Strict Geometry of Interaction Graph Models*

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Abstract. We study a class of *"wave-style" Geometry of Interaction* (GoI) λ-models based on the category *Rel* of sets and relations. Wave GoI models arise when Abramsky's GoI axiomatization, which generalizes Girard's original GoI, is applied to a traced monoidal category with the *categorical product* as tensor, using "countable power" as the traced strong monoidal functor !. Abramsky hinted that the category *Rel* is the basic setting for traditional denotational "static semantics". However, *Rel*, together with the cartesian product, apparently escapes Abramsky's original GoI construction. Here we show that *Rel* can be axiomatized as a *strict GoI situation*, i.e. a strict variant of Abramsky's GoI situation, which gives rise to a rich class of *strict graph models*. These are models of *restricted* λ-calculi in the sense of [HL99], such as Church's λ-I-calculus and the $\lambda\beta_{KN}$-calculus.
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Introduction

In [Abr96], Abramsky provides a categorical axiomatization/generalization of Girard's *Geometry of Interaction* (GoI) [Gir89], embracing previous axiomatic approaches, such as that based on *dynamic algebras* [DR93,DR95] and the one in [AJ94]. This generalization is based on traced monoidal categories, [JSV96], and it consists in building a compact closed category $\mathcal{G}(\mathcal{C})$ (*GoI category*) starting from a traced symmetric monoidal category $\mathcal{C}$. In [AHS02], the construction is extended to exponentials, which, in a general categorical setting, are captured by a strong monoidal functor ! on the traced category $\mathcal{C}$, together with some additional structure. Under these conditions on $\mathcal{C}$, the GoI category $\mathcal{G}(\mathcal{C})$ is a *weak linear category* (WLC), i.e. a weakening of a linear category (see [BBPH92]). Moreover, every reflexive object in a WLC gives rise to a *linear combinatory algebra* (LCA).

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Following [Abr96], there are two main instantiations of the GoI axiomatization. In the "particle-style" GoI, the tensor on the underlying category is a coproduct and the strong monoidal functor is a *countable* copower. Girard's GoI is an instance of this. Composition in the GoI category can be intuitively understood by simulating the flow of a particle around a network. Dually, in the "wave-style" GoI, the tensor is a product and the strong monoidal functor is a *countable* power. Composition in the GoI category is defined now statically and globally. In [Abr96], the category $(Rel, +)$ was suggested as the "basic setting" for particle/*dynamic semantics*, while $(Rel, \times)$ as the "basic setting" for wave/*static semantics*. This is clearly the case for the former which underlies many game categories in the [AJM00]-style, [Abr96], and contains as subalgebras those fruitfully used in [AL00,AL00a,AL01]. On the other hand, the thesis that $(Rel, \times)$ is the basic setting for static semantics is less immediate, also because $(Rel, \times)$ itself escapes the original GoI axiomatization.

The connection between traditional (static) semantics and wave GoI has not received much attention in the literature, apart from the investigations of some special wave-style models in [BDER97,AM99]. In the present paper and in [HL03], a companion paper to the present one, this connection has been taken seriously, and categories of relations in the wave-style have been explored, *vis-à-vis* graph models. In particular, in [HL03], a weaker axiomatization of the *GoI situation* of [AHS02] has been introduced, which still ensures that a GoI category is a WLC. This axiomatization allows to capture the case of Rel^*, i.e. the category of pointed sets and relations preserving the distinguished point, which otherwise escapes the original axiomatization. GoI algebras arising from Rel^* have then been shown to amount to (a variant of) graph models à la Scott-Plotkin-Engeler, [Sco80,Plo93].

In this paper, we focus directly on the basic setting for wave GoI, i.e. $(Rel, \times)$. This apparently escapes both the original axiomatization of the GoI situation of [AHS02] and the weaker axiomatization of [HL03]. In order to recover the case of *Rel*, we introduce a *strict* variant of the GoI situation. LCAs arising from a strict GoI situation are strict, i.e. application is *strict* w.r.t. a distinguished element $\perp$, and only a *restricted* form of K combinator is available, which allows to erase only non-$\perp$ elements. In particular, we show that GoI algebras on $(Rel, \times)$ are *strict graph models*. These are models of *restricted* λ-calculi, such as the λI-calculus and the $\lambda\beta_{KN}$-calculus of [HL99].

The paper consists of three parts.

In the first part (Section 1), we introduce strict versions of (linear) combinatory algebras, (linear) combinatory λ-models, and (linear) graph models, and we study the relations between them. In particular, we show that the somewhat unexpected fact that every graph model (GM) can be recovered from a linear graph model (LGM) via standardisation, proved in [HL03], extends also to the strict case. An important consequence is that all strict λ-theories induced by strict GM's can be captured by strict LGMs.

In the second part of the paper (Section 2), we study strict LGMs from a categorical point of view. In particular, we introduce the notion of *order-enriched* *strict* WLC, and we show that many strict LGMs can be captured as strict WLCs in the wave-style.

In the third part of the paper (Sections 3 and 4), we study *strict* wave GoI algebras. In particular, we introduce a strict variant for the axiomatization of the GoI situation of [AHS02]. Our axiomatization ensures that a GoI category is a *strict* WLC, and hence it allows to build a *strict* GoI algebra. This axiomatization allows us to capture the case of $(Rel, \times)$. In particular, we show that the class of GoI algebras induced by $(Rel, \times)$ are strict LGMs. Finally, we show that there are wave GoI models realizing non-sensible λ-theories, as was the case for Rel^*. This should be contrasted with the fact that, in [DFH99], it has been shown that game models capture only a very limited number of λ-theories, related to Böhm trees and Levy-Longo trees.

Basic categorical definitions used in the paper appear in Appendix A.

Notation. Let $\mathcal{P}^-_{<\kappa}$, for a cardinal κ, denote the powerset of all non-empty sets with cardinality less than κ. In particular, $\mathcal{P}^-_{<\omega}$ denotes the finite non-empty powerset. Let $\subseteq_{fne}$ denote finite non-empty subset inclusion. Let U, V be objects in a category $\mathcal{C}$. We denote by $\tau : U \lhd V : \tau'$ a retraction of U in V, i.e. $\tau' \circ \tau = id_U$. Let Rel be the category of sets and relations. Relations $f \subseteq A \times B$ will be also denoted by $f : A \multimap\!\!\!\!\rightarrow B$. Let $PO_\perp$ denote the category of partial orders with bottom element and monotone maps as morphisms. The sets of finite streams and the set of infinite streams on a set A will be denoted by $A_{<\omega}$ and A_ω, respectively. Streams in A_ω will be denoted by $\boldsymbol{a}, \boldsymbol{b}, \ldots$ The i-th component of a (finite) stream $\boldsymbol{a}$ is denoted by a_i; a^ω denotes the infinite stream whose components are all equal to a.

1 Strict Linear Combinatory Algebras and Graph Models

In this section, we introduce *strict* versions of *(linear) combinatory algebras*, *(linear) combinatory λ-models*, and *(linear) graph models*, and we study the relations between them. In particular, we show that every strict graph model is induced by a strict linear one.

We start by defining the notion of *strict combinatory algebra* (in the Curry style), which arises from a *strict applicative structure*, and where only a restricted form of the K combinator is available, which allows to erase only non-$\perp$ elements:

Definition 1 (Strict Combinatory Algebra). *i) A* strict applicative structure *is an applicative structure $(A, \cdot)$ with a distinguished element $\perp \in A$ and a* strict *notion of application, i.e. $x \cdot \perp = \perp$ and $\perp \cdot x = \perp$, for all $x \in A$.*
ii) A strict combinatory algebra (strict CA) *$\mathcal{A} = (A, \cdot)$ is a strict applicative structure together with distinguished elements (combinators) B, C, I, K, W satisfying the following equations (we associate $\cdot$ to the left): for all $x, y, z \in A$,*

$Bxyz = x(yz)$	*(Composition, Cut)*	$Cxyz = (xz)y$	*(Exchange)*
$Ix = x$	*(Identity)*	$Wxy = xyy$	*(Contraction)*
$Kxy = x$, *if* $y \neq \perp$	*(Restricted Weakening)*		

The notion of linear combinatory algebra of Abramsky refines the notion of combinatory algebra, in that it has an extra unary operation ! and a set of combinators, refining Curry's original set of combinators. Strict linear combinatory algebras are the linear counterpart of strict CA's:

Definition 2 (Strict Linear Combinatory Algebra). *A* strict linear combinatory algebra (strict LCA) $\mathcal{A} = (A, \cdot, !)$ *is a strict applicative structure* $(A, \cdot)$ *with a unary (injective) operation ! which is* strict, *i.e.* $!\perp = \perp$, *and combinators* $B, C, I, K, W, D, \delta, F$ *satisfying the following equations (we associate* $\cdot$ *to the left and we assume ! to have order of precedence greater than* $\cdot$*): for all* $x, y, z \in A$,

$Bxyz = x(yz)$	*(Composition, Cut)*	$Cxyz = (xz)y$	*(Exchange)*
$Ix = x$	*(Identity)*	$Kx!y = x$, *if* $y \neq \perp$	*(Restricted Weakening)*
$Wx!y = x!y!y$	*(Contraction)*	$D!x = x$	*(Dereliction)*
$\delta!x = !!x$	*(Comultiplication)*	$F!x!y = !(xy)$	*(Monoidal Functoriality)*

The *standardisation* procedure for building a standard combinatory algebra (CA) from a LCA extends to the strict case:

Proposition 1 (Standardisation). *Let* $(A, \cdot, !)$ *be a strict LCA. Then* $(A, \cdot_s)$, *where* $x \cdot_s y \triangleq x \cdot !y$, *is a strict CA with combinators* B_s, C_s, I_s, K_s, W_s *defined by:*

$C_s \triangleq D' \cdot C \qquad B_s \triangleq C \cdot (B \cdot (B \cdot B \cdot B) \cdot (D' \cdot I)) \cdot (C \cdot ((B \cdot B) \cdot F) \cdot \delta)$

$I_s \triangleq D' \cdot I \qquad K_s \triangleq D' \cdot K \qquad W_s \triangleq D' \cdot W$,

where $D' \triangleq C(BBI)(BDI)$ *is such that, for all* x, y, $D'x!y = xy$.

It is well known that λ-models à la Hindley-Longo can be characterized as *combinatory λ-models*, i.e. CAs with an extra *selector* combinator (see e.g. [Bar84]). One can give a notion of strict combinatory λ-model arising from a strict CA:

Definition 3 (Strict Combinatory λ-model). *A strict CA* $\mathcal{A} = (A, \cdot)$ *is a* strict combinatory λ-model *if there exists an extra* selector *combinator* ϵ *such that, for all* $x, y \in A$,

$$\epsilon xy = xy \quad \textit{and} \quad (\forall z.\ xz = yz) \Rightarrow \epsilon x = \epsilon y \ .$$

Here is the strict version of the notion of linear combinatory λ-model introduced in [HL03]:

Definition 4 (Strict Combinatory Linear λ-model). *A strict LCA* $\mathcal{A} = (A, \cdot, !)$ *is a* strict combinatory linear λ-model *if there exist* linear selector ϵ *and* selector *combinator* ϵ_s *such that, for all* $x, y \in A$,

$$\begin{array}{c} \epsilon xy = xy \quad \textit{and} \quad (\forall z.\ xz = yz) \Rightarrow \epsilon x = \epsilon y \\ \epsilon_s !x!y = x!y \quad \textit{and} \quad (\forall z.\ x!z = y!z) \Rightarrow \epsilon_s !x = \epsilon_s !y \end{array} \ .$$

Then we have:

Proposition 2. *Every strict combinatory linear λ-model gives rise by standardisation to a strict combinatory λ-model.*

Strict combinatory λ-models yield models of restricted λ-calculi, such as the $\lambda\beta_{KN}$-calculus of [HL99]. The $\lambda\beta_{KN}$-calculus is the λ-calculus, where the notion of β-reduction is substituted by the β_{KN}-reduction, which allows to erase an argument only when it is a closed β-normal form or a variable. I.e.: $(\lambda x.M)N \rightarrow_{\beta_{KN}} M[N/x]$, if $(\lambda x.M)N$ is either a $I\beta$-redex[3] or a $K\beta$-redex[4], whose argument is a closed normal form or a variable. Hence, in particular strict λ-models give models of the λI-calculus, the original calculus introduced by Church, where only strict λ-abstractions are available (i.e. the abstracted variable appears in the body). The notion of β_{KN}-reduction is correct w.r.t. observational equivalences induced by perpetual strategies (see [HL99] for more details).

Graph models à la Scott-Plotkin-Engeler and Abramsky's *linear graph models* are examples of combinatory (linear) λ-models. Here we present strict versions of them, which give rise to strict combinatory (linear) λ-models. Notice that, in the definition of graph models below, we allow retractions to be relations instead of simply partial functions, thus generalizing original definitions, and capturing also some constructions in [Sco80] and some weak variants of filter models of [BCD83].

Definition 5 (Strict (Linear) Graph Model). *A* strict graph model (S-GM) $\mathcal{U}$ *is an applicative structure* $(\mathcal{P}(U), \cdot_\tau)$*, where* U *is a (infinite) set with a retraction in* Rel $\tau : \mathcal{P}^-_{<\omega}(U) \times U \lhd U$*, and the application* $\cdot_\tau$ *is defined by:*

$$\text{for all } x, y \in \mathcal{P}(U), \quad x \cdot_\tau y \triangleq \{a \mid \exists u \subseteq_{fne} y. \exists b \in x.\ ((u,a),b) \in \tau\}\ .$$

A strict linear graph model (S-LGM) *is a structure* $\mathcal{U} = (\mathcal{P}(U), \cdot_{\tau_1}, !_{\tau_2})$*, where* U *is a (infinite) set with retractions in* Rel $\tau_1 : U \times U \lhd U$ *and* $\tau_2 : \mathcal{P}^-_{<\omega}(U) \lhd U$*, and linear application* $\cdot_{\tau_1}$ *and* $!_{\tau_2}$ *are defined by: for all* $x, y \in \mathcal{P}(U)$*,*

$$x \cdot_{\tau_1} y \triangleq \{a \mid \exists b \in y. \exists c \in x.\ ((b,a),c) \in \tau_1\}\ ,\quad !_{\tau_2} x \triangleq \{a \mid \exists u \subseteq_{fne} x.\ (u,a) \in \tau_2\}\ .$$

One can define combinators on strict (linear) graph models in such a way that:

Proposition 3. *Every strict (L)GM is a strict combinatory (linear) λ-model.*

As one expects, given a strict LGM, by standardisation, we get a strict GM:

Proposition 4. *Let* $\mathcal{U} \triangleq (\mathcal{P}(U), \cdot_{\tau_1}, !_{\tau_2})$ *be a S-LGM with retractions* $\tau_1 : U \times U \lhd U$ *and* $\tau_2 : \mathcal{P}^-_{<\omega}(U) \lhd U$*. Then by standardisation we get a S-GM* $\mathcal{S}(\mathcal{U}) \triangleq (\mathcal{P}(U), \cdot_\tau)$*, where the retraction* $\tau : \mathcal{P}^-_{<\omega}(U) \times U \lhd U$ *is defined by* $\tau_1 \circ (\tau_2 \times id_U)$*.*

The construction for building a LGM from a GM for which standardisation is an inverse, introduced in [HL03], extends to the strict case:

Proposition 5. *For any S-GM* $\mathcal{U}$*, there is a S-LGM* $\mathcal{L}(\mathcal{U})$ *such that* $\mathcal{S}(\mathcal{L}(\mathcal{U})) = \mathcal{U}$*.*

[3] I.e. the variable x occurs in the body M of the λ-abstraction.

[4] I.e. the variable x does not occur in the body M.

Proof. Let $\mathcal{U} \triangleq (\mathcal{P}(U), \cdot_\tau)$ be a S-GM. For any bijection $\xi : U \to \mathcal{P}^-_{<\omega}(U)$ (which exists by cardinality reasons), the retractions $\tau_1 : U \times U \lhd U$, $\tau_1 \triangleq \tau \circ (\xi \times id_U)$, and $\tau_2 : \mathcal{P}^-_{<\omega}(U) \lhd U$, $\tau_2 \triangleq \xi^{-1}$ induce a S-LGM $(\mathcal{P}(U), \cdot_{\tau_1}, !_{\tau_2})$. We take such LGM as $\mathcal{L}(\mathcal{U})$. □

From the point of view of strict λ-theories, we have the following important consequence:

Corollary 1. *The class of strict λ-theories induced by S-GMs coincides with the class of strict λ-theories induced by S-GMs obtained from S-LGMs via standardisation.*

Remark 1. Notice that, in the spirit of [Sco75], the above constructions of strict (L)GMs go through even if we consider $\mathcal{P}^-_{<\kappa}$, for any cardinal κ, in place of $\mathcal{P}^-_{<\omega}$.

2 Strict Weak Linear Categories and Linear Graph Models

In this section, we introduce a *strict* version of Abramsky's axiomatization of an LCA from a *weak linear category* (WLC). WLCs are the counterpart for linear combinatory algebras of the notion of linear category for linear λ-models (see [BBPH92]). In this section, we show that the category *Rel* with the cartesian product as tensor, together with suitable stream-based functors turns out to be a *strict* WLC. Moreover, the LCAs arising from the strict WLC *Rel* are S-LGMs.

We start by introducing a strict variant of Abramsky's notion of WLC. A strict WLC is a WLC which is also a *$PO_\perp$-enriched category*, [SP82], where strictness conditions of tensor, composition and ! are required, and only a restricted form of weakening is required. We recall that a $PO_\perp$-enriched category is a category where all spaces of morphisms form a partial order with $\perp$, and composition is monotone.

Definition 6 (Strict Weak Linear Category). *Let $(\mathcal{C}, \otimes)$ be a $PO_\perp$-enriched symmetric monoidal closed category together with a symmetric monoidal closed functor $! : \mathcal{C} \to \mathcal{C}$. Then $(\mathcal{C}, \otimes, !)$ is a* strict weak linear category *(strict WLC) if*

- *for all $f \in \mathcal{C}(A, B)$, for all C, D,*
  - $f \otimes \perp_{C,D} = \perp_{A\otimes C, B\otimes D}$ *and* $\perp_{C,D} \otimes f = \perp_{C\otimes A, D\otimes B}$
  - $\perp_{B,C} \circ f = \perp_{A,C}$ *and* $f \circ \perp_{C,A} = \perp_{C,B}$
  - $! \perp_{A,B} = \perp_{!A,!B}$*;*
- *there exist monoidal pointwise natural transformations:*
  $der : ! \Rightarrow Id$ (dereliction) $\quad \delta : ! \Rightarrow !!$ (comultiplication)
  $con : ! \Rightarrow ! \otimes !$ (contraction)*;*
- *there exists a monoidal pointwise natural transformation up-to-$\perp$ (i.e. the naturality condition is not required on $\perp$):*
  $weak : ! \Rightarrow \mathcal{K}_I$ (weakening) , *where $\mathcal{K}_I$ is the constant I functor.*

Definition 7. *A* reflexive *object in a (strict) WLC $\mathcal{C}$ is an object V in $\mathcal{C}$ with the following retracts:* $V \multimap V \lhd V \qquad !V \lhd V \qquad I \lhd V$.

The construction of [AHS02] of a LCA from a WLC can be tailored to the strict case:

Theorem 1. *Let $(\mathcal{C}, \otimes, !)$ be a strict WLC and V be a reflexive object in $\mathcal{C}$ with retracts $\theta_1 : V \multimap V \lhd V : \theta_1'$ and $\theta_2 : !V \lhd V : \theta_2'$. Then $(\mathcal{C}(I, V), \cdot, !)$ is a strict LCA, where $\cdot$ and $!$ are defined by: for $f, g \in \mathcal{C}(I, V)$,*

$$f \cdot g \triangleq ev \circ ((\theta_1' \circ f) \otimes g) \circ \rho_I^{-1} \qquad !f \triangleq \theta_2 \circ (!f) \circ \phi_I' \ ,$$

where $\rho_I : I \otimes I \simeq I$ and $\phi_I' : I \to !I$ is the morphism associated to the monoidal functor $!$ (see Appendix A).

We are now in the position of proving that the category *Rel* together with any of the following symmetric monoidal functors based on streams is a strict WLC:

Definition 8. *i) Let $(\ )_{<\omega} : Rel \to Rel$ be the functor defined by: for any set A, $(A)_{<\omega} \triangleq A_{<\omega}$, where $A_{<\omega}$ is the set of finite streams; for any $f : A \twoheadrightarrow B$, let $(f)_{<\omega} \triangleq f_{<\omega} : A_{<\omega} \twoheadrightarrow B_{<\omega}$, $(\boldsymbol{a}, \boldsymbol{b}) \in f_{<\omega}$ iff $|\boldsymbol{a}| = |\boldsymbol{b}|$ and $\forall i.\ (a_i, b_i) \in f$.*
ii) Let $(\ )_\omega : Rel \to Rel$ be defined by: for any set A, $(A)_\omega \triangleq A_\omega$, where A_ω is the set of (infinite) streams; the definition on arrows is similar to (i).
iii) Let $(\ )_\omega^f : Rel \to Rel$ be defined by: for any set A, $(A)_\omega^f \triangleq A_\omega^f$, where A_ω^f is the restriction of A_ω to the streams with finite codomain; the definition on arrows is similar to (i).

Proposition 6. *$(Rel, \times, !)$, where $!$ is either $(\ )_{<\omega}$ or $(\ )_\omega^f$ or $(\ )_\omega$, is a strict WLC.*

Proof. The proof of the fact that $(Rel, \times)$ is symmetric monoidal closed, with right adjoint of $\times$ the product $\times$ itself, and that the functors of Definition 8 are monoidal closed is routine. Moreover, one can easily check that *Rel* is $PO_\perp$-enriched w.r.t. set-theoretic inclusion, with the empty relation being $\perp$, satisfying the appropriate strictness conditions. Let $con : (\ )_\omega \Rightarrow (\ )_\omega \times (\ )_\omega$, $con_A : (A)_\omega \twoheadrightarrow (A)_\omega \times (A)_\omega$ be defined by $con_A \triangleq \{(a, (a', a'')) \mid a, a', a'' \in A_\omega \ \wedge \ \forall i \geq 0.\ a''_{2i+1} = a_i \ \wedge \ \forall i \geq 1.\ a''_{2i} = a'_i\}$. Weakening is a natural transformation only up-to the empty relation: $weak_A : A_\omega \to A$, $weak_A \triangleq \{(a^\omega, *) \mid a \in A\}$. We omit the definitions of the other natural transformations and the discussion of the other !-functors. $\square$

Theorem 2. *The LCAs generated by reflexive objects in the strict WLC* (Rel, $\times$, $!$)*, where $!$ is either $(\ )_{<\omega}$ or $(\ )_\omega^f$ or $(\ )_\omega$, are (isomorphic to) strict LGMs.*

Proof. We consider the case $! = (\ )_{<\omega}$. The other cases can be dealt with similarly. Using Theorem 1, one can easily check that any set U with retracts $\theta_1 : U \times U \lhd U$ and $\theta_2 : U_{<\omega} \lhd U$ gives rise to a LCA isomorphic, via the canonical isomorphism $\lambda_U : I \times U \to U$ (see Appendix A), to the LCA $(\mathcal{P}(U), \cdot, !)$, where application $\cdot$ is the application on a LGM w.r.t. coding θ_1, and $!f \triangleq \{\theta_2(\boldsymbol{a}) \mid \forall i \in \{1, \ldots, |\boldsymbol{a}|\}.\ a_i \in f\}$. Let $\xi : \mathcal{P}^-_{<\omega} U \multimap\!\!\rightarrow U_{<\omega}$ be the injective relation defined by $(u, \boldsymbol{a}) \in \xi$ iff, for all i, $a_i \in u$ and for all $b \in u$ there exists i s.t. $a_i = b$. Then $\tau_2 : \mathcal{P}^-_{<\omega} U \multimap\!\!\rightarrow U$, $\tau_2 \triangleq \theta_2 \circ \xi$ is s.t. $!_{\tau_2} f = \{\tau_2(u) \mid u \subseteq_{fne} f\} = !f$, i.e. the LCA $(\mathcal{P}(U), \cdot, !)$ is a strict LGM with codings θ_1, τ_2. Notice that the coding τ_2 in the proof above is forced to be non functional. $\square$

The situation is summarized in Table 1. Notice that, in the cases of infinite streams (possibly with finite codomain), the cardinality of U is at least $2^{\aleph_0}$. Moreover, for $! = (\ )_\omega$, we get an ω_1-strict LGM.

WLC	CODINGS		LCA
$(Rel, \times, (\ )_{<\omega})$	$\theta_1 : U \times U \lhd U,\ \theta_2 : U_{<\omega} \lhd U$	$\Rightarrow$	$\exists \xi : \mathcal{P}^-_{<\omega} U \lhd U_{<\omega}$ s.t. $(\mathcal{P}U, \cdot_{\theta_1}, !_{\theta_2 \circ \xi})$ strict LGM
$(Rel, \times, (\ )^f_\omega)$	$\theta_1 : U \times U \lhd U,\ \theta_2 : U^f_\omega \lhd U$	$\Rightarrow$	$\exists \xi : \mathcal{P}^-_{<\omega} U \lhd U^f_\omega$ s.t. $(\mathcal{P}U, \cdot_{\theta_1}, !_{\theta_2 \circ \xi})$ strict LGM
$(Rel, \times, (\ )_\omega)$	$\theta_1 : U \times U \lhd U,\ \theta_2 : U_\omega \lhd U$	$\Rightarrow$	$\exists \xi : \mathcal{P}^-_{<\omega_1} U \lhd U_\omega$ s.t. $(\mathcal{P}U, \cdot_{\theta_1}, !_{\theta_2 \circ \xi})$ ω_1-strict LGM

Table 1.

Remark 2. Notice that the category *Rel* fails to be a WLC in the original sense of [AHS02], since the "natural" definition of weakening (see the proof of Proposition 6 above) is *not* natural on the *empty* relation.
Somewhat unexpectedly, the powerset functor $\mathcal{P}^-_{<\omega}$ fails to induce a structure of (strict) WLC on *Rel*. Namely, the "natural" definition of dereliction, i.e. $der_A : \mathcal{P}^-_{<\omega}(A) \multimap\!\!\rightarrow A$, $der_A \triangleq \{(\{a\}, a) \mid a \in A\}$, is not natural.

3 The Geometry of Interaction Construction

The categorical axiomatization of the *Geometry of Interaction* (GoI) due to [Abr96,AHS02] is based on traced categories of [JSV96]. Any traced monoidal category gives rise, by the construction of [Abr96], to a *GoI category.* If moreover the traced category we start from has a strong traced monoidal functor together with suitable retractions, then the GoI category is a WLC, and hence every suitable reflexive object generates a (GoI) LCA. This situation, called *GoI situation*, is axiomatized and studied in [AHS02]. In this section, we introduce the notion of *strict GoI situation*, which gives rise to a GoI category which is a strict WLC. Our axiomatization of strict GoI situation is based on the weaker axiomatization of GoI situation given in [HL03]. In the next section, we will see that this notion of strict GoI situation captures the basic examples of wave models arising from *Rel*.

GoI categories arise from traced symmetric monoidal categories by the following construction:

Proposition 7 (Geometry of Interaction Construction, [Abr96]). *Given a traced symmetric monoidal category $\mathcal{C}$, we define a* GoI category $\mathcal{G}(\mathcal{C})$ *by:*

- **Objects**: *pairs of objects of* $\mathcal{C}$*, denoted by* (A^+, A^-)*, where* A^+ *and* A^- *are objects of* $\mathcal{C}$.
- **Arrows**: *an arrow* $f : (A^+, A^-) \to (B^+, B^-)$ *in* $\mathcal{G}(\mathcal{C})$ *is* $f : A^+ \otimes B^- \to A^- \otimes B^+$ *in* $\mathcal{C}$.
- **Identity**: $id_{(A^+,A^-)} = \sigma_{A^+,A^-}$.
- **Composition**: *it is given by* symmetric feedback. *Given* $f : (A^+, A^-) \to (B^+, B^-)$ *and* $g : (B^+, B^-) \to (C^+, C^-)$, $g \circ f : (A^+, A^-) \to (C^+, C^-)$ *is given by:* $g \circ f \triangleq Tr^{B^- \otimes B^+}_{A^+ \otimes C^-, A^- \otimes C^+}(\gamma' \circ (f \otimes g) \circ \gamma)$*, where* $\gamma \triangleq (id_{A^+} \otimes id_{B^-} \otimes \sigma_{C^-,B^+}) \circ (id_{A^+} \otimes \sigma_{C^-,B^-} \otimes id_{B^+})$ *and* $\gamma' \triangleq (id_{A^-} \otimes id_{C^+} \otimes \sigma_{B^+,B^-}) \circ (id_{A^-} \otimes \sigma_{B^+,C^+} \otimes id_{B^-}) \circ (id_{A^-} \otimes id_{B^+} \otimes \sigma_{B^-,C^+})$.
- **Tensor**: $(A^+, A^-) \otimes (B^+, B^-) \triangleq (A^+ \otimes B^+, A^- \otimes B^-)$*, and, for any* $f : (A^+, A^-) \to (B^+, B^-)$ *and* $g : (C^+, C^-) \to (D^+, D^-)$, $f \otimes g \triangleq (id_{A^-} \otimes \sigma_{B^+,C^-} \otimes id_{D^+}) \circ (f \otimes g) \circ (id_{A^+} \otimes \sigma_{C^+,B^-} \otimes id_{D^-})$.
- **Unit**: (I, I).

Then $\mathcal{G}(\mathcal{C})$ *is compact closed. Moreover,* $F : \mathcal{C} \to \mathcal{G}(\mathcal{C})$ *with* $F(A) = (A, I)$ *and* $F(f) = f$ *is a full and faithful embedding.*

In [AHS02], sufficient conditions are given on the traced monoidal category $\mathcal{C}$ for $\mathcal{G}(\mathcal{C})$ to be a WLC, and hence to give rise to a GoI LCA. Here we give a strict version of this construction. First of all notice that:

Lemma 1. *If* $(\mathcal{C}, \otimes)$ *is a* $PO_\perp$*-enriched traced symmetric monoidal category with monotone tensor and monotone trace operator, then* $\mathcal{G}(\mathcal{C})$ *is a compact closed* $PO_\perp$*-enriched category.*

Now we are in the position of introducing the notion of *strict GoI situation*. This is based on the weak axiomatization of GoI situation given in [HL03], which generalizes the original GoI situation of [AHS02], by requiring the existence of suitable retractions which are natural only in a weak sense. This generalization of a GoI situation still ensures that the GoI category is a WLC. It is necessary in order to capture categories based on relations, where some retractions in the definition of GoI situation fail to be natural in the full sense.

Definition 9 (Strict GoI Situation). *A* strict GoI situation *is a triple* $(\mathcal{C}, T, U)$*, where:*

- $\mathcal{C}$ *is a* $PO_\perp$*-enriched traced symmetric monoidal category with monotone tensor and monotone trace operator such that, for all* $f \in \mathcal{C}(A, B)$*, for all* C, D*,*
  - $f \otimes \perp_{C,D} = \perp_{A\otimes C, B\otimes D}$ *and* $\perp_{C,D} \otimes f = \perp_{C\otimes A, D\otimes B}$
  - $\perp_{B,C} \circ f = \perp_{A,C}$ *and* $f \circ \perp_{C,A} = \perp_{C,B}$
  - $Tr^C_{A,B}(\perp_{A\otimes C, B\otimes C}) = \perp_{A,B}$;
- $T : \mathcal{C} \to \mathcal{C}$ *is a traced strong symmetric monoidal functor such that, for all* A, B, $T(\perp_{A,B}) = \perp_{!A,!B}$*; moreover, there exist retractions* comultiplication, dereliction, contraction*, which satisfy a weak form of naturality, which we call* naturality up-to retraction*, and furthermore a retraction* weakening*, which is natural up-to-retraction and up-to-*$\perp$*, i.e.:*

1. $\{e_A : TTA \lhd TA : e'_A\}_A$ (Comultiplication) *is a family of monoidal retractions s.t., for all* $f : A \to B$, $e'_B \circ Tf \circ e_A = f$*;*
2. $\{d_A : A \lhd TA : d'_A\}_A$ (Dereliction) *is a family of monoidal retractions s.t., for all* $f : A \to B$, $d'_B \circ Tf \circ d_A = f$*;*
3. $\{c_A : TA \otimes TA \lhd TA : c'_A\}_A$ (Contraction) *is a family of monoidal retractions s.t., for all* $f : A \to B$, $c'_B \circ Tf \circ c_A = f$*;*
4. $\{w_A : I \lhd TA : w'_A\}_A$ (Weakening) *is a family of monoidal retractions s.t., for all* $f : A \to B$, $f \neq \perp_{A,B}$, $w'_B \circ Tf \circ w_A = f$.

– U *is an object of* $\mathcal{C}$*, called a* GoI reflexive object*, with retractions*
  1. $\theta_1 : U \otimes U \lhd U : \theta'_1$
  2. $I \lhd U$
  3. $\theta_2 : TU \lhd U : \theta'_2$.

Theorem 3. *Let* $(\mathcal{C}, T, U)$ *be a strict GoI situation. Then*
i) $(\mathcal{G}(\mathcal{C}), !)$ *is a strict WLC with* $! : \mathcal{G}(\mathcal{C}) \to \mathcal{G}(\mathcal{C})$ *defined as follows:* $!(A^+, A^-) \triangleq (TA^+, TA^-)$*, and, for* $f : (A^+, A^-) \to (B^+, B^-)$, $!f \triangleq TA^+ \otimes TB^- \overset{\simeq}{\to} T(A^+ \otimes B^-) \overset{Tf}{\to} T(A^- \otimes B^+) \overset{\simeq}{\to} TA^- \otimes TB^+$.
ii) $(\mathcal{C}(U,U), \cdot, !)$ *is a strict LCA, where for any* $f, g \in \mathcal{C}(U,U)$,
$f \cdot g \triangleq Tr^U_{U,U}((id_U \otimes g) \circ (\theta'_1 \circ f \circ \theta_1))$*, and* $!f \triangleq \theta_2 \circ Tf \circ \theta'_2$.

4 Strict Wave GoI Algebras and Linear Graph Models

We show that the traced category $(Rel, \times)$, together with stream functors, gives rise to a strict GoI situation. Moreover, the GoI algebras induced by $(Rel, \times)$ are strict linear graph models. Finally, we show that such wave models can induce restricted λ-theories in the sense of [HL99], where not all unsolvables of order 0 are equated.

4.1 GoI algebras on $(Rel, \times)$

The category $(Rel, \times)$ is traced with the trace operator $Tr^U_{A,B}(\ )$ defined by: for $f : A \times U \leftrightarrow B \times U$,

$$Tr^U_{A,B}(f) \triangleq \{(a, b) \mid \exists u.\ (a, u, b, u) \in f\}\ .$$

Both the functor of streams $(\ )_\omega$ and that of streams with finite codomain $(\ )^f_\omega$ induce on $(Rel, \times)$ a strict GoI situation. We focus on $(\ )^f_\omega$.

Proposition 8. *For any GoI reflexive object* U *in* Rel*,* $(Rel, \times, (\ )^f_\omega, U)$ *is a strict GoI situation.*

Proof. We only sketch the definitions of the retractions:
- $e_A : (A^f_\omega)^f_\omega \to A^f_\omega$, $e_A \triangleq \xi \circ \chi_A$, where $\chi_A : [\mathbf{N} \to_{fcod} [\mathbf{N} \to_{fcod} A]] \lhd [\mathbf{N} \times \mathbf{N} \to_{fcod} A]$ is a component of the retraction natural in A induced by curryfication, and $\xi \triangleq \lambda f \in [\mathbf{N} \times \mathbf{N} \to_{fcod} A].\lambda n \in \mathbf{N}.f(\epsilon^{-1}(n))$, where $\epsilon : \mathbf{N} \times \mathbf{N} \simeq \mathbf{N}$ is any bijective coding of pairs;

- $c_A : A^f_\omega \times A^f_\omega \to A^f_\omega$, $c_A(\boldsymbol{a}, \boldsymbol{a}') = \boldsymbol{a}''$, where, for all $i \geq 0$, $a''_{2i+1} = a_i$ and, for all $i \geq 1$, $a''_{2i} = a'_i$;
- $d_A : A \to A^f_\omega$, $d_A(a) = a^\omega$, $d'_A : A^f_\omega \to A$, $d'_A(a^\omega) = a$;
- $w_A : I \multimap A^f_\omega$, $w_A \triangleq \{(*, a^\omega) \mid a \in A\}$, $w'_A : A^f_\omega \to I$, $w'_A \triangleq \{(a^\omega, *) \mid a \in A\}$. $\square$

Notice that d in the proof above is natural up-to retraction, but not in the full sense; while w_A is natural only up-to the empty relation. Then we have:

Proposition 9. *i) The GoI category $\mathcal{G}(Rel, \times, (\ )^f_\omega)$ is a strict WLC.*
ii) Let U be a GoI reflexive object in Rel, with retractions $\theta_1 : U \times U \lhd U$, $\theta_2 : U^f_\omega \lhd U$. Then $(\mathcal{P}(U \times U), \cdot_{\theta_1}, !_{\theta_2})$ is a LCA, where, for all $x, y \in \mathcal{P}(U \times U)$
- $x \cdot_{\theta_1} y = \{(a, b) \mid \exists (c, d) \in y.\ (\theta_1(a, d), \theta_1(b, c)) \in x\}$;
- $!_{\theta_2} x = \{(\theta_2(\boldsymbol{a}), \theta_2(\boldsymbol{b})) \mid \boldsymbol{a}, \boldsymbol{b} \in U^f_\omega \wedge \forall i.(a_i, b_i) \in x\}$.

A crucial fact for our purposes is that all the GoI algebras of Proposition 9 give rise, for a suitable choice of the coding relations, to strict LGMs:

Theorem 4. *Let $\mathcal{U} = (\mathcal{P}(U \times U), \cdot_{\theta_1}, !_{\theta_2})$ be a GoI algebra induced by $(Rel, \times, (\ )^f_\omega)$, with $\theta_1 : U \times U \lhd U$, $\theta_2 : U^f_\omega \lhd U$. Then $\mathcal{U}$ coincides with the strict LGM $\mathcal{U}' = (\mathcal{P}(U'), \cdot_{\tau_1}, !_{\tau_2})$, where $U' \triangleq U \times U$, $\tau_1 : U' \times U' \lhd U'$ and $\tau_2 : \mathcal{P}^-_{<\omega}(U') \lhd U'$ are defined by:*

$$\tau_1 \triangleq \langle \theta_1 \circ (\pi_1 \circ \pi_2 \times \pi_2 \circ \pi_1), \theta_1 \circ (\pi_2 \circ \pi_2 \times \pi_1 \circ \pi_1) \rangle \ , \quad \tau_2 \triangleq (\theta_2 \times \theta_2) \circ \zeta \ ,$$

where $\zeta : \mathcal{P}^-_{<\omega}(U \times U) \multimap U^f_\omega \times U^f_\omega$ is the injective relation defined by $(u, (\boldsymbol{a}, \boldsymbol{b})) \in \zeta$ iff for all i, $(a_i, b_i) \in u$ and for all $(c, d) \in u$ there exists i such that $a_i = c$ and $b_i = d$.

Proof. Let $x, y \in \mathcal{P}(U \times U)$. By Proposition 9, $x \cdot_{\theta_1} y = \{(a, b) \mid \exists (c, d) \in y.\ (\theta_1(a, d), \theta_1(b, c)) \in x\} = \{(a, b) \mid \exists (c, d) \in y.\ \tau_1((c, d), (a, b)) \in x\}$, by definition of τ_1; i.e. $\cdot_{\theta_1}$ is the application on the LGM $\mathcal{U}'$. Moreover, by Proposition 9, $!_{\theta_2} x = \{(\theta_2(\boldsymbol{a}), \theta_2(\boldsymbol{b})) \mid \boldsymbol{a}, \boldsymbol{b} \in U^f_\omega \ \wedge \ \forall i.(a_i, b_i) \in x\} = \{(\theta_2(\boldsymbol{a}), \theta_2(\boldsymbol{b})) \mid \boldsymbol{a}, \boldsymbol{b} \in U^f_\omega \ \wedge \ \exists u \subseteq_{fne} x.\ \forall i.(a_i, b_i) \in u \ \wedge \ \forall (c, d) \in u\ \exists i.(a_i = c \wedge b_i = d)\} = \{(\theta_2 \times \theta_2)(\zeta(u)) \mid u \subseteq_{fne} x\} = \{\tau_2(u) \mid u \subseteq_{fne} x\}$, by definition of τ_2; i.e. $!_{\theta_2}$ is the ! operator on the LGM $\mathcal{U}'$. $\square$

Remark 3. Finally, notice that both the finite powerset functor $\mathcal{P}^-_{<\omega}$ and the finite stream functor $(\ )_{<\omega}$ fail to be strong monoidal, and hence they do not give a GoI situation on *Rel*.

4.2 Strict Wave GoI λ-theories

The class of strict λ-theories induced by wave models is quite rich, or at least it goes beyond theories where all unsolvable of order 0 are equated in the bottom element. Namely:

Lemma 2. *For any $k \geq 0$, there exists a GoI algebra on $(Rel, \times, (\ )^f_\omega)$ with k* self-singletons, *i.e. elements $a \in U$ such that $\theta_1(a, a) = a$ and $\theta_2(\{a\}) = a$.*

One can easily check that any self-singleton a belongs to the interpretation of a term iff it reduces to a closed λ-term (see [HR92]). Therefore:

Theorem 5. *There is a strict wave model in which* $[\![\lambda xy.\Delta\Delta xy]\!] \neq [\![\lambda xy.\Delta\Delta(xy)]\!]$.

5 Conclusions and Future Work

Building on [Abr96,AHS02,HL03], we have investigated the connections between graph models and wave GoI models arising in the basic setting of $(Rel, \times)$. The category $(Rel, \times)$ apparently fails to give a WLC and a GoI situation, because weakening is not well-behaved on the empty relation. However, we have shown that it induces a *strict* WLC and a *strict* GoI situation, where only a restricted form of weakening is available. Correspondingly, GoI algebras arising from $(Rel, \times)$ amount to a strict version of LGMs, where only a restricted K-combinator is present, allowing to erase non-$\perp$ elements. These are models for restricted λ-calculi, such as Church's λI-calculus and the $\lambda\beta_{KN}$-calculus of [HL99].

The notion of strict linear combinatory algebra in Definition 2 captures also models of *call-by-value* λ-calculus. However, strict combinatory λ-models in Definitions 3 and 4 rule this out. It would be interesting to explore GoI models for call-by-value λ-calculus or Moggi's computational λ-calculus.

Finally, here is a list of intriguing open questions. Definitive answers or even well-motivated conjectures to these appear to be rather difficult.
- Many classes of graph-like models have been considered in the literature, see [DH93]. Do they all induce the same λ-theories as the class of standard GMs?
- Can original LGMs and the generalizations of graph models in [DH93] be captured as possibly modified wave WLCs?
- Are all the theories of GMs induced by wave GoI algebras? Which theories escape GoI characterizations?
- Can standard graph models be "naturally" extracted from strict graph models?
- Finally, an interesting issue to be investigated is that of giving a logical characterization via *intersection types* to the graph models arising from wave GoI constructions. We feel that this will shed more light both on intersection types and on wave models.

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A Categorical Definitions

We collect some categorical definitions. For more details, see e.g. [AHS02,JSV96].

Definition 10 (Monoidal Category). *A* monoidal category *is a tuple* $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ *consisting of a category* $\mathcal{C}$*, an object* I *of* $\mathcal{C}$ *called the* unit*, a bifunctor* $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$*, and natural isomorphisms* α, λ, ρ *with components*
$\alpha_{A,B,C} : A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C \qquad \lambda_A : I \otimes A \simeq A \qquad \rho_A : A \otimes I \simeq A$
such that the two diagrams below commute.

$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[r, "\alpha"] \arrow[d, "id_A \otimes \alpha"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha"] & ((A \otimes B) \otimes C) \otimes D \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha"] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "\alpha \otimes id_D"']
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes (I \otimes B) \arrow[rr, "\alpha"] \arrow[dr, "id_A \otimes \lambda"'] & & (A \otimes I) \otimes B \arrow[dl, "\rho \otimes id_B"] \\
& A \otimes B &
\end{tikzcd}$$

A symmetric *monoidal category is a monoidal category equipped with a natural isomorphism* σ *with components* $\sigma_{A,B} : A \otimes B \to B \otimes A$ *such that* $\sigma_{B,A} \circ \sigma_{A,B} = id_{A \otimes B}$, $\rho_A = \lambda_A \circ \sigma_{A,I}$ *and the diagram below commutes*

$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "\alpha"] \arrow[d, "id_A \otimes \sigma"'] & (A \otimes B) \otimes C \arrow[r, "\sigma"] & C \otimes (A \otimes B) \arrow[d, "\alpha"] \\
A \otimes (C \otimes B) \arrow[r, "\alpha"] & (A \otimes C) \otimes B \arrow[r, "\sigma \otimes id_B"] & (\otimes A) \times B
\end{tikzcd}$$

A symmetric monoidal category $\mathcal{C}$ *is* closed *if, for each object* B *in* $\mathcal{C}$*, the functor* $\_ \otimes B : \mathcal{C} \to \mathcal{C}$ *has a right adjoint,* $B \multimap \_ : \mathcal{C} \to \mathcal{C}$*.*

Definition 11 (Traced Symmetric Monoidal Category). *A* traced symmentric monoidal category *is a symmetric monoidal category* $(\mathcal{C}, \otimes, I, \sigma)$ *with a family of functions* $Tr^U_{A,B} : \mathcal{C}(A \otimes U, B \otimes U) \to \mathcal{C}(A, B)$*, called a* trace*, satisfying the following conditions:*

- **Natural** *in* A*:* $Tr^U_{A,B}(f) \circ g = Tr^U_{C,B}(f \circ (g \otimes id_U))$*, where* $f : A \otimes U \to B \otimes U$*,* $g : C \to A$*;*
- **Natural** *in* B*:* $g \circ Tr^U_{A,B}(f) = Tr^U_{A,C}((g \otimes id_U) \circ f)$*, where* $f : A \otimes U \to B \otimes U$*,* $g : B \to C$*;*
- **Dinatural** *in* U*:* $Tr^U_{A,B}((id_B \otimes g) \circ f) = Tr^{U'}_{A,B}(f \circ (id_a \otimes g))$*, where* $f : A \otimes U \to B \otimes U'$*,* $g : U' \to U$*;*
- **Vanishing***:* $Tr^I_{A,B}(f) = f$ *and* $Tr^{U \otimes V}_{A,B}(g) = Tr^U_{A,B}(Tr^V_{A \otimes U, B \otimes U})$*, for* $f : A \otimes I \to B \otimes I$ *and* $g : A \otimes U \otimes V \to B \otimes U \otimes V$*;*
- **Superposing***:* $g \otimes Tr^U_{A,B}(f) = Tr^U_{C \otimes A, D \otimes B}(g \otimes f)$*, for* $f : A \otimes U \to B \otimes U$ *and* $g : C \to D$*;*
- **Yanking***:* $Tr^U_{U,U}(\sigma_{U,U}) = id_U$*.*

A *geometrical* representation of traced symmetric monoidal categories can be given using the language of "boxes and wires" (see [JSV96,AHS02] for more details).

Definition 12 (Monoidal Functor, Monoidal Natural Transformation). *A* monoidal functor *between monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is a triple (F, ϕ, ϕ'_I), where $F : \mathcal{C} \to \mathcal{D}$ is a functor, ϕ is a natural transformation with componenents $\phi_{A,B} : FA \otimes FB \to F(A \otimes B)$ and $\phi'_I : I \to FI$ is a morphism in $\mathcal{D}$ such that the following diagrams commute*

$$\begin{tikzcd}
FA \otimes (FB \otimes FC) \arrow[r, "id_{FA} \otimes \phi"] \arrow[d, "\alpha"'] & FA \otimes F(B \otimes C) \arrow[r, "\phi"] & F(A \otimes (B \otimes C)) \arrow[d, "F\alpha"] \\
(FA \otimes FB) \otimes FC \arrow[r, "\phi \otimes id_{FC}"] & F(A \otimes B) \otimes FC \arrow[r, "\phi"] & F((A \otimes B) \otimes C)
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes FA \arrow[r, "\lambda"] \arrow[d, "\phi'_I \otimes id_{FA}"'] & FA \\
FI \otimes FA \arrow[r, "\phi"] & F(I \otimes A) \arrow[u, "F\lambda"']
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes I \arrow[r, "\rho"] \arrow[d, "id_{FA} \otimes \phi'_I"'] & FA \\
FA \otimes FI \arrow[r, "\phi"] & F(A \otimes I) \arrow[u, "F\rho"']
\end{tikzcd}$$

A monoidal functor is strong *when ϕ is a natural isomorphism and ϕ' is an isomorphism.*

A monoidal functor $F : \mathcal{C} \to \mathcal{D}$, with $\mathcal{C}$ and $\mathcal{D}$ symmetric monoidal categories, is symmetric *if the following diagram commutes:*

$$\begin{tikzcd}
FA \otimes FB \arrow[r, "\phi_{A,B}"] \arrow[d, "\sigma_{FA,FB}"'] & F(A \otimes B) \arrow[d, "F\sigma_{A,B}"] \\
FB \otimes FA \arrow[r, "\phi_{B,A}"] & F(B \otimes A)
\end{tikzcd}$$

A symmetric monoidal functor $F : \mathcal{C} \to \mathcal{D}$ is traced *if, for all $f : A \otimes U \to B \otimes U$, $Tr^{FU}_{FA,FB}(\phi^{-1}_{B,U} \circ Ff \circ \phi_{A,U}) = F(Tr^{U}_{A,B}(f))$.*

A monoidal *natural transformation m between monoidal functors (F, ϕ, ϕ'_I) and (G, ψ, ψ'_I) is a natural transformation $m : F \Rightarrow G$ s.t. the following diagrams commute:*

$$\begin{tikzcd}
FA \otimes FB \arrow[r, "\phi_{A,B}"] \arrow[d, "m_A \otimes m_B"'] & F(A \otimes B) \arrow[d, "m_{A \otimes B}"] \\
GA \otimes GB \arrow[r, "\psi_{A,B}"] & G(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "\phi'_I"] \arrow[dr, "\psi'_I"'] & FI \arrow[d, "m_I"] \\
 & GI
\end{tikzcd}$$

A monoidal pointwise *natural transformation is a family of maps $m_A : FA \to GA$ s.t. the naturality diagram commutes for morphisms of the form $f : I \to A$, for all object A.*