Distributivity for endofunctors, pointed and co-pointed endofunctors, monads and comonads

Marina Lenisa [1]

Dipartimento di Matematica e Informatica
Università di Udine
I-33100 Udine, ITALY

John Power [2]

Laboratory for the Foundations of Computer Science, Division of Informatics
University of Edinburgh
King's Buildings, Edinburgh EH9 3JZ, SCOTLAND

Hiroshi Watanabe [3]

Osaka LERC
Electrotechnical Laboratory
Amagasaki 661-0974, JAPAN

Abstract

We generalise the notion of a distributive law between a monad and a comonad to consider weakened structures such as pointed or co-pointed endofunctors, or endofunctors. We investigate Eilenberg-Moore and Kleisli constructions for each of these possibilities. Then we consider two applications of these weakened notions of distributivity in detail. We characterise Turi and Plotkin's model of GSOS as a distributive law of a monad over a co-pointed endofunctor, and we analyse generalised coiteration and coalgebraic coinduction "up-to" in terms of a distributive law of the underlying pointed endofunctor of a monad over an endofunctor.

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1 Introduction

The notion of distributivity between a monad and a comonad plays an important role in the theory of coalgebraic methods in computer science [17,18]. But sometimes one has less data than that of a monad or a comonad, and sometimes one requires fewer commutativities, for instance having a pointed endofunctor or, dually, a co-pointed endofunctor. A pointed endofunctor has structure that is weaker than that of monad but stronger than that of an endofunctor. The notion was studied in Kelly [9], where he studied the free generation of a monad from a pointed endofunctor.

In computer science, distributivities involve such weakened notions have appeared recently in Turi and Plotkin's model of GSOS in programming languages without variable binding [22], in operations on (initial) algebras [3], and in work on generalised coiteration and coalgebraic coinduction "up-to"[13].

A distributive law of a monad (T, μ, η) over a comonad (D, δ, ϵ) on a category C consists of a natural transformation $TD \Rightarrow DT$ satisfying four commutative diagrams involving μ, η, δ and ϵ. The weakenings involving removing some of the axioms, as one only has some of the data.

Previously [17,18], we gave a systematic treatment of distributivity for a monad and a comonad. We analysed, in terms of a monad and a comonad in a 2-category, Eilenberg-Moore and Kleisli constructions. Here we extend that framework to pointed/co-pointed endofunctors and endofunctors, yielding a systematic treatment. We also analyse Eilenberg-Moore and Kleisli constructions for them, giving categories of algebras for pointed endofunctors, coalgebras for co-pointed endofunctors and Kleisli categories for pointed/co-pointed endofunctors in the 2-category of small categories.

An Eilenberg-Moore coalgebra for a comonad is a coalgebra $(X, k : X \to HX)$ which satisfies two commutative diagrams, one for each of the comultiplication and the counit. A coalgebra for a co-pointed endofunctor is a coalgebra for which only the diagram for the counit need commute.

In [17,18], a distributive law between a monad and a comonad was shown to be a 0-cell in a 2-category given by combining the monad and comonad constructions. Here we shall classify distributivities in a 2-categorical framework in the same manner. The abstract theory we give here is mathematically easy; but its value lies in providing substantial simplification and elucidation of applications. So we study two applications in detail.

One is an application to GSOS by Turi and Plotkin [22]. Given a syntax endofunctor Σ and a behaviour endofunctor B, they modelled a GSOS rule by an abstract operational rule, which is a natural transformation $\Sigma(B \times \mathrm{Id}) \Rightarrow BT$, where T is the free (syntax) monad on Σ. They showed that this natural transformation determines a distributivity λ of the monad T over the comonad D, the cofree comonad on B, giving a category λ-*Bialg* of λ-bialgebras that provides a combined operational and denotational model of the language.

In constructing their distributive law, they constructed a lifting of T to

B-coalg, the category of coalgebras for B, from the abstract operational rule $\Sigma(B \times \mathrm{Id}) \Rightarrow BT$, by using a special type of recursion (Theorem 5.1 in [22]).

We apply our framework to show that their abstract operational rule $\Sigma(B\times \mathrm{Id}) \Rightarrow BT$ is equivalent to a distributive law of the monad T over the co-pointed endofunctor $B \times \mathrm{Id}$ cofreely generated by B. This distributivity also implies Turi and Plotkin's distributive law of T over D. So we shall give a simple proof avoiding the recursion theorem.

Our other application lies in generalised coiteration and coalgebraic coinduction "up-to". We consider the work of [13]. Here, we express the key condition as a distributive law of the underlying pointed endofunctor of a monad T over an endofunctor F. This allows us to strengthen the main result of [13] and clarify the construction. The role of T and F is to provide an F-bisimulation up to T, and the result provides a sound coinduction principle for such bisimulations. We give a range of examples to show how this works.

In Section 2, we define the notion of pointed/co-pointed endofunctor and endofunctor in a 2-category. We refer an "endofunctor in a 2-category" as a 1-cell of a 2-category. So it needs not have anything to do with endofunctors. The same applies for pointed/co-pointed endofunctor. But we do make this convention through this paper. Then we introduce the 2-categories $\mathsf{pEndo}(K)$, $\mathsf{pEndo}^*(K)$, $\mathsf{copEndo}(K)$, $\mathsf{copEndo}^*(K)$, $\mathsf{Endo}(K)$ and $\mathsf{Endo}^*(K)$. We also explain the Eilenberg-Moore and Kleisli constructions and liftings to them. In Section 3, we show an analysis of distributive law of a monad over a co-pointed endofunctor by using our 2-categorical framework.

We introduce the notion of free pointed endofunctor and cofree co-pointed endofunctor in Section 4.

We give an application to Turi and Plotkin's work in Section 5. Their work can be captured as a distributivity of a monad over a co-pointed endofunctor. This observation simplifies the construction of distributive law of the free monad T on Σ over the cofree comonad D on B.

We finally investigate generalised coiteration and coalgebraic coinduction "up-to" in Section 6.

2 Eilenberg-Moore and Kleisli constructions in a 2-category

We need the basic notion of 2-categories (the definition, op, co-constructions, 2-functors and 2-adjunctions). Please see the Appendix 1 or Kelly and Street's review [10].

2.1 Monads and comonads

We give an overview of monads and comonads and the 2-categorical framework in which to treat them (for details and explicit definitions please see [17,18]). Let K be a 2-category.

Definition 2.1 A *monad* (C, T, μ, η) in a 2-category K consists of a 0-cell

C, a 1-cell T on it, two 2-cells $\mu : T^2 \Rightarrow T$ and $\eta : \mathrm{Id} \Rightarrow T$ subject to $\mu \cdot T\mu = \mu \cdot \mu T$, $\mu \cdot T\eta = \mathrm{id}_T$ and $\mu \cdot \eta T = \mathrm{id}_T$.

In the above, the usual definition of a monad on a small category is captured by the case of $K = Cat$.

A 2-category $\mathsf{Mnd}(K)$ of monads in K is as follows. A 0-cell is a monad in K, a 1-cell from (C, T, μ, η) to (C', T', μ', η') is a monad morphism, i.e., a 1-cell $J : C \to C'$ and a 2-cell $j : T'J \Rightarrow JT$ in K subject to two coherence conditions, and a 2-cell from (J, j) to (H, h) is a 2-cell $\alpha : J \Rightarrow H$ in K satisfying one coherence condition.

Using the op-construction, a 2-category $\mathsf{Mnd}^*(K)$ is defined by $\mathsf{Mnd}^*(K) = \mathsf{Mnd}(K^{\mathrm{op}})^{\mathrm{op}}$. Each 0-cell of this 2-category is also a monad in K.

If we reverse the directions of a 1-cell and 2-cells in the definition of a monad in K, we have a definition of a comonad in K.

In the same manner, we can define the 2-category $\mathsf{Cmd}(K)$ of comonads in K as $\mathsf{Cmd}(K) = \mathsf{Mnd}(K^{\mathrm{co}})^{\mathrm{co}}$. We can also define $\mathsf{Cmd}^*(K)$ as $\mathsf{Cmd}^*(K) = \mathsf{Cmd}(K^{\mathrm{op}})^{\mathrm{op}}$. Both 2-categories $\mathsf{Cmd}(K)$ and $\mathsf{Cmd}^*(K)$ have comonads in K as 0-cells.

2.1.1 Eilenberg-Moore/Kleisli constructions

For each of the 2-categories $\mathsf{Mnd}(K)$, $\mathsf{Mnd}^*(K)$, $\mathsf{Cmd}(K)$ and $\mathsf{Cmd}^*(K)$, there is a 2-functor Inc from the base 2-category K to the 2-category which sends each 0-cell X of K to the identity monad/comonad on the 0-cell X in K.

By using the four 2-functors Inc, we can characterise these four notions.

- the Eilenberg-Moore constructions for monads/comonads,
- the Kleisli constructions for monads/comonads.

For example, a 2-category K *admits Eilenberg-Moore constructions for monads* if the 2-functor $Inc : K \to \mathsf{Mnd}(K)$ has a right 2-adjoint. Similarly, K *admits Eilenberg-Moore constructions for comonads* if $Inc : K \to \mathsf{Cmd}(K)$ has a right 2-adjoint.

The validity and the reason for this definition is because Cat admits Eilenberg-Moore constructions for both monads and comonads. The right 2-adjoint functor to each 2-functor Inc sends each monad/comonad to the category of Eilenberg-Moore algebras/coalgebras for the monad/comonad.

Dually, we say a 2-category K *admits Kleisli constructions for monads* if the 2-functor $Inc : K \to \mathsf{Mnd}^*(K)$ has a left 2-adjoint. Similarly, K *admits Kleisli constructions for comonads* if the 2-functor $Inc : K \to \mathsf{Cmd}^*(K)$ has a left 2-adjoint.

The validity and the reason for this definition is because Cat admits Kleisli constructions for both monads and comonads. The left 2-adjoints send each monad/comonad to its Kleisli category respectively.

For the above four 2-functors Inc, we call each image 0-cell in K under the right 2-adjoint functor an *Eilenberg-Moore construction*, the image 0-cell under the left a *Kleisli construction*.

2.1.2 Liftings to Eilenberg-Moore/Kleisli constructions

Suppose a 2-category K admits Eilenberg-Moore constructions for monads, i.e., the 2-functor $Inc : K \to \mathsf{Mnd}(K)$ has a right 2-adjoint.

Given 0-cells $\mathbb{T} = (C, T, \mu, \eta), \mathbb{T}' = (C', T', \mu', \eta')$ in $\mathsf{Mnd}(K)$, a 1-cell $J : C \to C'$ *lifts to a 1-cell* $\bar{J} : \mathbb{T}\text{-Alg} \to \mathbb{T}'\text{-Alg}$ *on Eilenberg-Moore constructions* if $U^{T'}\bar{J} = JU^T$ holds, where U^T is the 1-cell part of the (C, T, μ, η)-component of the counit. When $K = Cat$, $U^T : \mathbb{T}\text{-Alg} \to C$ is the forgetful functor from the category of Eilenberg-Moore algebras for a monad (T, μ, η) to its base category C.

We can also define the notion of liftings for 2-cells. Suppose 1-cells $J, H : C \to C'$ lift to $\bar{J}, \bar{H}$ respectively on Eilenberg-Moore constructions. A 2-cell $\alpha : J \Rightarrow H : C \to C'$ *lifts to a 2-cell* $\bar{\alpha} : \bar{J} \Rightarrow \bar{H}$ *on Eilenberg-Moore constructions* if the equation $U^{T'}\bar{\alpha} = \alpha U^T$ holds.

The right 2-adjoint functor sends each 1-cell (J, j) in $\mathsf{Mnd}(K)$ to a 1-cell in K which is a lifting of J, and each 2-cell $\alpha : (J, j) \Rightarrow (H, h)$ in $\mathsf{Mnd}(K)$ to a lifting of $\alpha : J \Rightarrow H$. Moreover this map on 1-cells and 2-cells is bijective, so it gives an equivalence between lifting 1-cells in K and the 1-cells in $\mathsf{Mnd}(K)$, and lifting 2-cells in K and 2-cells in $\mathsf{Mnd}(K)$. These results yield an adjunction between the construction and its base 0-cell in K, and yield adjoint liftings [8,10].

We can also apply these results to Eilenberg-Moore constructions for comonads, by reversing the argument, to Kleisli constructions. For detail please see [17,18].

2.1.3 Distributive laws of a monad and a comonad

Definition 2.2 Given a monad (T, μ, η) and a comonad (D, δ, ϵ) on an object C of a 2-category K, a *distributive law of* T *over* D is a 2-cell $\lambda : TD \Rightarrow DT$ such that the four elementary diagrams involving each of μ, η, δ and ϵ commute.

This distributive law can be captured as a 0-cell in each of the 2-categories $\mathsf{CmdMnd}(K) \cong \mathsf{MndCmd}(K)$, $\mathsf{Cmd}^*\mathsf{Mnd}(K)$ and $\mathsf{Mnd}^*\mathsf{Cmd}(K)$.

2.2 Pointed endofunctors, co-pointed endofunctors

If we forget the multiplication and co-multiplication in the definition of monad and comonad, we can define the notion of a pointed/co-pointed endofunctor.

Definition 2.3 A *pointed endofunctor* in a 2-category K consists of a 0-cell C, a 1-cell $T : C \to C$, and a 2-cell $\eta : \mathrm{Id} \Rightarrow T$

Definition 2.4 A *co-pointed endofunctor* in a 2-category K consists of a 0-cell C, a 1-cell $D : C \to C$, and a 2-cell $\epsilon : D \Rightarrow \mathrm{Id}$

We can also define 2-categories of pointed endofunctors and co-pointed endofunctors and dual 2-categories for each of them.

Definition 2.5 For any 2-category K, the following data forms a 2-category $\mathsf{pEndo}(K)$ of pointed endofunctors:

- A 0-cell (C, T, η) is a pointed endofunctor in K.
- A 1-cell in $\mathsf{pEndo}(K)$ from (C, T, η) to (C', T', η') consists of a 1-cell $H : C \to C'$ together with a 2-cell $h : T'H \Rightarrow HT$ in K satisfying $h \cdot H\eta = \eta' H$.
- A 2-cell in $\mathsf{pEndo}(K)$ from (H, h) to (H', h') is a 2-cell $\alpha : H \Rightarrow H'$ in K satisfying $h' \cdot T'\alpha = \alpha T \cdot h$.

We can also give another definition of a 2-category of pointed endofunctors by using the op-construction as follows:

Definition 2.6 $\mathsf{pEndo}^*(K) = \mathsf{pEndo}(K^{\mathrm{op}})^{\mathrm{op}}$

For co-pointed endofunctors, we can also define 2-categories of co-pointed endofunctors by using co-constructions and op-constructions as follows:

Definition 2.7 We define $\mathsf{copEndo}(K) = \mathsf{pEndo}(K^{\mathrm{co}})^{\mathrm{co}}$, and $\mathsf{copEndo}^*(K) = \mathsf{copEndo}(K^{\mathrm{op}})^{\mathrm{op}}$.

Example 2.8 When $K = Cat$ a pointed endofunctor consists of an endofunctor $T : C \to C$ and a natural transformation $\eta : \mathrm{Id} \Rightarrow T$, and a co-pointed endofunctor consists of an endofunctor $D : C' \to C'$ and a natural transformation $\epsilon : D \Rightarrow \mathrm{Id}$.

2.2.1 Eilenberg-Moore/Kleisli constructions

As in Section 2.1.1, we can define Eilenberg-Moore/Kleisli constructions for pointed endofunctors and co-pointed endofunctors.

For each of the above 2-categories $\mathsf{pEndo}(K)$, $\mathsf{pEndo}^*(K)$, $\mathsf{copEndo}(K)$ and $\mathsf{copEndo}^*(K)$, there is a 2-functor Inc from the base 2-category K to the 2-category which sends each 0-cell X of K to the identity pointed/co-pointed endofunctor on it. For each of $\mathsf{pEndo}(K)$ and $\mathsf{copEndo}(K)$, the forgetful 2-functor U to K always has the 2-functor Inc as a left 2-adjoint.

Definition 2.9

- A 2-category K *admits Eilenberg-Moore constructions for pointed endofunctors* if the 2-functor $Inc : K \to \mathsf{pEndo}(K)$ has a right 2-adjoint.
- A 2-category K *admits Eilenberg-Moore constructions for co-pointed endofunctors* if the 2-functor $Inc : K \to \mathsf{Cmd}(K)$ has a right 2-adjoint.

Example 2.10 Cat admits Eilenberg-Moore constructions for pointed endofunctors. For the 2-functor $Inc : Cat \to \mathsf{pEndo}(Cat)$, the right 2-adjoint functor sends each pointed endofunctor (C, T, η) to the category of (T, η)-algebras for the pointed endofunctor. Here the category (T, η)-*alg* of (T, η)-*algebras for a pointed endofunctor* (T, η) on a category C is defined as follows:

- Each object is a T-algebra $(A, k : TA \to A)$ satisfying $k \circ \eta_A = \mathrm{id}_A$.
- Arrows are morphisms of T-algebras.

Similarly, Cat admits Eilenberg-Moore constructions for co-pointed endofunctors. The right 2-adjoint functor sends each co-pointed endofunctor (C, D, ϵ) to the category (D, ϵ)-*coalg* of (D, ϵ)-coalebras for the co-pointed endofunctor. Here the category (D, ϵ)-*coalg* of (D, ϵ)-*coalgebras for a co-pointed endofunctor* (D, ϵ) on a category C is defined as follows:

- Each object is a D-coalgebra $(A, k : A \to DA)$ satisfying $\epsilon_A \circ k = \mathrm{id}_A$.
- Arrows are morphisms of D-algebras.

Dually, for each of the $\mathsf{pEndo}^*(K)$ and $\mathsf{copEndo}^*(K)$, the forgetful 2-functor U to K always has the 2-functor Inc as a right 2-adjoint.

Definition 2.11

- A 2-category K *admits Kleisli constructions for pointed endofunctors* if the 2-functor $Inc : K \to \mathsf{pEndo}^*(K)$ has a left 2-adjoint.
- A 2-category K *admits co-Kleisli constructions for co-pointed endofunctors* if the 2-functor $Inc : K \to \mathsf{copEndo}^*(K)$ has a left 2-adjoint.

The 2-category Cat admits both Kleisli/co-Kleisli constructions for pointed/co-pointed endofunctors. By using the left adjoint 2-functors we can characterise both Kleisli categories for pointed endofunctors and co-Kleisli category for co-pointed endofunctors.

Proposition 2.12 *The Kleisli category* (T, η)-Kl *for a pointed endofunctor* (T, η) *is given as follows:*

- *Each object of* (T, η)-Kl *is the same as that of* C.
- *Let* a, b *be objects of* (T, η)-Kl. *The homset from* a *to* b *is given by the quotient set* $(T, \eta)\text{-Kl}(a, b) = \amalg_{n \in \mathbb{N}} C(a, T^n b)/\sim_{a,b}$, *where* $\sim_{a,b}$ *is the smallest equivalence relation on* $\amalg_{n \in \mathbb{N}} C(a, T^n b)$ *generated by* $f \sim_{a,b} \eta_{T^m b} \circ f \sim_{a,b} T\eta_{T^{m-1} b} \circ f \sim_{a,b} \cdots \sim_{a,b} T^{m-1}\eta_b \circ f$ *for each* $f : a \to T^m b$ *in* C *for* $m \in \mathbb{N}$.

We write the equivalence class of f *by* $[f]$. *For arrows* $[f] : a \to b$ *and* $[g] : b \to c$ *in* (T, η)-Kl, *represented by arrows* $f : a \to T^m b$ *and* $g : b \to T^n c$ *in* C *for some* $m, n \in \mathbb{N}$, *we define the composition* $[g][f]$ *in* (T, η)-Kl *by the equivalence class of the arrow* $(T^m g) \circ f : a \to T^{m+n} c$ *in* C. *The identity arrow on* a *seen as an object of* (T, η)-Kl *is the equivalence class of the identity arrow* id_a *in the category* C.

Proof. See Appendix 2. □

Remark 2.13 The characterisation of a *co-Kleisli category for a co-pointed endofunctor* is given by reversing the arrows in the above Proposition.

2.2.2 Liftings

Suppose K admits Eilenberg-Moore constructions for pointed endofunctors. For each pointed endofunctor (C, T, η) in K, we call the image, (T, η)-*alg* under the right 2-adjoint functor the *Eilenberg-Moore construction for the pointed endofunctor.*

Definition 2.14 Let (C, T, η) and (C', T', η') be 0-cells in $\mathsf{pEndo}(K)$. A 1-cell $J : C \to C'$ in K *lifts to a* 1*-cell* $\bar{J}$ *on Eilenberg-Moore constructions* if the following diagram commutes in K.

$$\begin{tikzcd}
(T,\eta)\text{-}alg \arrow[r, "\bar{J}"] \arrow[d, "U^{(T,\eta)}"'] & (T',\eta')\text{-}alg \arrow[d, "U^{(T',\eta')}"] \\
C \arrow[r, "J"'] & C',
\end{tikzcd}$$

where $U^{(T,\eta)}$ is the 1-cell part of the (T, η)-component of the counit.

Example 2.15 When $K = Cat$, each $U^{(T,\eta)} : (T, \eta)\text{-}alg \to C$ is the forgetful functor.

Definition 2.16 Suppose both 1-cells $J, H : C \to C'$ lift to $\bar{J}, \bar{H}$ respectively on Eilenberg-Moore constructions. A 2-cell $\alpha : J \Rightarrow H$ *lifts to a* 2*-cell* $\bar{\alpha} : \bar{J} \Rightarrow \bar{H}$ *on Eilenberg-Moore constructions* if the equation $U^{(T',\eta')}\bar{\alpha} = \alpha U^{(T,\eta)}$ holds.

Proposition 2.17 *The right* 2*-adjoint functor from* $\mathsf{pEndo}(K)$ *to* K *sends each* 1*-cell* $(J, j) : (C, T, \eta) \to (C', T', \eta')$ *to the lifting of* J*, and each* 2*-cell* $\alpha : (J, j) \Rightarrow (H, h)$ *to a lifting of* $\alpha : J \Rightarrow H$.

Outline of Proof. These results follows from the 2-naturality of the counit for 1-cells and 2-cells. □

Remark 2.18 The above results are the same as the results obtained in the case of $\mathsf{Mnd}(K)$ and $\mathsf{Cmd}(K)$ in Section 2.1.2 (or see [18]). Although every 1-cell in $\mathsf{pEndo}(K)$ determines a lifting 1-cell if Eilenberg-Moore constructions exist, the converse does not hold. This is in contrast to the situation for monads, and the reason is that we can not use the argument of adjoint lifting because, in general, there is no left adjoint to the 1-cell $U^{(T,\eta)} : (T, \eta)\text{-}alg \to C$ in K.

Hence we conclude

Corollary 2.19 *If* K *admits Eilenberg-Moore constructions for pointed endofunctors, then every* 1*-cell in* $\mathsf{pEndo}(K)$ *determines a lifting* 1*-cell. Every* 2*-cell in* $\mathsf{pEndo}(K)$ *determines a lifting* 2*-cell.*

Remark 2.20 Similarly we can define the notion of lifting to Eilenberg-Moore constructions for co-pointed endofunctors, and we can define the notion of lifting to Kleisli constructions using $\mathsf{pEndo}^*(K)$ and $\mathsf{copEndo}^*(K)$.

2.3 Endofunctors

Forgetting the unit and the counit of the definitions of pointed endofunctor and co-pointed endofunctor, we have just endofunctors. We can also provide a 2-categorical framework here.

Definition 2.21 For any 2-category K, the following data forms a 2-category $\mathsf{Endo}(K)$ of *endofunctors in* K:

- Each 0-cell (C, T) consists of a 0-cell C and a 1-cell $T : C \to C$ in K.
- A 1-cell from (C, T) to (C', T') consists of a 1-cell $J : C \to C'$ together with a 2-cell $j : T'J \Rightarrow JT$ in K.
- A 2-cell from (J, j) to (J', j') is a 2-cell $\alpha : J \Rightarrow J'$ in K subject to $\alpha T \cdot j = j' \cdot T'\alpha$.

We can also define another 2-category of endofunctors in K by using the op-construction as follows:

Definition 2.22 $\mathsf{Endo}^*(K) = \mathsf{Endo}(K^{\mathrm{op}})^{\mathrm{op}}$.

Proposition 2.23

- $\mathsf{Endo}(K^{\mathrm{co}})^{\mathrm{co}} = \mathsf{Endo}^*(K)$,
- *Hence* $\mathsf{Endo}(K^{\mathrm{co\ op}})^{\mathrm{co\ op}} = \mathsf{Endo}(K)$.

If, in Proposition 2.23, we replaced $\mathsf{Endo}(K)$ by $\mathsf{Mnd}(K)$, the result would not hold. This, along with the existence or otherwise of liftings and adjoints (see Remarks 2.18, 2.24, 2.31), is a fundamental difference in the treatment of monads and endofunctors: for monads, we have $\mathsf{Mnd}(K^{\mathrm{co}})^{\mathrm{co}} = \mathsf{Cmd}(K)$. The reason for the difference is that, in the precise sense in which monads are dual to comonads, the concept of endofunctor is self-dual.

2.3.1 Eilenberg-Moore/Kleisli constructions

Again as in Section 2.1.1, we can define Eilenberg-Moore/Kleisli constructions for endofunctors.

For each of the above 2-categories $\mathsf{Endo}(K)$ and $\mathsf{Endo}^*(K)$, there is a 2-functor Inc from the base 2-category K to the 2-category which sends each 0-cell X of K to the identity endofunctor on the 0-cell X in K. But this time, we have

Remark 2.24 For each of $\mathsf{Endo}(K)$ and $\mathsf{Endo}^*(K)$, the forgetful 2-functor U to K does not have the 2-functor Inc as a left or right 2-adjoint.

In consideration of Proposition 2.23, we define:

Definition 2.25

- A 2-category K *admits Eilenberg-Moore constructions for endofunctors* if the 2-functor $Inc : K \to \mathsf{Endo}(K)$ has a right 2-adjoint.
- K *admits Eilenberg-Moore constructions for co-endofunctors* if the 2-functor $Inc : K \to \mathsf{Endo}^*(K)$ has a right 2-adjoint.

Example 2.26 When $K = Cat$, K admits Eilenberg-Moore constructions for endofunctors. For the 2-functor $Inc : Cat \to \mathsf{Endo}(Cat)$, the right 2-adjoint functor sends each pointed endofunctor (C, T) to the category $T\text{-}alg$ of T-algebras for the endofunctor $T : C \to C$.

Similarly, Cat admits Eilenberg-Moore constructions for co-endofunctors. The right 2-adjoint functor sends each endofunctor (C, D) to the category $D\text{-}coalg$ of D-coalebras for the endofunctor $D : C \to C$.

For the constructions corresponding to Kleisli ones, we define:

Definition 2.27

- A 2-category K *admits Kleisli constructions for endofunctors* if the 2-functor $Inc : K \to \mathsf{Endo}^*(K)$ has a left 2-adjoint.
- K *admits co-Kleisli constructions for endofunctors* if the 2-functor $Inc : K \to \mathsf{Endo}(K)$ has a left 2-adjoint.

The 2-category of all small categories Cat admits Kleisli constructions for endofunctors, and also admits co-Kleisli constructions for endofunctors.

We can characterise the Kleisli category for a endofunctor in Cat.

Proposition 2.28 *The Kleisli category T-*Kl *for an endofunctor $T : C \to C$ is characterised as follows:*

- *Objects of T-*Kl *are the same as those of C.*
- *Let a, b be objects of T-*Kl*. A homset from a to b is given by the disjoint union $T\text{-}\mathrm{Kl}(a, b) = \amalg_{n \in \mathbb{N}} C(a, T^n b)$.*

*For arrows $f : a \to b$ and $g : b \to c$ in T-*Kl*, i.e., arrows $\hat{f} : a \to T^m b$ and $\hat{g} : b \to T^n c$ in C for some $m, n \in \mathbb{N}$, we define the composition gf in T-*Kl *by the arrow $(T^m \hat{g}) \circ \hat{f}$ in C. The identity arrow on a seen as an object of T-*Kl *is the identity arrow* id_a *in the category C.*

Outline of Proof. The proof is similar to Proposition 2.12. □

Remark 2.29 By reversing the arrows in the above Proposition, we can characterise the *co-Kleisli category T-*coKl *for an endofunctor T* on C.

2.3.2 Liftings

We can define the notions of lifting to these constructions as in Section 2.2.2. So we omit the definition.

- Eilenberg-Moore constructions for endofunctors,

- Eilenberg-Moore constructions for co-endofunctors,
- Kleisli constructions for endofunctors,
- co-Kleisli constructions for endofunctors.

The point is the following which is similar to the Proposition 2.17.

Proposition 2.30 *Every right or left 2-adjoint functor to Inc corresponding to each of the above constructions sends each 1-cell to a 1-cell in K that is a lifting for the construction, and each 2-cell to a lifting 2-cell.*

Remark 2.31 We remark that the correspondence is not bijective as in the pointed/co-pointed case (Remark 2.18). Although every 1-cell in each 2-category determines a lifting 1-cell if each of the above constructions exist, the converse does not hold. This is in contrast to the situation for monads, and the reason is that we can not use the argument of adjoint lifting.

Hence we conclude

Corollary 2.32 *If K admits Eilenberg-Moore constructions for endofunctors, then every 1-cell in $\mathsf{Endo}(K)$ determines a lifting 1-cell. Every 2-cell in $\mathsf{Endo}(K)$ determines a lifting 2-cell.*

2.4 Summary of the 2-categorical framework

Given a 2-category K, we have defined 2-categories $\mathsf{Mnd}(K)$, $\mathsf{Mnd}^*(K)$, $\mathsf{Cmd}(K)$, $\mathsf{Cmd}^*(K)$, $\mathsf{pEndo}(K)$, $\mathsf{pEndo}^*(K)$, $\mathsf{copEndo}(K)$, $\mathsf{copEndo}^*(K)$, $\mathsf{Endo}(K)$ and $\mathsf{Endo}^*(K)$. Modulo size, these constructions can themselves be made 2-functorial, yielding 2-functors Mnd, Mnd^*, Cmd, Cmd^*, pEndo, pEndo^*, $\mathsf{copEndo}$, $\mathsf{copEndo}^*$, Endo and Endo^* on 2-Cat.

For each of the 2-categories, we have a 2-functor Inc from K to it, which sends each 0-cell X in K to the identity on it. The existence of Eilenberg-Moore constructions is characterized by right 2-adjoints to Inc.

$$\begin{tikzcd} \heartsuit \arrow[d, dashed, shift left=2, "EM"] \\ K \arrow[u, shift left=2, "Inc"] \arrow[u, phantom, "\dashv"] \end{tikzcd} \qquad \begin{array}{cc} \mathsf{Mnd}(K), & \mathsf{Cmd}(K), \\ \mathsf{pEndo}(K), & \mathsf{copEndo}(K), \\ \mathsf{Endo}(K), & \mathsf{Endo}^*(K) \end{array}$$

Suppose the existence of a right 2-adjoint to Inc. The right 2-adjoint sends each 1-cell to a 1-cell in K and a 2-cell to one in K, which are liftings to the Eilenberg-Moore constructions.

We remark that for $\mathsf{Mnd}(K)$ and $\mathsf{Cmd}(K)$ this correspondence is bijective. Hence a lifting 1-cell or 2-cell to Eilenberg-Moore constructions is equivalent to a 1-cell or 2-cell.

The existence of Kleisli constructions is characterized by left 2-adjoints to Inc.

$$
\begin{array}{ll}
\mathsf{Mnd}^*(K), & \mathsf{Cmd}^*(K), \\
\mathsf{pEndo}^*(K), & \mathsf{copEndo}^*(K), \\
\mathsf{Endo}^*(K), & \mathsf{Endo}(K)
\end{array}
\qquad\qquad
\begin{tikzcd}
\Diamond \arrow[d, dashed, shift right=2, "Kl"'] \arrow[d, phantom, "\dashv"] \\
K \arrow[u, shift right=2, "Inc"']
\end{tikzcd}
$$

Suppose the existence of a left 2-adjoint to Inc. The left 2-adjoint sends each 1-cell to a 1-cell in K and a 2-cell to one in K, which are liftings to the Kleisli constructions.

We remark that for $\mathsf{Mnd}^*(K)$ and $\mathsf{Cmd}^*(K)$ this correspondence is bijective. Hence a lifting 1-cell or 2-cell to Kleisli constructions is equivalent to a 1-cell or 2-cell.

3 Distributive laws

Sometimes, the distributive law of a monad and a comonad as in Section 2.1.3 is too demanding, so we require fewer commutativities. We can weaken the condition of the distributive law of a monad and a comonad by replacing a monad by a pointed endofunctor or an endofunctor, and a comonad by a co-pointed endofunctor or an endofunctor. These variations of distributive laws are captured as 0-cells in 2-categories given by combining one of six 2-functors

$$\mathsf{Mnd},\ \mathsf{Mnd}^*,\ \mathsf{pEndo},\ \mathsf{pEndo}^*,\ \mathsf{Endo} \text{ and } \mathsf{Endo}^*$$

with one of six 2-functors

$$\mathsf{Cmd},\ \mathsf{Cmd}^*,\ \mathsf{copEndo},\ \mathsf{copEndo}^*,\ \mathsf{Endo}^* \text{ and } \mathsf{Endo},$$

like $\mathsf{MndCmd}(K)$, in our 2-categorical framework. Also the notions of Eilenberg-Moore constructions and Kleisli constructions help the analysis of distributive laws and relationships to liftings. Especially, Cat admits all Eilenberg-Moore constructions and Kleisli constructions, so the liftings always exist.

We explain by giving an analysis of a distributive law. The analysis of the other distributive laws is routine.

Here, let us consider a distributive law of a monad over a co-pointed endofunctor.

Definition 3.1 A *distributive law of a monad* (T, μ, η) *over a co-pointed endofunctor* (D, ϵ) is a 2-cell $\lambda : TD \Rightarrow DT$ such that the three elementary diagrams involving each of μ, η and ϵ commute.

The distributivity of a monad over a co-pointed endofunctor can be captured as a 0-cell in each of the 2-categories $\mathsf{Mnd}(\mathsf{copEndo}(K))$, $\mathsf{copEndo}(\mathsf{Mnd}(K))$, copEndo*(Mnd($K$)) and Mnd*(copEndo($K$)).

We analyse the 2-category Mnd(copEndo(K)) as a representative example.

3.1 Mnd(copEndo(K))

First, we can re-express the definition as follows:

Proposition 3.2 *For any 2-category K,* Mnd(copEndo(K)) *is isomorphic to the following:*

- *A 0-cell consists of a 0-cell C of K, a monad (T, μ, η) on it, a co-pointed endofunctor (D, ϵ) on it, and a distributive law $\lambda : TD \Rightarrow DT$ of the monad over the co-pointed endofunctor.*
- *A 1-cell $(J, j_t, j_d) : (C, T, D, \lambda) \to (C', T', D', \lambda')$ consists of a 1-cell $J : C \to C'$ in K together with a 2-cell $j_t : T'J \Rightarrow JT$ subject to the monad laws, together with a 2-cell in K of the form $j_d : JD \Rightarrow D'J$ subject to the law of co-pointed endofunctor, all subject to one coherence condition given by a hexagon*

$$\begin{tikzcd} & JTD \arrow[r, "J\lambda"] & JDT \arrow[dr, "j_dT"] & \\ T'JD \arrow[ur, "j_tD"] \arrow[dr, "T'j_d"'] & & & D'JT \\ & T'D'J \arrow[r, "\lambda'J"] & D'T'J. \arrow[ur, "D'j_t"'] & \end{tikzcd}$$

- *A 2-cell from (J, j_t, j_d) to (H, h_t, h_d) consists of a 2-cell from J to H in K subject to two conditions expressing coherence with respect to j_t and h_t and coherence with respect to j_d and h_d.*

Lemma 3.3 *For any 2-category K,* copEndo(Mnd(K)) *is isomorphic to* Mnd(copEndo(K))*.*

Now suppose a 2-category K admits Eilenberg-Moore constructions for both monads and co-pointed endofunctors, i.e., we have right 2-adjoint functors to both Inc 2-functors to Mnd(K) and copEndo(K). Since both copEndo and Mnd are 2-endofunctors on 2-Cat, they send adjunctions to adjunctions, so both copEndo(Inc) : copEndo(K) $\to$ copEndo(Mnd(K)) and Mnd(Inc) : Mnd(K) $\to$ Mnd(copEndo(K)) have right 2-adjoint 2-functors.

For given a distributive law $\lambda : TD \Rightarrow DT$ of a monad (T, μ, η) over a co-pointed endofunctor (D, ϵ) on object C, it is a 0-cell of both copEndo(Mnd(K))

and $\mathsf{Mnd}(\mathsf{copEndo}(K))$. Consider the images under the right 2-adjoint functors to $\mathsf{copEndo}(Inc) : \mathsf{copEndo}(K) \to \mathsf{copEndo}(\mathsf{Mnd}(K))$ and $\mathsf{Mnd}(Inc) : \mathsf{Mnd}(K) \to \mathsf{Mnd}(\mathsf{copEndo}(K))$.

The right 2-adjoint functor to $\mathsf{copEndo}(Inc)$ sends the distributive law λ to a co-pointed endofunctor $(T\text{-Alg}, \tilde{D}, \tilde{\epsilon})$ in K, where T-Alg is an Eilenberg-Moore construction for the monad (C, T, μ, η), $\tilde{D}$ is a 1-cell which is a lifting of D, and $\tilde{\epsilon}$ is a 1-cell which is a liftings of ϵ to Eilenberg-Moore constructions for monads.

Note that as this right 2-adjoint functor is basically given by the right 2-adjoint to $Inc : K \to \mathsf{Mnd}(K)$, we can retrieve the information of the distributive law λ from the co-pointed endofunctor $(T\text{-Alg}, \tilde{D}, \tilde{\epsilon})$ which consists of liftings to Eilenberg-Moore constructions for monads.

The right 2-adjoint functor to $\mathsf{Mnd}(Inc)$ sends the distributive law λ to a monad $((D,\epsilon)\textit{-coalg}, \tilde{T}, \tilde{\mu}, \tilde{\eta})$ in K, where (D,ϵ)*-coalg* is an Eilenberg-Moore construction for the co-pointed endofunctor (C, D, ϵ), $\tilde{T}$ is a 1-cell which is a lifting of T, and $\tilde{\mu}$ and $\tilde{\eta}$ are 2-cells that are liftings of μ and η respectively to Eilenberg-Moore constructions for co-pointed endofunctors.

Hence we conclude

Proposition 3.4 *If K admits Eilenberg-Moore constructions for monads and co-pointed endofunctors, then conditions (i) and (ii) are equivalent, and (i) determines (iii).*

(i) A distributive law λ of a monad (T, μ, η) over a co-pointed endofunctor (D, ϵ),

(ii) the lifting of a co-pointed endofunctor (D, ϵ) to T-Alg the Eilenberg-Moore construction for a monad (T, μ, η),

(iii) the lifting of a monad (T, μ, η) to (D, ϵ)-coalg the Eilenberg-Moore construction for a co-pointed endofunctor (D, ϵ).

4 Free pointed endofunctors and cofree co-pointed endofunctors

The following observation [9] is simple and almost trivial, but it is important.

Proposition 4.1 *Let C be a category with binary sums, and H be an endofunctor on it. Then the pair $(H + \mathrm{Id}, \mathrm{in}_r)$ consisting of the endofunctor $H + \mathrm{Id}$ and the inclusion $\mathrm{in}_r : \mathrm{Id} \Rightarrow H + \mathrm{Id}$ is the free pointed endofunctor on H [9], in the sense that for any pointed endofunctor (T, η) on C, composition with the inclusion $\mathrm{in}_l : H \Rightarrow H + Id$ gives a bijection between the collection of natural transformations $\alpha : H + \mathrm{Id} \Rightarrow T$ satisfying $\alpha \cdot \mathrm{in}_r = \eta$, and the set of natural transformations $\mathrm{Nat}(H, T)$.*

Moreover the isomorphism $(H + \mathrm{Id}, \mathrm{in}_r)$-alg $\cong H$-alg holds.

Proof. Let (T, η) be a pointed endofunctor on category C. Since the category

C has binary sums, for each natural transformation $\alpha : H + \mathrm{Id} \Rightarrow T$ there is a pair of natural transformation $\alpha_1 : H \Rightarrow T$ and $\alpha_2 : \mathrm{Id} \Rightarrow T$ such that $\alpha = [\alpha_1, \alpha_2]$. If α satisfies $\alpha \cdot \mathrm{in}_r = \eta$, we have $\alpha_2 = \eta$. Hence giving a natural transformation $\alpha : H + \mathrm{Id} \Rightarrow T$ satisfying $\alpha \cdot \mathrm{in}_r = \eta$ is equivalent to giving a natural transformation $\alpha_1 : H \Rightarrow T$. Since $\alpha \cdot \mathrm{in}_l = \alpha_1$ holds, it is easy to see the composition with in_l gives the above bijection.

A $(H+\mathrm{Id}, \mathrm{in}_r)$-algebra is a pair $(X, k : HX+X \to X)$ such that $k \circ \mathrm{in}_{rX} = \mathrm{id}_X$, so the structure morphism $k : HX+X \to X$ consists of an identity arrow on X and an arrow $\bar{k} : HX \to X$ in C. And every morphism of $(H+\mathrm{Id}, \mathrm{in}_r)$-*alg* from $(X, [k, \mathrm{id}_X] : HX + X \to X)$ to $(Y, [l, \mathrm{id}_Y] : HY + Y \to Y)$ consists of an arrow $f : X \to Y$ such that $[l, \mathrm{id}_Y] \circ Hf + f = f \circ [k, \mathrm{id}_X]$, equivalently $l \circ Hf = f \circ k$. Hence the isomorphism $(H + \mathrm{Id}, \mathrm{in}_r)$-*alg* $\cong$ H-*alg* holds. □

By using the dual argument, we have the following result.

Proposition 4.2 *Let C be a category with binary products, and let H be an endofunctor on it. Then the pair $(H \times \mathrm{Id}, \mathrm{pr}_r)$ consisting of the endofunctor $H \times \mathrm{Id}$ and the projection $\mathrm{pr}_r : H \times \mathrm{Id} \Rightarrow \mathrm{Id}$ is a cofree co-pointed endofunctor on H, in the sense that for any co-pointed endofunctor (D, ϵ) on C, composition with $\mathrm{pr}_l : H \times \mathrm{Id} \Rightarrow H$ gives a bijection between the collection of natural transformations $\alpha : D \Rightarrow H \times \mathrm{Id}$ satisfying $\mathrm{pr}_r \cdot \alpha = \epsilon$, and the set of natural transformations $\mathrm{Nat}(D, H)$.*

Moreover the isomorphism $(H \times \mathrm{Id}, \mathrm{pr}_r)$-coalg $\cong$ H-coalg holds.

5 Application 1: Distributivity in Turi and Plotkin's work

5.1 Review

We first review the work of Turi and Plotkin [22]. Given a cartesian category C, a signature endofunctor Σ and a behaviour endofunctor B on C, with Σ freely generating the syntax monad (T, μ, η), and given image finite GSOS (Generalized Structural Operational Rule), they showed, using the functoriality of GSOS, that each GSOS rule can be modelled by an abstract operational rule, which is a natural transformation $\rho : \Sigma(B \times \mathrm{Id}) \Rightarrow BT$. Next, each abstract operational rule $\rho : \Sigma(B \times \mathrm{Id}) \Rightarrow BT$ determines a distributive law $\lambda : TD \Rightarrow DT$ of the monad T over a comonad D, where the comonad D is cofreely generated by the behaviour functor B (assuming C and B have structure giving the existence of such D). The category λ-*Bialg* of λ-bialgebras provides combined operational and denotational models. They gave an analysis of those models. For details, see [22].

Their derivation of a distributive law from an abstract operational rule is given as follows:

(i) Given an abstract operational rule $\rho : \Sigma(B \times \mathrm{Id}) \Rightarrow BT$, first they show that the monad (T, μ, η) on C lifts to the category B-*coalg* of B-

coalgebras by using **structural recursion** ([22] Theorem 5.1).

(ii) Since the category *B-coalg* is isomorphic to *D-Coalg*, the category of Eilenberg-Moore coalgebras for the comonad (D, δ, ϵ) cofreely generated by B, the monad (T, μ, η) lifts to the category *D-Coalg*.

(iii) By using an adjoint lifting theorem (see [8,19,17,18]), a lifting of the monad T to *D-Coalg* is equivalent to a distributive law $\lambda : TD \Rightarrow DT$ of T over D.

The first step above is intricate because of the use of a structural recursion theorem. So we shall give a different proof.

5.2 Feedback from our framework

Now we use our framework to analyse the above. The fundamental observation is that an abstract operational rule can be seen as a distributive law of a monad over a co-pointed endofunctor. This allows a more general argument avoiding the special recursion theorem.

Proposition 5.1 *The co-pointed endofunctor* $(B \times \mathrm{Id}, \mathrm{pr}_2)$ *is the cofree co-pointed endofunctor of the endofunctor* B*.*

Outline of Proof. This follows from Proposition 4.2. $\square$

Let (H, ϵ) be a co-pointed endofunctor on a category C. We say a natural transformation $\rho : \Sigma H \Rightarrow HT$ *respects the structure of the co-pointed endofunctor* (H, ϵ) if the following diagram commutes:

$$\begin{array}{ccc} \Sigma H & \xrightarrow{\rho} & HT \\ {\scriptstyle \Sigma\varepsilon}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon T} \\ \Sigma & \xrightarrow[\theta]{} & T \end{array} \tag{1}$$

where $\theta : \Sigma \Rightarrow T$ is the unit exhibiting T as the free monad on the endofunctor Σ.

Proposition 5.2 *To give an abstract operational rule* $\rho : \Sigma(B \times \mathrm{Id}) \Rightarrow BT$ *is equivalent to giving a natural transformation* $\varrho : \Sigma(B \times \mathrm{Id}) \Rightarrow (B \times \mathrm{Id})T$ *which respects the structure of the co-pointed endofunctor* $(B \times \mathrm{Id}, \mathrm{pr}_2)$*.*

Proof. For each natural transformation $\varrho : \Sigma(B \times Id) \Rightarrow (B \times Id)T$ that respects the structure of $(B \times \mathrm{Id}, \mathrm{pr}_2)$, the second component must be

$$\Sigma(B \times Id) \xrightarrow{\Sigma \mathrm{pr}_2} \Sigma \xrightarrow{\theta} T.$$

So giving an natural transformation $\varrho : \Sigma(B \times Id) \Rightarrow (B \times Id)T$ that respects the structure of $(B \times \mathrm{Id}, \mathrm{pr}_2)$ is equivalent to giving the first component $\Sigma(B \times Id) \Rightarrow BT$, i.e., an abstract operational rule. □

Let (H, ϵ) be a co-pointed endofunctor on C.

Proposition 5.3 *Each natural transformation $\varrho : \Sigma H \Rightarrow HT$ respecting the structure of (H, ϵ) determines a distributive law of the free monad T on Σ over (H, ϵ).*

Proof. We first show that the endofunctor H lifts to an endofunctor $\bar{H}$ on the category Σ-*alg*, and the natural transformation $\epsilon : H \Rightarrow \mathrm{Id}$ lifts to $\bar{\epsilon} : \bar{H} \Rightarrow \mathrm{Id}$.

Define the action of $\bar{H} : \Sigma\text{-}alg \to \Sigma\text{-}alg$ as follows: A Σ-algebra $k : \Sigma X \to X$ is sent to $Hk^\sharp \circ \varrho_X$, where $k^\sharp : TX \to X$ is the corresponding Eilenberg-Moore algebra for the monad (T, μ, η) under the isomorphism $\Sigma\text{-}alg \cong T\text{-Alg}$. An arrow f of Σ-algebras from $k : \Sigma X \to X$ to $l : \Sigma Y \to Y$, i.e., an arrow $f : X \to Y$ in C satisfying $f \circ k = l \circ \Sigma f$, is sent to $Hf : HX \to HY$. Then the functor $\bar{H} : \Sigma\text{-}alg \to \Sigma\text{-}alg$ is a lifting of H.

Next, for each Σ-algebra $k : \Sigma X \to X$, we show that the X component $\varepsilon_X : HX \to X$ of ε gives a morphism of Σ-algebras from $\bar{H}k$ to k, i.e., $\varepsilon_X \bar{H}k = k\Sigma\varepsilon_X$: since the natural transformation ϱ respects the structure of (H, ϵ), each part of the following diagram commutes:

$$\begin{tikzcd}
\Sigma HX \arrow[r, "\varrho_X"] \arrow[d, "\Sigma\varepsilon_X"'] & HTX \arrow[r, "Hk^\sharp"] \arrow[d, "\varepsilon_{TX}"'] & HX \arrow[d, "\varepsilon_X"] \\
\Sigma X \arrow[r, "\theta_X"'] & TX \arrow[r, "k^\sharp"'] & X
\end{tikzcd}$$

Since the bottom arrow of above diagram is $k^\sharp \circ \theta_X = k$ and the top arrow is $\bar{H}k$, the arrow $\varepsilon_X : HX \to X$ is a morphism of Σ-algebras from $\bar{H}k$ to k.

We give $\bar{\varepsilon} : \bar{H} \Rightarrow Id$ by defining $k : \Sigma X \to X$ component to be ε_X. That this is natural follows from naturality of ε.

It is easy to see that $\bar{\varepsilon}$ is a lifting of ε to Σ-algebras.

Because Σ-*alg* is isomorphic to T-Alg, both the functor $\bar{H}$ and the natural transformation $\bar{\epsilon} : \bar{H} \Rightarrow \mathrm{Id}$ are liftings of H and ϵ to T-Alg.

By using Proposition 3.4, this condition is equivalent to a distributive law of a monad (T, μ, η) over a co-pointed endofunctor (H, ϵ). This concludes the proof of Proposition 5.3. □

¿From Propositions 5.2 and 5.3, we conclude that

Theorem 5.4 *Each abstract operational rule $\rho : \Sigma(B \times \mathrm{Id}) \Rightarrow BT$ determines a distributive law $\lambda : T(B \times \mathrm{Id}) \Rightarrow (B \times \mathrm{Id})T$ of the monad (T, μ, η) over the co-pointed endofunctor $(B \times \mathrm{Id}, \mathrm{pr}_2)$.*

5.2.1 A deduction of Turi and Plotkin's distributive laws

We apply the previous observations to the implication of distributivity in Turi and Plotkin's work, and modify the procedure. The procedure 1 in Section 5.1 can replaced as follows:

(a) Any abstract operational rule $\rho : \Sigma(B \times \mathrm{Id}) \Rightarrow BT$ determines a distributive law $\lambda : T(B \times \mathrm{Id}) \Rightarrow (B \times \mathrm{Id})T$ of the monad (T, μ, η) over the co-pointed endofunctor $(B \times \mathrm{Id}, \mathrm{pr}_2)$ by Theorem 5.4.

(b) Since the 2-category *Cat* admits Eilenberg-Moore constructions for co-pointed endofunctors, we can lift the monad (T, μ, η) to $(B \times \mathrm{Id}, \mathrm{pr}_2)$-*coalg* the category of coalgebras for the co-pointed endofunctor by Proposition 3.4.

(c) Since the co-pointed endofunctor $(B \times \mathrm{Id}, \mathrm{pr}_2)$ is cofreely generated by the endofunctor B (Proposition 5.1) and $(B \times \mathrm{Id}, \mathrm{pr}_2)$-*coalg* $\cong$ B-*coalg*, this means the monad (T, μ, η) lifts to the the category B-*coalg*. This is the lifting monad obtained by Turi and Plotkin.

Hence we can obtain the same distributive law of a monad over a comonad, but in a more structured manner not using a recursion theorem.

6 Application 2: Generalized coiteration and coalgebraic coinduction "up-to"

In this section, we illustrate how the categorical pattern of a pointed endofunctor (T, η) that distributes over an endofunctor F arises in the context of the coalgebraic approach to coinduction.

The gist of the coalgebraic approach to coinduction is that, under suitable conditions, equivalences induced by coiterative morphisms can be characterized as greatest coalgebraic F-bisimulations ([1,2,20]). The problem addressed in [13] and in this section, is that of extending this paradigm to generalized coiterative morphisms, thus providing coalgebraic coinduction principles for reasoning on them.

Generalized coiterative morphisms arise from the generalized coiteration scheme, which we call (T, η)-coiteration, where (T, η) is a pointed endofunctor. This scheme subsumes the standard coiteration scheme, and the corecursion scheme (see e.g. [5]), which in turn is dual to the scheme of primitive recursion. Standard coiteration is recovered by taking T to be the identity functor.

In [13], the notion of F-bisimulation "up-to-T", for T a monad, appeared generalising Aczel's original notion of F-bisimulation. The notion of F-bisimulation "up-to-T" can be viewed as the coalgebraic counterpart/generalization of the set-theoretic notion of bisimulation "up-to" due to Milner [14] (see [13] for more details). If the pointed endofunctor underlying the monad T distributes over F, then the equivalence induced by standard coiterative morphisms from F-coalgebras of the shape $(TX, \alpha : X \to FTX)$ can be characterized as the greatest F-bisimulation "up-to-T", thus providing a sound and complete principle of coalgebraic coinduction "up-to-T".

This result may be used to prove the soundness of a coinduction principle, in terms of F-bisimulations "up-to-T", for reasoning on the equivalence induced by a (T, η)-coiterative morphism, when (T, η) is the pointed endofunctor underlying the monad (T, μ, η). This substantiates the claim of the connection between generalized coiteration schemata and bisimulations "up-to".

In this section, we provide a range of concrete examples of (T, η)-coiterative morphisms and associated coinduction principles, including corecursive morphisms on streams, linear time semantics, and coiterative morphisms "up-to-context", thus emphasizing the pervasiveness of the abstract pattern of a distributive law described in Section 3.

Throughout this section, let C be a category, $F : C \to C$ an endofunctor, and (T, η) a pointed endofunctor on C.

6.1 (T, η)-coiteration

We start by recalling the definition of standard coiteration and corecursion schemata:

Definition 6.1 [Coiteration] Let F have final F-coalgebra $(\nu F, \alpha_{\nu F})$, and let (X, α_X) be an F-coalgebra. The *coiterative morphism* determined by the F-coalgebra (X, α_X) is the unique F-coalgebra morphism f from (X, α_X) to $(\nu F, \alpha_{\nu F})$, i.e. the unique morphism $f : X \to \nu F$ in C making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha_X"'] & \nu F \arrow[d, "\alpha_{\nu F}"] \\ FX \arrow[r, "Ff"'] & F(\nu F) \end{tikzcd}$$

Definition 6.2 [Corecursion] Let C be a category with coproducts, let F have final F-coalgebra $(\nu F, \alpha_{\nu F})$, and let $(X + \nu F, [\alpha_1, F(\mathrm{in}_2) \circ \alpha_{\nu F}])$ be an F-coalgebra, where $\mathrm{in}_2 : \nu F \to X + \nu F$ is the canonical sum injection. The *corecursive morphism* determined by the F-coalgebra $(X + \nu F, [\alpha_1, F(\mathrm{in}_2) \circ \alpha_{\nu F}])$ is defined by $f \circ \mathrm{in}_1$, where f is the coiterative morphism determined by the F-coalgebra $(X + \nu F, [\alpha_1, F(\mathrm{in}_2) \circ \alpha_{\nu F}])$. I.e.:

$$\begin{tikzcd} X \arrow[r, "\mathrm{in}_1"] & X + \nu F \arrow[r, "f"] \arrow[d, "{[\alpha_1, F(\mathrm{in}_2) \circ \alpha_{\nu F}]}"'] & \nu F \arrow[d, "\alpha_{\nu F}"] \\ & F(X + \nu F) \arrow[r, "Ff"'] & F(\nu F) \end{tikzcd}$$

Definition 6.3 [(T, η)-coiteration, [13]] Let $F : C \to C$ have final coalgebra $(\nu F, \alpha_{\nu F})$, and let (TX, α) be an F-coalgebra. We define the (T, η)-*coiterative* morphism $h : X \to \nu F$ as $f \circ \eta_X$, where $f : TX \to \nu F$ is the coiterative morphism induced by the F-coalgebra (TX, α). I.e.:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] & TX \arrow[r, "f"] \arrow[d, "\alpha"'] & \nu F \arrow[d, "\alpha_{\nu F}"] \\ & FTX \arrow[r, "Ff"'] & F(\nu F) \end{tikzcd}$$

The (T, η)-coiteration subsumes both standard coiteration, by taking T to be the identity functor, and corecursion, by taking T to be the pointed endofunctor underlying the corecursion monad T_F^+ defined as follows:

Definition 6.4 [Corecursion monad] Let C be a category with coproducts, and let $F : C \to C$ have final coalgebra $(\nu F, \alpha_{\nu F})$. Let (T_F^+, μ, η) be the monad defined by:

•

$$T_F^+ X = X + \nu F \ ,$$

$$\forall f : X \to Y \text{ morphism in } C.\ \ T_F^+ f = [\mathrm{in}_1 \circ f, \mathrm{in}_2] \ ;$$

- $\eta_X = \mathrm{in}_1 : X \to X + \nu_F$;
- $\mu_X = [\mathrm{id}, \mathrm{in}_2] : (X + \nu F) + \nu F \to X + \nu F$,

where in_1, in_2 are the canonical sum injections.

6.2 F-bisimulations "up-to-T" and coalgebraic coinduction "up-to-T"

We introduce the notion of F-bisimulation "up-to-T", for $T = (T, \mu, \eta)$, and we use this notion to prove soundness and completeness of a principle of coalgebraic coinduction "up-to-T" for establishing equivalences induced by coiterative morphisms. The principle of coalgebraic coinduction "up-to" is the counterpart of a suitable set-theoretical generalization of Milner's principle of bisimulation "up-to". This is discussed in [13]. In Section 6.3, we show that coalgebraic coinduction "up-to-T" is related to the (T, η)-coiteration scheme introduced above, in the sense that it can be used to derive a proof principle for establishing equivalences induced by (T, η)-coiterative morphisms, where (T, η) is the pointed endofunctor underlying the monad T.

We start by introducing the notion of F-bisimulation "up-to-T". We stress the fact that we assume T to be a monad, and not just a pointed endofunctor.

Definition 6.5 [F-bisimulation "up-to-T", [13]]

Let $F : C \to C$, let (T, μ, η) be a monad on C, and let (TX, α) and (TY, β) be F-coalgebras. An *F-bisimulation "up-to-T"* on the F-coalgebras (TX, α) and (TY, β) is a span $(\mathcal{R}, r_1, r_2)$ on TX and TY, such that there exists an arrow of C, $\gamma : \mathcal{R} \to FT(\mathcal{R})$, making the following diagram commute:

$$\begin{tikzcd}
TX \arrow[d, "\alpha"'] & \mathcal{R} \arrow[l, "r_1"'] \arrow[r, "r_2"] \arrow[d, "\gamma"'] & TY \arrow[d, "\beta"] \\
FTX & FT(\mathcal{R}) \arrow[l, "Fr_1^\sharp"] \arrow[r, "Fr_2^\sharp"'] & FTY
\end{tikzcd}$$

where $r_1^\sharp, r_2^\sharp$ are the unique extensions of r_1, r_2 given by the universality property of η in the adjunction between the Eilenberg-Moore category of T-algebras and the category C, i.e. $r_i^\sharp \circ \eta_{\mathcal{R}} = r_i$.

Aczel's original notion of F-bisimulation ([2]) is recovered by taking T to be the identity monad, and by considering the set-theoretical relation underlying the span, i.e. $\{(x, y) \in X \times Y \mid \exists u \in \mathcal{R} \,.\, \langle r_1, r_2 \rangle (u) = (x, y)\}$.

The following technical Theorem 6.6 is the core of this section. It allows to prove the soundness and completeness of a principle of Coinduction "up-to-T" (Principle 6.7 below).

Theorem 6.6 *Let C be a cocomplete category, let $F : C \to C$, let (T, μ, η) be a monad over C. If*
1) the pointed endofunctor (T, η) distributes over F,
2) $((\mathcal{R}, \gamma), r_1, r_2)$ is an F-bisimulation "up-to-T" on F-coalgebras (TX, α) and $(TY, \beta) \implies ((T(\mathcal{R}), \lambda \circ T\gamma), r_1^\natural, r_2^\natural)$ is an F-bisimulation "up-to-T" on F-coalgebras (TX, α) and (TY, β) ,
then
i) If $((\mathcal{R}, \gamma), r_1, r_2)$ is an F-bisimulation on (TX, α) and (TY, β), then $((\mathcal{R}, F(\eta_{\mathcal{R}}) \circ \gamma), r_1, r_2)$ is an F-bisimulation "up-to-T" on (TX, α) and (TY, β).
ii) For all $(\mathcal{R}, r_1, r_2)$ F-bisimulation "up-to-T" on (TX, α) and (TY, β), there exists $(\widetilde{\mathcal{R}}, \widetilde{r_1}, \widetilde{r_2})$ F-bisimulation on (TX, α) and (TY, β) such that $\mathcal{R} \leq \widetilde{\mathcal{R}}$.

Proof. The proof of item i) is immediate. In order to prove item ii), let $\widehat{R} : \omega \to C$ be a diagram given by $\widehat{R}(n) = T^n R$ and $\widehat{R}(n \to n+1) = \eta_{T^n R}$ for each $n \in \omega$. Let $\sigma : \widehat{R} \to \mathrm{colim}\widehat{R}$ be the colimiting cocone. Define $\widetilde{R} = \mathrm{colim}\widehat{R}$.

Given $\gamma : R \to FTR$, we first define arrows $\gamma_n : T^n R \to FT^{n+1}R$ for $n \geq 0$ inductively as follows:
$\gamma_0 = \gamma : R \to FTR$,
$\gamma_n = \lambda_{T^n R} \circ T(\gamma_{n-1})$ for each $n \geq 1$.
Next, we define a collection of arrows $\nu_n = F\sigma_{n+1} \circ \gamma_n$ for each n. Then ν becomes a cocone of $\widehat{R}$ because the following equation holds for each n.

$$\begin{aligned}
\nu_{n+1} \circ R(n \to n+1) &= (F(\sigma_{n+2}) \circ \lambda_{T^{n+1}R} \circ T\gamma_n) \circ \eta_{T^n R} \\
&= F(\sigma_{n+2}) \circ \lambda_{T^{n+1}R} \circ \eta_{FT^{n+1}(R)} \circ \gamma_n \\
&= F(\sigma_{n+2}) \circ F(\eta_{T^{n+1}(R)}) \circ \gamma_n && \text{by distributivity} \\
&= F(\sigma_{n+1}) \circ \gamma_n \\
&= \nu_n.
\end{aligned}$$

Let $\widetilde{\gamma} : \mathrm{colim}\widehat{R} \to F\mathrm{colim}\widehat{R}$ be the unique morphism from the colimiting cocone σ to the cocone $\nu : \widehat{R} \to F\mathrm{colim}\widehat{R}$.

The construction of the arrow $\widetilde{r_1} : \mathrm{colim}\widehat{R} \to TX$ from $r_1 : R \to TX$ is as follows. Let $\tau : \widehat{R} \to TX$ be the cocone over $\widehat{R}$ given inductively by $\tau_0 = r_1 : R \to TX$ and $\tau_n = {\tau_{n-1}}^\sharp : T^n R \to TX$ for each $n \geq 1$. We define the arrow $\widetilde{r_1} : \mathrm{colim}\widehat{R} \to TX$ to be the unique morphism from the limiting cocone of $\widehat{R}$.

The construction of $\widetilde{r_2} : \mathrm{colim}\widehat{R} \to TY$ from $r_2 : R \to TY$ is the same. Then the condition $\mathcal{R} \leq \widetilde{\mathcal{R}}$ follows immediately by construction.

Finally, we are in a position to prove that the following diagram commutes:

$$\begin{tikzcd}
TX \arrow[d, "\alpha"'] & \mathrm{colim}\widehat{R} \arrow[l, "\widetilde{r_1}"'] \arrow[d, "\widetilde{\gamma}"] \\
FTX & F\mathrm{colim}\widehat{R} \arrow[l, "F\widetilde{r_1}"]
\end{tikzcd}$$

We have two cocones to the object FTX over the diagram $\widehat{R}$ whose n-components are given by $\alpha \circ \widetilde{r_1} \circ \sigma_n$ and $F\widetilde{r_1} \circ \widetilde{\gamma} \circ \sigma_n$. If we show they are equal, then the universality of the colimiting cocone, the above diagram commutes.

Since $\widetilde{r_1}$ is the universal arrow from the cocone τ to the colimiting cocone, we have $\alpha \circ \widetilde{r_1} \circ \sigma_n = \alpha \circ \tau_n$.

For the second cocone, we have $F\widetilde{r_1} \circ \widetilde{\gamma} \circ \sigma_n = F\widetilde{r_1} \circ \nu_n = F\widetilde{r_1} \circ F\sigma_{n+1} \circ \gamma_n = F\tau_{n+1} \circ \gamma_n$.

So, in order to show the commutativity, we have to check that, $\alpha \circ \tau_n =$

$F\tau_{n+1} \circ \gamma_n$ holds for each $n \geq 0$. This follows immediately by induction on n, from the definitions of γ_n, τ_n and by hypothesis 2). Hence we have the commutatiity of the above diagram.

The same holds for $\widetilde{r_2}$. $\square$

Specializing Theorem 6.6 to endofunctors on the category Set, for simplicitly, we obtain an alternative characterization of the greatest F-bisimulation $\sim^F_{(TX,\alpha)}$ on the F-coalgebra (TX, α), which yields the following:

Principle 6.7 (Coalgebraic Coinduction "up-to-T") *Let $F : Set \to Set$ and let T be a monad over Set satisfying the hypothesis 1,2 of endofunctor distributes over F. If moreover F preserves weak pullbacks and has a final coalgebra, then the following principle is sound and complete*

$$\frac{(\mathcal{R}, r_1, r_2)\ F\text{-bisimulation "up-to-}T\text{" on } (TX,\alpha) \quad x\ \mathcal{R}\ y}{x \sim^F_{(TX,\alpha)} y},$$

where $\sim^F_{(TX,\alpha)}$ denotes the greatest F-bisimulation on the coalgebra (TX,α), and, by abuse of notation, $x\ \mathcal{R}\ y$ denotes the fact that the pair (x, y) belongs to the set-theoretical relation underlying the span $(\mathcal{R}, r_1, r_2)$, i.e. $(x, y) \in \{(x, y) \in TX \times TX \mid \exists u \in \mathcal{R}\ .\ \langle r_1, r_2\rangle(u) = (x, y)\}$.

6.3 Reasoning on equivalences of T-coiterative morphisms

In this section, the principle of Coalgebraic Coinduction "up-to-T" is put to use for reasoning on equivalences induced by (T, η)-coiterative morphisms, for (T, η) the pointed endofunctor underlying the monad T.

First, we need to recall the notions of image and inverse image of spans. Set-theoretically, these are to be intended as the (inverse) image of the subobject of $X_1 \times X_2$ determined by the relation underlying a span on X_1, X_2. See [4] for more details.

Definition 6.8 [(inverse) Image of spans] Let C be a category with products and pullpacks.

- The *image* of a span $(\mathcal{R}, r_1, r_2)$ on X_1, X_2 by $(f_1 : X_1 \to Y_1, f_2 : X_2 \to Y_2)$, denoted by $(f_1, f_2)^+(\mathcal{R}, r_1, r_2)$, is the span $(\mathcal{R}, f_1 \circ r_1, f_2 \circ r_2)$ on Y_1, Y_2.
- The *inverse image* of a span $(\mathcal{R}, r_1, r_2)$ on X_1, X_2 by $(f_1 : Y_1 \to X_1, f_2 : Y_2 \to X_2)$, denoted by $(f_1, f_2)^{-1}(\mathcal{R}, r_1, r_2)$, is the span $(\mathcal{P}, p_2; \pi_1, p_2; \pi_2)$ on Y_1, Y_2, where $(\mathcal{P}, p_1, p_2)$ is the pullback of $\langle r_1, r_2\rangle : \mathcal{R} \to X_1 \times X_2$ and $\langle f_1 \circ \pi_1, f_2 \circ \pi_2\rangle : Y_1 \times Y_2 \to X_1 \times X_2$.

If $(\mathcal{R}, r_1, r_2)$ is a span on X and $f : X \to Y$, $g : Y \to X$, we simply denote by $f^+(\mathcal{R}, r_1, r_2)$ the image of $(\mathcal{R}, r_1, r_2)$ by (f, f), and we denote by $g^{-1}(\mathcal{R}, r_1, r_2)$ the inverse image of $(\mathcal{R}, r_1, r_2)$ by (g, g).

Using the principle of Coalgebraic Coinduction "up-to-T", we can prove the following theorem

Theorem 6.9 (Coalgebraic Coinduction for T-coiterative Functions) *Let $F : Set \to Set$ and let (T, μ, η) be a monad on Set such that the pointed endofunctor (T, η) distributes over F. Let h be the T-coiterative morphism induced by the F-coalgebra (TX, α), i.e. $h = f \circ \eta_X$, where $f : TX \to \nu F$ is the coiterative morphism determined by (TX, α). Then the following principle is sound*

$$\frac{x\ \mathcal{R}\ y \quad \eta_X^{+}(\mathcal{R}, r_1, r_2)\ F\text{-}bisimulation\ \text{“}up\text{-}to\text{-}T\text{”}\ on\ (TX, \alpha)}{x \sim_h y} \ ,$$

where $\sim_h$ denotes the equivalence induced by the T-coiterative morphism h.

Proof. One can easily check, using Principle 6.7, that

$$\eta_X^{+}(\mathcal{R}, r_1, r_2)\ F\text{-bisimulation “up-to-}T\text{” on } (TX, \alpha) \Longrightarrow (R, r_1, r_2) \leq \eta_X^{-1}(\sim_f, \pi_1, \pi_2) \ ,$$

where $\sim_f$ denotes the equivalence induced by the coiterative morphism f. □

6.4 Examples

We discuss some concrete examples of generalized coiteration and we derive the corresponding coinduction principles “up-to”.

6.4.1 Corecursion on streams

Let $F_S : Set \to Set$ be the functor defined as follows:

$$F_S(X) = N \times X \ ,$$

$$\forall f : X \to Y \text{ morphism in } Set.\ F_S(f) = id_N \times f \ ,$$

where N denotes the set of natural numbers.

We recall that the final F_S-coalgebra is the set S_N of all (infinite) streams on natural numbers.

A simple example of a corecursive function from streams to streams which cannot be captured by the standard coiteration scheme is the following.

Definition 6.10 Let $h_0 : S_N \to S_N$ be the function which, given a stream of natural numbers s, yields the stream obtained by replacing the first element of s by the constant 0, i.e., for all streams (n, s),

$$h_0(n, s) = (0, s) \ .$$

One can easily check that

Proposition 6.11 *i) The function h_0 is not coiterative, in the sense that there is no way of endowing S_N with a structure of F_S-coalgebra $\alpha : S_N \to F_S(S_N)$ so that h_0 is the coiterative morphism determined by (S_N, α).*
ii) The function h_0 is the corecursive function determined by the F_S-coalgebra

$(S_N + S_N, [\alpha_1, F_S(in_2)])$, where $\alpha_1 : S_N \to N \times (S_N + S_N)$ is defined by $\alpha_1(n, s) = (0, in_2(s))$, i.e. $h_0 = f_0 \circ in_1$, where:

$$\begin{tikzcd}
S_N \arrow[r, "in_1"] & S_N + S_N \arrow[r, "f_0"] \arrow[d, "{[\alpha_1, F_S(in_2) \circ \alpha_{S_N}]}"'] & S_N \arrow[d, "\alpha_{S_N}"] \\
& F(S_N + S_N) \arrow[r, "F f_0"'] & F(S_N)
\end{tikzcd}$$

The essence of the corecursion scheme is that we can make a choice between the two possibilities offered by the two branches of the disjoint sum in the F-coalgebra. Notice how this is exploited in the definition of the F_S-coalgebra in Proposition 6.11 above.

Now we come to discussing the principle of coalgebraic coinduction (Principle 6.7) in the special case of corecursion on streams. One can check that $(\widehat{\mathcal{R}}, \widehat{r}_1, \widehat{r}_2)$ is an F_S-bisimulation "up-to-_ + S_N" on $(S_N + S_N, [\alpha_1, F_S(in_2)])$ if and only if the set-theoretic relation $\mathcal{R} \subseteq (S_N + S_N)^2$ underlying the span $(\widehat{\mathcal{R}}, \widehat{r}_1, \widehat{r}_2)$ is a Φ^+-bisimulation "up-to-$\cup \approx_2$", where

- $\Phi^+ : \mathcal{P}(T(S_N + S_N) \times T(S_N + S_N)) \to \mathcal{P}(T(S_N + S_N) \times T(S_N + S_N))$ is the monotone operator defined by
  $\Phi^+(\mathcal{R}) = \{(in_1(s), in_1(s')) \mid \pi_1(\alpha_1(s)) = \pi_1(\alpha_1(s')) \wedge \pi_2(\alpha_1(s))\ \mathcal{R}\ \pi_2(\alpha_1(s'))\} \cup$
  $\{(in_2(s), in_2(s')) \mid s = s'\}$;
- $\approx_2 = \{(in_2(s), in_2(s')) \mid s \in S_N\}$;
- $\mathcal{R}$ Φ^+-bisimulation "up-to-$\cup \approx_2$" means that $\mathcal{R} \subseteq \Phi^+(\mathcal{R} \cup \approx_2)$.

Then the principle of coalgebraic coinduction "up-to-_+S_N" for corecursive functions can be simply expressed as follows:

$$\frac{x\ \mathcal{R}\ y \quad in_1^+(\mathcal{R}) \subseteq \Phi^+(in_1^+(\mathcal{R}) \cup \approx_2)}{x \sim_h y} \ ,$$

where $in_1^+(\mathcal{R})$ denotes the set-theoretic image of $\mathcal{R}$ by in_1.

6.4.2 Linear time semantics

There is a straightforward natural definition of linear time semantics for CCS processes as $\mathcal{P}_{\mathtt{fin}}(\ )$-coiterative morphism, where $\mathcal{P}_{\mathtt{fin}}(\ )$ denotes the finite powerset monad defined as follows:

Definition 6.12 [Powerset monad] Let $(\mathcal{P}_{\mathtt{fin}}(\ ), \mu, \eta)$ be the monad defined by:

- $\mathcal{P}_{\mathtt{fin}}(\ ) : Set \to Set$ is the finite powerset functor in Set;
- $\eta_X = \{\ \} : X \to \mathcal{P}_{\mathtt{fin}}(X)$ is singleton constructor;

- $\mu_X : \mathcal{P}_{\mathtt{fin}}(\mathcal{P}_{\mathtt{fin}}(X)) \to \mathcal{P}_{\mathtt{fin}}(X)$ is defined by

$$\forall x \in \mathcal{P}_{\mathtt{fin}}(\mathcal{P}_{\mathtt{fin}}(X)).\ \mu_X(x) = \bigcup_{y \in x} y\ .$$

The notion of linear semantics that we consider here is that determined by all the possible (not necessarily complete) traces generated by a closed CCS process.

In order to define the linear trace semantics, we consider the $\mathcal{P}_{\mathtt{fin}}(\ )$-coiterative scheme obtained by taking F, in the generalized coiteration scheme, to be the functor $\mathcal{P}_{\mathtt{fin}}(A \times \_)$. Let $(\mathcal{P}_{\mathtt{fin}}(Proc^0_{CCS}), \alpha_l)$ be the $\mathcal{P}_{\mathtt{fin}}(A \times \_)$-coalgebra defined by:

$$\alpha_l(P) = \{(a, \{p_1 \mid \exists p \in P.\ p \stackrel{a}{\to} p_1\}) \mid a \in A\}\ .$$

Let $(\widetilde{P}, \alpha_{\widetilde{P}})$ denote the final coalgebra for $\mathcal{P}_{\mathtt{fin}}(A \times \_)$. Let $h_l : Proc^0_{CCS} \to \widetilde{P}$ be the $\mathcal{P}_{\mathtt{fin}}(\_)$-coiterative function determined by the coalgebra $(\mathcal{P}_{\mathtt{fin}}(Proc^0_{CCS}), \alpha_l)$, i.e. $h_l = f_l \circ \{\_\}$, where:

$$\begin{tikzcd}
Proc^0_{CCS} \arrow[r, "\{_\}"] & \mathcal{P}_{\mathtt{fin}}(Proc^0_{CCS}) \arrow[r, "f_l"] \arrow[d, "\alpha_l"'] & \widetilde{P} \arrow[d, "\alpha_{\widetilde{P}}"] \\
& \mathcal{P}_{\mathtt{fin}}(A \times \mathcal{P}_{\mathtt{fin}}(Proc^0_{CCS})) \arrow[r, "\mathcal{P}_{\mathtt{fin}}(id_A \times f_l)"'] & \mathcal{P}_{\mathtt{fin}}(A \times \widetilde{P})
\end{tikzcd}$$

Proposition 6.13 *The equivalence induced by the $\mathcal{P}(\_)$-coiterative morphism $h_l : Proc^0_{CCS} \to \widetilde{P}$ is the linear trace equivalence.*

A natural question that one can ask at this point is whether it is possible to define a (interesting) trace semantics as a final semantics, i.e. as a standard coiterative morphism, or the trace semantics is intrinsically non-coiterative. All the attempts in the literature of giving a trace semantics by finality actually hide a definition by $\mathcal{P}_{\mathtt{fin}}$-coiteration.

Here we just notice that the function h_l escapes the standard coiteration scheme. In fact, there is no way of endowing $Proc^0_{CCS}$ with a $\mathcal{P}_{\mathtt{fin}}(A \times \_)$-coalgebra structure in order to recover the linear-time semantics. On the other hand, notice that the powerset of the set of traces of CCS processes is a proper subset of the final coalgebra of $\mathcal{P}_{\mathtt{fin}}(A \times \_)$, which in turn consists of all branching-time processes. Therefore, a related issue is that of looking for different functors whose final coalgebras are tighter to the powerset of the set of CCS-traces.

6.4.3 Coiteration and coinduction "up-to-context"

Another interesting coiteration scheme is the one which we call context-coiteration. This is related to the (set-theoretic) principle of bisimulation "up-to-context", which has been used quite extensively in providing smooth proofs of important properties of CCS and π-calculus (see e.g. [15,21]), and, more recently, also for reasoning on various observational (contextual) equivalences on λ-calculus (see e.g. [7,11,12,6]).

We focus on CCS processes. We recall the definition of context on CCS processes:

Definition 6.14 [CCS contexts] The class of *CCS contexts* is defined as follows:

$$C[\,] \ ::= \ p \ \mid \ [\,] \ \mid \ a.C[\,] \ \mid \ C[\,]|C[\,] \ \mid \ C[\,]+C[\,] \ \mid \ \mathrm{rec}\ x.C[\,]\ ,$$

where $p \in \mathrm{Proc}_{CCS}$.

Definition 6.15 [Context monad] Let $T_C : Set \to Set$ be the endofunctor defined as follows:

$$T_C(X) = C[X] = \{C[x] \mid x \in X \ \wedge \ C[\,] \text{ is a closed CCS context}\}\ ,$$

$$\forall f : X \to Y \text{ morphism in } Set.\ T_C(f) = C[x] \mapsto C[f(x)]\ .$$

The functor T_C can be endowed with a structure of monad by defining

- $\eta_X : X \to T_C(X)$ by $x \mapsto [x]$
- $\mu_X : T_C^2(X) \to T_C(X)$ by $C_1[C_2[x]] \mapsto C_1[C_2[\,]/[\,]][x]$.

This time we discuss a special instance of the Principle of Coalgebraic Coinduction 6.7 for reasoning on standard coiterative morphisms.

Let us consider the functor F to be $\mathcal{P}_{\mathtt{fin}}(A \times \_)$, and the $\mathcal{P}_{\mathtt{fin}}(A \times \_)$-coalgebra $(C[\mathrm{Proc}^0_{CCS}], \alpha_s)$, where $\alpha_s : C[\mathrm{Proc}^0_{CCS}] \to \mathcal{P}(A \times C[\mathrm{Proc}^0_{CCS}])$ is defined by:

$$\alpha_s(C[p]) = \{(a, C[p_1]) \mid p \xrightarrow{a} p_1\}\ .$$

One can check that, a span $(\widehat{\mathcal{R}}, \widehat{r}_1, \widehat{r}_2)$ is a $\mathcal{P}_{\mathtt{fin}}(A \times \_)$-bisimulation "up-to-$C[\,]$" on the $\mathcal{P}_{\mathtt{fin}}(A \times \_)$-coalgebra $(C[\mathrm{Proc}^0_{CCS}], \alpha_s)$ if and only if the set-theoretic relation $\mathcal{R}$ underlying the span is a Φ_s-bisimulation "up-to-context" for the operator $\Phi_s : \mathcal{P}(T_C(Proc^0_{CCS}) \times T_C(Proc^0_{CCS})) \to \mathcal{P}(T_C(Proc^0_{CCS}) \times T_C(Proc^0_{CCS})$ defined by:

$$\begin{aligned}\Phi_s(\mathcal{R}) = \{(u,v) \mid\ & (\forall (a,u_1) \in \alpha_s(u).\ \exists (a,v_1) \in \alpha_s(v).\ u_1\ \mathcal{R}\ v_1) \ \wedge \\ & (\forall (a,v_1) \in \alpha_s(v).\ \exists (a,u_1) \in \alpha_s(u).\ u_1\ \mathcal{R}\ v_1)\}\ .\end{aligned}$$

I.e. the following principle "up-to-context" is sound and complete:

$$\frac{x\ \mathcal{R}\ y \quad \mathcal{R} \subseteq \Phi_s(C[\mathcal{R}])}{x \approx_s y},$$

where $\approx_s$ is the equivalence induced by the coiterative morphism determined by the coalgebra $(C[\mathrm{Proc}^0_{CCS}], \alpha_s)$, i.e. strong bisimilarity.

References

[1] P. Aczel, Non-well-founded sets, CSLI Lecture Notes 14, Stanford (1988)

[2] P. Aczel and N. Mendler, A Final Coalgebra Theorem, Proc. CTCS'89, Lecture Notes in Comp Science 389, D. Pitt et al (eds.) Springer-Verlag (1989), 357–365

[3] A. Corradini, M. Grosse-Rhode and R. Heckel, Structured Transition Systems as Lax Coalgebras, Proc. first Workshop of CMCS'98, Electric Notes in Theoretical Computer Science, Vol. 11 (1998), `http://www.elsevier.nl/locate/entcs/volume11.html`

[4] P. Freyd and A. Scedrov, Categories, Allegories, North-Holland, Amsterdam (1990)

[5] H. Geuvers, Inductive and Coinductive types with Iteration and Recursion, Proc. 1992 Workshop on Types for Proofs and Programs, Bastad, Sweden, eds. B. Nordstrom, K. Petersson and G.Plotkin (1992), 183–207,

[6] F. Honsell and M. Lenisa, Coinductive Characterizations of Applicative Structures, Math. Struct. in Comp. Science 9 (1999), 403–435.

[7] D. Howe, Proving congruence of bisimulation in functional programming languages, Inf. and Comp. 124(2) (1996), 103–112.

[8] P.T. Johnstone, Adjoint lifting theorems for categories of algebras, Bull. London Math. Soc. 7 (1975), 294–297

[9] G.M. Kelly, A Unified Treatment Of Transfinite Constructions For Free Algebras, Free Monoids, Colimits, Associated Sheaves, And So On, Bull. Austral. Math. Soc. vol. 22(1980), 1–83

[10] G.M. Kelly and R. Street, Review of the elements of 2-categories, Sydney Category Seminar, Lecture Notes in Mathematics 420 (1974), 75–103

[11] S. Lassen, Relational Reasoning About Contexts, Higher Order Operational Techniques in Semantics, A.Gordon and A.Pitts (eds.), Cambridge University Press (1996).

[12] M. Lenisa, Themes in Final Semantics, Ph.D. Thesis TD-6/98 (1998), Dipartimento di Informatica, Università di Pisa, Italy.

[13] M. Lenisa, From Set-theoretic Coinduction to Coalgebraic Coinduction: some results, some problems, Proc. CMCS'99, Electric Notes in Theoretical Computer Science, Vol.19 (1999), `http://www.elseier.nl/locate/entcs/volume19.html`

[14] R. Milner, Calculi for synchrony and asynchrony, Theoretical Computer Science 25 (1983), 267–310

[15] R. Milner, J.Parrow and D.Walker, A calculus of mobile processes, Inf. and Comp. 100 (1992), 1–77.

[16] J. Power and D. Turi, A Coalgebraic Foundation for Linear Time Semantics, Proc. CTCS'99, Electric Notes in Theoretical Computer Science, vol.24 (1999)

[17] J. Power and H. Watanabe, Distributivity for a monad and a comonad, Proc. second Workshop of CMCS'99, Electric Notes in Theoretical Computer Science, Vol. 19 (1999),
`http://www.elsevier.nl/locate/entcs/volume19.html`

[18] J. Power and H. Watanabe, Combining a monad and a comonad, ETL Technical Report TR-99-33 (submitted to TCS)

[19] R. Street, The formal theory of monads, J. Pure Appl. Algebra 2 (1972), 149–168

[20] J. Rutten and D. Turi, Initial algebra and final coalgebra semantics for concurrency, Proc. REX workshop A Decade of Concurrency – Reflections and Perspectives, Lecture Notes in Comp Science 803, J. de Bakker et al (eds.) Springer-Verlag (1994), 530–582

[21] D. Sangiorgi, On the bisimulation proof method, Math. Struct. In Comp. Science 98(8) (1998), 447–478.

[22] D. Turi and G. Plotkin, Towards a mathematical operational semantics, Proc. LICS 97 (1997), 280–291

Appendix 1

2-*category,* op-*construction,* co-*construction*

Definition 6.16 A 2-category K consists of

- a set of 0-*cells* or *objects*,
- for each pair of 0-cells X and Y, a category $K(X,Y)$ called the *homcategory from* X to Y,
- for each triple of 0-cells X, Y and Z, a *composition* functor $\circ : K(Y,Z) \times K(X,Y) \to K(X,Z)$,
- for each 0-cell X, an object id_X of $K(X,X)$, or equivalently, a functor $\mathrm{id}_X : 1 \to K(X,X)$, called the *identity on* X,

such that the following diagrams of functors commute

$$\begin{tikzcd}
K(Z,W) \times K(Y,Z) \times K(X,Y) \arrow[r, "\circ \times K(X,Y)"] \arrow[d, "K(Z,W) \times \circ"'] & K(Y,W) \times K(X,Y) \arrow[d, "\circ"] \\
K(Z,W) \times K(X,Z) \arrow[r, "\circ"'] & K(X,W)
\end{tikzcd}$$

$$\begin{tikzcd}
K(X,Y) \times K(X,X) \arrow[dr, "\circ"'] & K(X,Y) \arrow[l, "K(X,Y) \times id_X"'] \arrow[r, "id_Y \times K(X,Y)"] \arrow[d, "="] & K(Y,Y) \times K(X,Y) \arrow[dl, "\circ"] \\
& K(X,Y) &
\end{tikzcd}$$

In the definition of a 2-category, the objects of each $K(X,Y)$ are often called 1-*cells* and arrows of each $K(X,Y)$ are often called 2-*cells*.

Example 6.17 The leading example of a 2-category is Cat, in which the 0-cells are small categories and $Cat(C,D)$ is defined to be the functor category $[C,D]$. In this paper, we treat Cat as through Set is a 0-cell of Cat. Technically, the existence of two strongly inaccessible cardinals together with a careful variation in the use of the term small allows that.

There are two dual constructions which make other 2-categories.

Definition 6.18 For any 2-category K, the opposite 2-category K^{op} is defined as follows:

- 0-cells are the same as that of K,
- $K^{\mathrm{op}}(X,Y) = K(Y,X)$.

Definition 6.19 The 2-category K^{co} is defined as follows:

- 0-cells are the same as those of K,
- $K^{\mathrm{co}}(X,Y) = K(X,Y)^{\mathrm{op}}$.

Hence the op-construction reverses the 1-cells of K, and the co-construction the 2-cells of K.

Appendix 2

We show the Propostion 2.12 in the following form.

Proposition 6.20 *Cat admits Kleisli constructions for pointed endofunctors. The left* 2*-adjoint functor sends each pointed endofunctor* (T, η) *on a category* C *to* (T, η)-Kl *the Kleisli category for the pointed endofunctor.*

Proof. Let (T, η) be a pointed endofunctor on a category C and $J : C \to (T, \eta)\text{-Kl}$ be a functor sending each object a in C to a and an arrow $f : a \to b$ in C to $[f] : a \to b$ in (T, η)-Kl.

We define a natural transformation $j : JT \Rightarrow J$ by the arrow $j_a = [\mathrm{id}_{Ta}] : Ta \to a$ in (T, η)-Kl for each object a in C. These data give rise to a 1-cell $(J, j) : (C, T, \eta) \to Inc((T, \eta)\text{-Kl})$ in $\mathsf{pEndo}^*(Cat)$.

Now we show that (J, j) is the unit for a left 2-adjoint to $Inc : Cat \to \mathsf{pEndo}^*(Cat)$. Given a category X and an arrow $(K, k) : (C, T) \to Inc(X)$ in $\mathsf{pEndo}^*(Cat)$, define $\bar{K} : (T, \eta)\text{-Kl} \to X$ on objects by setting $\bar{K}a = Ka$, and on arrows by sending $[f] : a \to b$ in (T, η)-Kl, which is represented by an arrow $f : a \to T^n b$ in C for some $n \in \mathbb{N}$, to $k_b \circ k_{Tb} \circ \cdots \circ k_{T^{n-1}b} \circ Kf$. Then the following diagram commutes in $\mathsf{pEndo}^*(Cat)$.

$$
\begin{tikzcd}
(C, T) \arrow[r, "{(J, j)}"] \arrow[dr, "{(K, k)}"'] & Inc((T, \eta)\text{-Kl}) \arrow[d, "Inc(\bar{K})"] \\
& Inc(X)
\end{tikzcd}
$$

Since each arrow $[f] : a \to b$ in (T, η)-Kl, which is represented by an arrow $f : a \to T^n b$ in C for some $n \in \mathbb{N}$, can be written as composites $k_b \circ k_{Tb} \circ \cdots \circ k_{T^{n-1}b} \circ Jf$ in (T, η)-Kl, the unicity of such $\bar{K}$ follows immediately.

It remains to verify the 2-dimensional property that every 2-cell $\alpha : (K, k) \to (L, l)$ in $\mathsf{pEndo}^*(Cat)$ extends uniquely to a natural transformation $\hat{\alpha} : \bar{K} \to \bar{L}$ such that $Inc(\hat{\alpha}) \circ (J, j) = \alpha$ holds. If we define $\hat{\alpha}$ by $\hat{\alpha}_a = \alpha_a$ for each object a in (T, η)-Kl, then this is the unique natural transformation which satisfies the 2-dimensional property. $\square$