Coalgebraic Coinduction in (Hyper)set-theoretic Categories ⋆

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Abstract

This paper is a contribution to the *foundations* of *coinductive types* and *coiterative functions*, in *(Hyper)set-theoretical Categories*, in terms of *coalgebras*. We consider *atoms* as first class citizens. First of all, we give a *sharpening*, in the way of cardinality, of Aczel's Special Final Coalgebra Theorem, which allows for good estimates of the cardinality of the final coalgebra. To these end, we introduce the notion of *κ-Y-uniform functor*, which subsumes Aczel's original notion. We give also an n-ary version of it, and we show that the resulting class of functors is closed under many interesting operations used in Final Semantics.

We define also *canonical wellfounded* versions of the final coalgebras of functors uniform on maps. This leads to a *reduction* of *coiteration* to *ordinal induction*, giving a possible answer to a question raised by Moss and Danner.

Finally, we introduce a *generalization* of the notion of *F-bisimulation* inspired by Aczel's notion of *precongruence*, and we show that it allows to extend the theory of *categorical bisimulations* also to functors *non*-weakly preserving pullbacks.

Examples, non-examples, and open questions are frequent in the paper.

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Introduction

In recent years, much effort has been devoted to developing adequate *categorical foundations* to *coinduction*, based on *coalgebras*, [Acz88,AM89,RT93,RT94], etc.. This approach amounts, essentially, to viewing *coinductive types* as *final coalgebras*, and to construing *coiterative functions* in terms of *coalgebras*.

This research enterprise has been very successful. It has provided a new approach to the semantics of programming languages and process calculi, *i.e. Final Semantics* [Acz88,Acz93,RT93,HL95,Jac96,Len96,HLMP98,Len98,TR98], and it has supported new tools for the analysis of *dynamical systems* [Rut96].

It is surprising how much of Final Semantics can be carried out already in purely *set-theoretic categories*. Far from being too poor, set-theoretic categories allow to model, in fact, *more* than other categories. For instance, we can treat directly and fruitfully *term models* (e.g. of λ-calculus [HL95] and of higher order imperative concurrent languages [Len96]), as well as equivalences which do not have continuous semantics (e.g. weak-equivalences on CCS and π-calculus, see [Acz93,HLMP98,Len98]).

Following the seminal work of Aczel [Acz88,Acz93], very natural ambient categories for Final Semantics are the categories of classes of *hypersets*, i.e. non-wellfounded sets satisfying the Antifoundation Axiom X_1 of [FH83] (called AFA in [Acz88]).

However, despite the vast amount of published literature on final semantics in categories of sets and hypersets, e.g. [Acz88,AM89,Bar93,Bar94,RT93,RT94] [BM96,MD96,Tur96,TR98], a definitive, *final*, picture is just only beginning to emerge [Moss00]. Each paper utilizes its peculiar definition, and little attention has been given to comparing the various results or to achieving generality.

In this paper, we try to give a *general* form of Aczel's *Special Final Coalgebra Theorem*, which is the main tool for Final Semantics in categories of hypersets. In particular, we give a *sharpening*, in the way of cardinality, of Aczel's original theorem, which provides good estimates of the cardinality of the final coalgebra. Furthermore, we refrain from reductionistic seductions, and we take *Urelementen* seriously, as first class citizens. These are crucial for providing smooth definitions, and doing without them often brings in spurious difficulties.

The functors we shall focus on are the *κ-Y-uniform functors*. This notion is essentially a reformulation of Aczel's original notion of functor uniform on maps to our setting. Inspired by Paulson [Pau99], we give an n-ary version of this notion, and show that the resulting class of functors is closed under many interesting operations for Final Semantics. These operations appear to be very promising in the treatment of mutually corecursively specified datatypes.

We prove various properties of κ-Y-uniform functors, including the fact that such functors are uniquely determined by their value on objects. Throughout the paper, we try to assess the generality of our results: discussing examples, non-examples, and raising many open questions. Whenever possible, we compare our notions and results to others appearing in the literature on Final Coalgebra Theorems, but more work needs to be done.

In this paper, we try to capitalize on the properties of (hyper)set-theoretic categories. And furthermore, we try to maintain a set-theoretic conceptual standpoint, i.e. we try to analyze seriously, from a set-theoretic perspective, the categorical concepts in use.

In this spirit, we propose two contributions to the *categorical theory of bisimulations*. The first consists in a *canonical wellfounded* version of the final coalgebra of a functor uniform on maps. This leads to a *reduction* of *coiteration* to *ordinal induction*, thus giving a possible answer to a question raised in [MD96].

The second contribution, which builds on [AM89], consists in a *generalization* of the notion of F-*bisimulation*, which allows to extend the theory of *categorical bisimulations* also to functors *non*-weakly preserving pullbacks.

More specifically, the paper is organized as follows. In Section 1, we discuss *hypersets* and Antifoundation Axioms. In Section 2, we briefly recall the coalgebraic foundations of coinductive types and coiterative functions. In Section 3, we introduce the notion of, possibly n-ary, κ-Y-uniform functor, and give our sharpened version of the Special Final Coalgebra Theorem. In Subsection 3.2, we give a number of *closure theorems* for the class of κ-Y-uniform functors. In Section 4, we give some properties of κ-Y-uniform functors, and we compare our definitions and results to others which have been introduced in the literature. In Section 5, we give a *simple explanation* of coinduction in terms of induction. In Section 6, we give our coalgebraic account alternative to F-*bisimulations*. Finally, in Section 7, we discuss briefly some remaining challenges to the paradigm of Coalgebraic Coinduction.

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1 (Hyper)set-theoretic Preliminaries

In this section, we present some basic definitions and facts concerning non-wellfounded set theories ([FH83,Acz88]). We work in a Zermelo-Frænkel-like (or, even better, Gödel-Bernays-like) set-theory with *Urelementen* (*atoms*). The formalistic reader can cast everything we say in the set theory $ZFC_0^- +$ *Extensionality up-to U* (ZFC_0^- is ZFC without Extensionality and Foundation) or equivalently in $A^0BCN +$ *Extensionality up-to U* (A^0BCN is Gödel-Bernays set-theory $ABCN$ without Extensionality). The axiom of *Extensionality up-to U* is defined as follows:

Axiom 1.1 (Extensionality up-to U)
$(\forall u \in U.\ \forall z.\ z \notin u)$ *and*
$(\exists x \notin U.\ \forall z.\ z \notin x)$ *and*
$(\forall x, y \notin U.\ \forall z.\ (z \in x \;\Leftrightarrow\; z \in y) \implies x = y)$.

Definition 1.2 • V_U denotes the universal class of objects, i.e. sets and atoms in U.

- A set $x \in \mathcal{P}(V_U)$ is *transitive* if $x \subseteq \mathcal{P}(x) \cup U$.
- The *transitive closure* of an object $x \in V_U$ is defined by

$$TC(x) = \bigcap \{y \mid y \text{ transitive } \wedge\ x \in y\} \ ;$$

- Let $U_0 \subseteq U$. V_{U_0} denotes the class of objects x, such that $(TC\{x\} \cap U) \subseteq U_0$.

Now we introduce the *Free Construction Principle FCU* (see e.g. [FHL94]). This axiom is a version of the Antifoundation Axiom X_1 of Forti-Honsell, [FH83] (see also [Acz88]), for a set theory with a class U of atoms. I.e. X_1 is equivalent to FCU, when $U = \emptyset$.

Axiom 1.3 (FCU) *For every function $f : X \to \mathcal{P}(X \cup U) \cup (U \setminus X)$, there is a* unique *function $g : X \to V_U$ which makes the following diagram commute*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & \mathcal{P}(X \cup (U \setminus X)) \cup (U \setminus X) \arrow[dl, "\mathcal{P}(g \cup id_{U\setminus X}) \cup id_{U \setminus X}"] \\ V_U & \end{tikzcd}$$

I.e., for all $x \in X$,

$$g(x) = \begin{cases} f(x) & \textit{if } \ f(x) \in U \setminus X \\ \{g(y) \mid y \in f(x) \cap X\} \cup \{u \mid u \in f(x) \setminus X\} & \textit{otherwise} \ . \end{cases}$$

Notice that the Axiom X_1 is precisely, word-by-word, equivalent to the fact that the non-wellfounded universe $V_\emptyset$ is a *final coalgebra* for the functor $\mathcal{P}(\_)$ in the category of non-wellfounded classes. This is not so for the formulation of [Acz88].

As is the case for the antifoundation axiom X_1, also the axiom FCU yields a form of *strong extensionality*:

Axiom 1.4 (SExt up-to U) *Two sets x, y are equal if and only if there exists a Φ^+_U-*bisimulation *$\mathcal{R}$ such that $x \,\mathcal{R}\, y$, where Φ^+_U is the following operator on relations of the universe of sets $\mathcal{P}(V_U)$:*
$\Phi^+_U(\mathcal{R}) = \{(x, y) \mid x \cap U = y \cap U \ \wedge$
$\forall x_1 \in (x \setminus U).\ \exists y_1 \in (y \setminus U).\ x_1 \,\mathcal{R}\, y_1 \ \wedge\ \forall y_1 \in (y \setminus U).\ \exists x_1 \in (x \setminus U).\ x_1 \,\mathcal{R}\, y_1\}$.

We have in fact:

Proposition 1.5

$$ZFC_0^-(U) \ + \ \textit{SExt up-to } U \ + \ FCU \implies \textit{SExt up-to } U \ .$$

The Axiom SExt up-to U amounts to the *strong extensionality* of $\mathcal{P}(V_U)$ viewed as final coalgebra for the functor $\mathcal{P}(\_ + U)$.

It will be very useful to be able to bound the cardinality of the image of X under g in the axiom FCU. First of all, we need the following definition:

Definition 1.6 • Let X be a class, let $f : X \to V_U$, and let $x \in X$. We define the *f-transitive closure* as follows:

$$TC_f(x) = \bigcap\{y \mid y \text{ transitive } \wedge \ x \in y \ \wedge \ \forall z \ (z \in y \cap X \ \Rightarrow \ f(z) \subseteq y)\} \ .$$

• Let $U_0 \subseteq U$. Let $HC_\kappa(U_0)$ denote the class of sets over atoms in U_0 whose hereditary cardinal is less than κ. I.e.:

$$x \in HC_\kappa(U_0) \iff (|TC(x)| < \kappa \ \wedge \ U \cap TC(x) \subseteq U_0) \ .$$

The following proposition is an easy consequence of FCU.

Proposition 1.7 *Assume FCU. Let $U_0 \subseteq U$, let X be a class such that $X \cap U_0 = \emptyset$, and let $f : X \to \mathcal{P}(X \cup U_0) \cup U_0$. If, for all $y \in X$, $|TC_f(y)| < \kappa$, then the image of X under the function g given by FCU, denoted by $g^+(X)$* [1]*, is such that $g^+(X) \subseteq HC_\kappa(U_0)$.*

Finally, we recall the notions of *substitution* and *substitution extension*, which will be useful in Section 3:

Definition 1.8 [Substitution] Let $X \subseteq U$. A *substitution* σ is a function $\sigma : X \to V_U$.

Proposition 1.9 (Substitution Extension) *Assume FCU. Let $X \subseteq U$. The* substitution extension *of a substitution $\sigma : X \to V_U$ is the unique function $\widehat{\sigma} : V_U \to V_U$ such that*

$$\widehat{\sigma}(a) = \begin{cases} \sigma(a) & \textit{if } \ a \in X \\ a & \textit{if } \ a \in U \setminus X \\ \{\widehat{\sigma}(b) \mid b \in a\} & \textit{otherwise} \ . \end{cases}$$

Proof. Let Z, W be such that $(Z \cup W) \cap U = \emptyset$ and $Z \cap W = \emptyset$, and let $\alpha : Z \to V_U$, $\beta : W \to V_U$ be two bijections. Let $f : Z \to \mathcal{P}(Z \cup U) \cup U$ be defined by

$$f(z) = \begin{cases} u & \text{if } \alpha(z) = u \in U \\ \{z' \mid \alpha(z') \in \alpha(z)\} & \text{otherwise} \ . \end{cases}$$

The function g_f obtained by applying FCU is such that $g_f(\alpha^{-1}(x)) = x$. Similarly, the function g_h induced by the function $h : W \to \mathcal{P}(W \cup U) \cup U$

[1] It is often denoted also by $\mathcal{P}(g)(X)$.

below satisfies $g_h(\beta^{-1}(x)) = x$.

$$h(w) = \begin{cases} u & \text{if } \beta(w) = u \in U \\ \{w' \mid \alpha(w') \in \alpha(w)\} & \text{otherwise .} \end{cases}$$

Let $t : Z \cup W \to \mathcal{P}(Z \cup W \cup U) \cup U$ be defined by

$$t(y) = \begin{cases} h(y) & \text{if } y \in W \\ h(w) & \text{if } y \in Z \ \wedge \ \alpha(y) \in X \ \wedge \ \sigma(x) = \beta(w) \\ u & \text{if } y \in Z \ \wedge \ \alpha(y) \in U \setminus X \\ \{y' \mid \alpha(y') \in \alpha(y)\} & \text{otherwise .} \end{cases}$$

Now $\widehat{\sigma} = g_t \circ \alpha^{-1}$, where the function g_t is obtained by applying FCU to t. □

1.1 (Hyper)set-theoretic Categories

In this paper, we shall discuss *(hyper)set-theoretic categories.* We shall not give an explicit definition of this notion. Informally, these are categories where objects are sets and morphisms are set-theoretic functions. The main examples we shall focus on are the following:

Definition 1.10 • Let $Set(U)$ ($Set^*(U)$) be the category whose objects are the *(non-)wellfounded sets* belonging to a Universe of $ZF_0^-(FCU)$, and whose arrows are the set-theoretic (total) functions. We denote by $FinSet(U)$ ($Finset^*(U)$) the corresponding subcategories whose objects are the *finite sets.*

- Let $Class(U)$ ($Class^*(U)$) be the category whose objects are the *classes* of (non-)wellfounded sets belonging to a Universe of $ZF_0^-(FCU)$, and whose arrows are the functional classes.
- Let $\mathcal{HC}_\kappa(U)$ ($\mathcal{HC}^*_\kappa(U)$) be the category whose objects are the (non-)wellfounded sets whose *hereditary cardinal* is less than κ, κ inaccessible, and whose arrows are the set-theoretic functions.
- Let $Card$ ($CARD$) be the category whose objects are the cardinals (including Ord), and whose arrows are the set-theoretic functions.

Most of the definitions and results in this paper are given just for $Class^*(U)$. But, many of them, *mutatis mutandis*, can be stated also for $\mathcal{HC}^*_\kappa(U)$ and $Set^*(U)$. For lack of space, we do not elaborate on these.

1.1.1 An aside on Operators on Sets vs Functors

The notion of functor is somewhat orthogonal to the notion of set, hence it appears a little elusive when investigated from a purely set-theoretical perspective.

Although it can be somewhat marginal to the purposes of the present paper, for the sake of conveying this kind of feeling, we raise the following open problems, as far as we know:

(i) Which conditions does an operator $T : \mathrm{Obj}(Class^*(U)) \to \mathrm{Obj}(Class^*(U))$ need to satisfy to be the object part of an endofunctor in $Class^*(U)$?
A *necessary* condition is that T be *monotone w.r.t. cardinality*, i.e., for $x, y \neq \emptyset$, $|x| \leq |y| \;\Rightarrow\; |T(x)| \leq |T(y)|$. Notice that the following two operators do not satisfy this condition:
- $X \mapsto \bigcup X$;
- $X \mapsto \{x \mid x \cap X = \emptyset\}$.

(ii) Let $F : Class^*(U) \to Class^*(U)$ be an endofunctor on $Class^*(U)$ such that, for all class A, $FA = A$. Is it the case that F is isomorphic to the identity functor, i.e. there is a natural isomorphism from F to Id?
Partial answer: Yes on $FinSet(U)$.

2 Categorical Foundations of Coinductive Types and Coiterative Functions in terms of Coalgebras

In this section, we recall the categorical description of *coinductive types* and *coiterative functions* ([Acz88,AM89]) based on *coalgebras*. We work in a (hyper)-set-theoretic category $\mathcal{C}$. Throughout this section, let $F : \mathcal{C} \to \mathcal{C}$ be a functor on $\mathcal{C}$.

We start by introducing the category of F-coalgebras:

Definition 2.1 [$\mathcal{C}^F$] The category $\mathcal{C}^F$ of F-coalgebras is defined as follows. The objects of $\mathcal{C}^F$ are F-coalgebras, i.e. pairs (X, α_X), where $\alpha_X : X \to FX$ is an arrow in $\mathcal{C}$. The morphisms in $\mathcal{C}^F$, $f : (X, \alpha_X) \to (Y, \alpha_Y)$, are morphisms $f : X \to Y$ of the category $\mathcal{C}$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha_X"'] & Y \arrow[d, "\alpha_Y"] \\ F(X) \arrow[r, "F(f)"'] & F(Y) \end{tikzcd}$$

Unique morphisms into final coalgebras are called *coiterative morphisms*:

Definition 2.2 [Coiteration Scheme] Let $F : \mathcal{C} \to \mathcal{C}$, let (X, α_X) be an F-coalgebra, and let $(\nu F, \alpha_{\nu F})$ be a final F-coalgebra. The *coiterative morphism* is the unique F-coalgebra morphism $f : (X, \alpha_X) \to (\nu F, \alpha_{\nu F})$.

To put it briefly, coinductive data types can be construed coalgebraically as final F-coalgebras, for an endofunctor F, which can be “read off” from the *introduction rules* of the type. Correspondingly, *coiterative morphisms* can be viewed as the unique morphisms into a final coalgebra. For extensions of this approach which cover the class of corecursive morphisms see [Len99].

Example 2.3 Let us consider the coinductive type of *streams* on natural numbers, $\boldsymbol{N}$-*Stream*, with the usual introduction rule *cons* : $\mathbf{N} \times \boldsymbol{N}\text{-}Stream \rightarrow \boldsymbol{N}\text{-}Stream$. $\boldsymbol{N}$-*Stream* turns out to be final coalgebra for the functor $F_S : \mathcal{C} \rightarrow \mathcal{C}$ defined as follows:

$$\forall X \text{ object in } \mathcal{C}.\ F_S(X) = \mathbf{N} \times X\ ,$$

$$\forall f : X \rightarrow Y \text{ morphism in } \mathcal{C}.\ F_S(f) = id_{\mathbf{N}} \times f\ .$$

More precisely, ($\boldsymbol{N}$-*Stream*, $\langle H, T\rangle$) is final F_S-coalgebra, where $H : \boldsymbol{N}\text{-}Stream \rightarrow \mathbf{N}$ and $T : \boldsymbol{N}\text{-}Stream \rightarrow \boldsymbol{N}\text{-}Stream$ are the *head* and *tail* functions, respectively. As an example of a coiterative morphism, we show how to define as a coiterative morphism the function $h : \mathbf{N} \rightarrow \boldsymbol{N}\text{-}Stream$ such that $f(n) = (n, f(n), f^2(n), \ldots)$, for $f : \mathbf{N} \rightarrow \mathbf{N}$. Endow the set of natural numbers with a structure of F_S-coalgebra $\alpha_{\mathbf{N}} : \mathbf{N} \rightarrow \mathbf{N} \times \mathbf{N}$ as follows

$$\alpha_{\mathbf{N}}(n) = (n, f(n))\ .$$

One can easily check that the function h is the coiterative morphism from the F_S-coalgebra $(\mathbf{N}, \alpha_{\mathbf{N}})$ into the final coalgebra of streams.

3 The Special Final Coalgebra Theorem for κ-Y-uniform functors

A natural, interesting, and general problem is that of defining *significant* classes of endofunctors on (hyper)set-theoretic categories which have *final coalgebra*. Since the seminal book of Peter Aczel [Acz88], various authors have addressed this issue: Aczel and Mendler [AM89], Rutten and Turi [RT93,RT94], Barr [Bar93,Bar94], Rutten [Rut96], Barwise and Moss [BM96], Moss and Danner [MD96], and Moss [Moss00].

Probably, the largest class of such functors is difficult to capture and/or not very informative.

In this paper, we address the more significant question of defining classes of functors whose final coalgebra does exist, but which has, moreover, a *conceptually independent characterization*. After all, especially in applications, we are interested in computing and studying the properties of the elements of a coinductive type, and in developing formal tools for reasoning about them. Hence, an unsuggestive description of the type is of little value in this respect. This is the very reason for focusing on *(hyper)set-theoretic categories*. The crucial result in this area is Aczel's *Special Final Coalgebra Theorem* ([Acz88]). This theorem shows that a functor which is *uniform on maps* has a final coalgebra, which can be characterized as the maximal fixed point, in the hyperset universe, of the set theoretical operator consisting of the object part of the functor itself. In the literature, there are stronger theorems on the existence of final coalgebras, but they do not offer very perspicuous descriptions of the final coalgebra. They either show the existence of the final coalgebra using "abstract nonsense" or through a quotient.

This section is divided into two subsections. In Subsection 3.1, we discuss and strengthen Aczel's *Special Final Coalgebra Theorem*, in the way of cardinality. We obtain this result for *κ-Y-uniform functors*, this being a generalization of Aczel's notion of functor uniform on maps. Using this result we can obtain good estimates on the cardinality of the final coalgebra of such functors. In Subsection 3.2, following Paulson [Pau99], we discuss the n-ary version of the notion of *κ-Y-uniform functor*. We prove, moreover, that this class of functors is closed under many interesting operations, which allow, for instance, to treat smoothly mutually corecursive datatypes.

We start with some easy remarks concerning Final Coalgebras in (hyper)set-theoretic categories:

- When discussing final F-coalgebras, the value of $F(\emptyset)$ does not matter, in the sense that $(\emptyset, \epsilon)$, where $\epsilon : \emptyset \to F(\emptyset)$ is the empty-function, is always an *initial* coalgebra, whatever is the value of $F(\emptyset)$. Hence, we can assume $F(\emptyset) = \emptyset$ w.l.o.g..
- If $F(A) = B$, for $|A| = |B| = 1$, then (A, α_A) is *always* a final coalgebra, α_A being the unique function from A to B.

The following is an easy, but useful, proposition, which allows to reflect properties of final coalgebras between pairs of functors. In view of this proposition, interesting classes of functors which provide *conceptually independent* characterizations of their final coalgebra are even more valuable:

Proposition 3.1 *Let $F, G : \mathcal{C} \longrightarrow \mathcal{C}$ be endofunctors on $\mathcal{C}$, and let $t : F \stackrel{\cdot}{\longrightarrow} G$ be a natural transformation. If $(\nu G, \alpha_{\nu G})$ is a final G-coalgebra, and $t_{\nu G}$ is a bijection, then $(\nu G, t_{\nu G}^{-1} \circ \alpha_{\nu G})$ is a final F-coalgebra.*

Proof. Straightforward, by diagram chasing. □

As we will see in Subsection 3.1, Proposition 3.1 allows to compute the final coalgebra of important functors which are *not* uniform on maps, but which are *naturally isomorphic* to functors which *are* uniform on maps. An example is the identity functor.

3.1 The Special Final Coalgebra Theorem

Before presenting the main result of this section, i.e. a sharpening, in the way of cardinality, of the Special Final Coalgebra Theorem, we introduce the notion of *κ-Y-uniform functor*, which generalizes, in various ways, Aczel's notion of functor *uniform on maps.*

Definition 3.2 [κ-Y-uniform functor]

Let $F : Class^*(U) \to Class^*(U)$. F is *κ-Y-uniform* if

- $Y \subseteq U$;
- $\kappa \in$ Ord $\cup \{\text{Ord}\}$;

- $\forall A \neq \emptyset.\ \exists \delta_A : X_A \to A$ bijective . $(X_A \subseteq U \ \wedge\ X_A \cap Y = \emptyset \ \wedge\ \exists \phi_A : F(A) \to \mathcal{P}(V_{X_A \cup Y}) \cup Y)$ such that
  - $\forall f : A \to B$, $\forall a \in F(A)$, $F(f)(a) = \widehat{f \circ \delta_A} \circ \phi_A(a)$, i.e.:

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\phi_A"'] & F(B) \arrow[d, "\iota_{FB}"] \\ \mathcal{P}(V_{X_A \cup Y}) \cup Y \arrow[r, "\widehat{f \circ \delta_A}_{|\mathcal{P}(V_{X_A \cup Y}) \cup Y}"'] & V_U \end{tikzcd}$$

  - $\forall A.\ \forall \alpha : A \to F(A).\ \forall x \in X_A.\ |\mathrm{TC}_{h_\alpha}(x)| < \kappa$,
    where $h_\alpha : V_{X_A \cup Y} \to V_{X_A \cup Y}$ is defined by

$$h_\alpha(x) = \begin{cases} \phi_A(\alpha(\delta_A(x))) & \text{if } x \in X_A \\ x & \text{otherwise .} \end{cases}$$

A few remarks are in order.

κ-Y-uniform functors differ from Aczel's functors uniform on maps [Acz88] in two respects. They are not defined only on *pure sets*, but on the whole of $Class^*(U)$, and they take object values in $Class^*(Y)$. The parameter Y denotes, precisely, the class of atoms which can appear in the transitive closure of the object part of the functor F. The parameter κ provides a bound on the cardinality of the transitive closure of an element of the final coalgebra (see Theorem 3.7).

Perhaps the most remarkable property of κ-Y-uniform functors is that they are determined by their value on objects. Moreover, the family of ϕ_A's extends to a natural transformation between F and a suitable functor. Hence κ-Y-uniform functors fall into the pattern of [TR98,Moss00]. Other properties of κ-Y-uniform functors on maps are discussed in Section 4.

The definition of κ-Y-uniform functor, as given in Definition 3.2, is a little restrictive. As it is given, $U \setminus Y$ has to be a proper class. Hence there exist perfectly reasonable functors (e.g. the constant functor whose value is U), which cannot be κ-Y-uniform for any Y. At the price of a slight complication in the definition we could overcome this difficulty. We decided to keep the simpler definition, which is more intuitive, since all the extra functors would be in any case uniformly isomorphic to a κ-Y-uniform functor as in Definition 3.2.

We give now the two properties of κ-Y-uniform functors mentioned earlier:

Proposition 3.3 *Let $F : Class^*(U) \to Class^*(U)$ be κ-Y-uniform. Then, for all $A \neq \emptyset$, and for all bijective $\delta_A : X_A \to A$, there exists a unique $\phi_A : F(A) \to \mathcal{P}(V_{X_A \cup Y}) \cup Y$ such that $\phi_A = ((\widehat{\delta_A})_{|F(X_A)})^{-1}$.*

Proof. Taking f in the definition of κ-Y-uniformity to be the identity $id_A : A \to A$, we get $\widehat{\delta_A} \circ \phi_a = \iota_{F(A)}$. Hence, in particular both $\widehat{\delta_A}_{|\mathrm{img}(\phi_A)}$ and ϕ_A are injective. Moreover, by taking f in the definition of κ-Y-uniformity to be

δ_A^{-1}, we get $\iota_{F(X_A)} \circ F(\delta_A^{-1}) = \widehat{\delta_A^{-1} \circ \delta_A} \circ \phi_A$. Then, since $\widehat{\delta_A^{-1} \circ \delta_A} = id$, we derive $\mathrm{img}(\phi_A) = F(X_A)$. And therefore $\phi_A = ((\widehat{\delta_A})_{|F(X_A)})^{-1}$. □

Corollary 3.4 *Let $F : Class^*(U) \to Class^*(U)$ be κ-Y-uniform. Then the value of F is completely determined by its value on objects.*

Now we show that, for any family of bijective functions $\{\delta_A : X_A \to A\}_{A \neq \emptyset}$, the family $\{\phi_A : F(A) \to \mathcal{P}(V_{X_A \cup Y}) \cup Y\}_{A \neq \emptyset}$ gives rise to a natural transformation from F to L, where L is the functor defined as follows:

Definition 3.5 Let $L : Class^*(U) \to Class^*(U)$ be the functor defined by
for any class A, $L(A) = \mathcal{P}(V_{X_A \cup Y}) \cup Y$,
for any morphism $f : A \to B$, $L(f) = (\widehat{\delta_B^{-1} \circ f \circ \delta_A})_{|\mathcal{P}(V_{X_A \cup Y}) \cup Y}$.

Proposition 3.6 *Let $F : Class^*(U) \to Class^*(U)$ be κ-Y-uniform. Then, for all families of bijective functions $\{\delta_A : X_A \to A\}_{A \neq \emptyset}$, the family $\{\phi_A : F(A) \to L(A)\}_A$, where $\phi_\emptyset$ is any function with domain $F(\emptyset)$ and codomain $L(\emptyset)$* [2] *, is a natural transformation from F to L.*

Proof. We have to show that, for all $f : A \to B$, $\phi_B \circ F(f) = \widehat{\delta_B^{-1} \circ f \circ \delta_A} \circ \phi_A$. By definition of κ-Y-uniformity, $F(f) = \widehat{f \circ \delta_A} \circ \phi_A$. Then, in particular, $F(f) = (\delta_B \circ \widehat{\delta_B^{-1} \circ f \circ \delta_A}) \circ \phi_A$. Moreover, by the uniformity condition applied to id_B, we get $\phi_B = ((\widehat{\delta_B})_{|F(X_B)})^{-1}$ and $\mathrm{img}(\widehat{\delta_B}) = F(B)$. Then $\phi_B \circ F(f) = ((\widehat{\delta_B})_{|F(X_B)})^{-1} \circ (\delta_B \circ \widehat{\delta_B^{-1} \circ f \circ \delta_A}) \circ \phi_A = (\widehat{\delta_B^{-1} \circ f \circ \delta_A}) \circ \phi_A$. □

The interest of κ-Y-uniform functors lies in the following:

Theorem 3.7 *Let $F : Class^*(U) \longrightarrow Class^*(U)$ be a functor that is κ-Y-uniform. Then F has final coalgebra $(\nu F, id)$, where νF is the greatest fixed point of the operator on classes given by the object part of F, i.e. $\nu F = \nu A.F^{obj}(A)$, and $\nu F \subseteq HC^*_\kappa(Y)$.*

Proof. (sketch)
Let $\alpha_A : A \to F(A)$ be an F-coalgebra. Then the unique F-coalgebra morphism $f_A : (A, \alpha_A) \to (\nu F, \mathrm{id})$ is the function $g \circ \delta_A^{-1}$, where g is the uniuqe function given by FCU for $h_A : V_{X_A \cup Y} \to V_{X_A \cup Y}$ defined by

$$h_A(x) = \begin{cases} \phi_A(\alpha_A(\delta_A(x))) & \text{if } x \in X_A \\ x & \text{otherwise .} \end{cases}$$

The cardinality constraints follow from the estimate on the cardinality of sets generated by X_1 (see Proposition 1.7). □

The notion of κ-Y-uniformity on maps can be generalized to n-ary functors as follows:

[2] As we have seen in the introduction to Section 3, the value of $F(\emptyset)$ does not matter, when discussing final coalgebras, and hence we can assume $F(\emptyset) = \emptyset$ w.l.o.g..

Definition 3.8 [κ-Y-uniform functors (n-ary version)]

Let $F : \underbrace{Class^*(U) \times \ldots \times Class^*(U)}_{n} \to Class^*(U)$. F is *κ-Y-uniform* if

- $Y \subseteq U$;
- $\kappa \in$ Ord $\cup \{\text{Ord}\}$;
- $\forall A_1, \ldots, A_n \neq \emptyset.\ \forall i.\ \exists \delta_{A_i} : X_{A_i} \to A_i$ bijective $(X_{A_1}, \ldots, X_{A_n} \subseteq U \ \wedge\ \forall i.\ X_{A_i} \cap Y = \emptyset \ \wedge\ \forall i \neq j.\ X_{A_i} \cap X_{A_j} = \emptyset \ \wedge\ \exists \phi_{A_1,\ldots,A_n} : F(A_1, \ldots, A_n) \to \mathcal{P}(V_{X_{A_1} \cup \ldots \cup X_{A_n} \cup Y}) \cup Y$ such that
  · $\forall f_1 : A_1 \to B_1, \ldots, f_n : A_n \to B_n$, $\forall a \in F(A_1, \ldots, A_n)$, $F(f_1, \ldots, f_n)(a) = ((\widehat{\bigcup_{i=1}^n f_i \circ \delta_{A_i}}) \circ \phi_{A_1,\ldots,A_n})(a)$.
  · $\forall A_1, \ldots, A_n.\ \forall \alpha_1 : A_1 \to F(A_1, \ldots, A_n) \ldots \alpha_n : A_n \to F(A_1, \ldots, A_n).\ \forall a \in F(A_1, \ldots, A_n).\ |\mathrm{TC}_{h_{A_1,\ldots,A_n}}(\phi_{A_1,\ldots,A_n}(a))| < \kappa$,
  where $h_{A_1,\ldots,A_n} : V_{X_{A_1} \cup \ldots \cup X_{A_n} \cup Y} \to V_{X_{A_1} \cup \ldots \cup X_{A_n} \cup Y}$ is defined by

$$h_{\alpha_1,\ldots,\alpha_n}(x) = \begin{cases} \phi_{A_1,\ldots,A_n} \circ (\bigcup_{i=1}^n \alpha_i \circ \delta_{A_i})(x) & \text{if } x \in \bigcup_{i=1}^n A_i \\ x & \text{otherwise .} \end{cases}$$

Example 3.9 Let us consider the functor F in Example 2.3:

$$F(A) = \mathbf{N} \times A \ .$$

Then we immediately have:
$\phi_A((n, a)) = (n, x_a)$
$\kappa = \aleph_0$
$Y = \emptyset$.
And hence we can conclude that the cardinality of the streams of natural numbers is $2^{\aleph_0}$.

The following cardinality result is useful in computing the cardinality of final coalgebras:

$|HC_\kappa(U)| \leq exp_{\kappa+1}(|U|)$, where $exp_0(|U|) = |U|$, $exp_{\alpha+1}(|U|) = 2^{exp_\alpha(|U|)}$, and $exp_\lambda(|U|) = \sup_{\gamma<\lambda} exp_\gamma(|U|)$.

3.2 Closure theorems for the class of κ-Y-uniform functors

In this section, we present significant *non-examples* of κ-Y-uniform functors, and we give closure theorems for the class of n-ary κ-Y-uniform functors.

The following are *non-examples* of κ-Y-uniform functors:

(i) The *Identity functor* is not κ-Y-uniform, for any κ, Y. In fact, we reach a contradiction by taking $f_1(a) = \emptyset \neq f_2(a)$, for some $a \in A$, and $\emptyset \in B$.

(ii) Consider a *strict linear ordering*, L, of V. Let $F_L : Class^*(U) \to Class^*(U)$ be the functor defined as follows:
for any class A,

$$F_L(A) = L_{|A} \equiv \{(a_1, a_2) \mid a_1, a_2 \in A \ \wedge\ a_1 L a_2\} \ ;$$

for any functional class $f : A \to B$, $F_L(f) : F_L(A) \to F_L(B)$ is defined by

$$F_L(f)(a_1, a_2) = \begin{cases} (f(a_1), f(a_2)) & \text{if } (f(a_1), f(a_2)) \in F_L(B) \\ (f(a_2), f(a_1)) & \text{if } (f(a_2), f(a_1)) \in F_L(B)\ . \end{cases}$$

A simple argument allows to show that F_L is not κ-Y-uniform, for any κ, Y.

(iii) The *contravariant* powerset functor composed with itself, $\mathcal{P}^c \circ \mathcal{P}^c$, is not κ-Y-uniform, because the value of $\mathcal{P}^c \circ \mathcal{P}^c(f)$ depends on the codomain of f.

Notice that both the Identity functor and the functor F_L are *naturally isomorphic* to functors which *are* $\aleph_0$-Y-uniform on maps. In particular:

(i) Id $\simeq$ Sing, where $Sing(A) = \{\{a\} \mid a \in A\}$. Sing is easily shown to be $\aleph_0$-Y-uniform, by considering $\phi_A = \{\{x_a\} \mid x_a \in X_A\}$.

(ii) $F_L \simeq H_L$, where H_L is the $\aleph_0$-Y-uniform functor defined by

$$H_L(A) = \{((a_1, a_2), (a_2, a_1)) \mid a_1, a_2 \in A\}\ .$$

The class κ-Y-*Unif* of, possibly n-ary, endofunctors on $Class^*(U)$ which are κ-Y-uniform is closed under many interesting operations. To simplify matters and notation, we shall discuss the case of *Ord*-Y-uniform functors.

Theorem 3.10 (i) Constant *functors (i.e. functors mapping all objects on a fixed class A_0 and all morphisms on id_{A_0}) are in* $\mid A_0 \mid$*-Y-Unif, if and only if $TC(A_0) \cap U \subseteq Y$ and $U \setminus Y$ is a proper class.*

(ii) *The class* Ord-Y-Unif *is closed under* generalized compositon. *I.e.: if* $F_1, \dots, F_k : \underbrace{Class^*(U) \times \ldots \times Class^*(U)}_{n} \to Class^*(U)$, $F_i \in$ Ord-Y-Unif *or F_i is a projection,* $G : \underbrace{Class^*(U) \times \ldots \times Class^*(U)}_{k} \to Class^*(U)$, *and* $G \in$ Ord-Y-Unif, *then* $G(F_1, \dots, F_n) \in$ Ord-Y-Unif.

(iii) *Consider a formula ϕ of set-theory, such that*

$$\vdash \forall u_1, \dots, u_n \exists! x \in \mathcal{P}(V_U).\ \phi(u_1, \dots, u_n, x)\ .$$

Define the endofunctor

$$F_\phi : \underbrace{Class^*(U) \times \ldots \times Class^*(U)}_{n} \to Class^*(U)$$

as follows:
for all objects $A_1, \dots, A_n$,

$$F_\phi(A_1, \dots, A_n) = \{x \mid \exists u_1 \in A_1, \dots, u_n \in A_n.\ \phi(u_1, \dots, u_n, x)\}\ ;$$

for all $f_1 : A_1 \to B_1, \dots, f_n : A_n \to B_n$,

$$F_\phi(f_1, \dots, f_n)(\iota x.\phi(u_1, \dots, u_n, x)) = \iota y.\phi(f(u_1), \dots, f(u_n), y)\ .$$

Then the functor F_Φ is Ord-$\emptyset$-uniform. This construction subsumes many interesting functors uniform on maps. E.g. finite and infinitary products, *finite and infinitary* disjoint sums, *and* quotients.

(iv) *Let* $F : \underbrace{Class^*(U) \times \ldots \times Class^*(U)}_{n} \to Class^*(U)$.
Define the functor $\nu_1 F : \underbrace{Class^*(U) \times \ldots \times Class^*(U)}_{n-1} \to Class^*(U)$ *as follows:*
- for all objects $A_2, \ldots, A_n$, $\nu_1 F(A_2, \ldots, A_n)$ *is the final coalgebra of the unary functor* $X \mapsto F(X, A_2, \ldots, A_n)$*;*
- for all $f_2 : A_2 \to B_2, \ldots, f_n : A_n \to B_n$,

$$\nu_1 F(f_2, \ldots, f_n) = \tau_{\vec{B}}^{-1} \circ F(g, f_2, \ldots, f_n) \circ \tau_{\vec{A}} \ ,$$

where $\tau_{\vec{A}} : \nu X.F(X, \vec{A}) \to F(\nu X.F(X, \vec{A}), \vec{A})$ *is the isomorphism, and* $g : \nu X.F(X, \vec{A}) \to \nu X.F(X, \vec{B})$ *is the unique coalgebra morphism from the coalgebra* $F(id_{\nu X.F(X,\vec{A})}, f_2, \ldots, f_n) \circ \tau_{\vec{A}} : \nu X.F(X, \vec{A}) \to F(\nu X.F(X, \vec{A}), \vec{B})$ *into the final coalgebra* $\nu X.F(X, \vec{B})$.
If F *is Ord-*Y*-uniform on maps, then also* $\nu_1 F$ *is Ord-*Y*-uniform on maps.*

All closure properties listed above could be stated more sharply taking into account the parameter κ. For example, in the case of item iii, F_ϕ is κ-Y-uniform on maps if $|TC\{x \mid \phi(u_1, \ldots, u_n, x)\}| < \kappa$.

Example 3.11 Consider the following mutual coinductive specification of two datatypes:

$$P_1 = A \times P_2 \times P_1 \qquad P_2 = B \times P_1 \times P_2 \ .$$

The intended solution can be expressed in terms of final coalgebras for appropriate functors. One can easily see that

$$P_1 = \nu X.(A \times (\nu Y.B \times X \times Y) \times X) \qquad P_2 = \nu Y.(B \times (\nu X.A \times Y \times X) \times Y) \ .$$

Since $Class^*(U)$ is a *boolean algebra*, another interesting consequence of the closure under the operation of final coalgebra (item iv above) is that *Conway identities* hold, i.e.

$$\nu X.\nu Y.F(X, Y) = \nu X.F(X, X) = \nu Y.\nu X.F(X, Y)$$
$$\nu X.F(G(X)) = F(\nu X.G(F(X))) \ .$$

This allows us to derive many isomorphisms between *mutually defined* coinductive types, e.g.:

$$\nu X.\nu Y.\mathbf{N} \times X \times Y \simeq \nu X.\mathbf{N} \times X \times X \ .$$

4 Comparison with the Literature

In this section, we compare briefly our definitions and results to those appearing in the literature on Final Coalgebra Theorems. It is remarkable how many apparently different notions have been considered in this context, see e.g. [Acz88,AM89,Bar93,Bar94,BM96,MD96,RT93,TR98,Moss00]. Unfortunately, there is very little published work discussing the precise relations between such notions. This section can be viewed also as a preliminary and modest contribution in this direction.

We start by recalling some crucial properties of functors, which have been defined in the literature. These properties have been defined originally only for functors defined either on $Class$ or on $Class^*(\emptyset)$, which is called by Aczel $Class^*$. Of course, they make perfect sense also for any $Class^*(U)$ for arbitrary U.

Definition 4.1 [Acz88,AM89] Let $F : Class^*(U) \to Class^*(U)$ be a functor.

- F is *inclusion preserving* if
$$\forall A, B.\ A \subseteq B \Longrightarrow\ (F(A) \subseteq F(B)\ \wedge\ F(\iota_{A,B}) = \iota_{F(A),F(B)})\ ,$$
where $\iota_{A,B} : A \to B$ is the inclusion map from A to B.
- F is *set based*-[Acz88] if, for each class A and each $a \in F(A)$, there exist a set $A_0 \subseteq A$ such that $a \in F(A_0)$.
- F is *set based*-[AM89] if, for each class A and each $a \in F(A)$, there exist a set $A_0 \subseteq A$ and $a_0 \in F(A_0)$ such that $a = F(\iota_{A_0,A})(a_0)$.
- F is *standard* if it is inclusion preserving and set based-[Acz88].
- F is *uniform on maps* if $\forall A \neq \emptyset.\ \exists \delta_A : X_A \to A$ bijective . $(X_A \subseteq U\ \wedge\ \exists \phi_A : F(A) \to V_{X_A})$ such that
  · $\forall f : A \to V$, $\forall a \in F(A)$, $F(f)(a) = (\widehat{f \circ \delta_A} \circ \phi_A(a))$, i.e.:

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\phi_A"'] & F(V) \arrow[d, "\iota_{F(V)}"] \\ \mathcal{P}(V_{X_A}) \arrow[r, "\widehat{f \circ \delta_A}"'] & V \end{tikzcd}$$

Notice that, if a functor is inclusion preserving, then its value on a morphism depends only on the graph of the morphism, and not on the target. So, there are indeed many examples of functors which are not uniform on maps.

A simple, but not so completely well-known proposition is:

Proposition 4.2 *Let $F : Class^*(U) \to Class^*(U)$.*
i) F standard $\Longrightarrow$ F set based-[AM89].
ii) F is set based-[AM89] $\Longrightarrow$ F is naturally isomorphic to a standard functor up to the value on $\emptyset$.
iii) F uniform on maps $\Longrightarrow$ F standard.

As far as κ-Y-uniform functors, we have the following:

Proposition 4.3 *Let $F : Class^*(U) \to Class^*(U)$.*
i) F is κ-Y-uniform on maps $\Longrightarrow$ F is standard.
ii) If F is uniform on maps $\Longrightarrow$ F is Ord-$\emptyset$-uniform on maps.

Notice that, By Proposition 4.3i), one could have equivalently considered only functions $f : A \to V$, in the definition of κ-Y-uniformity.

The notion of κ-Y-uniform functor is probably closely related to the ntion of *map uniform* functor introduced by Moss and Danner ([MD96]), but further work is required to make that connection explicit.

Here is a list of further questions concerning uniform functors:

- Clearly, the restriction to $Class^*(\emptyset)$ of a κ-Y-uniform functor F whose values on objects are in $Class^*(\emptyset)$, induces a functor uniform on maps in the sense of Aczel, and by Proposition 3.6 also a uniform functor in the sense of Moss [Moss00]. If moreover the values are wellfounded classes, then it induces a uniform functor in the sense of [TR98]. But is it the case that any uniform functor in the sense of Moss [Moss00] can be extended conservatively to a κ-Y-uniform functor on $Class^*(U)$? We conjecture that this is the case, but more work needs to be done.
- Can we prove a counterpart of Corollary 3.4 for functors $F : Class^*(\emptyset) \to Class^*(\emptyset)$ uniform on maps in the sense of Aczel (i.e. Definition 4.1), or for uniform functors in the sense of [Moss00]? That is, are such functors uniquely determined by their values on objects? We conjecture that this is so.
- What is the exact relationship between uniform endofunctors in $Class^*(\emptyset)$ uniform in the sense of [Moss00], and endofunctors in $Class^*(\emptyset)$ uniform on maps in the sense of Aczel (i.e. Definition 4.1)? In view of Proposition 3.6, we conjecture that they are equivalent.

Besides the Special Final Coalgebra Theorem, Aczel proved, in [Acz88], also an alternative *Final Coalgebra Theorem.* While the *Special Final Coalgebra Theorem* gives an independent characterization of the final coalgebra as the maximal fixed point of the set theoretical operator underlying the functor F, the *Final Coalgebra Theorem* is more abstract and defines the final coalgebra as a quotient. Aczel and Mendler, in [AM89], reduced the hypotheses in [Acz88] for the *Final Coalgebra Theorem* to the simple fact that the functor is *set based*-[AM89] (see Definition 4.1).

Theorem 4.4 ([AM89]) *Every* set based-*[AM89] functor has a final coalgebra.*

In view of Proposition 4.2, the *Special Final Coalgebra Theorem*, i.e. Theorem 3.7, does not add anything to the *Final Coalgebra Theorem*, i.e. Theorem 4.4, as far as the existence of final coalgebras for functors. There are indeed many functors which are not uniform on maps, but are nonetheless standard, and hence set-based, e.g. the Identity functor and the functor F_L of Section 3.2. However, in view of Proposition 3.1, we do not know if the former is indeed more general than the latter.

Here are some further questions:

(i) is there a set-based functor which is not naturally isomorphic, on non-empty classes, to a functor uniform on maps?

(ii) is there an inclusion preserving functor which is not uniform on maps?

(iii) are inclusion preserving functors determined by the value on objects?

Clearly, the rôle of hypersets is prominent only in *Special Final Coalgebra Theorems*. One can easily see that the proof of the *Final Coalgebra Theorem*, i.e. Theorem 4.4, works just as well in *Class*-like categories, irrespectively of which foundation/anti-foundation axiom it satisfies, if any. This proof is reminiscent of the absolute proof of relative consistency of X_1 in [FH83], (or that of AFA). Essentially, it consists in solving a *universal problem* by taking a *quotient*. Therefore, the sense of the *Special Final Coalgebra Theorem* can be explained as follows. The consistency proof of the Axiom X_1 is the "master" proof of many *Final Coalgebra Theorems*. Once we have proved that a final coalgebra for the powerset functor exists, i.e. the hyperset universe, we do not need to redo that proof for all the functors which are uniform on maps, and we can fully describe the final coalgebras of such functors (hyper)set-theoretically.

Two last, important, results concerning the existence of final coalgebras in set-theoretic categories are the one proved by Barr in [Bar93,Bar94] and the one proved by Rutten in [Rut96]. The former concerns *κ-accessible functors*, for κ regular and such that $\lambda < \kappa \Longrightarrow 2^\lambda \leq \kappa$, while the latter concerns functors which have a *set of generators* and which *preserve weak pullbacks*. Both results provide information also on the cardinality of the final coalgebra, just as our Theorem 3.7. More work is necessary to work out the exact connections of these results with Theorem 4.4. At the moment we can only say that there exist functors which are uniform on maps but *do not preserve weak kernel pairs*, see the example in Section 6 derived from [AM89].

We conclude this section with one last question:

- **Question** : Is there an endofunctor which does not have final coalgebra in $Class^*(U)$?

5 Viewing Coinductive Datatypes Inductively

It is often the case that coinductive types can be implemented functionally as *wellfounded* classes, in such a way that coiteration can be explained away using induction. The classical example is the following:

Example 5.1 Elements of the coinductive datatype ***N**-Stream* can be viewed also as functions in $[\mathbf{N} \to \mathbf{N}]$. Then the example of the *coiterative function* from natural numbers into streams, $h : \mathbf{N} \to \mathbf{N}$-Stream of Section 2 can be modeled by the currification of the *iterative* function $g : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ defined by:

$$\begin{cases} g(0, n) = n \\ g(m+1, n) = f(g(m, n)) \ . \end{cases}$$

Alternatively, one could have modeled streams via their finite approximations, i.e. as wellfounded objects belonging to $\Pi_{n \in \mathbf{N}} \mathbf{N}^n$.

Many questions can be raised in this area, which have not received completely satisfactory answers. For instance: can something like the above be done always? In many circumstances, final coalgebras for a given functor can be viewed as *completions under some topological construction* of the initial algebras for that functor. E.g. see [Bar93], for the case of ω-continuous, $\aleph_0$-accessible functors, and the papers on Hyperuniverses [FHL94,FH96,FHL96], for the κ-compact powerset functor. But this is not the full story.

A very stimulating discussion of the general issue of viewing *coinductive* datatypes *inductively*, and reducing *coiteration* to *recursion* appears in [MD96]. Many special cases are dealt with there in depth, but the general problem is left as an open question.

In this section, we show how the inductive treatment of streams in Example 5.1 can be generalized to all coinductive types, when the coinductive type is determined by a κ-Y-uniform functor. Namely, we show that *coinduction* can be expressed, in general, in terms of *ordinal recursion* over a suitable *implementation of the coinductive type* as a *space of functions.*

To clarify the procedure, we shall present first the result for the "master" of all coinductive types, i.e. the final coalgebra of the functor $\mathcal{P}(\ )$, or equivalently the universe V of a pure (i.e. without atoms) theory of sets satisfying the axiom X_1.

This implementation of the universe of hypersets as a "space of functions" yields as a by-product a *canonical wellfounded representation* of the universe of hypersets. The idea comes from the following:

Theorem 5.2 ([FH84]) $ABCX_1 \vdash V \simeq \Pi$, *where* Π *denotes the universe of wellfounded sets.*

Proof. Define:
$G(0) = \lambda x.\emptyset$
$G(\alpha + 1) = \mathcal{P}(G(\alpha))$
$G(\beta) = \lambda x.\ \lambda\alpha < \beta.G(\alpha)(x)$ for limit β.
An easy application of strong extensionality implies that for all $x \in V$ there exists α such that $G(\alpha)(x)^{-1}\{x\}$ is a singleton. Then one can prove by induction on ordinals that the image of all the $G(\ )_\alpha$'s is wellfounded. Hence an injective mapping $In : V \to \Pi$ can be defined by putting $In(x) = G(\alpha)(x)$ for the least ordinal α for which $G(\alpha)^{-1}(x) = \{x\}$. $\square$

Not to clutter up the argument in taking care of proper classes, we shall consider $HC_\kappa = HC_\kappa(\emptyset)$, for κ inaccessible, instead of V. This is the final coalgebra of the functor $\mathcal{P}_{<\kappa}(\ )$.

We shall implement HC_κ as a set of "κ"-indexed sequences as follows, consider $G(\kappa) : HC_\kappa \to \Pi^\kappa$. It is straightforward to check that $G(\kappa)$ is injective. Then calling *Hyper* the image of $G(\kappa)$, and putting, for $a, b \in$ *Hyper*,

$$a \epsilon b \iff \forall\alpha < \kappa.\ a_\alpha \in b_{\alpha+1}\ ,$$

we get a *well-founded canonical representation* of the hyperset universe up to κ, namely:

Theorem 5.3 $\langle Hyper, \epsilon\rangle \simeq \langle HC_\kappa, \in\rangle$

Coiterative functions can now be defined by *ordinal induction* into $\langle Hyper, \epsilon\rangle$. Suppose that the coiterative function $h : A \to HC_\kappa$ is defined by the following commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & \mathcal{P}_{<\kappa}(A) \arrow[d, "\mathcal{P}_{<\kappa}(h)"] \\ HC_\kappa \arrow[r, "\mathrm{id}"'] & HC_\kappa \end{tikzcd}$$

we can define the function $h^* : A \to Hyper$ as
$h^* = G_{|A}(|A|)$, where the $G(\alpha)$'s are *inductively* defined on ordinals as follows:
$G(0) = \lambda x.\emptyset$
$G(\alpha+1) = \mathcal{P}(G(\alpha)) \circ f$
$G(\beta) = \lambda x.\ \lambda\alpha < \beta.G(\alpha)(x)$, for limit β

It is easy to check that, $\forall x, y \in A$ and $\alpha < \beta < |A|$, $G(\beta)(x) = G(\beta)(y) \Rightarrow G(\alpha)(x) = G(\alpha)(y)$. Hence there exists a bijective function δ such that the following diagram comutes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h^*"'] & \mathcal{P}_{<\kappa}(A) \arrow[d, "\mathcal{P}_{<\kappa}(h^*)"] \\ Hyper \arrow[r, "\delta"'] & Hyper \end{tikzcd}$$

And hence h^* satisfies the analogue of the corecursive property of h.

Mutatis mutandis, we can give wellfounded implementations of other coinductive types determined by functors which are κ-Y-uniform, and we can define, by ordinal induction, coiterative functions into these.

Let $F : Class^*(U) \to Class^*(U)$ be κ-Y-uniform, let $\langle A, F : A \to F(A)\rangle$ be an F-coalgebra. The unique function h into the *wellfounded* final F-coalgebra is defined inductively as follows:
$h_A = G_{|A}(|A|)$, where the $G(\alpha)$'s are *inductively* defined on ordinals as follows:
$G(0) = \lambda x.\emptyset$
$G(\alpha+1) = F(G(\alpha)) \circ f$
$G(\beta) = \lambda x.\ \lambda\alpha < \beta.G(\alpha)(x)$ for limit β.

Again, using the fact that $F(G(\alpha)) = (\widehat{G(\alpha) \circ \delta_A}) \circ \phi_A$, it is easy to check by ordinal induction that $\forall x, y \in A$ and $\alpha < \beta < |A|$, $G(\beta)(x) = G(\beta)(y) \Rightarrow G(\alpha)(x) = G(\alpha)(y)$. Hence there exists a bijective function δ_A such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h_A"'] & F(A) \arrow[d, "F(h_A)"] \\ \mathrm{img}(h_A) \arrow[r, "\delta_A"'] & F(\mathrm{img}(h_A)) \end{tikzcd}$$

It is straightforward now to see that putting $H = \bigcup_{\langle A,F:A\to F(A)\rangle} h_A$ and $\nu_w F = \mathrm{img} H$. We have that $\langle \nu_w F, \bigcup_{\langle A,F:A\to F(A)\rangle} \delta_A\rangle$ is a wellfounded final F-coalgebra.

Of course it is nice when we can find conceptually independent characterizations of $\nu_w F$. We do not discuss this issue here.

We conclude this section by raising the following issue. Are coinductive types really necessary? Why are they more convenient/efficient? There are various lines of argument here. For lack of space we just mention two. Coinductive types are a well-behaved instances of higher order objects. Moreover their structure reflects often a more efficient implementation. This is the case for stream implementations of real numbers, where in most algorithms it is far more common to compute the "next value" rather than accessing randomly the values of the function.

6 An alternative coalgebraic account of bisimulation

In this section, we generalize the theory of F-bisimulations ([AM89]) to functors which do not necessarily preserve weak pullbacks.

The success of *Final Semantics* relies on the beautiful theory of *F-bisimulations*, see [AM89,RT93,RT94,Rut96]. We briefly recall the main definitions and results:

Definition 6.1 [F-bisimulation, [AM89]] A relation $\mathcal{R} \subseteq X \times Y$ is an *F-bisimulation* on the F-coalgebras (X, α_X) and (Y, α_Y), if there exists $\gamma : \mathcal{R} \to F(\mathcal{R})$, such that $(\mathcal{R}, \gamma)$ is an F-coalgebra, and both projections $\pi_1 : \mathcal{R} \to A$, $\pi_2 : \mathcal{R} \to B$ are coalgebra morphisms, i.e.:

$$\begin{tikzcd} X \arrow[d, "\alpha_X"'] & \mathcal{R} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\gamma"] & Y \arrow[d, "\alpha_Y"] \\ F(X) & F(\mathcal{R}) \arrow[l, "F(\pi_1)"] \arrow[r, "F(\pi_2)"'] & F(Y) \end{tikzcd}$$

When the two F-coalgebras (X, α_X) and (Y, α_Y) in the definition above coincide, we will simply say that $\mathcal{R}$ is an *F-bisimulation* on the F-coalgebra (X, α_X).

In [TP97], a generalization of the original notion of F-bisimulation, which works in a generic category, is given, by considering, instead of set-theoretic relations, generic *spans* on the objects X and Y, i.e. objects $\mathcal{R}$ with two ordered arrows $r_1 : \mathcal{R} \to X$, $r_2 : \mathcal{R} \to Y$.

Definition 6.2 [F-bisimulation, [TP97]] Let F be an endofunctor on the category $\mathcal{C}$. A span $(\mathcal{R}, r_1, r_2)$ on objects X, Y is an *F-bisimulation* on the F-coalgebras (X, α_X) and (Y, α_Y), if there exists an arrow of $\mathcal{C}$, $\gamma : \mathcal{R} \to F(\mathcal{R})$, such that $((\mathcal{R},\gamma),r_1,r_2)$ is a coalgebra span, i.e.

$$\begin{tikzcd} X \arrow[d, "\alpha_X"'] & \mathcal{R} \arrow[l, "r_1"'] \arrow[r, "r_2"] \arrow[d, "\gamma"] & Y \arrow[d, "\alpha_Y"] \\ F(X) & F(\mathcal{R}) \arrow[l, "F(r_1)"] \arrow[r, "F(r_2)"'] & F(Y) \end{tikzcd}$$

Every F-bisimulation à la Aczel-Mendler is a span F-bisimulation. Moreover, the main theorem of Final Semantics (see Theorem 6.3 below), holds whatever definition of F-bisimulation we consider. We emphasize the fact that in both cases the hypothesis "F preserves weak pullbacks" is required. The following is the main theorem underpinning Final Semantics:

Theorem 6.3 ([AM89]) *If F* preserves weak pullbacks*, and it has final F-coalgebra $(\nu F,\alpha_{\nu F})$, then, for any F-coalgebra (X,α_X), the kernel of the coiterative morphism $f:(X,\alpha_X)\to(\nu F,\alpha_{\nu F})$ is the union of all F-bisimulations on (X,α_X), i.e.:*

$$f(x)=f(x') \iff \exists\, \mathcal{R}\subseteq X\times X.\ \ \mathcal{R}\ \ F\textit{-bisimulation} \ \wedge\ x\,\mathcal{R}\,x' \ .$$

The following proposition is an essential tool for reasoning on F-bisimulations:

Proposition 6.4 ([AM89,RT93,RT94,Rut96]) (i) *The* graph *of the coiterative morphism determined by the functor F and the F-coalgebra (X,α_X) is an F-bisimulation on the F-coalgebra (X,α_X).*

(ii) *Any F-bisimulation on a final F-coalgebra is a subset of the* identity relation*;*

(iii) *The* composition *of F-bisimulations is an F-bisimulation.*

(iv) *The* union *of all F-bisimulations is an F-bisimulations.*

Unfortunately, in Theorem 6.3, as well as in Proposition 6.4(iii), the hypothesis F *preserves weak pullbacks* is crucial. This requirement is indeed too strong, as the following example shows.

Example 6.5 [AM89] Let $F : Class^* \to Class^*$ be the functor defined as follows:

$$F(X)=\{(x,y,z)\in X^3 \mid |\{x,y,z\}|<3\}$$
$$F(f)(x,y,z)=(f(x),f(y),f(z)) \ .$$

F is uniform on maps and νF is any singleton set, e.g. $x=\{x\}$.
But, let (A,α_A) be defined as follows: $A=\{0,1\}$, $\alpha_A(0)=(0,0,1)$, $\alpha_A(1)=(0,1,1)$. One can easily see that the kernel of the unique morphism into νF is not preserved. Furthermore, the kernel of the unique morphism from (A,α_A) into νF is not an F-bisimulation.

In view of the above discussion, it is therefore extremely important to get rid of the extra hypothesis "F preserves weak pullbacks". But, for this we need some alternative to the notion of F-bisimulation. To this end, we could draw inspiration from the notion of *F-precongruence*.

Definition 6.6 [F-precongruence, [AM89]] An *F-precongruence* $\mathcal{R}$ on an F-coalgebra (X, α_X) is a relation $\mathcal{R} \subseteq X \times X$ such that $\mathcal{R} \subseteq \Phi^F(\mathcal{R})$, where $\Phi^F : \mathcal{P}(X \times X) \to \mathcal{P}(X \times X)$ is the monotone operator defined by:

$$\Phi^F(\mathcal{R}) = \{(x, x') \in X \times X \mid F(q_{\mathcal{R}}) \circ \alpha(x) = F(q_{\mathcal{R}}) \circ \alpha(x')\} ,$$

where $q_{\mathcal{R}} : X \to X/\mathcal{R}^e$, $\mathcal{R}^e$ is the least equivalence relation containing $\mathcal{R}$, and $q_{\mathcal{R}}$ is the quotient function.

Following [AM89], we call *F-congruence* an F-precongruence which is an *equivalence* relation.

The relationship between the notion of F-bisimulation and that of F-precongruence is clarified by the following proposition:

Proposition 6.7 ([AM89]) *i) Any F-bisimulation on an F-coalgebra (X, α_X) is an F-precongruence on (X, α_X).*
ii) If F preserves weak pullbacks, then any F-congruence on (X, α_X) is an F-bisimulation on (X, α_X).

Notice that the relation $A \times A$ in the Example 6.5 *is* a precongruence.

However, F-precongruences are defined only on a *single* F-coalgebra. And so, to be able to use it in place of F-bisimulations, we have to generalize it.

Therefore, we introduce the following generalization of Definition 6.6, which makes sense in a generic category which has both pullbacks and pushouts. Notice that it does not mention any notion of "least equivalence containing the given relation":

Definition 6.8 [Generalized F-precongruence] Let $\mathcal{C}$ be a category with both pullbacks and pushouts, and let $F : \mathcal{C} \to \mathcal{C}$. An *$F$-precongruence* on F-coalgebras (X, α_X) and (Y, α_Y) is a span $(\mathcal{R}, r_1 : \mathcal{R} \to X, r_2 : \mathcal{R} \to Y)$ such that

$$(\mathcal{R}, r_1 : \mathcal{R} \to X, r_2 : \mathcal{R} \to Y) \ \leq \ (P, p_1 : P \to X, p_2 : P \to Y) ,$$

where[3] $(P, p_1 : P \to X, p_2 : P \to Y)$ is the pullback of $(F(q_1) \circ \alpha_X : X \to F(Q), F(q_2) \circ \alpha_Y : Y \to F(Q))$, and $(Q, q_1 : X \to Q, q_2 : Y \to Q)$ is the pushout of $(r_1 : \mathcal{R} \to X, r_2 : \mathcal{R} \to Y)$.

Generalized F-congruences are defined as generalized F-precongruences which are equivalences (see [FS90] for more details on the categorical notion of equivalence).

The notion of *generalized precongruence* generalizes that of [AM89], in the sense of Proposition 6.9 below. The proof is given for set-theoretic categories only. For more general categories we need an extra hypothesis where indicated.

[3] We recall that, for $(\mathcal{R}, r_1 : \mathcal{R} \to X, r_2 : \mathcal{R} \to Y)$, $(\mathcal{R}', r'_1 : \mathcal{R} \to X, r'_2 : \mathcal{R} \to Y)$ spans on X, Y, $\mathcal{R} \leq \mathcal{R}'$ iff there exists $f : \mathcal{R} \to \mathcal{R}'$ such that $\forall i = 1, 2.\ r_i = r'_i \circ f$.

Proposition 6.9 *Let $\mathcal{C}$ be a set-theoretic category, and let $F : \mathcal{C} \to \mathcal{C}$.*
i) Any F-bisimulation on F-coalgebras (X, α_X) and (Y, α_Y) is an F-precongruence on (X, α_X) and (Y, α_Y).
ii) If F preserves weak pullbacks, then any F-congruence on F-coalgebras (X, α_X) and (Y, α_Y) is an F-bisimulation on (X, α_X) and (Y, α_Y).

Proof. i) Assume that $(\mathcal{R}, r_1, r_2)$ is an F-bisimulation on (X, α_X) and (Y, α_Y). Using the universality property of the pullback, all the diagrams in the picture below commute, and hence, in particular, $(\mathcal{R}, r_1, r_2)$ is an F-precongruence:

$$\begin{tikzcd}
 & P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
X \arrow[d, "\alpha_X"'] & \mathcal{R} \arrow[l, "r_1"'] \arrow[r, "r_2"] \arrow[u, "\exists f"] \arrow[d, "\gamma"] & Y \arrow[d, "\alpha_Y"] \\
F(X) \arrow[dr, "F(q_1)"'] & F(\mathcal{R}) \arrow[l, "F(r_1)"'] \arrow[r, "F(r_2)"] & F(Y) \arrow[dl, "F(q_2)"] \\
 & F(Q) &
\end{tikzcd}$$

ii) Assume $(\mathcal{R}, r_1, r_2)$ is an F-congruence. Then the following is weak pullback diagram [4]:

$$\begin{tikzcd}
\mathcal{R} \arrow[r, "r_2"] \arrow[d, "r_1"'] & Y \arrow[d, "q_2"] \\
X \arrow[r, "q_1"'] & Q
\end{tikzcd}$$

Since F preserves weak pullbacks, then all the diagrams in the following picture commute, and hence in particular $(\mathcal{R}, r_1, r_2)$ is an F-bisimulation.

$$\begin{tikzcd}
\mathcal{R} \arrow[dr, "\exists g"] \arrow[drr, bend left, "r_2"] \arrow[ddr, bend right, "r_1"'] & & & \\
 & P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, "\exists\delta"] & Y \arrow[dr, "\alpha_Y"] & \\
 & X \arrow[dr, "\alpha_X"'] & F(\mathcal{R}) \arrow[r, "F(r_2)"] \arrow[d, "F(r_1)"] & F(Y) \arrow[d, "F(q_2)"] \\
 & & F(X) \arrow[r, "F(q_1)"'] & F(Q)
\end{tikzcd}$$

□

The notion of generalized pre-congruence can indeed replace that of F-bisimulation in Final Semantics as is shown by the following generalization of Theorem 6.3. Again the proof is given only in set theoretic categories. For more general categories we neeed an extra hypothesis where indicated.

Theorem 6.10 *Let $\mathcal{C}$ be a set-theoretic category, let $F : \mathcal{C} \to \mathcal{C}$ have final*

[4] Here we are using the fact that we are working on a set-theoretic category.

F-coalgebra $(\nu F, \alpha_{\nu F})$, and let (X, α_X) be any F-coalgebra. Then:
i) if $(\mathcal{R}, r_1, r_2)$ is an F-precongruence on (X, α_X), then $\mathcal{R} \leq K_f$, where K_f is the kernel of the coiterative morphism $f : (X, \alpha_X) \to (\nu F, \alpha_{\nu F})$;
ii) K_f is an F-precongruence on (X, α_X).

Proof. i) Assume that $(\mathcal{R}, r_1, r_2)$ is an F-precongruence. Let $\tau : Q \to \nu F$ be the unique morphism making the following diagram commute:

$$\begin{tikzcd}
\mathcal{R} \arrow[r, "r_2"] \arrow[d, "r_1"'] & X \arrow[d, "q_2"] \arrow[ddr, bend left, "f"] & \\
X \arrow[r, "q_1"'] \arrow[drr, bend right, "f"'] & Q \arrow[dr, "\tau"] & \\
& & \nu F
\end{tikzcd}$$

By definition of (P, p_1, p_2), $F(q_1) \circ \alpha_X \circ p_1 = F(q_2) \circ \alpha_X \circ p_2$, and then also $\alpha_{\nu F}^{-1} \circ \tau \circ F(q_1) \circ \alpha_X \circ p_1 = \alpha_{\nu F}^{-1} \circ \tau \circ F(q_2) \circ \alpha_X \circ p_2$. Hence, since all the diagrams (2)–(5) commute in the picture below, we have $f \circ p_1 = f \circ p_2$, and hence, by the universality property of K_f, $(P, p_1, p_2) \leq (K_f, \pi_1, \pi_2)$, and we are done.

$$\begin{tikzcd}
P \arrow[dr, "\exists g"] \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & & & \\
& K_f \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "(1)"] & X \arrow[d, "f"] \arrow[dr, "\alpha_X"] & & \\
& X \arrow[r, "f"'] \arrow[dr, "\alpha_X"'] & \nu F \arrow[d, phantom, "(2)"] \arrow[r, phantom, "(3)"] & F(X) \arrow[d, "F(f)"] \arrow[ddr, bend left, "F(q_2)"] & \\
& & F(X) \arrow[r, "F(f)"] \arrow[drr, bend right, "F(q_1)"'] & F(\nu F) \arrow[ul, "\alpha_{\nu F}^{-1}"'] \arrow[d, phantom, "(4)"] \arrow[r, phantom, "(5)"] & {} \\
& & & {} & F(Q) \arrow[ul, "\tau"']
\end{tikzcd}$$

ii) Since K_f is the kernel of f, then $f \circ \pi_1 = f \circ \pi_2$, and hence, since the diagrams (1)–(5) in the picture below commute, we have: $F(\tau) \circ F(q_1) \circ \alpha_X \circ \pi_1 = F(\tau) \circ F(q_2) \circ \alpha_X \circ \pi_2$ (where τ is defined as in the proof of item i).

$$\begin{tikzcd}
K_f \arrow[drr, "\pi_2"] \arrow[ddr, "\pi_1"'] & & & \nu F \arrow[r, "\alpha_{\nu F}"] \arrow[dr, phantom, "(5)"] & F(\nu F) \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"] & X \arrow[ur, "f"] \arrow[r, "\alpha"] & F(X) \arrow[ur, "F(f)"] \arrow[dd, "F(q_2)"'] \arrow[r, phantom, "(3)"] & {} \\
& X \arrow[dl, "f"'] \arrow[d, "\alpha_X"] \arrow[r, phantom, "(1)"] & {} & & \\
\nu F \arrow[d, "\alpha_{\nu F}"'] \arrow[r, phantom, "(2)"] & F(X) \arrow[dl, "F(f)"] \arrow[rr, "F(q_1)"] \arrow[d, phantom, "(4)"] & & F(Q) \arrow[uuur, bend right, "F(\tau)"'] \arrow[dlll, bend left, "F(\tau)"] & \\
F(\nu F) & {} & & &
\end{tikzcd}$$

Using the fact that $\mathcal{C}$ is a set-theoretic category, one can prove that τ is mono. Moreover, in a set-theoretic category one can easily prove also that, if τ is mono, then also $F(\tau)$ is mono. This completes the proof. $\square$

We are confident that, using the definition of F-precongruence instead of F-bisimulation, we can extend the scope of all other results in the theory of F-coalgebras, which rely on the assumption that F preserves weak pullbacks.

We end this section by remarking that, in set-theoretic categories, a set-theoretic relation $\mathcal{R}$ is an F-precongruence if and only if $\mathcal{R}' \subseteq \Phi^F(\mathcal{R}')$, where $\mathcal{R}' \subseteq (X+Y) \times (X+Y)$ is defined by $\mathcal{R}' = \{(\mathrm{in}_1^{X+Y}(x), \mathrm{in}_2^{X+Y}(y)) \mid x \; \mathcal{R} \; y\}$, and $\Phi^F : \mathcal{P}((X+Y) \times (X+Y)) \to \mathcal{P}((X+Y) \times (X+Y))$ is the generalized version of the operator Φ^F of Definition 6.6, i.e.:

$$\Phi^F(\mathcal{R}) =$$
$$\{(x,y) \in X \times Y \mid F(q_{\mathcal{R}'}) \circ F(\mathrm{in}_1^{X+Y}) \circ \alpha_X(x) = F(q_{\mathcal{R}'}) \circ F(\mathrm{in}_2^{X+Y}) \circ \alpha_Y(y)\} \; ,$$

where $q_{\mathcal{R}'} : X+Y \to X+Y/\, \mathcal{R}^e$ is the least equivalence containing $\mathcal{R}'$, and $q_{\mathcal{R}'}$ is the quotient function.

This provides a natural way to go from *categorical* bisimulations to *set-theoretic* bisimulations (cfr. [Rut98]). It would be interesting to give a *compositional explanation* of Φ^F in terms of the structure of F (cfr. [HJ98,Len99]).

7 Final Remarks

A wide range of coinductive phenomena have been successfully captured coalgebraically in (hyper)set-theoretic categories.

First of all, *final semantics* have been given for many programming languages and process calculi, both first-order and higher-order, with various *observational equivalences*, e.g.:

- *Branching* and *linear*, *strong* and *weak* equivalences on CCS-like processes (see e.g. [Acz88,Acz93,Len98]);
- *λ-calculus.* Standard denotational semantics for λ-calculus is defined solving *mixed-variance* domain equations. In order to define a final semantics, one can consider the following *purely covariant* functors (see [HL95,HL99]):
$$F(X) = (\Lambda^0 \to X) + (\Lambda^0 \to X) \; ,$$
$$G(X) = \mathcal{P}(X \times X) + \mathcal{P}(X \times X) \; ;$$
- *Objects* (see [Rei95,Jac96]);
- *Probabilistic transition systems* (see [RV97]);
- *Process calculi* with higher-order *imperative* features (see [Len96]);
- *π-calculus* (see [HLMP98]).

Moreover, the paradigm of coalgebraic coinduction has been extended to cover more general forms of coiteration and coinduction "up-to" ([Len99,LPW00]).

Still, there are situations where set-theoretic coinduction does not seem to be directly expressible in coalgebraic terms. Here is a list of possible challenges still remaining for coalgebraic coinduction:

- In [Rei95,Jac96], it shown how fruitful it is to model objects coalgebraically. However binary methods, such as *object equality* appear to be problematic, in that they give rise to contravariant functors. It would be interesting to explore if the idea of viewing "functions as graphs", underpinning the functor $G(\ )$, defined above for λ-calculus, works also in this case.
- As we have shown in Section 3.2, it is possible to treat mutually corecursively defined datatypes. However a full blown theory is still lacking.
- There are various kinds of set-theoretical bisimulations which have not been yet captured coalgebraically, e.g. Milner's bisimulation "up-to-$\approx$" and *non uniform* bisimulations [Len99].
- Mixed induction-coinduction is very problematic.
- Is it always possible, or useful, to specify coalgebraically arbitrary coinductive predicates on coinductive types, as it has been done with the *equivalences induced by coiterative morphisms*? Simple examples would be the predicates which define *infinite* or *fair streams*, or *truly non-wellfounded sets*.

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