Processes and Hyperuniverses*

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Abstract. We show how to define domains of processes, which arise in the denotational semantics of concurrent languages, using hypersets, i.e. non-wellfounded sets. In particular we discuss how to solve recursive equations involving set-theoretic operators within *hyperuniverses* with atoms. Hyperuniverses are transitive sets which carry a uniform topological structure and include as a clopen subset their exponential space (i.e. the set of their closed subsets) with the exponential uniformity. This approach allows to solve many recursive domain equations of processes which cannot be even expressed in standard Zermelo-Fraenkel Set Theory, e.g. when the functors involved have negative occurrences of the argument. Such equations arise in the semantics of concurrrent programs in connection with function spaces and higher order assignment. Finally, we briefly compare our results to those which make use of complete metric spaces, due to de Bakker, America and Rutten.

Introduction

In the Semantics of Concurrent Languages, the word *process* refers to the denotation of a concurrent program (or statement, or state), especially in those contexts where parallelism is reduced to non-determinism + sequentiality. Processes have been defined, or described, in the following three settings:
- *Complete Metric Spaces* (cms's) [5, 3];
- *Partial Orders* (cpo's) [12, 1, 11];
- *Non-Wellfounded Sets* (hypersets) [2, 13].

A completely satisfactory analysis of the expressive power of these approaches has not yet been carried out. This paper focuses on the description of processes as hypersets and on the relationship between semantics based on hypersets and semantics based on cms's.

Mathematical semantics should provide *compositional* interpretations. Compositionality is important in that it allows to factorize, i.e. decompose, the semantics of complex objects into that of their components. This, in turn, allows

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for the development of structured methods for reasoning about programs. Techniques based on cms's and cpo's have been widely used for defining compositional semantics for concurrent languages. *Banach's Fixed Point Theorem* and the *Least Fixed Point Theorem* are used, in cms's and cpo's settings respectively, to ensure the existence of solutions of recursive domain equations and well-posedness of definitions.

Since the seminal work of Aczel and Rutten [2, 13], also hypersets have been used for defining compositional semantics; the existence of domains of processes and well-posedness of definitions being ensured by some *Anti-Foundation Axiom*. This approach leads naturally to processes which give *branching-time semantics*, i.e. processes which represent abstract trees of computation steps. Hypersets belonging to universes which satisfy *strong extensionality properties* allow to define processes up to *bisimilarity*. This approach yields compositional semantics for languages whose operational semantics is specified in suitable formats ([13]), where *bisimilarity* of states is a *congruence* w.r.t. the syntactic operators. A definite advantage of using hypersets is that one can maintain the *naive set theoretic intuition* in construing processes. However the approach *à la* Aczel-Rutten is apparently weaker than those based on cpo's or cms's, when carried out in a Set Theory which abides by the *limitation of size principle*, e.g. *ZF*. In fact, there are simple domain equations of processes, involving function spaces, which cannot be solved using *small* hypersets; the reason being that *some interesting classes are not sets*. Such processes arise e.g. when dealing with language features like "second order assignment", see [4]. In order to maintain the set theoretic intuition and to overcome some of the current limitations of the existing hyperset approaches, we propose *hyperuniverses* as a *new setting* for defining processes as hypersets. In this paper we give evidence that the constructions usually carried out in the context of complete metric spaces can be recast more intuitively in the context of hypersets and we show that a very large class of domain equations of processes can be solved in this setting.

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1 Hyperuniverses

Hyperuniverses are models of Set Theories which *strongly contradict the Axiom of Foundation.* They have *strong extensionality* properties and *very strong comprehension* properties. They combine two lines of research: *Free Construction Principles (Antifoundation Axioms)* and *Comprehensive Theories of Sets.*

Research on *Antifoundation Axioms* goes back to Mirimanoff (10's), Scott (50's), Boffa (60's), who studied how to represent *arbitrary extensional relations* by the membership relation. A comprehensive treatment of these axioms based on a general *Free Construction Principle* appears in [8]. A very interesting axiom, first introduced there, is the axiom X_1, later popularized by Aczel as AFA ([2]), which allows to maintain the strong extensionality properties of wellfounded sets still providing a very rich collection of hypersets.

Research on *Comprehensive Theories* of sets originates with Frege's inconsistent formulation of the naive *General Comprehension Principle*, which allows the formation of the set $\{x \mid \phi(x)\}$ for any well-defined set-theoretic property ϕ. Russell's paradox arises instantiating ϕ with $x \notin x$. Set theories *à la* Zermelo (e.g. ZF) weaken the Comprehension Principle and substitute it by the *Separation Axiom*, which allows the formation of sets $\{x \in a \mid \phi(x)\}$, for any already existing set a. These theories have a very limited self-descriptive power, in that they do not even allow the existence of a universal set. *Comprehensive Theories of Sets* were investigated among others by Quine (30's), Church (50's), Malitz (70's). More recently Forti, Hinnion, Honsell and Weydert ([7, 9, 14]) considered a *very strong version* of comprehension consistent with extensionality, namely the *Generalized Positive Comprehension Principle*:

$GPC \qquad \{x \mid \phi(x)\} \quad$ *is a set, if* ϕ *is a* Generalized Positive Formula.[3]

The *Comprehension Schema GPC* is quite powerful (see [7]). *Inter alia*, it allows the formation of the *identity* $\Delta = \{(x,y) \mid x = y\}$, the *membership* $E = \{(x,y) \mid x \in y\}$, the *projections* π_1, π_2, the set-theoretic *inclusion* $\subseteq$, the *singleton* map $Sing$, the *universe* V, and, for any given x and y, also of the sets $\{x\}$, $\{x,y\}$, $x \cup y$, $x \cap y$, $x \times y$, $x \circ y$, $\bigcup x$, $\bigcap x$, $dom(x)$, $cod(x)$, x^{-1}, $\mathcal{P}(x)$, $\Diamond(x) = \{y \mid y \cap x \neq \emptyset\}$, $\widehat{x}(y) = \{z \mid \exists w \in y.(w,z) \in x\}$, $\mathcal{F}(x) = \{y \mid x \in y\}$, $x \star y = \{(u,(v,w)) \mid (u,v) \in x \ \wedge \ (u,w) \in y\}$.

Hyperuniverses are models of comprehensive theories of sets even stronger than GPC [7, 9, 14]. The following definition of hyperuniverse generalizes of the notion introduced in [10] to a Set Theory with *Urelements* (i.e. atoms).

Definition 1. Let N be a transitive set and let U be the set of Urelements of N. Then N is a *κ-hyperuniverse* if the following conditions hold:[4]
i) the complements of the elements of $N \setminus U$ are a κ-additive, κ-compact, 0-dimensional uniform topology;
ii) U is a clopen subset of N and $N \setminus U$ is the exponential space of N endowed with the exponential κ-uniformity (which induces the Vietoris κ-topology).

The above definition can be generalized by dropping either κ-compactness or 0-dimensionality or both. Notice that, if κ-compactness is not assumed, then the exponential κ-uniformity induces a *strictly finer* topology than Vietoris κ-topology; hence the universe lacks many useful comprehension properties.

We work in a Set Theory with atoms since they are needed in the applications for modeling atomic actions. More precisely we work within the Set Theory $ZFC_0^-(U)$, which consists of the following axioms: *Pairing, Union, Power Set,*

[3] The class of *Generalized Positive Formulæ GPF* is the least class of formulæ which includes the atomic formulæ $x \in y$, $x = y$, and is closed under the connectives $\wedge, \vee$, the quantifiers $\forall x, \exists x$, the bounded quantifiers $\forall x \in y, \exists x \in y$ and "pseudo-implication", i.e.: if $\phi \in GPF$, θ is a formula which can contain also the connective $\neg$ and $FreeVar(\theta) = \{x\}$, then $\forall x.(\theta \rightarrow \phi) \in GPF$.

[4] Throughout the paper κ denotes a regular cardinal. For definitions and concepts from the theory of uniform spaces we refer to Appendix A (see also[6]).

Replacement, Infinity, Choice, and Extensionality up to the set of atoms U. Since we want to consider structures which are *equal* and not simply isomorphic to their exponential space, we have to give up the Axiom of Foundation and assume, instead of it, the axiom FCU of *Unique Free Construction with respect to a set U of atoms*, which generalizes the Axiom X_1 of [8, 9] to Set Theories with atoms:

FCU *Let X be a set such that $X \cap U = \emptyset$. For every function $f : X \to \mathcal{P}(X \cup U)$ there is a unique function $g : X \to V$ verifying*

$$g(x) = (f(x) \cap U) \cup \{g(y) \mid y \in f(x) \cap X\}, \quad \forall x \in X.$$

Some antifoundation axioms yield interesting characterizations of equality between sets. In particular FCU yields the *Super Strong Extensionality Axiom up to the set of atoms U*:

$SSExtU$ *Two sets x, y are equal iff there is a bisimulation R s.t. xRy.*[5]

This axiom states that two sets are equal if and only if they belong to the greatest fixed point of the monotone operator $(\ )^+_U$, which corresponds to the *bisimilarity relation* in Concurrency, when atoms are taken as actions. In Non-wellfounded Set Theory this greatest fixed point is the *coarsest* equivalence consistent with extensionality. In Concurrency Theory, it is the *finest* equivalence considered, since bisimilar processes cannot be separated by observations.

2 The Non-wellfounded Approach

Natural *branching-time* compositional semantics can be given to a large class of concurrent languages in the non-wellfounded set theory $ZFC_0^-(U)+FCU$, see [2, 13]. A natural semantics, which assigns the same process to bisimilar states, is definable in fact for any *Labelled Transition System* $T = (S, A, \to)$ by a direct application of the Axiom FCU.

Theorem 1. *Let $T = (S, A, \to)$ be a LTS and let $A \subseteq U$. There is a unique model $\mathcal{M}_T : S \to V$ such that* $\mathcal{M}_T[\![s]\!] = \{(a, \mathcal{M}_T[\![s']\!]) \mid s \xrightarrow{a} s'\}$.

In [13], Rutten shows how this technique yields a compositional semantics when the labelled transition system is specified in a suitable format which ensures that *bisimilarity* on states is a *congruence*. In particular, processes are modeled as elements of the class $\overline{X}$ which is the greatest solution of the equation $X = \mathcal{P}(A \times X)$ in $ZFC_0^-(U)+FCU$, where the set of actions A consists of atoms. Assuming suitable "guardedness" conditions, an equivalent "branching time" semantics is definable also using metric spaces, see [5]. In this case the domain of processes is the solution of the equation $X \cong \mathcal{P}_{co}(A \times X_{\frac{1}{2}})$ in the category of complete metric spaces and non-distance increasing functions [3]. [6] Therefore, in the cases

[5] A *bisimulation* is a relation R s.t. $R \subseteq (R)^+_U$, where $(R)^+_U = \{(x, y) \mid x \cap U = y \cap U$ and $\forall t \in (x \setminus U).\exists s \in (y \setminus U).\ tRs$ and $\forall s \in (y \setminus U).\exists t \in (x \setminus U).\ sRt\}$.

[6] $\mathcal{P}_{co}(A \times X_{\frac{1}{2}})$ is the set of all closed subsets of the complete metric space $A \times X_{\frac{1}{2}}$, where $X_{\frac{1}{2}}$ is obtained from X by contracting the distance by a factor $\frac{1}{2}$.

dealt in [13], there is a clear-cut correspondence between processes for branching-time denotational semantics defined by metric techniques and processes defined in terms of hypersets in a Zermelo-like Set Theory. A compositional "branching time" semantics can still be given *à la* Aczel-Rutten, also to concurrent languages with complex features such as "process creation" and "sharing of variables". However, the equations which specify the processes used in the metric setting for modeling features such as "second order assignment", [4], cannot be *even expressed à la* Aczel-Rutten. In fact, one necessitates of a space of processes P which solves the system of equations

$SemState \cong (IVar \rightarrow^0 Val) \times (PVar \rightarrow^0 P)$

$P \cong \{p_0\} \cup (SemState \rightarrow^0 \mathcal{P}_{co}(SemState \times P_{\frac{1}{2}}))$.

The problem in mimicking this construction using hypersets is that class operators like $\Phi(X) = \ldots(X \rightarrow \ldots)\ldots$, where the argument occurrs in a negative position, are not well defined in the theory $ZFC_{\circ}^{-}(U) + FCU$, since otherwise proper classes would be elements. One should remark, however, that a less abstract semantics can be given also in this case *à la* Aczel-Rutten.

One of the goals of this paper is to show that even when dealing with "large" reflexive processes one is not really forced to give up the set theoretic intuition. In fact we can consider a very comprehensive set-theoretic framework, namely a *hyperuniverse*, i.e. a transitive universe where *many interesting classes can be safely taken to be sets.*

3 A Construction of κ-Hyperuniverses

In this section we outline the construction of a κ-hyperuniverse. This amounts essentially to truncating at a suitable cardinal κ the construction which extensionalizes a binary structure (see [8]).

Let $\kappa >\mid U \mid$ be a *strongly inaccessible weakly compact* cardinal. Define the equivalences $\approx_\alpha = (\approx_{U,\alpha} \cup \Delta_U)$ for all $\alpha \in Ord$ by

$\approx_{U,0} = (V \setminus U) \times (V \setminus U)$, $\approx_{U,\alpha+1} = (\approx_{U,\alpha})^+_U$, $\approx_{U,\lambda} = \bigcap_{\gamma<\lambda}(\approx_{U,\gamma})$ (λ limit).

The nested chain of equivalences $\{\approx_\alpha \mid \alpha < \kappa\}$ induces a κ-*ultrametric* δ_κ on the quotient set $\mathcal{N}^*_\kappa(U) \equiv V/\approx_\kappa$:[7]

put $\delta_\kappa(x, y) = 0$, if $x \approx_\kappa y$, and

$\delta_\kappa(x, y) = \epsilon_\alpha$, otherwise, where $\alpha = min\{\gamma \mid x \not\approx_\gamma y\}$;

here $< \epsilon_\alpha \mid \alpha < \kappa >$ is a suitable *strictly decreasing* κ-sequence , *converging* to 0 in any *non standard model* of the real numbers with cofinality κ (if $\kappa = \omega$, simply take $\epsilon_n = \frac{1}{2^n}$).

Consider on $\mathcal{N}^*_\kappa(U)$ the κ-*membership* relation $\in_\kappa$ defined by:

$$[x] \in_K [y] \quad \text{iff} \quad (\forall\alpha < \kappa \exists z \in y \quad x \approx_\alpha z) \vee (x \in y \cap U);$$

apply the axiom FCU to $(\mathcal{N}^*_K(U), \in_K)$ and get the isomorphic transitive structure $(\mathcal{N}_K(U), \in)$.

[7] it amounts to a *contraction* of Hausdorff's κ-distance corresponding to the discrete metric on U.

The metric, topological and set theoretic properties of the space $\mathcal{N}_\kappa(U)$ are summarized in the following proposition (see [7, 9, 10]):

Proposition 2. *Let $(\mathcal{N}_\kappa(U), \in)$ be the space constructed above. Then*
- $\mathcal{N}_\kappa(U)$ is a κ-metrizable κ-hyperuniverse.
- $\mathcal{N}_\kappa(U) \models GPC, FCU$, hence also SSExtU.
In particular

- *$\mathcal{N}_\kappa(U) \models$ pairing, power set, union, infinity, choice.*
- *The restrictions to $\mathcal{N}_\kappa(U)$ of the classes $\Delta, E, V, \pi_1, \pi_2, \subseteq$ are in $\mathcal{N}_\kappa(U)$.*
- *$\mathcal{N}_\kappa(U)$ is closed under the following operations:*
  $\bigcup(\,),\ (\,)\cap(\,),\ dom(\,),\ cod(\,),\ Sing(\,),\ (\,)^{-1},\ \widehat{(\,)}(\,),\ (\,)\star(\,),\ \Diamond(\,).$
- *$\mathcal{N}_\kappa(U)$ is closed under unions, subsets and products of size less than κ.*

Most of the *set theoretic* closure properties of κ-hyperuniverses follow from their *topological* properties, especially *κ-compactness.* In particular a function f belongs to $\mathcal{N}_\kappa(U)$ if and only if it is continuous with closed domain. Unfortunately $\mathcal{N}_\kappa(U)$ is not closed under the set x^y of continuous functions with domain y and range included in x (unless y is discrete). In fact the topology induced by Hausdorff's distance on graphs is *strictly coarser* than that induced by the supremum-norm. However the closure of the set of continuous functions $\overline{x^y}$ belongs to $\mathcal{N}_\kappa(U)$, as well as the sets of all *β-expansive* or *β-contractive* functions for any $\beta < \kappa$ (see Definition 2), hence, in particular, the set $x \to^0 y$ of all non-distance-increasing functions.

Surprisingly, ω-hyperuniverses are ubiquitous in Mathematics and Theoretical Computer Science. In particular, the special ω-hyperuniverse $\mathcal{N}_\omega(\emptyset)$ has been studied and manufactured in many different ways:

- $\mathcal{N}_\omega(\emptyset)$ is isomorphic to the *Stone space of the free modal Boolean algebra* on the empty set of generators ([11]).
- $\mathcal{N}_\omega(\emptyset)$ can be seen also as the *Cantor-1 space*, i.e. the union of Cantor's "discontinuum", obtained by the standard middle-third-removal construction, plus the centres of all the removed intervals.
- The *unique solution of the domain equation* $X \cong \mathcal{P}_{cl}(X_{\frac{1}{2}})$ in the setting of complete metric spaces ([5]), can be made into the hyperuniverse $\mathcal{N}_\omega(\emptyset)$ by an application of FCU, turning the isometry into the identity.
- Abramsky, Mislove, Moss and Oles [1, 11] investigated $\mathcal{N}_\omega(\emptyset)$ in the setting of cpo's, as the *space of maximal elements* of the solution in the category of SFP's of a suitable domain equation. For example Abramsky considered the equation $X \cong \mathcal{P}_P(X_\perp) \oplus 1_\perp$, where $\mathcal{P}$ denotes Plotkin's Power Domain constructor, $\oplus$ the coalesced sum and $1_\perp$ the lifted singleton cpo.

Taking into account the existence of Urelements, we have:

Proposition 3. *The ω-hyperuniverse $\mathcal{N}_\omega(U)$, U finite and discrete, is* uniformly isomorphic (isometric) *to the unique solution, in the category of cms's, of* $X \cong U \overline{\cup} \mathcal{P}_{cl}(X_{\frac{1}{2}})$, *where $\overline{\cup}$ denotes the disjoint union of metric spaces.*

The set U of atoms considered up to now is a discrete κ-compact space, hence of size less than κ. Sometimes, in the semantics of concurrent languages, more structured spaces of atoms are needed, e.g. for modeling "infinite actions", "real time", or "random choices". We can handle infinite discrete atomic actions or random choices already in $\mathcal{N}_\kappa(U)$, where U is discrete, for $\kappa > \omega$. This is an interesting application of such hyperuniverses to semantics. However, an alternative method, more general and more suitable for real time, say, is to embed isomorphically more structured spaces of urelements in a hyperuniverse. The following theorem of [9], which generalizes the construction above, is a first step in this direction.

Theorem 4. *If U carries a κ-*ultrametric *structure, then there exists a κ-ultrametric space $\mathcal{N}_K(U)$ such that U is a clopen subspace of $\mathcal{N}_K(U)$ and $\mathcal{N}_K(U) \setminus U$ is the exponential space of $\mathcal{N}_K(U)$, endowed with the exponential uniformity determined by the κ-ultrametric structure.*
Moreover, if κ is a strongly inaccessible weakly compact *cardinal and U is κ-compact (κ-bounded), then $\mathcal{N}_K(U)$ is κ-compact (κ-bounded).* [8]
Finally, if U is (ultra-)metric complete, then $\mathcal{N}_\omega(U)$ is (ultra-)metric complete; if U is κ-metrizable and $\kappa > \omega$, then $\mathcal{N}_K(U)$ is κ-ultrametrizable.

For instance, the theorem above implies that, choosing as set of Urelements the ultrametric space $U \equiv (\mathbf{N}, \delta)$ of natural numbers endowed with the discrete metric, the very same construction outlined before yields, for $\kappa = \omega$, a complete ultrametric space $\mathcal{N}_\omega(U)$. We believe that the atoms of a κ-hyperuniverse can carry a much more general topological structure, namely any *paracompact* topology. But this appealing mathematical generality does not seem up to now to have enough semantical motivation. Finally, we can stop the construction also at a *non-regular limit ordinal* $\lambda > \kappa = cof\lambda$. The transitive set $\mathcal{N}_\lambda(U)$ thus obtained cannot be a hyperuniverse, since κ-compactness is lost. However, we still obtain a complete metric space, when λ has countable cofinality:

Theorem 5. ([9]) *i) If U is an (ultra-)metric complete space and $cof\lambda = \omega$, then $\mathcal{N}_\lambda(U)$ is (ultra-)metric complete.*
ii) If either $\lambda > cof\lambda > \omega$ or λ is regular but not weakly compact, then $\mathcal{N}_\lambda(U)$ is not complete.

4 Fixed Point Theorems in κ-Hyperuniverses

In this section we show that κ-hyperuniverses are adequate frameworks for providing semantics for concurrent languages. In particular we give some general *Fixed Point Theorems* which allow to solve in this setting those recursive (domain) equations which are used for defining processes.

The first theorem can be seen as a generalization of Tarski's Fixed Point Theorem; in general it does not yield uniqueness of fixed points.

[8] *Caveat*: in the *non-κ-compact* case $\mathcal{N}_K(U)$ is not so rich in closure properties, e.g. it is not closed w.r.t. $\bigcup x$, $dom(x)$, etc..

Theorem 6. *Let $f \in \mathcal{N}_\kappa(U)$ be a function and let $\leq \in \mathcal{N}_\kappa(U)$ be a partial order such that f is monotone w.r.t. $\leq$. Put $D = \{x \in dom(f) \mid x \leq f(x)\}$ and assume that any λ-chain in D with $\lambda < \kappa$ has an upper bound in D. Then, for all $x \in D$, there is $y \geq x$ such that $y = f(y)$.*

Notice that theorem 6 applies whenever $\leq$ is κ-inductive on $dom(f)$, i.e. any λ-chain with $\lambda < \kappa$ has a l.u.b.. In case $\leq$ is $\subseteq$ ($\supseteq$), Theorem 6 gives exactly the translation of Tarski's Fixed Point Theorem into κ-hyperuniverses, namely:

Theorem 7. *If $f \in \mathcal{N}_\kappa(U)$ is a* monotone *function on the complete lattice of closed sets $\mathcal{N}_\kappa(U) \setminus U$, then f has a complete lattice of fixed points.*

Using the above theorems, a plethora of reflexive domain equations, involving many different set-theroretic operators, are immediatly solvable in the setting of κ-hyperuniverses. In the metric setting, where Banach's Fixed Point Theorem is the main tool, it is more complicated and problematic to handle many of these equations. In fact, in order to solve recursive domain equations, the contractiveness of the functors involved in the equations is needed (see [3]). E.g., in hyperuniverses, we can apply directly Theorem 7 to the following functions:

$F_1(X) = (X_0 \cup X) \times \mathcal{P}(X) \times \Diamond(X)$, $\qquad F_2(X) = X_0 \cup cod(X) \cup X^{-1}$.

Also Banach's Fixed Point Theorem [9] has a very significant analogue in the setting of ω-hyperuniverses. First we need to give the notion corresponding to the "k-Lipschitz" condition in the context of hyperuniverses:

Definition 2. Let $f \in \mathcal{N}_\kappa(U)$ be a function and let $\nu < \kappa$.
We say that f is *ν-contractive* if $\forall \alpha < \kappa \; (x \approx_\alpha y \;\Rightarrow\; f(x) \approx_{\alpha+\nu} f(y))$.
We say that f is *ν-expansive* if $\forall \alpha < \kappa \; (x \approx_{\alpha+\nu} y \;\Rightarrow\; f(x) \approx_\alpha f(y))$.

According to the above definition, 0-contractive (or equivalently 0-expansive) functions are exactly the non-distance-increasing functions w.r.t. the natural distance on $\mathcal{N}_\kappa(U)$. Therefore we shall denote with $x \to^0 y$ the set of non-distance-increasing functions with domain x and range included in y. Clearly, if a function f is α-contractive (α-expansive) then it is β-contractive for all $\beta < \alpha$ (β-expansive for all $\beta > \alpha$). Remark that, conversely, if α is a limit ordinal then β-contractiveness for all $\beta < \alpha$ is equivalent to α-contractiveness.

In the following theorem we use the notation of Section 1.

Theorem 8. *Let $f_1, f_2 \in \mathcal{N}_\kappa(U)$ be functions such that f_1 is ν_1-contractive and f_2 is ν_2-contractive. Put $\nu = min\{\nu_1, \nu_2\}$ for $\nu_1, \nu_2 < \kappa$; then*
- the function $f_2 \circ f_1$ is $(\nu_1 + \nu_2)$-contractive;
- the function $f_1 \star f_2$ is $(\nu + 2)$-contractive;
- the functions $f_1 \times f_2$, $f_1 \cup f_2$ and $f_1 \to^0 f_2$ are ν-contractive; [10]

[9] *if (X, δ) is a complete metric space and $f : X \to X$ is k-Lipschitz for $k < 1$ (i.e. $\delta(f(x), f(y)) \leq k\delta(x, y)\ \forall x, y \in X$), then f has a unique fixed point.*

[10] The functions $f_1 \times f_2$, $f_1 \cup f_2$ and $f_1 \to^0 f_2$ are defined pointwise. E.g. $(f_1 \to^0 f_2)(x) = f_1(x) \to^0 f_2(x), \;\; \forall x \in dom(f_1) \cap dom(f_2)$.

- *if ν is a successor ordinal or $\kappa = \omega$, then both $f_1 \times f_2$ and $f_1 \to^0 f_2$ are $(\nu + 2)$-contractive.*

Moreover, in every hyperuniverse:
- *Id, $(\ )^{-1}$ and all permutations and groupings of n-tuples are 0-contractive;*
- *$\mathcal{P}$, $\Diamond$, $\mathcal{F}$, Sing are 1-contractive;*
- *the constants functions are α-contractive for all $\alpha < \kappa$.*

Theorem 8 can be easily extended also to ν-expansive functions by considering *expansive degrees as negative contractive degrees.* Of special interest, in view of the applications of the following theorem, is the possibility of extending the additivity of contractive degrees under composition also to mixed cases: e.g. if f is ν-contractive, g is μ-expansive and $\nu \geq \mu$, then $g \circ f$ is $(\nu - \mu)$-contractive. Clearly, in $\mathcal{N}_\omega(U)$, 1-contractive functions are exactly all k-Lipschitz functions for $k < 1$. Thus we can give our analogue of Banach's Fixed Point Theorem:

Theorem 9. *Any* 1*-contractive function $f \in \mathcal{N}_\omega(U)$ such that $cod(f) \subseteq dom(f)$ has a unique fixed point.*

We conjecture that the theorem above should cover all cases which are dealt by America-Rutten in [3] using the properties of contractive and hom-contractive functors. On the ground of Theorem 8, the condition "f is 1-contractive" is satisfied by a large class, indeed, of functions $f \in \mathcal{N}_\omega(U)$. E.g. we can solve the equation $X = (X \to^0 X) \cup U$, since $Id \to^0 Id$ is 2-contractive.

5 Concluding Remarks

A first benefit deriving from working in hyperuniverses is that, much as in the non-wellfounded approach *à la* Aczel-Rutten, we can utilize directly naive set-theoretic intuitions; whereas more sophisticated notions and techniques from Category Theory are involved in the cpo and cms settings. Moreover we have two different techniques for solving reflexive domain equations of processes. The first, based on Theorems 8 and 9, works essentially in metric hyperuniverses and possibly generalizes and simplifies the technique of [3]. In fact, contraction coefficients are implicit, and domain equations do not mention contractivity explicilty. The second technique, based on Theorem 6, does not necessitate at all that the hyperuniverse have a metric structure, since it utilizes successive iterations of a "monotone" function w.r.t. a suitable partial order. There are equations, e.g. $X = \Diamond X$ or $X = \{f : \mathcal{N}_\kappa(U) \to^0 \mathcal{N}_\kappa(U) \mid \widehat{f}(X) \subseteq U\}$, which are solvable in an arbitrary hyperuniverse, but do not seem to be handled directly using the formulation of the technique [3].

The possibility of defining processes within the whole transfinite hierarchy of κ-hyperuniverses is not merely a mathematical generalization. In κ-hyperuniverses, for $\kappa > \omega$, we can discuss infinite sets of actions and define infinitely branching processes, without having to endow the space of actions with unnatural metrics. Moreover, we do not need to assume so called "guardedness" conditions in the definitions of recursive processes. Finally, these κ-hyperuniverses can provide, hopefully, more "elbow room" for dealing directly with fair processes.

We conclude the paper by raising some interesting and natural questions, which should deserve further investigation:

1. Can we prove that we recover in our framework all compact solutions of equations given using the technique of [3]?
2. Conversely, are there instances of Theorems 6 and 9 which cannot be dealt within a suitable category of Complete Metric Spaces?
3. Can the intuition that a hyperuniverse is a "universal domain of processes" be made precise?
4. Can we easily derive compositionality of semantics given using hypersets for other concurrent languages, besides those in [13]?

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Appendix A

Throughout this Appendix κ denotes a regular cardinal.

Definition 3. i) A tree T is *κ-ary* if any subchain of T has less than κ immediate successors;
ii) κ is *strongly inaccessible* and *weakly compact* if every κ-ary tree of size κ has a κ-branch.

We recall below some basic definitions from General Topology (see [6]).

Definition 4. i) A *filter* on X is a non empty set $\mathcal{F} \subset \mathcal{P}(X)$ such that:
a) $A \in \mathcal{F}$ and $B \supseteq A \quad \Rightarrow \quad B \in \mathcal{F}$;
b) $A, B \in \mathcal{F} \quad \Rightarrow \quad A \cap B \in \mathcal{F}$.
(The filter generated by a set S is the least filter containing S).
ii) A filter is *κ-complete* if it is closed under intersections of length less than κ.
iii) A *uniformity* on X is a filter $\mathcal{U}$ of *entourages*, i.e. symmetric and reflexive binary relations on X satisfying the following conditions:
a) $\bigcap \mathcal{U} = \Delta_X \equiv \{(x,y) \in X \times X \mid x = y\}$ (separation);
b) $\forall U \in \mathcal{U}\ \exists V \in \mathcal{U}.\ V \circ V \subseteq U$ (triangular inequality),
where $\circ$ is the composition of relations.
iv) The *topology* $\tau_{\mathcal{U}}$ induced by the uniformity $\mathcal{U}$ has as open sets $\{A \subseteq X \mid \forall x \in A\ \exists U \in \mathcal{U}\ \ U(x) \subseteq A\}$, where $U(x) = \{y \mid (x,y) \in U\}$.
v) A *κ-uniformity* is a uniformity which is *a κ-complete filter.* A topology is *κ-additive* (shortly a κ-topology) if the intersection of less than κ open sets is open. If $\mathcal{U}$ is a κ-uniformity then $\tau_{\mathcal{U}}$ is a κ-topology.
vi) A κ-uniformity $\mathcal{U}$ on X is *κ-bounded* if $\forall U \in \mathcal{U}\ \ \exists S \subseteq X$ such that $\mid S \mid < \kappa$ and $\{U(s) \mid s \in S\}$ is a cover of X. Notice that ω-bounded is usually called totally bounded.
vii) A *(κ-)uniformity basis* $\mathcal{B}$ is a set of entourages such that the filter generated by $\mathcal{B}$ is a (κ-)uniformity. The ($\kappa-$)uniformity *generated* by the ($\kappa-$)metric δ has a basis $\mathcal{B} = \{U_\alpha \mid \alpha < \kappa\}$, where $U_\alpha = \{(x,y) \in X \times X \mid \delta(x,y) < \epsilon_\alpha\}$.

Definition 5. i) A topology is *0-dimensional* if it has a basis of clopen (i.e. simultaneously closed and open) sets.
ii) A κ-topology is *κ-compact* if every open cover of X has a subcover of cardinality less than κ.
iii) The *exponential space* $\mathcal{P}_{cl}(X)$ of X is the space of all closed subsets of X.
iv) The *Vietoris (κ-)topology* on the exponential space $\mathcal{P}_{cl}(X)$ is the coarsest (κ-)topology such that $\mathcal{B}(A) = \mathcal{P}_{cl}(X) \cap \mathcal{P}(A)$ is open for every open set A of X and $\mathcal{B}(F)$ is closed for every closed set F of X. A subbasis for this topology is given by the sets $\mathcal{B}(A)$ and $\mathcal{D}(A) = \{C \in \mathcal{P}_{cl}(X) \mid A \cap C \neq \emptyset\}$, for A open.
v) The *exponential uniformity* $\mathcal{U}^+$ on the exponential space $\mathcal{P}_{cl}(X)$ corresponding to the κ-uniformity $\mathcal{U}$ on X has a basis consisting of the sets

$$U^+ = \{(A,B) \in \mathcal{P}_{cl}(X) \times \mathcal{P}_{cl}(X) \mid (\forall x \in A \exists y \in B.(x,y) \in U) \wedge (\forall y \in B \exists x \in A.(y,x) \in U)\}, \text{ where } U \in \mathcal{U}.$$

$\mathcal{U}^+$ is a κ-uniformity whenever $\mathcal{U}$ is.

Appendix B

We sketch the proofs of theorems 1, 6 and 8.

Proof of theorem 1.

Existence and uniqueness of $\mathcal{M}_T[\![\ ]\!]$ follow by applying FCU to
$f : X \cup \mathcal{P}(X \cup A) \cup \mathcal{P}^2(X \cup A) \to \mathcal{P}(X \cup \mathcal{P}(X \cup A) \cup \mathcal{P}^2(X \cup A) \cup A)$,
where $X = \{x_s \mid s \in S\}$,
$f_{|X}(x_s) = \{(a, x_{s'}) \mid s \xrightarrow{a} s'\}$,
$f_{|\mathcal{P}(X \cup A) \cup \mathcal{P}^2(X \cup A)} = id_{|\mathcal{P}(X \cup A) \cup \mathcal{P}^2(X \cup A)}$
((x, y) denotes a Kuratowski pair, i.e. it is a shorthand for $\{\{x\}, \{x, y\}\}$).
The axiom $SSExtU$ guarantees that we model programs up to bisimilarity.

Proof of theorem 6.

Pick $x \in D$ and define the κ-sequence $\{x_\alpha\}_{\alpha<\kappa}$ by $x_0 = x$, $\ x_{\alpha+1} = f(x_\alpha)$, and, for limit $\lambda < \kappa$, $\quad x_\lambda = y$ for some $y \in D$ s.t. $y \geq x_\gamma\ \forall \gamma < \lambda$.
By κ-compactness there is a κ-subsequence $\{x_{i_\alpha}\}_{\alpha<\kappa}$ converging to $\overline{x}$. By hypothesis $x_{i_\alpha} \leq f(x_{i_\alpha}) = x_{i_\alpha+1} \leq x_{i_{\alpha+1}}$, for all $\alpha < \kappa$. Since f is continuous, we have $\overline{x} \leq f(\overline{x}) \leq \overline{x}$.

Proof of theorem 8.

All the assertions, but the one concerning $f_1 \to^0 f_2$, need only straightforward computations, which we omit. In order to show that, if $x \approx_\alpha y$, then $(f_1 \to^0 f_2)(x) \approx_{\alpha+\nu} (f_1 \to^0 f_2)(y)$, we assume first that $\nu = \mu + 1$ is a successor ordinal. We fix a system of representatives for the equivalence classes modulo $\approx_{\alpha+\nu}$, then we pick $g \in (f_1 \to^0 f_2)(x)$, i.e. $g : f_1(x) \to^0 f_2(x)$ and we define $h : f_1(y) \to^0 f_2(y)$ as follows. Given $t \in f_1(y)$ we pick $s \in f_1(x)$ such that $t \approx_{\alpha+\mu} s$, which exists since $f_1(x) \approx_{\alpha+\nu_1} f_1(y)$. Now we put $h(t) = z$, where z is the representative in $f_2(y)$ of the class of $g(s)$ modulo $\approx_{\alpha+\mu}$, which exists since $f_2(x) \approx_{\alpha+\nu_2} f_2(y)$. The function h so defined is constant on the equivalence classes modulo $\approx_{\alpha+\mu}$, since g is 0-contractive. Therefore h is itself 0-contractive: points which are $\approx_{\alpha+\mu}$-equivalent have equal images, whereas points which are $\approx_\beta$ for $\beta < \alpha + \mu$ are mapped onto $\approx_\beta$-equivalent points, since g is 0-contractive and $f_i(x) \approx_{\alpha+\nu_i} f_i(y)$ for $i = 1, 2$. The fact that $h \approx_{\alpha+\nu+1} g$ is immediate from our choice of h, taking into account that Kuratowski's pairs have the property that if $u \approx_\gamma u'$ and $v \approx_\gamma v'$ then $(u, v) \approx_{\gamma+2} (u', v')$.

When $\nu > 0$ is a limit ordinal, the proof above can be readily modified taking into acount that $\approx_{\alpha+\nu}$ is $\bigcap_{\beta<\nu} \approx_{\alpha+\beta+2}$. When $\nu = 0$ and α is a successor, again we can repeat the same argument as above, taking the predecessor of α instead of $\alpha + \mu$. The remaining case $\nu = 0$ and α limit follows again, utilizing the fact that $\approx_\alpha$ is $\bigcap_{\beta<\alpha} \approx_{\beta+2}$.