Axiomatic Characterizations of Hyperuniverses and Applications*

Marco Forti†
Dipartimento di Matematica Applicata "U. Dini"
Università di Pisa, Italy.
forti@dm.unipi.it

Furio Honsell‡
Dipartimento di Matematica e Informatica
Università di Udine, Italy.
honsell@dimi.uniud.it

Marina Lenisa
Dipartimento di Informatica
Università di Pisa, Italy.
lenisa@di.unipi.it

Abstract

Hyperuniverses are topological structures exhibiting strong closure properties under formation of subsets. They have been used both in Computer Science, for giving denotational semantics *à la* Scott-de Bakker, and in Mathematical Logic, in order to show the consistency of set theories which do not abide by the "limitation of size" principle. We present correspondences between set-theoretic properties and topological properties of hyperuniverses. We give existence theorems and discuss applications and generalizations to the *non κ-compact* case.

*Work partially supported by 40% and 60% MURST grants, CNR grants, and EEC Science MASK, and BRA Types 6453 contracts.

†Member of GNSAGA of CNR.

‡The main results of this paper have been communicated by this author at the "11^{th} Summer Conference on General Topology and Applications" August 1995, Portland, Maine.

Introduction

Natural frameworks for dicussing Selfreference and other circular phenomena are extremely useful in areas such as the Foundations of Mathematics and Computer Science (see e.g. [2, 8, 11, 20, 24]), or the Semantics of Natural and Programming Languages (see e.g. [1, 3, 4, 15, 19, 21, 22]).

Hyperuniverses can play the rôle of such frameworks. From the foundational point of view, hyperuniverses are transitive models of non-standard set theories which do not abide by the "limitation of size" principle and hence they provide comprehensive but intuitive "universes of sets" (see [8, 11, 12, 13]). Many classes having the size of the universe, such as those necessary for an adequate self-description, are sets of the hyperuniverse; moreover it is closed under many basic set-theoretic operations. From the semantical point of view, hyperuniverses provide a uniform setting for solving many recursive domain equations such as those which arise in giving denotational semantics to programming languages *à la* Scott-de Bakker (see [14, 15]).

In this paper we discuss various classes of generalized hyperuniverses which exhibit different topological properties. We investigate their closure properties under set theoretic operations and we provide axiomatic characterizations of many of them. In view of the applications, we consider hyperuniverses with *atoms* (*Urelements*). Atoms are needed to model non-reductionist foundational theories, where many important mathematical concepts (operations, relations, properties, etc) are taken as primitive and are not represented by ingenious, but artificial, set theoretic codings (see [11]). In the semantics of programming languages, atoms are convenient in modeling directly primitive data types (such as natural numbers with the discrete topology or real numbers with the euclidean metric) or simply actions whose internal structure is not analyzed further at the level of abstraction under consideration. Such structures could be embedded in atomless hyperuniverses only assuming specific axioms which postulate particular non-wellfounded sets. In fact, the uniform structure of the closure of the wellfounded sets is uniquely determined (see [12]).

In view of the above remarks, we work in a weakly extensional Zermelo-Frænkel-like set theory. Moreover, since hyperuniverses are necessarily non-wellfounded, we give up the Axiom of Foundation in the ground theory, and we assume instead a suitable *Free Construction Principle* as in [11, 12]. Thus the axioms of our ground theory are:

$\mathsf{ZF}_0^-\mathsf{C}$ *Pairing, Union, Powerset, Infinity, Replacement and Choice* as in Zermelo-Frænkel Set Theory ZFC;

ExtA *Weak Extensionality with respect to a (possibly empty) set A of atoms:*

$$\forall xz\ (x \in A \to z \notin x)\ \&\ \exists x \notin A.\ \forall y.\ y \notin x\ \&$$

$$\forall xy\ (x \notin A\ \&\ y \notin A\ \&\ \forall z\ (z \in x \leftrightarrow z \in y) \to x = y).$$

FCA *Unique Free Construction with respect to A:*
Given a function $f : X \to \mathcal{P}(X \cup A)$, where $X \cap A = \emptyset$, there exists a unique function $g : X \to V$ such that for all $x \in X$

$$g(x) = (f(x) \cap A) \cup \{g(y) \mid y \in f(x) \cap X\}$$

The Axiom FCA, used in [15], generalizes to a set theory with atoms the Axiom X_1 of [9], which is equivalent to the Axiom AFA of [2].

We call *sets* objects which do not belong to A. The way the axiom ExtA is phrased allows us to introduce the *empty set* $\emptyset$ as the unique $x \notin A$ such that $\forall y.\ y \notin x$. We take the *subset* relation $\subseteq$ to be defined only between sets. Therefore the *powerset* operator $\mathcal{P}(\ )$ applied to atoms yields the empty set. We refer to [17] for standard definitions and results in Set Theory.

Hyperuniverses have been used to show the consistency of the Comprehension Principle GPC of [8]. The axiom schema GPC asserts the existence of the *set* $\{x \mid \phi\}$ for any *generalized positive* formula ϕ. Since we allow for the existence of a set of atoms, we extend, in this paper, the language used to define the class *GPF* of generalized positive formulæ, to include a constant denoting the set of atoms A. Hence the class *GPFA* is defined as the least class of formulæ which includes the *atomic* formulæ, is closed under *conjunction, disjunction, existential* and *universal quantification* and also under the following rules of *bounded quantification* (which, strictly speaking, are non-positive):

if ϕ is $GPFA$, then both $\forall x(x \in y \to \phi)$ and $\forall x(\theta \to \phi)$ are $GPFA$, where θ is any formula with only x free.

The *Generalized Positive Comprehension Schema with Atoms* becomes:

GPCA $\quad \exists y \forall x(x \in y \leftrightarrow \phi)$ where ϕ is $GPFA$ and y is not free in ϕ.

In Section 1 we give the basic axiomatic characterizations of hyperuniverses. In particular we present and discuss various correspondences between set-theoretic properties and topological properties of hyperuniverses. In Section 2 we introduce and discuss the weaker notion of quasi-hyperuniverse. In Section 3 we give a construction for generating quasi-hyperuniverses, and we isolate conditions for obtaining hyperuniverses. In Section 4 we discuss briefly applications of hyperuniverses to semantics of programming languages, presenting general fixed point and final coalgebra theorems. Final remarks appear in Section 5. In the appendix we list definitions and basic facts from Set Theory and General Topology used in the paper.

The authors are grateful to Franck van Breugel, Jan Pelant and Jan Rutten for helpful discussions, and to the referee for useful remarks and suggestions.

1 Characterization Theorems

In this Section we define *topological hyperuniverses* and we investigate their set-theoretic properties. For topological notions we refer to the appendix and [5, 23]. Throughout the paper we assume that κ is an *infinite, regular* cardinal.

Definition 1.1 (*Topological Hyperuniverse*) Let A be a set of atoms. A set $N \subseteq \mathcal{P}(N) \cup A$ is a *κ-topological hyperuniverse* if the elements of $N \cap \mathcal{P}(N)$ are the closed subsets of a κ-additive, Hausdorff topology τ on N, such that A is a clopen subset of N and τ induces on $N \setminus A$ the Vietoris κ-topology.

Notice that every κ-topological hyperuniverse is *κ-compact* (see Proposition 1 of the appendix). The "formidable comprehensiveness" of topological hyperuniverses is illustrated by the following theorem, which generalizes Theorem 2 of [14] to structures with atoms.

Theorem 1.1 *Let N be a κ-topological hyperuniverse. Then*
i) Any subset of N of size less than κ belongs to N. The cartesian product of less than κ elements of $N \cap \mathcal{P}(N)$ belongs to N. If $X \in N \cap \mathcal{P}(N)$, then $\bigcup X$, $N \cap \mathcal{P}(X) = \Box(X)$, $\{Y \in N \mid X \cap Y \neq \emptyset\} = \Diamond(X)$, $dom(X)$ and $cod(X)$ are elements of N.
ii) N is a transitive model of GPCA*. In particular $N \cap \mathcal{P}(N)$ is closed under*

all Gödel operations, excluding complement. Moreover, both the relations of membership and inclusion between members of N are members of N, as well as the identity, the singleton map, and all projections, permutations and regroupings of n-tuples.
iii) A function f belongs to N if and only if it is continuous with closed domain, and then f is a closed uniformly continuous map. More generally, if $X \subseteq N$, a function $g : X \longrightarrow N$ is extendable to a function $f \in N$ if and only if g is uniformly continuous on X (the domain of the closure of g being then the closure of X).
iv) For any $X \in N \cap \mathcal{P}(N)$ and for any $Y \subseteq N$, the function space $Y^X \cap N$ is the set $C(X,Y)$ of all uniformly continuous functions from X into Y, endowed with the Vietoris κ-topology on their graphs. Moreover, a set of functions $F \subseteq N^X$ is an element of N if and only if F is equicontinuous and X is closed. In particular, if $\mid Y \mid > 1$, then $C(X,Y)$ belongs to N if and only if X is discrete (i.e. if $\mid X \mid < \kappa$), and then $C(X,Y) = Y^X$.
v) Internal and external cardinalities agree below κ. Moreover κ is the set of all ordinals of N and its closure is the set $\overline{\kappa} = \kappa \cup \{\overline{\kappa}\} \in N$. Finally $\kappa \to (\kappa)^2_2$ and the cardinality of N is greater than or equal to 2^κ.
vi) A point $X \in N \cap \mathcal{P}(N)$ is isolated if and only if it is made up of less than κ isolated points. Then $T = TC(X)$ is isolated and all elements of $\Pi_\kappa(T)$ are isolated. Moreover, if $T \in N$ is transitive and clopen, then $\Pi(T) \cap N \subseteq \Pi_\kappa(T)$, and equality holds if and only if $\mid T \mid < \kappa$.

Proof. The proofs of i)-v) follow closely the corresponding ones of Theorem 2 in [14], taking into account the presence of atoms and the fact that we have not assumed that the topology is 0-dimensional. Hence, in case $\kappa = \omega$, the topology of N is induced by a uniformity $\mathcal{U}$, which cannot have in general a basis made out of equivalences (see the appendix). To complete the proof of v), we need only to prove that $\kappa \to (\kappa)^2_2$. Clearly, κ is regular by assumption. Moreover, κ is strong limit, since, if $X \subseteq N$ and $|X| = \xi < \kappa$, then X is closed and discrete; hence $\mathcal{P}(X)$ is closed and discrete, whence $2^\xi < \kappa$. Finally κ has the *tree property* since the hyperuniverse N includes κ and induces on κ the same topology as the κ-hyperuniverse N_κ of [11, 12]; if κ does not have the tree property, then the same counterexamples of [11] Sec.2 yield that N is not κ-compact.

The first assertion of vi) follows immediately from the fact that a set $x \in N \cap \mathcal{P}(N)$ is a cluster point of N if and only if it contains a cluster point of N. This can be proved as in [14]. Assume that x has no cluster point. Then $\mid x \mid < \kappa$ and all elements of x are isolated in N. Therefore there is an entourage $U \in \mathcal{U}$ such that, for all $y \in x$, $U(y) = \{y\}$. Hence x itself is isolated, being the only element of $U^+(x)$. On the other hand, if a cluster point of N, say y, belongs to x, then for any open entourage $U \in \mathcal{U}$ we have that $U(y) \neq \{y\}$. Hence we can pick an element $y_U \in U(y) \setminus \{y\}$, and put $x_U = (x \setminus U(y)) \cup \{y_U\}$. Clearly $x_U \in U^+(x)$, hence x is not isolated. By inaccessibility of κ, any element of $\Pi_\kappa(T)$ has size less than κ if $|T| < \kappa$, and hence it is isolated if T is. What remains to be proved is that, if $x \in \Pi(T) \setminus \Pi_\kappa(T)$ and T is clopen, then x is not closed. One can easily see by transfinite induction that, if there is such a set x in N, then there exists also an $x \in N \cap (\Pi_{\kappa+1}(T) \setminus \Pi_\kappa(T))$. Clearly $x \cap \Pi_\alpha(T) \neq x$ for all $\alpha < \kappa$. For all $\alpha < \kappa$, pick $y_\alpha \in x \setminus \Pi_\alpha(T)$: the set $Y = \{y_\alpha \mid \alpha < \kappa\}$ cannot have cluster points in $\Pi_\kappa(T) \cap N$, since $\Pi_\alpha(T) \cap N$ is open for all $\alpha < \kappa$. Hence either Y has size less than κ, whence $x \not\subseteq \Pi_\kappa(T)$, or x is not closed. $\square$

The remarkable fact that few set-theoretic properties can capture such a rich variety of subsets suggests the following definition.

Definition 1.2 (*Hyperuniverse*) Let A be a set of atoms. A set $N \subseteq \mathcal{P}(N) \cup A$ is a *hyperuniverse* if the following conditions are satisfied:

H0 $A, N \setminus A \in N$ (*atoms*);

H1 if $X \in N$, then $\{X\} \in N$ (*singletons*);

H2 if $X, Y \in N \cap \mathcal{P}(N)$, then $X \cup Y \in N$ (*binary union*);

H3 if $X \in \mathcal{P}(N \setminus A)$, then $\bigcap X \in N$ (*intersection*);

H4.1 if $X \in N \cap \mathcal{P}(N)$, then $N \cap \mathcal{P}(X) = \Box(X) \in N$ (*powerset*);

H4.2 if $X \in N \cap \mathcal{P}(N)$, then $\{Y \in N \mid X \cap Y \neq \emptyset\} = \Diamond(X) \in N$ (*dual powerset*);

H5 if $X \in N \cap \mathcal{P}(N)$, then $\bigcup X \in N$ (*internal union*);

H6 if $X, Y \in N \cap \mathcal{P}(N)$ and $X \cap Y = \emptyset$, then $\exists Z, W \in N \cap \mathcal{P}(N)$ such that $Z \cup W = N$, $X \cap Z = \emptyset$ and $Y \cap W = \emptyset$ (*separation*).

Actually, the conditions **H1-H5** in the above definion are quite natural and *prima facie* rather weak. Everybody should agree that a set-theoretic structure sufficiently self-descriptive and comprehensive to deserve the name "hyperuniverse" would satisfy all of them and moreover some kind of "weak complementation", such as the properties **H6.2** or **QC** below, both of which imply **H6**.

Definition 1.3 (*Additivity*) The cardinal $\kappa = min\{|X| \mid X \subseteq \mathcal{P}(N)\,, \bigcup X \notin N\}$ is called the *additivity* of the hyperuniverse N.

Clearly the additivity of a hyperuniverse is at least ω because of **H2**. Moreover, by property **H5**, the definition of additivity is equivalent to that of *fullness*, namely $\kappa = min\{|X| \mid X \subseteq N\,, X \notin N\}$

Theorem 1.2 *N is a κ-topological hyperuniverse if and only if N is a hyperuniverse of additivity κ.*

Proof First we show that a κ-topological hyperuniverse N satisfies the properties **H0-6**. Properties **H1, H2, H4.1-2, H5** are instances of the generalized positive comprehension schema GPCA, hence they follow from (ii) of Theorem 1.1. For sake of completeness, we give here a direct proof. Property **H0** holds by definition. Properties **H2** and **H3** trivially hold for the closed sets of any topology. Property **H6** merely rephrases in terms of closed sets the separation axiom T_4, which follows by κ-compactness. Then also Property **H1** follows at once. Properties **H4.1-2** hold since the sets $\Box(x)$ and $\Diamond(x)$ are closed in the Vietoris κ-Topology if x is closed. Moreover, since $\in$ is closed whenever the space is uniformizable (see e.g. [18] Ch.1 §17 Sec.IV Thm.1) and the domain of a closed set is closed, by κ-compactness, $\bigcup x = dom(\in \cap (N \times x))$ is in N for all $x \in N \cap \mathcal{P}(N)$. Finally the additivity of N is obviously κ.

Next we show that a hyperuniverse N according to Definition 1.2 satisfies the properties of Definition 1.1, where κ is taken to be the additivity of N according to Definition 1.3.

Clearly the complements, relative to N, of the elements of $N \cap \mathcal{P}(N)$ are the open sets of a κ-additive topology τ. This topology is T_4 by Property **H6** and it induces on $N \cap \mathcal{P}(N)$ a topology finer than the Vietoris κ-topology by Properties **H4.1-2**. Hence it suffices to prove that the Vietoris κ-topology is κ-compact, since then also the topology τ is κ-compact. As a topological space, N is in fact homeomorphic to the closed subspace of $N \cap \mathcal{P}(N)$ consisting of the singletons of the elements of N. Now, the Vietoris κ-topology is Hausdorff, since τ is T_4, and therefore it cannot be strictly coarser than τ on $N \cap \mathcal{P}(N)$.

Assume that the Vietoris κ-topology is not κ-compact: then there exists a regular cardinal $\mu \geq \kappa$, and a μ-sequence of closed sets $g_\alpha, \alpha < \mu$, which we can assume strictly decreasing and continuous at limits, such that $g_0 = N$ and $g_\mu = \bigcap_{\alpha<\mu} g_\alpha = \emptyset$.

Put $G_\beta = \{g_\alpha \mid \alpha < \beta\}$, $\mathcal{G} = \{\overline{G_\beta} \mid \beta < \mu\}$, $\mathcal{F} = \{(g_\beta, \overline{G_\beta}) \mid \beta < \mu\}$.

Now we show that $\mathcal{F}$ is closed while $\mathcal{G}$ is not, and since we have that $\mathcal{G} \subset \bigcup\bigcup \mathcal{F}$ and G_μ is a cluster point of $\mathcal{G}$ not belonging to $\bigcup\bigcup \mathcal{F}$, we obtain a contradiction. We break up the proof in several steps.

1. *$\forall \alpha < \kappa\ \forall \gamma < \alpha$ there is a neighbourhood of g_α disjoint from $\{g_\delta \mid \delta \leq \gamma$ or $\delta > \alpha\}$; hence only g_α's with limit α and $cof\,\alpha \geq \kappa$ can be cluster points.*

Proof. Pick $z \in g_\gamma \setminus g_\alpha$ and put $W = \Box(N \setminus \{z\}) \cap \Diamond(N \setminus g_{\alpha+1})$. Clearly W is an open neighbourhood of g_α disjoint from the set $\{g_\delta \mid \delta \leq \gamma \text{ or } \delta > \alpha\}$.

2. *G_β is closed if and only if g_β is not a cluster point; otherwise $\overline{G_\beta} = G_{\beta+1}$.*

Proof. Only non-empty sets can be limit points of g_β's. Now if $x \notin G_\mu$ is a non-empty set, then there exists δ such that $x \not\subseteq g_\alpha$ if and only if $\alpha \geq \delta$. Then $\delta = \beta + 1$, since the μ-sequence $\{g_\alpha\}_{\alpha<\mu}$ is continuous at limits. Thus take $U = (N \setminus \Box(g_\delta)) \cap \Box(N \setminus \{z\})$, where $z \in g_\beta \setminus x$. Clearly U is an open neighbourhood of x disjoint from G_μ. Hence G_μ is closed. By 1., if $\alpha > \beta$ then g_α has a neighbourhood disjoint from G_β. On the other hand, if g_β is a cluster point, then, by 1., β is a limit ordinal and $cof(\beta) \geq \kappa$, hence $g_\beta = \bigcap_{\alpha<\beta} g_\alpha$. But this amounts to the fact that in the Vietoris κ-topology g_β is a limit point of G_β.

3. *If X is a cluster point of $\mathcal{G}$, then there is a limit λ such that $X = \overline{G_\lambda}$.*

Proof. Since G_μ is closed, $\Box(G_\mu)$ is a closed set including $\mathcal{G}$, hence $X \subseteq G_\mu$. First we show that X is an "initial segment" of G_μ. Assume that there exist $\alpha < \beta < \mu$ such that $g_\alpha \notin X$ and $g_\beta \in X$: then put $U = \Box(N \setminus \{g_\alpha\}) \cap \Diamond(W)$, where W is an open neighbourhood of g_β disjoint from $\{g_\delta \mid \delta \leq \alpha\}$, which exists by 1.. Clearly U is an open neighbourhood of X, which is disjoint from $\mathcal{G}$. If X has no largest γ such that $g_\gamma \in X$, then $X = G_\lambda$ for limit λ and being closed it is also equal to $\overline{G_\lambda}$. On the other hand, suppose that there exists a largest γ such that $g_\gamma \in X$. By 1. and 2., we need only to show that g_γ is a cluster point of G_μ. Assuming the contrary, let W be an open neighbourhood of g_γ such that $G_\mu \cap W = \{g_\gamma\}$ and let U be an open neighbourhood of g_γ including X and disjoint from $\{g_\delta \mid \delta > \gamma\}$, which exists by 1.. Clearly $\Diamond(W) \cap \Box(U)$ is a neighbourhood of X disjoint from $\mathcal{G}$.

4. $G_\mu = \overline{G_\mu}$ *is a cluster point of* $\mathcal{G}$ *not belonging to* $\mathcal{G}$.

Proof. The fact that G_μ is closed was proved in 2., and clearly G_μ does not belong to $\mathcal{G}$. In order to show that it is a cluster point of $\mathcal{G}$, pick a neighbourhood of G_μ of the form $W = \bigcap_{\alpha<\eta<\kappa} \Box(U_\alpha) \cap \bigcap_{\beta<\lambda<\kappa} \Diamond(V_\beta)$, where the U_α's are open sets including G_μ and the V_β's are open sets intersecting G_μ. For each $\beta < \lambda$, pick $\gamma_\beta < \mu$ such that $g_{\gamma_\beta} \in V_\beta \cap G_\mu$ and put $\gamma = (sup_{\beta<\lambda}\gamma_\beta) + 1$. Then $\gamma < \mu$, since μ is regular, and clearly $G_\gamma \in W$.

5. $\mathcal{F}$ *is closed.*

Proof. A cluster point of $\mathcal{F}$ has to be a pair (x, X), where x is a cluster point of G_μ and X is a cluster point of $\mathcal{G}$. By 1. and 3. there are limit ordinals α and λ such that $x = g_\alpha$ and $X = \overline{G_\lambda}$. If $\alpha = \lambda$, then $(x, X) \in \mathcal{F}$. Otherwise, assuming $\alpha < \lambda$, let U be a neighbourhood of g_α disjoint from $\{g_\delta \mid \delta > \alpha\}$, existing by 1., and let V be a neighbourhood of $\overline{G_\lambda}$ disjoint from $\{\overline{G_\beta} \mid \beta \leq \alpha\}$, say $V = \Diamond(W)$, where W is an open neighbourhood of $g_{\alpha+1}$ disjoint from $\{g_\delta \mid \delta \leq \alpha\}$. Clearly $U \times V$ is an open neighbourhood of (x, X) disjoint from $\mathcal{F}$, contradiction. The case $\alpha > \lambda$ is dealt with similarly.

$\Box$

We point out the following facts:

Remark 1.1 Notice that the notion of hyperuniverse is not "first order", but rather *topological* or *set-theoretical*, since condition **H3** refers to the intersection of *arbitrary* families of subsets. A "first order approximation" of property **H3** can be obtained by introducing the schema

H3.θ $\forall x_1 \ldots \forall x_n\ \exists y\ \forall z\ (z \in y \leftrightarrow \forall x\ (\theta(x, x_1, \ldots, x_n) \to z \in x))$

where θ is *any* formula whose free variables are among $x, x_1, \ldots, x_n$ (see [6]).

Remark 1.2 One can easily see from the proof above that condition **H5**, which is crucial in obtaining κ-compactness, can be replaced equivalently by any of the following:

H5.1 if $G \in N \cap \mathcal{P}(N^2)$, then $dom(G) \in N$ (*domain*);

H5.2 if $G \in N \cap \mathcal{P}(N^2)$, then $cod(G) \in N$ (*codomain*);

H5.3 if $F, G \in N \cap \mathcal{P}(N^2)$, then $F \circ G \in N$ (*composition*);

H5.4 if $f \in N$ is a function and $X \in N$, then $\{f(x) \,|\, x \in X\} \in N$ (*replacement*);

and also by the apparently weaker

H5.5 if $G \in N \cap \mathcal{P}(N^2)$ is one-to-one, then $G \circ G^{-1} \in N$ (*identity's restrictions*).

Remark 1.3 Clearly, the axiom of complementation cannot be assumed consistently. Weak forms of "complementation" can be achieved via "separation" axioms. The property **H6** amounts to the Axiom of Separation T_4, which implies that the Vietoris topology is regular, hence Hausdorff. Equivalently one could have postulated the weaker property corresponding to T_3:

H6.1 if $x \in N$, $Y \in N \cap \mathcal{P}(N)$ and $x \notin Y$, then there exist $Z, W \in N \cap \mathcal{P}(N)$ such that $Z \cup W = N$, $x \notin Z$ and $Y \cap W = \emptyset$ (*regularity*).

Moreover, if $\kappa > \omega$, then N is 0-dimensional, hence it satisfies the property:

H6.2 if $X, Y \in N \cap \mathcal{P}(N)$ and $X \cap Y = \emptyset$, then $\exists Z \in N \cap \mathcal{P}(N)$ such that $N \setminus Z \in N$, $X \cap Z = \emptyset$ and $Y \subseteq Z$ (*0-dimensionality*).

The property **H6.2** has the flavour of an *approximation property*, and it amounts to asserting that the complementable sets of the Universe are enough to distinguish every set, namely:

QC Every set is the intersection of a family of complementable sets (*quasi-complementation*).

A surprisingly strong property can be obtained from the Axiom **QC** by postulating that families of size κ of complementable sets are sufficient to approximate sets:

Theorem 1.3 *N is a κ-ultrametrizable hyperuniverse if and only if N satisfies the properties* **H0-5** *and*

QC$^{\leq\kappa}$ *every element of $\mathcal{P}(N) \cap N$ is the intersection of at most κ complementable elements of N.*

Proof Clearly $\mathbf{QC}^{\leq\kappa}$ implies **H6**. By κ-compactness, any nested family of κ clopen subsets of N^2, whose intersection is Δ_N, is a uniformity basis for N, hence N is κ-metrizable. $\square$

Clearly the clopen subsets of N include all clopen sets of atoms and are stable with respect to complementation, $\Box$'s, $\Diamond$'s, $\bigcup_{<\kappa}$ and $\bigcap_{<\kappa}$. A strong "Minimality Principle" for hyperuniverses can be obtained by adding to the clopen approximability property **QC** the following *Axiom of Constructibility*:

CD the clopen subsets of N are the least subset of $\mathcal{P}(N)$ including a clopen subbasis of A, N, and stable with respect to $\Box$'s, $\Diamond$'s, $\bigcup_{<\kappa}$ and $\bigcap_{<\kappa}$ (*clopen describability*).

Remark 1.4 In hyperuniverses, Choice and Ordering Principles are strictly connected to metrizability, as shown in [14]. In fact we have the following theorem, whose proof is a mere rephrasing, to the case with atoms, of that of Theorems 3 and 4 of [14].

Theorem 1.4 *A topological κ-hyperuniverse N is κ-metrizable if and only if it satisfies any of the following properties:*

H7.1 *there exists $R \in N \cap \mathcal{P}(N^2)$ which is a total ordering of N* (ordering principle)*;*

H7.2 *there exists $F \in N$, $F : N^2 \to N$ such that $F(x,y) = F(y,x) \in \{x,y\}$ for all $x,y \in N$* (weak selection)*;*

H7.3 *there exists $F \in N$, $F : \Diamond(N) \to N$ such that $F(x) \in x$ for all $x \in \Diamond(N)$* (universal choice).

Remark 1.5 Given a binary relation $R \subseteq \mathcal{P}((N \setminus A)^2)$, define the relation:

$$R^+ = \{(x,y) \mid x \cap A = y \cap A \ \ \& \\ \forall t \in (x \setminus A).\exists s \in (y \setminus A).\ \ tRs \ \& \ \forall s \in (y \setminus A).\exists t \in (x \setminus A).\ \ tRs\}.$$

Compare the definition of R^+ with the definition of exponential uniformity of Definition 3(v). A *bisimulation* on N is a relation R such $R \subseteq R^+$. One can introduce the axiom of *strong extensionality up to atoms*, as follows:

H8 two sets are equal if they belong to some bisimulation.

Then a standard κ-compactness argument yields

Theorem 1.5 *A hyperuniverse N satisfies the property* **H8** *if and only if no element of the uniformity of N is a bisimulation.*

2 Quasi-hyperuniverses

The κ-compactness property of κ-hyperuniverses is crucial in showing strong comprehension properties, but it is clearly very demanding. On one hand, if $\kappa = \omega$, it excludes important spaces of atoms such as the natural numbers or the reals. On the other hand, if $\kappa > \omega$, κ-compactness necessitates "large cardinal" assumptions. In this section we generalize the notion of hyperuniverse to *non-compact* spaces.

Definition 2.1 (κ*-topological Quasi-Hyperuniverse*) Let A be a set of atoms. A set $N \subseteq \mathcal{P}(N) \cup A$ is a κ*-topological quasi-hyperuniverse* if the elements of $N \cap \mathcal{P}(N)$ are the closed subsets of a κ-additive T_1 topology τ on N, such that A is a clopen subset of N.

The interest of quasi-hyperuniverses appears once suitable topological properties are assumed. Clearly the κ-topological hyperuniverses are precisely those κ-topological quasi-hyperuniverses whose topology is κ-compact.

Definition 2.2 (*Special Quasi-Hyperuniverses*) Let N be a κ-topological quasi-hyperuniverse, τ the natural topology on N, υ the corresponding Vietoris κ-topology on $N \setminus A$, and τ_+ the topology induced by τ on $N \setminus A$:
i) N is κ*-Vietoris* if τ_+ is finer than υ;
ii) N is κ*-Bolzano* if every subset of N of size κ has a cluster point;
iii) N is κ*-Tychonoff* if τ is induced by a κ-uniformity $\mathcal{U}$ such that τ_+ is induced by the exponential uniformity $\mathcal{U}^+$;
iv) N is κ*-Hausdorff* if τ is induced by a bounded κ-metric ρ such that τ_+ is induced by the Hausdorff κ-distance ρ_H.

Various classes of κ-topological quasi-hyperuniverses can be characterized by purely set-theoretic properties, as was the case for κ-hyperuniverses.

Theorem 2.1 *N is a κ-topological quasi-hyperuniverse if and only if it has additivity $\geq \kappa$ and satisfies the properties* **H0-3**. *Moreover:*

i) N is κ-Vietoris if and only if it satisfies **H0-4**;
ii) N is both κ-Bolzano and κ-Vietoris if and only if it has additivity κ and it satisfies the properties **H0-4** *and*

H5$^{\leq\kappa}$ *if $X \in N \cap \mathcal{P}(N)$ and $|X||\bigcup X| \leq \kappa$, then $\bigcup X \in N$;*

iii) if N is κ-Tychonoff, then N satisfies **H4.1** *and* **H6.1** (**H6.2** *if $\kappa > \omega$*).

Proof
i) Immediate.
ii) We call κ-Bolzano a space which satisfies the property that every subset of size at least κ has a cluster point. The validity of **H0-4** in κ-Bolzano quasi-hyperuniverses is proved as in Theorem 1.2. The validity of **H5$^{\leq\kappa}$** can be shown following the lines of Theorem 1.2. Simply take into account that a closed subset of a κ-Bolzano space is κ-Bolzano, that κ-Bolzano sets of size less than or equal to κ are closed, and that images under continuous functions of κ-Bolzano subsets are κ-Bolzano.

In order to prove the reverse implication, we recall that the κ-Bolzano Property is equivalent to the non-existence of a strictly decreasing κ-sequence of closed subsets with empty intersection. Therefore we can proceed as in the proof of Theorem 1.2 and take $\mu = \kappa$, thus obtaining that $\mathcal{F}, \mathcal{G}$ and all G_β's have size not exceeding κ.
iii) Property **H6.1** is immediate (and if $\kappa > \omega$ also **H6.2**, since then any κ-uniformity has a basis of equivalences). To show **H4.1** we prove that $\Diamond(G)$ is open for G open. Let $X \cap G \neq \emptyset$ and pick $x \in X \cap G$. Let $U \in \mathcal{U}$ be such that $U(x) \subseteq G$. Then $U^+(X) \subseteq \Diamond(G)$. In fact, if $Y \in U^+(X)$, then there is $y \in Y$ such that $(y, x) \in U$ and hence $y \in G \cap Y$. $\square$

Theorem 2.2 *Let N be a κ-topological quasi-hyperuniverse:*
i) N is κ-Hausdorff if and only if N is κ-Tychonoff with respect to a κ-metrizable uniformity;
ii) if N is paracompact (in particular if N is κ-Hausdorff or 0-dimensional) and κ-Tychonoff w.r.t. the universal uniformity, then N is κ-Vietoris;
iii) if N is κ-Bolzano, κ-Vietoris, and κ-metrizable, then N is a κ-hyperuniverse, hence also κ-Hausdorff;
iv) there exist κ-Hausdorff quasi-hyperuniverses which are not κ-Vietoris.

Proof
i) Use the fact that if a bounded κ-metric ρ induces the uniformity $\mathcal{U}$, then the Hausdorff κ-metric ρ_H induces the exponential uniformity $\mathcal{U}^+$.
ii) We only need to show property **H4.2**. Since N is paracompact, the universal uniformity $\mathcal{U}$ includes all open entourages of the diagonal ([5], Theorem 5.1.12). We show that $\Box(G)$ is open for G open. Let $X \in N$ be such that $X \subseteq G$. For each $x \in X$, pick $U_x \in \mathcal{U}$ such that $U_x(x) \subseteq G$. Put

$$U = (G \setminus X)^2 \cup \bigcup_{x \in X} (U_x(x))^2 ,$$

then $U^+(X) \subseteq \Box(G)$. In fact, if $(Y, X) \in U^+$, then $\forall t \in Y. \exists s \in X (t, s) \in U$, hence there exists $x \in X$ such that both $t, s \in U_x(x)$; therefore $t \in G$.
iii) Since N is κ-metrizable, the κ-Bolzano property implies κ-compactness of the topology τ. Now, the Vietoris κ-topology on N is Hausdorff and coarser than τ; hence it coincides with τ on $N \cap \mathcal{P}(N)$. Finally, if ρ is any bounded κ-metric inducing τ on N, then ρ_H induces the Vietoris κ-topology on $N \cap \mathcal{P}(N)$.
iv) Let ${}^*\mathbf{R}$ be a (non-standard) model of the *reals* of cofinality κ, let A be isometric to ${}^*\mathbf{R}$ with the eucledean κ-metric ρ, and let $\mathcal{U}$ be the κ-uniformity induced by ρ. In the quasi-hyperuniverse $\mathcal{N}_\kappa(\mathcal{U})$ of Theorem 3.1 below the set $\Diamond(\Delta_A)$ is not closed. E.g. pick an increasing unbounded κ-sequence $< x_\alpha >_{\alpha<\kappa}$ in ${}^*\mathbf{R}^+$ and put $X = \{(x, x + \frac{1}{x}) \mid x \in {}^*\mathbf{R}^+\}$. Clearly the κ-sequence $< X \cup \{(x_\alpha, x_\alpha)\} >_{\alpha<\kappa}$ converges to X in the Hausdorff κ-metric.
$\Box$

Although non-κ-compact quasi-hyperuniverses lack many comprehension properties, we still have the following theorem, whose proof we omit since it is similar to that of Theorem 1.1:

Theorem 2.3 *Let N be a κ-Tychonoff quasi-hyperuniverse. Then*
i) Any subset of N of size less than κ belongs to N. The cartesian product and the union of less than κ elements of $N \cap \mathcal{P}(N)$ belong to N.
ii) $N \cap \mathcal{P}(N)$ is closed under arbitrary intersections and $\Box$'s (and also $\Diamond$'s if N is κ-Vietoris). Moreover both the relations of membership and inclusion between members of N are members of N, as well as the identity, the singleton map, and all projections, permutations and regroupings of n-tuples.

iii) All continuous functions with closed domain belong to N.[1]
iv) Internal and external cardinalities agree below κ. Moreover κ is the set of all ordinals of N and its closure is the set $\overline{\kappa} = \kappa \cup \{\overline{\kappa}\} \in N$. Finally the cardinality of N is greater than or equal to 2^κ.

3 Existence Theorems

In this section we give existence theorems for various kinds of quasi-hyperuniverses. More precisely we describe a construction for obtaining quasi-hyperuniverses starting from any uniform structure on the set A of atoms, which generalizes the one for κ-ultrametric spaces given in [11]. Under suitable assumptions, which are specified in the theorems below, this construction produces quasi-hyperuniverses satisfying several of the axioms of Section 1.

Let $\mathcal{U}$ be a uniformity on the set of atoms A. For any cardinal μ, let $e_0(\mu) = \mu + 1$, $e_{\alpha+1}(\mu) = \mu + 2^{e_\alpha(\mu)}$, and $e_\lambda(\mu) = \Pi_{\alpha<\lambda} e_\alpha(\mu)$ for limit λ. Put $\nu = e_{\kappa+1}(|A|)$ and $S = H(\nu^+)$, where $H(\nu^+) = \{x| \; |TC(x)| \leq \nu\}$. Since we are assuming FCA, S is a set (see e.g. [10] I). We endow the set $S \cup A$ with a "pseudo-uniform" topology. To this end, define inductively, for each $U \in \mathcal{U}$, the κ-sequence of entourages $< U_\alpha >_{\alpha<\kappa}$ as follows: $U_0 = U \cup (S \times S)$, $U_{\alpha+1} = U \cup (U_\alpha^+ \cap (S \times S))$ and $U_\lambda = \bigcap_{\alpha<\lambda} U_\alpha$ for limit λ.[2] Similarly to [11], we define on $S \cup A$ the relations $\approx_\mathcal{U}$ ($\mathcal{U}$*-equivalence*) and $\in_\mathcal{U}$ ($\mathcal{U}$*-membership*) as follows:

$$x \approx_\mathcal{U} y \text{ if and only if } \forall U \in \mathcal{U} \;\; \forall \alpha < \kappa \; (x, y) \in U_\alpha;$$

$$x \in_\mathcal{U} y \text{ if and only if} \begin{cases} \exists x' \approx_\mathcal{U} x \; \exists y' \approx_\mathcal{U} y. \; x' \in y' \\ \text{ or equivalently} \\ \forall U \in \mathcal{U} \; \forall \alpha < \kappa \; \exists x' \in y. \; (x, x') \in U_\alpha \end{cases}$$

The relation $\approx_\mathcal{U}$ is indeed an equivalence and it induces the identity on atoms. Clearly, it is reflexive and symmetric. Moreover, since $\mathcal{U}$ is a uniformity, for all $U \in \mathcal{U}$ there exists $W \in \mathcal{U}$ such that $W \circ W \subseteq U$ and, by induction, the same inclusion holds at level α, using the fact that $W^+ \circ W^+ \subseteq (W \circ W)^+$. Hence $\approx_\mathcal{U}$ is transitive.

[1] *Caveat*: also continuous functions with non-closed domain, and even non-continuous functions, can have closed graphs and hence can be in N.

[2] See Definition 3(v) of the Appendix for the notation U^+.

The family $\{U_\alpha | \alpha < \kappa\ ,\ U \in \mathcal{B}\}$, where $\mathcal{B}$ is any basis of the uniformity $\mathcal{U}$, is a *pseudo-κ-uniformity basis* on $S \cup A$, which allows for nice topological characterizations of both $\mathcal{U}$-membership and $\mathcal{U}$-equivalence. It is easily seen that $x, y \in S$ are $\mathcal{U}$-equivalent if and only if they have the same closure. Moreover, the $\mathcal{U}$-members of any $x \in S$ are exactly the *ordinary* members of its closure (compare to [11], Lemma 1.1).

Define the function $f : S \to \mathcal{P}(S \cup A)$ by $f(x) = \{y \in S \cup A \mid y \in_{\mathcal{U}} x\}$ and let g be the unique function given by the Axiom FCA. The transitive set $\mathcal{N}_\kappa(\mathcal{U}) = A \cup cod(g)$ provides a *transitive collapse* of the *quotient structure* $(S \cup A, \in_{\mathcal{U}})/ \approx_{\mathcal{U}}$. Put $\sigma = g \cup \Delta_A$: a straightforward generalization of [11] yields

Theorem 3.1 *The function $\sigma : S \cup A \to \mathcal{N}_\kappa(\mathcal{U})$ is the unique function satisfying the following properties:*
i) for all $x \in S \cup A$, $x \approx_{\mathcal{U}} \sigma(x)$;
ii) for all $x, y \in S \cup A$, $\sigma(y) = \sigma(x)$ if and only if $x \approx_{\mathcal{U}} y$;
iii) for all $x, y \in S \cup A$, $\sigma(y) \in \sigma(x)$ if and only if $x \in_{\mathcal{U}} y$.
In particular $\in_{\mathcal{U}}$-membership on $\mathcal{N}_\kappa(\mathcal{U})$ agrees with ordinary membership. The entourages $\{U_\alpha \cap (\mathcal{N}_\kappa(\mathcal{U}) \times \mathcal{N}_\kappa(\mathcal{U})) \mid U \in \mathcal{U},\ \ \alpha < \kappa\}$ are the basis of a κ-uniformity $\mathcal{W}$ on $\mathcal{N}_\kappa(\mathcal{U})$ inducing on A the original uniformity. For all $x \subseteq \mathcal{N}_\kappa(\mathcal{U})$ the closure of x is $\sigma(x)$. Moreover $\mathcal{W}^+$ is the uniformity induced by $\mathcal{W}$ on $\mathcal{N}_\kappa(\mathcal{U}) \setminus A$. Hence $\mathcal{N}_\kappa(\mathcal{U})$ is a κ-Tychonoff quasi-hyperuniverse, including as a clopen subset the space $N_\kappa = \{x \in \mathcal{N}_\kappa(\mathcal{U}) \mid TC(x) \cap A = \emptyset\}$.[3]

Proof First of all we show that $|\mathcal{N}_\kappa(\mathcal{U})| \leq \nu$ and hence $\mathcal{P}(\mathcal{N}_\kappa(\mathcal{U})) \subseteq S$, in particular $\sigma(x) \in S$ for all $x \in S$. To this end we introduce the decreasing κ-sequence of equivalence relations $< E_\alpha >_{\alpha \leq \kappa}$ as follows: $E_0 = \Delta_A \cup (S \times S)$, $E_{\alpha+1} = \Delta_A \cup (E_\alpha^+ \cap (S \times S))$, and $E_\lambda = \bigcap_{\alpha<\lambda} E_\alpha$ for limit λ. Clearly $E_\alpha \subseteq U_\alpha$ for all $\alpha < \kappa$ and all $U \in \mathcal{U}$, hence $\approx_{\mathcal{U}}$ is coarser than E_κ. Call ν_α the number of equivalence classes *modulo* E_α. An easy computation yields: $\nu_0 =| A | +1$, $\nu_{\alpha+1} =| A | +2^{\nu_\alpha}$, and $\nu_\lambda \leq \Pi_{\alpha<\lambda}\nu_\alpha$ for limit λ; hence $\nu_\kappa \leq \nu$.

In order to prove item i) we show, by induction on α, that for all $U \in \mathcal{U}$ and all $x \in S \cup A$, $(x, \sigma(x)) \in U_\alpha$. The initial and the limit steps are true by definition. In order to show the successor step, given $U \in \mathcal{U}$, pick $W \in \mathcal{U}$ such that $W \circ W \subseteq U$. Now, for all $s \in x$, we have that $\sigma(s) \in \sigma(x)$ and $(s, \sigma(s)) \in U_\alpha$ by induction hypothesis. On the other hand, for any $t \in \sigma(x)$,

[3] The space N_κ was introduced in [8].

there is $y \in_{\mathcal{U}} x$ such that $t = \sigma(y)$. Pick $s \in x$ such that $(s, y) \in W_\alpha$: then $(s, t) \in W_\alpha \circ W_\alpha$, therefore $(x, \sigma(x)) \in U_\alpha^+$.
The remaining assertions are straightforward, taking into account that $\sigma(a) = a$ if a is an atom and $\sigma(x) = \{\sigma(y) \mid y \in_{\mathcal{U}} x\}$ for $x \in S$. $\square$

Taking into account the characterisations of Theorem 2.2, we obtain the following embedding theorems in special quasi-hyperuniverses:

Corollary 3.1 *Any κ-additive Tychonoff space X is homeomorphic to the space of atoms of a κ-Tychonoff quasi-hyperuniverse N. If moreover X is paracompact, then N can be taken κ-Vietoris. Finally, N can be taken κ-Hausdorff if and only if X is κ-metrizable.*

If $\kappa = \omega$ and the topology on the space X is compact, the previous construction yields a compact hyperuniverse. Actually we have:

Theorem 3.2 *A topological space X is homeomorphic to the space of atoms of a ω-hyperuniverse N if and only if X is compact. Moreover:*
i) N can be taken (ultra-)metrizable if and only if X is (ultra-)metrizable;
ii) N can be taken 0-dimensional if and only if X is 0-dimensional;
iii) in all cases N can be chosen so as to satisfy the axioms **H8** *and* FCA*;*
iv) if X is 0-dimensional, then N can be chosen so as to satisfy the axioms **QC** *and* **CD***.*

Proof Assertions i) and ii) follow from the fact that the construction outlined in this section preserves compactness, (ultra-)metrizability, and 0-dimensionality. In order to prove iii), one can adapt the proof of Theorem 5 in [12], so as to take care of the presence of atoms. To show iv), notice that X, being compact and 0-dimensional, has a uniformity basis $\mathcal{B}$ made up by equivalences having a finite number of clopen classes. The proof then runs exactly as in the case of uncountable κ (see below). $\square$

If $\kappa > \omega$, even if the topolgy on the space X is κ-compact, the previous construction does not provide in general a κ-compact hyperuniverse. By Theorem 1.1.(v), if $\kappa \not\rightarrow (\kappa)^2_2$, then there are no κ-hyperuniverses. However we have:

Theorem 3.3 *Let κ be an uncountable regular cardinal. Every κ-additive Tychonoff space X is homeomorphic to the space of atoms of a κ-Tychonoff, κ-Vietoris, 0-dimensional quasi-hyperuniverse N. Moreover:*
i) if κ is strongly compact, then N can be taken to be a κ-hyperuniverse if and only if X is κ-compact;
ii) N can be taken to be a κ-Hausdorff hyperuniverse if and only if $\kappa \to (\kappa)^2_2$, X is κ-metrizable and κ-compact;
iii) in all cases the hyperuniverse N can be chosen so as to satisfy the axioms **H8**, FCA, **QC** *and* **CD**.

Proof Since $\kappa > \omega$, the universal uniformity $\mathcal{U}$ on any κ-Tychonoff space has a basis $\mathcal{B}$ made up of equivalences. Then for all $U \in \mathcal{B}$ and $\alpha < \kappa$ the U_α's are equivalences, hence the quasi-hyperuniverse $N = N_\kappa(\mathcal{U})$ is κ-Tychonoff, κ-Vietoris, and 0-dimensional. Moreover, if the atom space is κ-compact, then every U_α has less than κ clopen classes.

In order to achieve i), we need only to prove that N is κ-compact. Let $\mathcal{C}$ be a family of closed subsets of N such that the intersection of less than κ members of $\mathcal{C}$ is non-empty: we have to show that $\mathcal{C}$ has non-empty intersection. Let $\mathcal{D}$ be the least κ-complete filter including $\mathcal{C}$, and let $\mathcal{F}$ be a κ-complete ultrafilter refining $\mathcal{D}$, which exists since κ is strongly compact. For each $U \in \mathcal{B}$ and $\alpha < \kappa$ there is exactly one U_α-class in $\mathcal{F}$, say F^U_α. Moreover, either all F^U_α's are included in A (and we call $\mathcal{F}$ *atom-like*), or all of them are disjoint from A (and we call $\mathcal{F}$ *set-like*). By κ-compactness of A, if $\mathcal{F}$ is atom-like, then $\bigcap F^U_\alpha \neq \emptyset$. We show now that the same holds for set-like ultrafilters. Let $\mathcal{S}$ be the set of all set-like κ-complete ultrafilters on N, and define the function $f : \mathcal{S} \to \mathcal{P}(\mathcal{S} \cup A)$ by

$$f(x) = \{\mathcal{G} \in \mathcal{S} \mid \forall U \in \mathcal{B}\ \forall \alpha < \kappa\ \exists s \in G^U_\alpha\ \exists x \in F^U_{\alpha+1}.\ s \in x\} \cup \{a \in A \mid \forall U \in \mathcal{B}\ \forall \alpha < \kappa.\ a \in F^U_\alpha\}.$$

Let g be the unique function given by the axiom FCA: an easy cardinality argument proves that the range of g is a transitive set of size not exceeding κ, hence $g : \mathcal{S} \to S$. The same inductive argument used in the proof of Theorem 3.1 yields that $(\sigma(g(\mathcal{F})), F^U_\alpha) \in U_\alpha$ for all $U \in \mathcal{B}$ and all $\alpha < \kappa$. Therefore $\sigma(g(\mathcal{F})) \in \bigcap F^U_\alpha \subseteq \bigcap \mathcal{C}$, and i) is proved.

The proof of ii) is given in [11], where the construction of $N_\kappa(\mathcal{U})$ is performed using the canonical basis of the uniformity $\mathcal{U}$ induced by a κ-ultrametric on A.

In order to prove iii), one can adapt the proof of Theorem 5 in [12] so as to take care of atoms, as for the case of $\kappa = \omega$; one then obtains that FCA holds in N, hence also **H8** follows.

Let $\mathcal{A}$ be the least family of subsets of N which includes all equivalence classes modulo U for all $U \in \mathcal{B}$, and is closed under $\Box$'s, $\Diamond$'s, $\bigcup_{<\kappa}$ and $\bigcap_{<\kappa}$. Denoting by $[x]^U_\alpha$ the equivalence class of x modulo U_α, we have by definition $[x]^U_{\alpha+1} = \bigcap_{s\in x} \Diamond([s]^U_\alpha) \cap \Box(\bigcup_{s\in x}[s]^U_\alpha)$ and, for limit λ, $[x]^U_\lambda = \bigcap_{\alpha<\lambda}[x]^U_\alpha$. Since all U_α's have less than κ classes, it follows that all classes belong to $\mathcal{A}$. Now, by κ-compactness, every clopen set is the union of less than κ classes, and the property **CD** follows. Finally **QC** holds in any 0-dimensional space. $\Box$

4 Applications

4.1 Fixed Point Theorems in Hyperuniverses

In this subsection we generalize the fixed point theorems of [15] to arbitrary hyperuniverses. Thereby, also general hyperuniverses can be used as frameworks for providing semantics for concurrent languages, in that they allow to solve naturally the recursive (domain) equations used for defining processes.

The first theorem can be seen as a generalization of Tarski's Fixed Point Theorem; in general it does not yield uniqueness of fixed points.

Theorem 4.1 *Let N be a κ-hyperuniverse, let $f \in N$ be a function and let $\leq \in N$ be a partial order such that f is monotone w.r.t. $\leq$. Put $D = \{x \in dom(f) \mid x \leq f(x)\}$ and assume that any λ-chain in D with $\lambda < \kappa$ has an upper bound in D. Then, for all $x \in D$, there is $z \geq x$ such that $z = f(z)$.*

Proof Given $x \in D$, pick a κ-sequence $< x_\alpha >_{\alpha<K}$ in D such that $x_0 = x$, $x_{\alpha+1} = f(x_\alpha)$, and, for limit $\lambda < \kappa$, $\quad x_\lambda \geq x_\gamma \; \forall\gamma < \lambda$.
If the κ-sequence is not eventually constant, by κ-compactness it has a cluster point z, say. Since f is uniformly continuous, $(z, f(z))$ is a cluster point of $\{(x_\alpha, x_{\alpha+1})\}_{\alpha<\kappa}$. Similarly $(f(z), z)$ is a cluster point of $\{(x_{\alpha+1}, x_\beta)\}_{\alpha+1<\beta<\kappa}$. Hence both $(z, f(z))$ and $(f(z), z)$ are in $\leq$. $\Box$

It is interesting to notice that Theorem 4.1 holds also for *κ-Bolzano quasi-hyperuniverses*, since only κ-sequences are involved in its proof.

Using the above theorem, a plethora of reflexive domain equations, involving many different set-theroretic operators, are immediatly solvable in the setting of κ-hyperuniverses. Theorem 4.1 applies in particular whenever $(D, \leq)$ is a κ-complete upper semilattice (i.e. every subset of size $< \kappa$ has a l.u.b.), and it provides a κ-complete upper semilattice of fixed points. In case $\leq$ is $\subseteq$ ($\supseteq$), it amounts to a special case of Tarski's Fixed Point Theorem. It yields immediately fixed points to e.g. the following functions:

$F_1(X) = (X_0 \cup X) \times \mathcal{P}(X) \times \Diamond(X), \qquad F_2(X) = X_0 \cup cod(X) \cup X^{-1}.$

It is interesting to consider also Banach-type fixed point theorems [4] in the context of general hyperuniverses. To this end we need to introduce notions corresponding to the "k-Lipschitz" condition.

Definition 4.1 Let N be a κ-Tychonoff quasi-hyperuniverse, let $\mathcal{B}$ be a basis for the κ-uniformity $\mathcal{U}$ of N and let $\nu : \mathcal{B} \to \mathcal{U}$ be a monotone map, such that $\{\nu(U) \mid U \in \mathcal{B}\}$ is again a uniformity basis.
A function $f \in N$ is *ν-controlled* if $\forall U \in \mathcal{B}.\ (x, y) \in U \Rightarrow (f(x), f(y)) \in \nu(U)$.
The function $f \in N$ is *$\mathcal{B}$-nonexpansive* if f is $\Delta_{\mathcal{B}}$-controlled.
A function $f \in N$ is *$\mathcal{B}$-contractive* if f is ν-controlled for some ν satisfying $\bigcap_{\alpha<\kappa} U_\alpha = \Delta_N$, where $U_0 = N^2$, $U_{\alpha+1} = \nu(U_\alpha)$ and $U_\lambda = \bigcap_{\alpha<\lambda} U_\alpha$.

In every κ-Tychonoff quasi-hyperuniverse N, the set $X \to_\nu Y$ of all ν-controlled functions from to X to Y is an element of N for any ν. Hence in particular the set of all $\mathcal{B}$-nonexpansive functions belongs to N. Theorem 4.1 allows to solve domain equations involving function spaces such as $X = \{f : N \to_\nu N \mid f(x) \in A \text{ for all } x \in X\}$.

A map ν, as in the definition of $\mathcal{B}$-contractive functions, can exist in a κ-hyperuniverse N only if N is κ-metrizable. Hence Banach's Fixed Point Theorem can be stated as follows:

Theorem 4.2 *Let N be a hyperuniverse and let $\mathcal{B}$ be a uniformity basis for N. Any $\mathcal{B}$-contractive $f \in N$ such that $cod(f) \subseteq dom(f)$ has a unique fixed point.*

The above theorem is surprisingly powerful: it allows to solve domain equations which involve "mixed variance" constructors (e.g. $X = A \cup$

[4] *If (X, δ) is a complete metric space and $f : X \to X$ is k-Lipschitz for $k < 1$ (i.e. $\delta(f(x), f(y)) \leq k\delta(x, y)\ \forall x, y \in X$), then f has a unique fixed point.*

$(X \rightarrow_{\Delta_{\mathcal{B}}} X))$. Notice that Theorem 4.2 holds for any *complete* ω-Hausdorff quasi-hyperuniverse: it would be interesting to compare its strength to that of [3].

4.2 Hyperuniversal Categories

A categorical viewpoint is often very useful for giving and investigating the semantics of programs. In fact, in providing semantics to programming languages, one has to introduce simultaneously both *domains* of denotations as well as *interpretation functions*. In this subsection we associate to hyperuniverses suitable categories and we attempt to recast, in the setting of hyperuniverses, the *final semantics* approach of [2, 22, 16]. Final coalgebra theorems are crucial tools in this approach. They allow to prove uniformly the existence of interpretation functions and they lead to coinductive characterizations of significant equivalences on programs.

Using the strong closure properties of hyperuniverses, we can immediately view them as categories as follows:

Definition 4.2 (*Hyperuniversal Categories*) Let N be a κ-Tychonoff quasi-hyperuniverse and let $\mathcal{B}$ be a uniformity basis for N.
i) The category $\mathbf{C}_N$ having the elements of N as objects and the functions of N as arrows is called the *external category* associated to N.
ii) The category $\mathbf{C}_N^{\mathcal{B}}$ having the elements of N as objects and the $\mathcal{B}$-nonexpansive functions of N as arrows is called the *internal category* associated to N, $\mathcal{B}$.

Using the hyperuniversal categories defined above, and the fixed points theorems of Subsection 4.1, we recover two *final coalgebra theorems* in the style of [2, 22] for suitable endofunctors. First of all we give a *special final coalgebra theorem*:

Theorem 4.3 *Let N be a κ-topological quasi-hyperuniverse satisfying* FCA*. Let $F = (F_{obj}, F_{arr})$ be an endofunctor on $\mathbf{C}_N$, which is inclusion preserving and uniform on maps.*[5] *Then F_{obj} has a greatest fixed point X, such that (X, Δ_X) is the final F-coalgebra.*

[5] I.e. if $j : A \to B$ is an inclusion map, then $F_{arr}(A, B)(j) : F_{obj}(A) \to F_{obj}(B)$ is also an inclusion map. The technical notion of "uniform on maps functor" is given in [2].

Proof The function F_{obj} has a greatest fixed point X by Theorem 4.1. Using Axiom FCA, one can adapt the proof of Theorem 4.22 of [22] and show that (X, Δ_X) is the final F-coalgebra. $\square$

Actually, in proving the above theorem one simply uses Tarski's Fixed Point Theorem. Using Theorem 4.1 in its full generality, one can get many "weakly" final coalgebras theorems. Using instead Theorem 4.2, we can state a *metric final coalgebra theorem*:

Theorem 4.4 *Let N be a hyperuniverse and let $\mathcal{B}$ be a uniformity basis for N. Let $F = (F_{obj}, F_{arr})$ be an endofunctor on $\mathbf{C}_N^{\mathcal{B}}$, such that F_{obj} and all F_{arr}'s are $\mathcal{B}$-contractive. Then F_{obj} has a unique fixed point X, such that (X, Δ_X) is the final F-coalgebra.*

Proof The function F_{obj} has a unique fixed point X by Theorem 4.2. The F-coalgebra (X, Δ_X) can be seen to be final by modifying the proof of Theorem 5.5 of [22]. $\square$

Remark 4.1 The category $\mathbf{C}_N$ of Definition 4.2i) is indeed *external*, in the sense that the hom-sets $\mathcal{C}_N(A, B)$ in general *are not* elements of N. On the other hand, the hyperuniversal category $\mathbf{C}_N^{\mathcal{B}}$ of Definition 4.2ii) is *internal* in the sense that all *hom-sets* are *objects* of $\mathbf{C}_N^{\mathcal{B}}$ and, moreover, the operations of *domain* and *codomain* of arrows, the operation which assigns to every object the *identity* arrow, and all *composition operations* $Comp_{ABC} : \mathbf{C}_N^{\mathcal{B}}(B, C) \times \mathbf{C}_N^{\mathcal{B}}(A, B) \to \mathbf{C}_N^{\mathcal{B}}(A, C)$, are themselves *arrows* of $\mathbf{C}_N^{\mathcal{B}}$. It would be interesting to pursue further the possibility of adequately internalizing categorical notions in hyperuniverses.

5 Final Remarks

1. Interesting game theoretic accounts of extensionality in hyperuniverses can be given by reformulating the game originally introduced in [7]. For $x, y \in V$, we define the following two-players game $G_{x,y}$. If x, y are both non-empty sets, then Player I picks a set E_1 such that $dom(E_1) = x$ and $cod(E_1) = y$. Presented with E_1, Player II picks a pair $(x_1, y_1) \in E_1$. The game is over if Player I is presented with a pair not consisting of two non-empty sets. In this case, Player I wins if presented with $(x, y) \in \Delta_{A \cup \{\emptyset\}}$, otherwise Player II wins. If the game goes on indefinitely, then Player I wins. One can easily see that the Axiom of strong extensionality **H8** can be rephrased as follows:

H8.1 Two sets x and y are equal if and only if Player I has a winning strategy in the game $G_{x,y}$.

Using the notation of Section 3 one can see that, whenever two sets x, y are in the relation E_{n+1} of the proof of Theorem 3.1, then Player I can postpone defeat in $G_{x,y}$ for n moves. Hence we can characterize the equivalence $\approx_{\mathcal{U}}$ on S, in the case $\kappa = \omega$, as follows:

Theorem 5.1 *Let $\mathcal{U}$ be a uniformity on A having a countable basis. For all $x, y \in S$, $x \approx_{\mathcal{U}} y$ if and only if for all $n \in \omega$ Player I can postpone defeat for n moves in $G_{x,y}$.*

In particular the following theorem holds in $\mathcal{N}_\omega(\mathcal{U})$:

Theorem 5.2 *If Player I can postpone defeat arbitrarily, then Player I has a winning strategy in $G_{x,y}$. In particular $\mathcal{N}_\omega(\mathcal{U})$ satisfies the axiom:*

H8.2 *Two sets x and y are equal if and only if for all $n \in \omega$ Player I can postpone defeat for n moves in $G_{x,y}$.*

2. In this paper we consider at the outset theories with atoms, because they seem more natural in view of the applications. However, topological spaces could be embedded faithfully also in atomless hyperuniverses by suitably modifying the theorems in [13]. This amounts to use some kind of non-wellfounded sets in place of *atoms*, e.g. *self-singletons*. Hence, one is forced to give up the axiom FCA and to assume instead in the ground universe more involved Anti-foundation Axioms. In general the resulting hyperuniverses do not satisfy precious structural principles such as the principle of clopen describability **CD** and the strong extensionality axiom **H8**.

3. The κ-Bolzano condition follows from the "Maximal Union Principle" of [24], which asserts that the cardinality of the hyperuniverse is *minimal* with respect to its additivity. Namely we have:

Theorem 5.3 *Any κ-Vietoris quasi-hyperuniverse of cardinality 2^κ is κ-Bolzano. In particular N is a κ-metrizable κ-hyperuniverse if and only if N has cardinality 2^κ and satisfies the conditions* **H0-4** *together with any of* **H7.1-3**.

Proof We proceed by contradiction. Assume that $I \subseteq N$ is a set of κ isolated points. Then $\mathcal{P}(I) \subseteq N$ consists of clopen sets. Then every filter over I is closed, and hence is in N. To see this, suppose that J is a cluster point of a filter $\mathcal{F}$. Then $J \subseteq I$ since $\Box(I) = \mathcal{P}(I)$ is closed and hence $\Box(J)$ is a clopen neighbourhood of J. Therefore there exists $F \in \mathcal{F}$ such that $F \subseteq J$. Since there are 2^{2^κ} filters over I, the hyperuniverse N cannot have cardinality 2^κ. $\square$

4. It would be interesting to investigate further *approximation properties* in quasi-hyperuniverses, such as those which have been dealt with in [12] for the special case of the hyperuniverses N_κ:

H9.1 Every set is approximated by sets wellfounded over A (*weak foundation*).

H9.2 Every set is approximated by isolated sets (*pseudo-foundation*).

H9.3 Every set is approximated by sets of hereditary cardinal $< \kappa$ (*small approximation*).

For instance, under the hypothesis on X and κ of Theorems 3.2 and 3.3, one can replace FCA by **H9.1** in item iii) of each of the two theorems: simply take the closure in $\mathcal{N}_\kappa(\mathcal{U})$ of the set $A \cup WF$, where WF consists of the wellfounded sets over A.

Appendix

Here, we recall some basic definitions and facts from Set Theory and General Topology. For more details and complete proofs we refer to [5, 17].

Throughout this appendix let X be a set, let τ be a topology on X, and let κ be an infinite regular cardinal.

Definition 1 Let κ be an infinite regular cardinal.
i) κ has the *partition property* $\kappa \to (\kappa)^2_2$ if for any function $p : [\kappa]^2 \to$

$\{0,1\}$ there is a subset $H \subseteq \kappa$ of size κ such that p is constant on $[H]^2$; or equivalently if every tree of size κ, in which each chain has less than κ immediate successors, has a κ-branch.
ii) κ is *strongly compact* if any κ-complete filter on a κ-complete field of sets is included in a κ-complete ultrafilter.

Notice that ω is strongly compact by the Boolean Prime Ideal Theorem. Notice also that if κ is strongly compact, then $\kappa \to (\kappa)^2_2$, and hence κ is strongly inaccessible.

Definition 2 i) A *filter* on X is a non empty set $\mathcal{F} \subset \mathcal{P}(X)$ such that:
a) $A \in \mathcal{F}$ and $B \supseteq A \quad \Rightarrow \quad B \in \mathcal{F}$;
b) $A, B \in \mathcal{F} \quad \Rightarrow \quad A \cap B \in \mathcal{F}$.
(The filter generated by a set S is the least filter containing S).
ii) A filter is *κ-complete* if it is closed under intersections of length $< \kappa$.
iii) A *uniformity* on X is a filter $\mathcal{U}$ of *entourages* (i.e. symmetric and reflexive binary relations on X) satisfying the following conditions:
a) $\bigcap \mathcal{U} = \Delta_X \equiv \{(x,y) \in X \times X \mid x = y\}$ (separation);
b) $\forall U \in \mathcal{U}\ \exists V \in \mathcal{U}.\ V \circ V \subseteq U$ (triangular inequality),
where $\circ$ is the composition of relations.
iv) The *topology* $\tau_{\mathcal{U}}$ induced by the uniformity $\mathcal{U}$ has as open sets $\{A \subseteq X \mid \forall x \in A\ \exists U \in \mathcal{U}\ \ U(x) \subseteq A\}$, where $U(x) = \{y \mid (x,y) \in U\}$.
v) A *κ-uniformity* is a uniformity which is *a κ-complete filter.* A topology is *κ-additive* (shortly a κ-topology) if the intersection of less than κ open sets is open. If $\mathcal{U}$ is a κ-uniformity then $\tau_{\mathcal{U}}$ is a κ-topology.
vi) A *(κ-)uniformity basis* $\mathcal{B}$ is a set of entourages such that the filter generated by $\mathcal{B}$ is a (κ-)uniformity.
vii) A κ-metric (resp. κ-ultrametric) on X is a symmetric function $\rho : X^2 \to {}^*\mathbf{R}$ such that ${}^*\mathbf{R}$ has cofinality κ, satisfying $\rho(x,y) = 0 \Rightarrow x = y$ and the triangular inequality (resp. $\rho(x,z) \leq max\{\rho(x,y),\ \rho(y,z)\}$) inequality.
viii) The (κ)-uniformity *generated* by the (κ-)metric ρ has a canonical basis $\mathcal{B} = \{U_\alpha \mid \alpha < \kappa\}$, where $U_\alpha = \{(x,y) \in X \times X \mid \delta(x,y) < \epsilon_\alpha\}$ and $< \epsilon_\alpha >_{\alpha<\kappa}$ is a strictly decreasing κ-sequence with infimum 0.

Definition 3 i) A topology is *0-dimensional* if it has a basis of clopen (i.e. simultaneously closed and open) sets.
ii) A κ-topology is *κ-compact* if every open cover of X has a subcover of

cardinality less than κ.
iii) The *exponential space* $\mathcal{P}_{cl}(X)$ of X is the space of all closed subsets of X.
iv) The *Vietoris (κ-)topology* on the exponential space $\mathcal{P}_{cl}(X)$ is the coarsest (κ-)topology such that $\Box(A) = \mathcal{P}_{cl}(X) \cap \mathcal{P}(A)$ is open for every open set A of X and $\Box(F)$ is closed for every closed set F of X. A natural subbasis for this topology is given by the sets $\Box(A)$ and $\Diamond(A) = \{C \in \mathcal{P}_{cl}(X) \mid A \cap C \neq \emptyset\}$, for A open.
v) The *exponential uniformity* $\mathcal{U}^+$ on the exponential space $\mathcal{P}_{cl}(X)$ corresponding to the κ-uniformity $\mathcal{U}$ on X has a basis consisting of the sets

$$U^+ = \{(A, B) \in \mathcal{P}_{cl}(X) \times \mathcal{P}_{cl}(X) \mid (\forall x \in A \exists y \in B.(x, y) \in U) \wedge (\forall y \in B \exists x \in A.(y, x) \in U)\}, \text{ where } U \in \mathcal{U}.$$

$\mathcal{U}^+$ is a κ-uniformity whenever $\mathcal{U}$ is.
vi) The *Hausdorff κ-metric* ρ_H on $\mathcal{P}_{cl}(X)$ corresponding to the bounded κ-metric ρ on X is defined as

$$\rho_H(A, B) = max\{sup_{x \in A} inf_{y \in B} \rho(x, y), sup_{y \in B} inf_{x \in A} \rho(y, x)\}.$$

Proposition 1 (see [5] 2.7.20 and 3.12.26) *Let τ be a κ-topology on X and let υ be the Vietoris κ-topology on $\mathcal{P}_{cl}(X)$. Then:*
i) υ is T_1 if and only if τ is T_1;
ii) υ is Hausdorff if and only if τ is T_3;
iii) υ is T_3 if and only if τ is T_4;
iv) υ is T_4 if and only if τ is κ-compact;
v) υ is κ-compact if and only if τ is κ-compact.

Proposition 2 *Let τ be a (κ-)topology on X.*
i) If τ is κ-compact, and σ is a κ-topology on X strictly coarser than τ, then σ is not Hausdorff.
ii) There can be more than one *or possibly* no *uniformity which induces τ. However, if (X, τ) is κ-compact, then there is* exactly one *κ-uniformity inducing τ.*
iii) If $\mathcal{U}$ induces τ on X, then the topology induced on $\mathcal{P}_{cl}(X)$ by the exponential uniformity $\mathcal{U}^+$ can be different from the Vietoris topology. If (X, τ) is κ-compact then the Vietoris κ-topology is induced by the exponential κ-uniformity.
iv) If the topology τ is induced by a uniformity $\mathcal{U}$, then τ is 0-dimensional if and only if $\mathcal{U}$ has a basis of equivalences. This is always the case if $\mathcal{U}$ is a κ-uniformity for some $\kappa > \omega$. If $\mathcal{U}$ is generated by a bounded κ-metric ρ,

then $\mathcal{U}^+$ is generated by the corresponding Hausdorff κ-metric ρ_H.
v) The topology τ is κ-metrizable (resp. κ-ultrametrizable) if and only if τ is induced by a κ-uniformity $\mathcal{U}$ having a nested basis of size κ (resp. consisting of equivalences). If κ is uncountable, then any κ-metrizable space is κ-ultrametrizable.

References

[1] S.Abramsky, *A Cook's tour of finitary non-wellfounded sets*, unpublished manuscript, 1988.

[2] P.Aczel, *Non-well-founded sets*, CSLI Lecture Notes **14**, Stanford 1988.

[3] P.America, J.J.M.M.Rutten, Solving domain equations in a category of complete metric spaces, *J. Comp. Sys. Sci.* **39** (1989), 343-375.

[4] J.W.de Bakker, J.I.Zucker, Processes and denotational semantics of concurrency, *Info. Contr.* **54** (1982), 70-120.

[5] R.Engelking, *General Topology*, Polish Scientific Publishers, Warszawa 1977.

[6] O.Esser, *An interpretation of the Zermelo-Frænkel set theory and the Kelley-Morse set theory in a positive theory*, manuscript, 1995.

[7] T.Forster, *Axiomatizing set Theory with a Universal Set*, manuscript, 1983.

[8] M.Forti, R.Hinnion, The consistency problem for positive comprehension principles, *J. Symb. Logic* **54** (1989), 1401-1418.

[9] M.Forti, F.Honsell, Set theory with free construction principles, *Ann. Scuola Norm. Sup. Pisa*, Cl. Sci. (4) **10** (1983), 493-522.

[10] M.Forti, F.Honsell, Axioms of choice and free construction principles I, *Bull. Soc. Math. Belg.* **36** (B) (1984), 69-79; II, Ibid. **37** (B) (1985), fasc.2, 1-12; III, Ibid. **39** (B) (1987), 259-76.

[11] M.Forti, F.Honsell, Models of Selfdescriptive Set Theories, in *Partial Differential Equations and the Calculus of Variations, Essays in Honor of Ennio De Giorgi*, I (F.Colombini et al. eds.), Birkhäuser, Boston 1989, 473-518.

[12] M.Forti, F.Honsell, Weak Foundation and Antifoundation Properties of Positively Comprehensive Hyperuniverses, in *L'Anti-fondation en Logique et en Théorie des Ensembles*, (R.Hinnion ed.), Cahiers du Centre de Logique **7**, Louvain-la-Neuve 1992, 31-43.

[13] M.Forti, F.Honsell, *A general construction of Hyperuniverses*, to appear in *Theoretical Computer Science* **156**.

[14] M.Forti, F.Honsell, Choice Principles in Hyperuniverses, *Annals of Pure and Applied Logic* **77** (1996), 35-52.

[15] M.Forti, F.Honsell, M.Lenisa, Processes and Hyperuniverses, in *Mathematical Foundations of Computer Science 1994* (I.Prívara et al., eds.), Springer Lecture Notes in Computer Science **841**, Berlin 1994, 352-361.

[16] F.Honsell, M.Lenisa, Final Semantics for Untyped Lambda Calculus, in *TLCA '95* (M.Dezani, G.Plotkin eds.), Springer Lecture Notes in Computer Science **902**, Berlin 1995, 249-265.

[17] T.J.Jech, *Set theory*, Academic Press, New York 1978.

[18] K.Kuratowski *Topology*, Academic Press, New York 1966.

[19] M.Lenisa, Final Semantics for a Higher Order Concurrent Language, in *CAAP '96* (H.Kirchner ed.), Springer Lecture Notes in Computer Science **1059**, Berlin 1996, 202-218 .

[20] R.J.Malitz, *Set theory in which the axiom of foundation fails*, Ph.D. Thesis, UCLA, Los Angeles 1976 (unpublished).

[21] M.W.Mislove, L.S.Moss, F.J.Oles, Non-wellfounded Sets Modeled as Ideal Fixed Points, *Information and Computation* **93**, (1991), 16-54.

[22] J.J.M.M.Rutten, D.Turi, On the Foundations of Final Semantics: Non-Standard Sets, Metric Spaces, Partial Orders, *REX Conference Proceedings* (J.de Bakker et al. eds.), Springer Lecture Notes in Computer Science **666**, 1993, 477-530.

[23] F.W.Stevenson, W.J.Thron, Results on ω_μ-metric spaces, *Fundam. Math.* **65** (1969), 317-324.

[24] E.Weydert, *How to approximate the naive comprehension scheme inside of classical logic,* Bonner mathematische Schriften **194**, Bonn 1989.