^{\Psi_{\mathsf{Eq}_\sigma}} N}{\lambda x.P \widehat{\approx}^{\Psi_{\mathsf{Eq}_\sigma}} N}$

From the definition of $\approx^{\Psi_{\mathsf{Eq}_\sigma}}$ it follows that there exists Q such that $N \Downarrow_\sigma \lambda x.Q$. By Lemma 6.6, $N \approx^{\Psi_{\mathsf{Eq}_\sigma}} \lambda x.Q$, hence, by transitivity of $\approx^{\Psi_{\mathsf{Eq}_\sigma}}$, $\lambda x.N' \approx^{\Psi_{\mathsf{Eq}_\sigma}} \lambda x.Q$. In particular, using again Lemma 6.6, it is easy to check that $N' \approx^{\Psi_{\mathsf{Eq}_\sigma}} Q$. In fact: in order to show $N' \approx^{\Psi_{\mathsf{Eq}_\sigma}} Q$, we have to show that $\forall P \exists R.\ P \approx^{\Psi_{\mathsf{Eq}_\sigma}} R \ \wedge \ N'[P/x] \approx^{\Psi_{\mathsf{Eq}_\sigma}} Q[R/x]$. But, $\lambda x.N' \approx^{\Psi_{\mathsf{Eq}_\sigma}} \lambda x.Q$ implies $\forall P \exists R.\ P \approx^{\Psi_{\mathsf{Eq}_\sigma}} R \ \wedge \ (\lambda x.N')P \approx^{\Psi_{\mathsf{Eq}_\sigma}} (\lambda x.Q)R$. Then the thesis follows from Lemma 6.6. Hence, from $P \widehat{\approx}^{\Psi_{\mathsf{Eq}_\sigma}} N'$ and $N' \approx^{\Psi_{\mathsf{Eq}_\sigma}} Q$, using iv) of Lemma 6.3, we get $P \widehat{\approx}^{\Psi_{\mathsf{Eq}_\sigma}} Q$.

- $M \equiv M_1 M_2 : \ \exists N_1, N_2$ s.t. $\qquad \dfrac{M_1 \widehat{\approx}^{\Psi_{\mathsf{Eq}_\sigma}} N_1 \quad M_2 \widehat{\approx}^{\Psi_{\mathsf{Eq}_\sigma}} N_2 \quad N_1 N_2 \approx^{\Psi_{\mathsf{Eq}_\sigma}} N}{M_1 M_2 \widehat{\approx}^{\Psi_{\mathsf{Eq}_\sigma}} N}$

We assume w.l.o.g. that $N_1 N_2 \in \Lambda^0$, otherwise one can apply the Substitutivity Lemma. We deal with the case $\sigma = l$, the other case is similar. Since $M_1 M_2 \Downarrow_l$, there exist P, P' such that

$$\frac{M_1 \Downarrow_l \lambda x.P' \quad P'[M_2/x] \Downarrow_l \lambda x.P}{M_1 M_2 \Downarrow_l \lambda x.P} \ .$$

By induction hypothesis, since $M_1 \widehat{\approx}^{\Psi_{\mathsf{Eq}_l}} N_1$ and $M_1 \Downarrow_l \lambda x.P'$, there exists Q' such that $N_1 \Downarrow_l \lambda x.Q'$ and $P' \widehat{\approx}^{\Psi_{\mathsf{Eq}_l}} Q'$. By the Substitutivity Lemma, $P'[M_2/x] \widehat{\approx}^{\Psi_{\mathsf{Eq}_l}} Q'[N_2/x]$. Hence, by induction hypothesis, there exists Q'' such that $Q'[N_2/x] \Downarrow_l \lambda x.Q''$ and $P \widehat{\approx}^{\Psi_{\mathsf{Eq}_l}} Q''$. Moreover, from $N_1 N_2 \approx^{\Psi_{\mathsf{Eq}_l}} N \wedge N_1 N_2 \Downarrow_l \lambda x.Q''$, using Lemma 6.7ii), $\exists Q.\ N \Downarrow_l \lambda x.Q \approx^{\Psi_{\mathsf{Eq}_l}} \lambda x.Q''$. But $\lambda x.Q'' \approx^{\Psi_{\mathsf{Eq}_l}} \lambda x.Q \implies Q'' \approx^{\Psi_{\mathsf{Eq}_l}} Q$, by definition of $\approx^{\Psi_{\mathsf{Eq}_l}}$. Finally, by Lemma 6.3iv), $P \widehat{\approx}^{\Psi_{\mathsf{Eq}_l}} Q'' \ \wedge \ Q'' \approx^{\Psi_{\mathsf{Eq}_l}} Q \implies P \widehat{\approx}^{\Psi_{\mathsf{Eq}_l}} Q$. □

Next we discuss the "eager" strategies, notice again the use of $\Downarrow^o_\sigma$.

Theorem 6.9. *Let $M, N \in \Lambda^0$, and let $\sigma \in \{h, p\}$. Then*

$$M \widehat{\approx}^{\Psi_{\mathsf{Eq}_\sigma}} N \ \wedge \ M \Downarrow^o_\sigma \lambda x.P \implies \exists Q.(N \Downarrow^o_\sigma \lambda x.Q \ \wedge \ P \widehat{\approx}^{\Psi_{\mathsf{Eq}_\sigma}} Q) \ .$$

Proof. The proof proceeds by induction of the derivation of $M \Downarrow^o_\sigma \lambda x.P$. For simplicitly, we work out in detail only the case of $\sigma = h$, the other case being similar.

- The only rule applied in the derivation of $M \Downarrow^o_h \lambda x.P$ is the first rule in Definition 4.12. Then M is solvable, i.e. $M \equiv \lambda x_1 \ldots x_n.x_i M_1 \ldots M_k$. Hence $\exists N_1, \ldots, N_k, N^0, \ldots, N^{k-1}$, $N'_1, \ldots, N'_n$ s.t.

$$\dfrac{\dfrac{\dfrac{\dfrac{x_i \approx^{\Psi_{\mathsf{Eq}_h}} N^0}{x_i \approxeq^{\Psi_{\mathsf{Eq}_h}} N^0} \quad M_1 \approxeq^{\Psi_{\mathsf{Eq}_h}} N_1 \quad N^0 N_1 \approx^{\Psi_{\mathsf{Eq}_h}} N^1}{x_i M_1 \approxeq^{\Psi_{\mathsf{Eq}_h}} N^1}}{\vdots}}{\dfrac{x_i M_1 \ldots M_{k-1} \approxeq^{\Psi_{\mathsf{Eq}_h}} N^{k-1} \qquad M_k \approxeq^{\Psi_{\mathsf{Eq}_h}} N_k \qquad N^{k-1} N_k \approx^{\Psi_{\mathsf{Eq}_h}} N'_n}{x_i M_1 \ldots M_k \approxeq^{\Psi_{\mathsf{Eq}_h}} N'_n}}$$

and

$$\dfrac{x_i M_1 \ldots M_k \approxeq^{\Psi_{\mathsf{Eq}_h}} N'_n \quad \lambda x_n . N'_n \approx^{\Psi_{\mathsf{Eq}_h}} N'_{n-1}}{\lambda x_n . x_i M_1 \ldots M_k \approxeq^{\Psi_{\mathsf{Eq}_h}} N'_{n-1}}$$

$$\vdots$$

$$\dfrac{\lambda x_2 \ldots x_n . x_i M_1 \ldots M_k \approxeq^{\Psi_{\mathsf{Eq}_h}} N'_1 \qquad \lambda x_1 . N'_1 \approx^{\Psi_{\mathsf{Eq}_h}} N}{\lambda x_1 \ldots x_n . x_i M_1 \ldots M_k \approxeq^{\Psi_{\mathsf{Eq}_h}} N}$$

In order to show that $\exists Q.\ N \Downarrow_h^o \lambda x_1.Q \ \wedge \ Q \approxeq^{\Psi_{\mathsf{Eq}_h}} \lambda x_2 \ldots x_n.x_i M_1 \ldots M_k$, it is sufficient to prove that $\lambda x_1.N'_1 \Downarrow_h^o$. Then the thesis follows using Theorem 4.13 and Lemma 6.3iv), and using the fact that $\lambda x_2 \ldots x_n.x_i M_1 \ldots M_k \approxeq^{\Psi_{\mathsf{Eq}_h}} N'_1$. Then we prove $\lambda x_1.N'_1 \Downarrow_h^o$.

$x_i \approx^{\Psi_{\mathsf{Eq}_h}} N^0 \implies x_i N_1 \approx^{\Psi_{\mathsf{Eq}_h}} N^0 N_1$

$x_i N_1 \approx^{\Psi_{\mathsf{Eq}_h}} N^0 N_1 \ \wedge \ N^0 N_1 \approx^{\Psi_{\mathsf{Eq}_h}} N^1 \implies x_i N_1 \approx^{\Psi_{\mathsf{Eq}_h}} N^1$

$\vdots$

$x_i N_1 \ldots N_k \approx^{\Psi_{\mathsf{Eq}_h}} N^{k-1} N_k \ \wedge \ N^{k-1} N_k \approx^{\Psi_{\mathsf{Eq}_h}} N'_n \implies x_i N_1 \ldots N_k \approx^{\Psi_{\mathsf{Eq}_h}} N'_n$.

Using Theorem 4.13, $N'_n \Downarrow_h^o \Rightarrow \lambda x_n.N'_n \Downarrow_h^o \Rightarrow N'_{n-1} \Downarrow_h^o \Rightarrow \ldots N'_1 \Downarrow_h^o \Rightarrow \lambda x_1.N'_1 \Downarrow_h^o$.

- $M \equiv \lambda x.P$ and M is not a head normal form then:

$\exists N'$ s.t. $\qquad \dfrac{P \approxeq^{\Psi_{\mathsf{Eq}_h}} N' \quad \lambda x.N' \approx^{\Psi_{\mathsf{Eq}_h}} N}{\lambda x.P \approxeq^{\Psi_{\mathsf{Eq}_h}} N}$

W.l.o.g., we assume that $\lambda x.N' \in \Lambda^0$, otherwise one can apply the Substitutivity Lemma. Since $\lambda x.P \Downarrow_h^o \lambda x.P$, then, by definition of $\Downarrow_h^o$, $\exists P'.\ P[P'/x] \Downarrow_h^o$. From $P \approxeq^{\Psi_{\mathsf{Eq}_h}} N'$ and $P' \approxeq^{\Psi_{\mathsf{Eq}_h}} P'$, by the Substitutivity Lemma, $P[P'/x] \approxeq^{\Psi_{\mathsf{Eq}_h}} N'[P'/x]$. Hence, by induction hypothesis, $N'[P'/x] \Downarrow_h^o$, and then $\lambda x.N' \Downarrow_h^o \lambda x.N'$ and $P \Downarrow_h^o N'$. But, since $\lambda x.N' \approx^{\Psi_{\mathsf{Eq}_h}} N$, then $\exists Q.\ N \Downarrow_h^o \lambda x.Q$ and, using Lemma 6.7i), $\lambda x.N' \approx^{\Psi_{\mathsf{Eq}_h}} \lambda x.Q$. Using Lemma 6.6, we get $N' \approx^{\Psi_{\mathsf{Eq}_h}} Q$. Finally, from $P \approxeq^{\Psi_{\mathsf{Eq}_h}} N'$ and $N' \approx^{\Psi_{\mathsf{Eq}_h}} Q$, we get $P \approxeq^{\Psi_{\mathsf{Eq}_h}} Q$.

- $M \equiv (\lambda x.M_1) M_2 \ldots M_k$: then, by hypothesis $\exists V$ s.t.

$$\dfrac{M_1[M_2/x] M_3 \ldots M_k \Downarrow_h^0 V}{(\lambda x.M_1) M_2 \ldots M_k \Downarrow_h^0 V} \quad k \geq 2$$

and $\exists N_1, \ldots, N_k, N^1, \ldots, N^{k-1}$ s.t.

$$\dfrac{\dfrac{M_1 \approxeq^{\Psi_{\mathsf{Eq}_h}} N_1 \quad \lambda x.N_1 \approx^{\Psi_{\mathsf{Eq}_h}} N^1}{\lambda x.M_1 \approxeq^{\Psi_{\mathsf{Eq}_h}} N^1} \qquad M_2 \approxeq^{\Psi_{\mathsf{Eq}_h}} N_2 \quad N^1 N_2 \approx^{\Psi_{\mathsf{Eq}_h}} N^2}{(\lambda x.M_1)M_2 \approxeq^{\Psi_{\mathsf{Eq}_h}} N^2}$$

$$\vdots$$

$$\dfrac{(\lambda x.M_1)M_2 \ldots M_{k-1} \approxeq^{\Psi_{\mathsf{Eq}_h}} N^{k-1} \qquad M_k \approxeq^{\Psi_{\mathsf{Eq}_h}} N_k \qquad N^{k-1} N_k \approx^{\Psi_{\mathsf{Eq}_h}} N}{(\lambda x.M_1)M_2 \ldots M_k \approxeq^{\Psi_{\mathsf{Eq}_h}} N}$$

Hence

$N^{k-2} N_{k-1} \approx^{\Psi_{\mathsf{Eq}_h}} N^{k-1} \ \wedge\ N^{k-1} N_k \approx^{\Psi_{\mathsf{Eq}_h}} N \implies N^{k-2} N_{k-1} N_k \approx^{\Psi_{\mathsf{Eq}_h}} N$

$N^{k-3} N_{k-2} \approx^{\Psi_{\mathsf{Eq}_h}} N^{k-2} \ \wedge\ N^{k-2} N_{k-1} N_k \approx^{\Psi_{\mathsf{Eq}_h}} N \implies N^{k-3} N_{k-2} N_{k-1} N_k \approx^{\Psi_{\mathsf{Eq}_h}} N$

$\vdots$

$N^1 N_2 \approx^{\Psi_{\mathsf{Eq}_h}} N^2 \ \wedge\ N^2 N_3 \ldots N_k \approx^{\Psi_{\mathsf{Eq}_h}} N \implies N^1 N_2 \ldots N_k \approx^{\Psi_{\mathsf{Eq}_h}} N$

$\lambda x.N_1 \approx^{\Psi_{\mathsf{Eq}_h}} N^1 \ \wedge\ N^1 N_2 \ldots N_k \approx^{\Psi_{\mathsf{Eq}_h}} N \implies (\lambda x.N_1) N_2 \ldots N_k \approx^{\Psi_{\mathsf{Eq}_h}} N.$

W.l.o.g., we assume that $(\lambda x.N_1)N_2 \ldots N_k \in \Lambda^0$, otherwise one can apply the Substitutivity Lemma. In order to show that $\exists Q.N \Downarrow_h^o \lambda x.Q \wedge P \approxeq^{\Psi_{\mathsf{Eq}_h}} Q$, it is sufficient to show that $\exists Q'.(\lambda x.N_1)N_2 \ldots N_k \Downarrow_h^o \lambda x.Q' \ \wedge\ P \approxeq^{\Psi_{\mathsf{Eq}_h}} Q'$. Then the thesis follows using Lemma 6.6 and Lemma 6.7i). Hence we need only to show that $\exists Q'.N_1[N_2/x] \ldots N_k \Downarrow_p^o \lambda x.Q' \ \wedge$ $P \approxeq^{\Psi_{\mathsf{Eq}_h}} Q'$. Using the Substitutivity Lemma, we get $M_1[M_2/x] \ldots M_k \approxeq^{\Psi_{\mathsf{Eq}_h}} N_1[N_2/x] \ldots N_k$. Finally, applying the induction hypothesis, we get the thesis. □

7. Set-theoretical applicative structures

In this section we discuss briefly set-theoretical applicative structures, and their connections with Eq_D-cartesian applicative structures. First of all, we introduce the crucial definition

Definition 7.1. *A set D is a* set-theoretical applicative structure over a set U_D, *if*

1 $U_D \cap (D \times D) = \emptyset$

2 $D \subseteq \mathcal{P}((D \times D) \cup U_D)$;

3 $\forall d_1, d_2 \in D.\ \exists! d_3 \in D.\ (d_2, d_3) \in d_1$.

In the above definition the set U_D plays the rôle of a set of *Urelementen* used for tagging different copies of the same graph. The condition on U_D allows to introduce the notion of set-theoretical applicative structure even in an atomless universe.

Definition 7.2. *A* functional *set-theoretical applicative structure is a set-theoretical applicative structure D such that $D \subseteq D^D$.*

The above definitions are justified by the following obvious fact:

Proposition 7.3. *Let D be a set-theoretical applicative structure. Then the structure $(D, \bullet_D)$, where*

$$d_1 \bullet_D d_2 = d_3 \iff (d_2, d_3) \in d_1 \ ,$$

is an applicative structure.

The following proposition shows that the existence of set-theoretical applicative structures is sensitive to the foundation/antifoundation properties of the universe.

Proposition 7.4.

1 The Foundation Axiom implies that there are no set-theoretical applicative structures.
2 BAFA *(see (Aczel 1988)) implies that all functional applicative structures are applicatively isomorphic to an extensional set-theoretical applicative structure.*
3 $\mathsf{ZF}^\circ\mathsf{X}_1$ *(see Appendix A) implies that there are no non-trivial functional set-theoretical applicative structures.*

Proof. 1) Immediate, since a set-theoretical applicative structure is a non-wellfounded set.
2) Immediate from the definition of BAFA.
3) Immediate from the strong extensionality of a universe satisfying X_1. □

Hence the theory of set-theoretical applicative structures is most interesting when the universe satisfies the Axiom X_1 of (Forti and Honsell 1983) (see Appendix A). In this case, the universe itself satisfies the well known *strong extensionality* property, which amounts exactly to the fact that the universe is a final $\mathcal{P}(\ )$-coalgebra (see Appendix A).

The following proposition illuminates on the connections between set-theoretical applicative structures and Ψ_{Eq_D}-cartesian coinduction principles in a universe satisfying X_1.

Proposition 7.5. *Assume* $\mathsf{ZF}^\circ\mathsf{X}_1$.

1 Any applicative structure of cardinality κ is applicatively isomorphic to a set-theoretical applicative structure D over a set U_D of cardinality κ.
2 Let D be a set-theoretical applicative structure over the set U_D. Then $(D, \bullet_D)$ is an Eq_D*-cartesian applicative structure, for* Eq_D *defined by* $d \ \mathsf{Eq}_D \ d' \Leftrightarrow d \cap U_D = d' \cap U_D$. *Moreover (D, id_D) is a strongly extensional $H^{\Psi_{\mathsf{Eq}_D}}_{|U_D|}$-coalgebra in Set*.*
3 Any Eq_D*-cartesian applicative structure is applicatively isomorphic to a set-theoretical applicative structure over a set U_D whose cardinality is the cardinality of the equivalence classes of* Eq_D.

Proof. Straightforward using the definitions. □

It is somewhat funny to point out that the very last proposition of this paper was actually what triggered the whole investigation carried out in the paper itself.

8. Concluding remarks

— In (Honsell and Lenisa 1995), we had given already a proof of the fact that the theory $\approx_v$ is Eq_v-cartesian. The proof which derives from the general method of Section 5 is conceptually simpler. The results concerning the strategies l and v, and techniques in Section 6 are essentially those of (Lenisa 1998).

— In this paper we did not address fully all possible natural questions which can arise in connection with the operator Ψ_{Eq_D}, let alone all the other operators of Section 2. For instance, one could ask whether $\approx_n$ is Eq_n-cartesian. Although we confidently conjecture that this is the case, the proof could be extremely technical, since in defining $\Downarrow^o_n$ for abstractions one should test termination on infinitely many closed terms.

Some interesting observations concerning the other operators can be made readily.

For example, the $\Phi^n_{\mathsf{Eq}_\sigma}$-coinduction principle, for $n > 1$, is clearly unsound for lazy strategies, but it is sound for those strategies which yield extensional term models such as h, n, o. More general results, however, seem extremely hard to establish.

Appendix A. Non-wellfounded sets

Non-wellfounded sets are elements of a Universe of a Zermelo Fräe nkel-like set-theory $\mathsf{ZF}^\circ\mathsf{X}_1$. $\mathsf{ZF}^\circ\mathsf{X}_1$ is the theory consisting of the axioms *extensionality*, *Pairing*, *Union*, *Power Set*, *Replacement*, *Infinity*, *Choice*, and the Antifoundation Axiom X_1 of (Forti and Honsell 1983) (or equivalently, the Antifoundation Axiom AFA of (Aczel 1988)).

Let V denote the Universe of sets (without atoms).

Definition 1. (X_1) *Let X be a set. For every function $f : X \to \mathcal{P}(X)$, there is a* unique *function $g : X \to V$ which makes the following diagram commute*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & \mathcal{P}(X) \arrow[dl, "g^+"] \\ V & \end{tikzcd}$$

I.e., for all $x \in X$, $g(x) = \{g(y) \mid y \in f(x)\}$.

It is interesting to point out that X_1 express precisely the fact that the universe V is *final* coalgebra for the functor $\mathcal{P}(\cdot)$.

The Antifoundation Axiom X_1 yields immediately a coinductive characterization of equality between sets, i.e. strong extensionality (Forti and Honsell 1983; Aczel 1988).

Proposition 2. (strong extensionality) *Two sets x, y are equal if and only if there exists a* Φ^+-bisimulation $\mathcal{R}$ *such that $x\ \mathcal{R}\ y$, where Φ^+ is the following operator on binary relations on the universe V:*

$$\Phi^+(\mathcal{R}) = \{(x, y) \mid \forall x_1 \in x.\ \exists y_1 \in y.\ x_1\ \mathcal{R}\ y_1\ \wedge\ \forall y_1 \in y.\ \exists x_1 \in x.\ x_1\ \mathcal{R}\ y_1\}\ .$$

Notice that Φ^+-bisimulations are called *id-admissible relations* in (Forti and Honsell 1983).

Appendix B. Coalgebraic description of coinduction

In this appendix we recall the categorical *coalgebraic* description of coinduction. This arises from the *Final Semantics Paradigm* introduced by Aczel (Aczel 1988; Aczel and Mendler 1989), and further developed by Rutten and Turi (Rutten and Turi 1993; Rutten 1996; Turi 1996).

Coalgebraically, coinduction principles for reasoning on the possibly circular and infinite objects of a data type X arise when X can be viewed as *F-coalgebra* for a suitable endofunctor $F : \mathcal{C} \to \mathcal{C}$.

Definition 3. *Let $F : \mathcal{C} \to \mathcal{C}$ be a functor. An F-*coalgebra *is a pair (X, α_X), where $\alpha_X : X \to F(X)$ is a morphism of $\mathcal{C}$.*
The F-coalgebras are the objects of a category whose morphisms between F-coalgebras

(X, α_X) and (Y, α_Y) are morphisms $f : X \to Y$ of the category $\mathcal{C}$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha_X"'] & Y \arrow[d, "\alpha_Y"] \\ F(X) \arrow[r, "F(f)"'] & F(Y) \end{tikzcd}$$

The categorical counterpart of the the set-theoretic notion of maximal fixpoint is the notion of *final F-coalgebra*.

The categorical counterpart of the set-theoretic notion of bisimulation is the notion of *F-bisimulation*. We give the definition of *F-bisimulation* in the category Set^*:

Definition 4. *An F-bisimulation on the F-coalgebras (X, α_X) and (Y, α_Y) is a set-theoretic relation $R \subseteq X \times Y$ such that there exists an arrow of $\mathcal{C}$, $\gamma : \mathcal{R} \to F(\mathcal{R})$, making the following diagram commute:*

$$\begin{tikzcd} X \arrow[d, "\alpha_X"'] & \mathcal{R} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\gamma"] & Y \arrow[d, "\alpha_Y"] \\ F(X) & F(\mathcal{R}) \arrow[l, "F(\pi_1)"] \arrow[r, "F(\pi_2)"'] & F(Y) \end{tikzcd}$$

Definition 5. (strong extensionality) *An F-coalgebra, (U, α_U), is* strongly F-extensional *if for all $u, u' \in U$,*

$$u = u' \iff \exists\, \mathcal{R}\ \ F\text{-}bisimulation\ on\ (U, \alpha_U).\ u\, \mathcal{R}\, u'\ .$$

We recall the crucial theorem of the Final Semantics Paradigm, which allows to characterize coinductively the equivalence induced by the unique F-coalgebra morphism into the final F-coalgebra (Aczel 1988; Rutten and Turi 1993):

Theorem 6. *Let F preserve weak pullbacks, and let (X, α_X) be an F-coalgebra. If there exists the final F-coalgebra (U, α_U), then the equivalence $\sim_f$ induced by the unique morphism $f : (X, \alpha_X) \to (U, \alpha_U)$ can be characterized as follows:*

$$\sim_f \;=\; \bigcup \{\mathcal{R} \mid \mathcal{R}\ is\ an\ F\text{-}bisimulation\ on\ (X, \alpha_X)\}\ .$$

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