Final Semantics for a Higher Order Concurrent Language*

Marina Lenisa

Dipartimento di Matematica e Informatica, Università di Udine, Italy.
Dipartimento di Informatica, Università di Pisa, Italy.
lenisa@dimi.uniud.it

Abstract. We show that adequate semantics can be provided for imperative *higher order concurrent* languages simply using *syntactical final* coalgebras. In particular we investigate and compare various behavioural equivalences on higher order processes defined by finality using *hypersets* and *c.m.s.'s.* Correspondingly, we derive various coinduction and mixed induction-coinduction proof principles for establishing these equivalences.
Keywords: second order assignment, F-coalgebra, F-bisimulation, final semantics, operational semantics, hyperset, complete metric space, coinduction, mixed induction-coinduction.

Introduction

In this paper we show that a *syntactical* "final approach" to semantics can be applied successfully to higher order concurrent languages. Such languages are usually modeled by means of *recursive* domains of metric processes (see e.g. [2]). This paper builds on [13] and is a companion to [10], where the "final approach" is applied to λ-calculus.

More specifically, using final syntactical coalgebras, we provide compositional semantics to a particular *concurrent* language with *higher order imperative* features, called $\mathcal{L}_{pas_2}$. This language is obtained by adding a parallel operator to the language with *second order assignment* $\mathcal{L}_{as_2}$, introduced and studied by de Bakker and van Breugel in [2]. It contains two kinds of *location*, one for storing values and one for storing procedures. We present and investigate *various* logically independent, but ultimately equivalent, final descriptions of the branching behavioural equivalence for $\mathcal{L}_{pas_2}$. We feel that $\mathcal{L}_{pas_2}$ is a good example, representative of an interesting class of concurrent languages with higher order features and global variables. For instance, all the results that we show for $\mathcal{L}_{pas_2}$ can be proved also for the concurrent language with second order communication $\mathcal{L}_{co_2}$ introduced and studied in [2], and further investigated in [3]. The methodology outlined in this paper for investigating $\mathcal{L}_{pas_2}$ should be, therefore, quite general.

We use extensively categorical tools and constructions. We shall mostly utilize a categorical setting of *non-wellfounded sets* which extends that of [1] to a

* Work supported by CNR, EEC Science grant MASK, and MURST 40% grant.

universe with atoms. One could have used also other categorical settings, such as *c.p.o.'s*, *c.m.s.'s*, or *hyperuniverses* ([7]), but the choice of non-wellfounded sets seems very natural, since it allows to define directly syntactical coalgebras. Clearly, non-wellfounded sets *per se* are not stricly necessary, but their use simplifies the notation and hence clarifies the arguments. Technically, non-wellfounded sets (also known as hypersets [5]) are elements of a Universe of $ZF_0^-(U)FCU$. This is a Zermelo Fraenkel like set theory with extensionality "up to atoms" and with the Axiom of Foundation replaced by the "unique" antifoundation axiom FCU, which generalizes to a setting with atoms the antifoundation axiom X_1 introduced in [6]. X_1 is also known as AFA [1].

An interesting result of this paper is the coincidence of the equivalence defined using syntactical coalgebras (hyperset equivalence) and that using recursively defined metric processes *à la* de Bakker-van Breugel (metric equivalence). The proof of this fact is technically rather involved. It can be viewed as a "full abstraction" result of the metric semantics w.r.t. the more syntax-based hyperset semantics.

We present a collection of logically independent *proof principles* for establishing behavioural equivalences. These comprise coinduction principles, mixed induction-coinduction principles, and "bisimulation up to" coinduction principles. Some of these principles are derived using a technique based on Freyd's algebraically compact categories in the line of Pitts [11].

Finally, we remark that none of the equivalences induced by the various semantics considered in this paper comes from a SOS system in tyft/tyxt format (see [9] for a description of this format).

The paper is organized as follows. In Section 1, we present the concurrent higher order language $\mathcal{L}_{pas_2}$. In Section 2, we define, by finality in the hyperset setting, a branching covariant semantics, $\mathcal{M}$, for $\mathcal{L}_{pas_2}$. In Section 3, we show that the semantics $\mathcal{M}$ is compositional, using two dual techniques. In Section 4, we show that the hyperset equivalence induced by $\mathcal{M}$ coincides with that obtained using the metric denotational semantics à la de Bakker. In Section 5, we give yet another final characterization of the hyperset equivalence. In Section 6, we derive a list of coinduction and induction-coinduction principles for $\mathcal{L}_{pas_2}$. Final remarks appear in Section 7. Basic definitions appear in Appendix A. Some technical definitions and results used in Section 4 appear in Appendix B.

The author would like to thank Fabio Alessi, Franck van Breugel, Gianluigi Ferrari, Furio Honsell, Jan Rutten, and Daniele Turi for illuminating discussions.

1 A Concurrent Language with Second Order Assignment

We introduce the higher order language $\mathcal{L}_{pas_2}$, which combines imperative and concurrent features:

Definition 1. Let $(v\in)IVar$ be the set of *individual variables* and let $(x\in)PVar$ be the set of *procedure variables.* The set of statements $(s\in)Statement$ of the language $\mathcal{L}_{pas_2}$ is defined by

$$s ::= \; E \; \mid \; v := e \; \mid \; s;s \; \mid \; s+s \; \mid \; s \,||\, s \; \mid \; x \; \mid \; x := s \;.$$

The operational semantics of $\mathcal{L}_{pas_2}$ is given by a transition system whose configurations are pairs of *syntactical states* and statements. Syntactical states are *global* and keep track of the value of both individual variables and procedure variables. These are defined as follows:

Definition 2. The set of syntactical states $SynState$ is defined by
$(\sigma \in)\ SynState = (IVar \to Val) \times (PVar \to Statement)$.

Let $v \in IVar$, $x \in PVar$. Let the states $\sigma[\alpha/v]$ and $\sigma[s/x]$ be short for $(\pi_1(\sigma)[\alpha/v], \pi_2(\sigma))$ and $(\pi_1(\sigma), \pi_2(\sigma)[s/x])$ respectively. The transition system $\mathcal{T}_{pas_2}$ is defined in Plotkin's *SOS* style as follows:

Definition 3 ($\mathcal{T}_{pas_2}$). The transition relation $\to$ is the smallest subset of $(SynState \times Statement) \times (SynState \times Statement)$, satisfying

$(\sigma,\ v := e) \to (\sigma[\mathcal{V}(e)(\sigma)/v],\ E)\ \ {}_{:=_1},$

where $\mathcal{V} : Expression \times Synstate \to Val$ is a function which associates values to expressions according to a syntactical state;

$$\frac{(\sigma,\ s) \to (\sigma_1,\ s_1)}{(\sigma,\ s;s') \to (\sigma_1,\ s_1;s')}\ ;_1 \qquad \frac{s \not\to \quad (\sigma,\ s') \to (\sigma_1',\ s_1')}{(\sigma,\ s;s') \to (\sigma_1',\ s_1')}\ ;_2$$

$$\frac{(\sigma,\ s) \to (\sigma_1,\ s_1)}{(\sigma,\ s+s') \to (\sigma_1,\ s_1)}\ +_1 \qquad \frac{(\sigma,\ s') \to (\sigma_1',\ s_1')}{(\sigma,\ s+s') \to (\sigma_1',\ s_1')}\ +_2$$

$$\frac{(\sigma,\ s) \to (\sigma_1,\ s_1)}{(\sigma,\ s \mid\mid s') \to (\sigma_1,\ s_1 \mid\mid s')}\ \mid\mid_1 \qquad \frac{(\sigma,\ s') \to (\sigma_1',\ s_1')}{(\sigma,\ s \mid\mid s') \to (\sigma_1',\ s \mid\mid s_1')}\ \mid\mid_2$$

$(\sigma,\ x) \to (\sigma,\ \sigma(x))\ \ {}_{var}$

$(\sigma,\ x := s) \to (\sigma[s/x],\ E)\ \ {}_{:=_2},$

where $s \not\to$ means that, for every σ, there is no transition starting from $(\sigma,\ s)$.

We feel that in Definition 3 we have captured the "essence" of a higher order concurrent imperative language with global variables. Small variations, e.g. in the *var* or $\mid\mid$ rules should not matter. We do not consider an explicit recursion operator μ, since it can be simulated using second order assignment. The expressiveness of the language could be increased by adding, say, an `if then else` construct. The constructions and the results of this paper would go through, without any substantianl modification, even with such extensions.

2 A Final Semantics in the Hyperset Setting for $\mathcal{L}_{pas_2}$

In this section we define, by finality, an operational semantics $\mathcal{M}$ in the hyperset setting, by endowing the syntactical set $Statement$ with a suitable coalgebra structure. We work in $Class^*$, the category whose objects are the classes of the Universe of $ZF_0^-(U)FCU$ and whose arrows are the functional classes. (For the antifoundation axiom FCU see Appendix A.) The particular nature of the transition system involved (finitely branching) makes *proper classes* unnecessary. All the results in this paper could have been carried out under suitable cardinality restrictions (see e.g. [4]). This being irrelevant for our purposes, the more straightforward approach with proper classes was preferred.

Definition 4. i) Let $F : Class^* \to Class^*$ be the endofunctor defined by
$F(X) = (SynState \to \mathcal{P}_{ne}((IVar \to Val) \times (PVar \to X) \times X)) + \{E\}$
$F(f) = (id \to \mathcal{P}_{ne}(id \times (id \to f) \times f)) + id,$
where $\mathcal{P}_{ne}$ denotes the set of all non-empty subsets and $+$ denotes the disjoint sum.
ii) Let $(Statement,\ \alpha)$ be the F-coalgebra defined by

$$\alpha(s) = \begin{cases} \{(\sigma,\ \{(\sigma_1,\ s_1) \mid (\sigma,\ s) \to (\sigma_1,\ s_1)\}) \mid \sigma \in SynState\} & \text{if } s \to \\ E & \text{if } s \not\to \end{cases}$$

where $s \to$ means that, for every σ, there is a transition starting from (σ, s).
iii) Let $\approx$ be the greatest F-bisimulation on the F-coalgebra $(Statement, \alpha)$. (For the definitions of F-coalgebra and F-bisimulation see Appendix A.)

Lemma 1. *The functor F has a greatest fixed point $\overline{X}$ such that $(\overline{X}, id)$ is a final F-coalgebra.*

Proof. One can easily see that F is set-continuous, inclusion preserving and uniform on maps. Hence the thesis follows from Corollary 4.23 of [13], suitably extended to a universe with atoms.

□

The hyperset semantics $\mathcal{M}$ is defined as follows

Definition 5. Let $\mathcal{M}$ be the unique morphism from the coalgebra $(Statement, \alpha)$ to the coalgebra $(\overline{X}, id)$, i.e. the unique function making the following diagram commmute

$$\begin{tikzcd} Statement \arrow[r, "\alpha"] \arrow[d, "\mathcal{M}"'] & F(Statement) \arrow[d, "F(\mathcal{M})"] \\ \overline{X} \arrow[r, "id"] & F(\overline{X}) \end{tikzcd}$$

Hence

$$\mathcal{M}(s) = \begin{cases} \{(\sigma,\ \{((\pi_1(\sigma_1),\ \{(x,\ \mathcal{M}(\pi_2(\sigma_1)(x))) \mid x \in PVar\}), \\ \qquad \mathcal{M}(s_1)) \mid (\sigma,\ s) \to (\sigma_1,\ s_1)\}) \mid \sigma \in SynState\} & \text{if } s \to \\ E & \text{if } s \not\to . \end{cases}$$

The greatest F-bisimulation on the F-coalgebra $(Statement, \alpha)$, $\approx$, is exactly the equivalence induced by $\mathcal{M}$:

Theorem 2. *For all $s, s' \in Statement$,* $\quad \mathcal{M}(s) = \mathcal{M}(s') \iff s \approx s'.$

Proof. One can easily see that F weakly preserves kernel pairs, hence, by Corollary 3.9 of [13] (suitably extended to a universe with atoms), the equivalence induced by $\mathcal{M}$ is $\approx$.

$\square$

From now onwards we shall refer to $\approx$ as the *hyperset equivalence.*

Now we give a simple set-theoretic description of the categorical notion of F-bisimulation on the F-coalgebra $(Statement, \alpha)$.

Notation. Let $R \subseteq Statement \times Statement$, $\sigma, \sigma' \in SynState$. We will use the shorthand $\sigma R \sigma'$ for: $\pi_1(\sigma) = \pi_1(\sigma')$ and $\forall x \in PVar.\ (\pi_2(\sigma)(x) R \pi_2(\sigma')(x))$.

Definition 6. Let $\mathcal{T} : \mathcal{P}(Statement \times Statement) \to \mathcal{P}(Statement \times Statement)$ be the operator defined as follows:
$\mathcal{T}(R) = \{(s, s') \mid \forall\sigma.(((\sigma, s) \to (\sigma_1, s_1) \ \Rightarrow\ \exists\sigma'_1, s'_1.((\sigma, s') \to (\sigma'_1, s'_1) \wedge \sigma_1 R \sigma'_1 \wedge s_1 R s'_1)) \wedge ((\sigma, s') \to (\sigma'_1, s'_1) \ \Rightarrow\ \exists\sigma_1, s_1.((\sigma, s) \to (\sigma_1, s_1) \wedge \sigma_1 R \sigma'_1 \wedge s_1 R s'_1)))\}$.

Lemma 3. *i) $R \subseteq Statement \times Statement$ is an F-bisimulation on the coalgebra $(Statement, \alpha)$ if and only if $R \subseteq \mathcal{T}(R)$.*
ii) The greatest F-bisimulation on the coalgebra $(Statement, \alpha)$, $\approx$, is the greatest fixed point of $\mathcal{T}(R)$.

Proof. i) The proof follows immediately from the definition of F-bisimulation.
ii) The proof follows from item i of this lemma and from the monotonicity of the operator $\mathcal{T}(R)$.

$\square$

3 Compositionality of the Hyperset Semantics

In this section we prove that $\mathcal{M}$ is compositional, i.e., for all $s, s' \in Statement$,

$$\mathcal{M}(s) = \mathcal{M}(s') \implies \forall C[\,].\mathcal{M}(C[s]) = \mathcal{M}(C[s'])\ ,$$

where $C[\,]$ denotes a context of $\mathcal{L}_{pas_2}$, i.e.
$C[\,] ::= [\,] \mid E \mid v := e \mid C[\,]; C[\,] \mid C[\,] + C[\,] \mid C[\,] \parallel C[\,] \mid x \mid x := C[\,]$.

We illustrate two dual methods for showing compositionality of the non well-founded semantics $\mathcal{M}$, which we call *a posteriori* and *a priori*, respectively.

In the *a posteriori* approach, one shows, possibly by coinduction, that $\approx$ is a congruence with respect to the syntactical operators (i.e. $\approx$ is an equivalence relation s.t. $s \approx s' \implies \forall C[\,].C[s] \approx C[s']$). Then, using Theorem 2, one can immediately define, *a posteriori*, semantical operators on the image of the interpretation function $\mathcal{M}$. Hence compositionality of $\mathcal{M}$ is equivalent to saying that the equivalence induced is a congruence. Notice that, in order to show that $\approx$ is a congruence, we cannot apply the standard technique on tyft/tyxt formats as described e.g. in [9].

The *a priori* approach consists in defining *a priori* operators coinductively and independently, on the semantical domain $\overline{X}$, and then showing by induction on statements that $\widetilde{op}_n(\mathcal{M}(s_1), \ldots, \mathcal{M}(s_n)) = \mathcal{M}(op_n(s_1, \ldots, s_n))$, for all

$s_1, \ldots, s_n$, where $\widetilde{op}_n$ is the semantical operator corresponding to the n-ary syntactical operator op_n.

One can easily check that the semantical operators defined *a posteriori* coincide with the semantical operators defined *a priori* on the image of $\mathcal{M}$.

A Posteriori Approach.

Theorem 4 (Congruence). *The greatest F-bisimulation $\approx$ is a congruence w.r.t. the syntactical operators of $\mathcal{L}_{pas_2}$, i.e., if $u \approx u'$ and $v \approx v'$, then*
1. $u + v \approx u' + v'$ **2**. $u; v \approx u'; v'$ **3**. $u \mid\mid v \approx u' \mid\mid v'$ **4**. $x := u \approx x := u'$.

Proof. Let R_u and R_v be the F-bisimulations which relate u and u', and v and v', respectively. Proceed by coinduction, i.e. by defining a suitable F-bisimulation R which, in the first three cases, relates $u\ op\ v$ and $u'\ op\ v'$, for $op \in \{+, ;, \mid\mid\}$, and, in the last case, relates $x := u$ and $x := u'$.

1. Let $R = R_u \cup R_v \cup \{(u + v,\ u' + v')\}$.

2. Let $R = R_u \cup R_v \cup \bigcup_n T_;^n$, where $T_;^0 = \{(u; v,\ u'; v')\}$, $T_;^{n+1} = T_;(T^n)$, and $T_;$ is the operator defined on pairs of statements of the shape $(w_1; w_2,\ w_1'; w_2')$ as follows: $T_;(\{(w,\ w') \mid w \equiv w_1; w_2 \ \wedge\ w' \equiv w_1'; w_2'\}) = \{(\overline{w}; w_2,\ \overline{w'}; w_2') \mid \exists\sigma\ ((\sigma,\ w_1) \to (\sigma_1,\ \overline{w})\ \wedge\ (\sigma,\ w_1') \to (\sigma_1',\ \overline{w'})\ \wedge\ \forall x.\sigma_1(x) R_u \sigma_1'(x)\ \wedge\ \overline{w} R_u \overline{w'})\}$.

3. $R = R_u \cup R_v \cup \bigcup_n T_{||}^n$, where $T_{||}^0 = \{(u \mid\mid v,\ u' \mid\mid v')\}$, $T_{||}^{n+1} = T_{||}(T^n)$ and $T_{||}$ is the operator defined on pairs of statements of the shape $(w_1 \mid\mid w_2,\ w_1' \mid\mid w_2')$ as follows: $T_{||}(\{(w,\ w') \mid w \equiv w_1 \mid\mid w_2 \ \wedge\ w' \equiv w_1' \mid\mid w_2'\}) =$
$\{(\overline{w} \mid\mid w_2,\ \overline{w'} \mid\mid w_2') \mid \exists\sigma\ ((\sigma,\ w_1) \to (\sigma_1,\ \overline{w})\ \wedge\ (\sigma,\ w_1') \to (\sigma_1',\ \overline{w'})\ \wedge\ \forall x.\sigma_1(x) R_u \sigma_1'(x)\ \wedge\ \overline{w} R_u \overline{w'})\} \cup \{(w_1 \mid\mid \overline{w},\ w_1' \mid\mid \overline{w'}) \mid \exists\sigma\ ((\sigma,\ w_2) \to (\sigma_1,\ \overline{w})\ \wedge\ (\sigma,\ w_2') \to (\sigma_1',\ \overline{w'})\ \wedge\ \forall x.\sigma_1(x) R_v \sigma_1'(x)\ \wedge\ \overline{w} R_v \overline{w'})\}$.

4. Let $R = R_u \cup \{(x := u, x := u'), (E, E)\}$.

$\square$

Theorem 5. *The hyperset semantics $\mathcal{M}$ is compositional.*

Proof. By theorems 2, 4 one can define semantical operators
$\tilde{;}, \tilde{+}, \widetilde{||} : cod(\mathcal{M}) \times cod(\mathcal{M}) \to cod(\mathcal{M})$ $\qquad \widetilde{:=}_x : cod(\mathcal{M}) \to cod(\mathcal{M})$
as follows:

$\mathcal{M}(s) \tilde{;} \mathcal{M}(s') = \mathcal{M}(s; s')$ $\qquad \mathcal{M}(s) \tilde{+} \mathcal{M}(s') = \mathcal{M}(s + s')$

$\mathcal{M}(s) \widetilde{||} \mathcal{M}(s') = \mathcal{M}(s \mid\mid s')$ $\qquad \widetilde{:=}_x(\mathcal{M}(s)) = \mathcal{M}(x := s)$.

$\square$

A Priori Approach.

Proposition 6. *The following semantical operators are well defined:*

i) $\tilde{;} : \overline{X} \times \overline{X} \to \overline{X}$

$$p \tilde{;} q = \begin{cases} E & \text{if } p = E \wedge q = E \\ q & \text{if } p = E \wedge q \neq E \\ \{(\sigma,\ \{(\eta,\ p_1 \tilde{;} q) \mid (\eta, p_1) \in p(\sigma)\}) \mid \sigma \in SynState\} & \text{otherwise.} \end{cases}$$

ii) $\widetilde{+} : \overline{X} \times \overline{X} \to \overline{X}$

$$p \mathbin{\widetilde{+}} q = \begin{cases} p & \textit{if } q = E \\ q & \textit{if } p = E \\ p \cup q & \textit{otherwise.} \end{cases}$$

iii) $\widetilde{||} : \overline{X} \times \overline{X} \to \overline{X}$

$$p \mathbin{\widetilde{||}} q = \begin{cases} E & \textit{if } p = E \wedge q = E \\ p & \textit{if } p \neq E \wedge q = E \\ q & \textit{if } p = E \wedge q \neq E \\ \{(\sigma,\ \{(\eta,\ p_1 \mathbin{\widetilde{||}} q) \mid (\eta, p_1) \in p(\sigma)\} \cup & \\ \qquad \{(\eta,\ p \mathbin{\widetilde{||}} q_1) \mid (\eta, q_1) \in q(\sigma)\}) \mid \sigma \in SynState\} & \textit{otherwise.} \end{cases}$$

iv) $\widetilde{:=}_x : \overline{X} \to \overline{X}$
$\widetilde{:=}_x\ p = \{(\sigma, \{((\pi_1(\sigma), \{(y, \pi_2(\sigma)(y)) \mid y \in PVar \ \wedge\ y \neq x\} \cup \{(x, p)\}), E)\}) \mid \sigma \in SynState\}$.

Proof. We have only to check that the operators $\widetilde{;}$ and $\widetilde{||}$ are well defined.
Let $\widehat{;}$ be the unique coalgebra morphism from the F-coalgebra $(\overline{X} \times \overline{X},\ \beta_;)$ to the final F-coalgebra $(\overline{X},\ id)$, where $\beta_; : \overline{X} \times \overline{X} \to F(\overline{X} \times \overline{X})$ is defined by

$$\beta;(p,q) = \begin{cases} E & \text{if } p = E \wedge q = E \\ \{(\sigma,\ \{(\eta_E, (E, q_1)) \mid (\eta, q_1) \in q(\sigma)\}) \mid \sigma \in SynState\} & \text{if } p = E \wedge q \neq E \\ \{(\sigma,\ \{(\eta_E, (p_1, q)) \mid (\eta, p_1) \in p(\sigma)\}) \mid \sigma \in SynState\} & \text{otherwise,} \end{cases}$$

where $\eta_E \in (IVar \to Val) \times (PVar \to \overline{X} \times \overline{X})$, $\eta_E(v) = \eta(v)$, for all $v \in IVar$, and $\eta_E(x) = (E, \eta(x))$, for all $x \in PVar$.
Then $\widetilde{;}$ coincides with $\widehat{;}$, since the coalgebra morphism $f_; : (\overline{X}, id) \to (\overline{X}, id)$, defined by $f_;(p) = E \mathbin{\widehat{;}} p$, must be the identity on $\overline{X}$, by finality of $(\overline{X}, id)$.
The operator $\widetilde{||}$ is dealt with similarly.

$\square$

Theorem 7. *The hyperset semantics $\mathcal{M}$ is compositional.*

Proof. By induction on statements, one can easily show that:
$\mathcal{M}(s) \mathbin{\widetilde{;}} \mathcal{M}(s') = \mathcal{M}(s; s')$ $\qquad$ $\mathcal{M}(s) \mathbin{\widetilde{+}} \mathcal{M}(s') = \mathcal{M}(s + s')$
$\mathcal{M}(s) \mathbin{\widetilde{||}} \mathcal{M}(s') = \mathcal{M}(s \mathbin{||} s')$ $\qquad$ $\widetilde{:=}_x(\mathcal{M}(s)) = \mathcal{M}(x := s)$.

$\square$

Corollary 3.1 A posteriori *semantical operators coincide with* a priori *semantical operators on the image of $\mathcal{M}$.*

Finally, we prove an interesting property of $\approx$, which implies that $\mathcal{M}$ is determined "up to" $\approx$-equivalent syntactical states. It amounts to say that $\approx$ is a fixed point of a suitable "contravariant" operator.

Lemma 8. *i) Let $R \subseteq Statement \times Statement$ be reflexive and symmetric. If*

$$sRs' \implies \forall \sigma, \sigma' .(\sigma R \sigma' \implies ((\sigma,\ s) \to (\sigma_1,\ s_1) \implies \exists \sigma_1', s_1' \ ((\sigma',\ s') \to (\sigma_1',\ s_1') \ \wedge \ \sigma_1 R \sigma_1' \ \wedge \ s_1 R s_1'))),$$

then R is an F-bisimulation.
Vice versa, an F-bisimulation which is an equivalence relation and a congruence w.r.t. the syntactical operators of $\mathcal{L}_{pas_2}$, satisfies the property above.
ii) The relation $\approx$ is such that, $\forall s, s' \in Statement$, $s \approx s' \iff$
$\forall \sigma, \sigma' .(\sigma \approx \sigma' \implies$
$((\sigma,\ s) \to (\sigma_1,\ s_1) \implies \exists \sigma_1', s_1' \ ((\sigma',\ s') \to (\sigma_1',\ s_1') \ \wedge \ \sigma_1 \approx \sigma_1' \ \wedge \ s_1 \approx s_1')) \ \wedge$
$((\sigma',\ s') \to (\sigma_1',\ s_1') \implies \exists \sigma_1, s_1 \ ((\sigma,\ s) \to (\sigma_1,\ s_1) \ \wedge \ \sigma_1 \approx \sigma_1' \ \wedge \ s_1 \approx s_1'))).$

Proof. i) The first part of the proof is immediate. In order to show the converse, let R be an F-bisimulation which is an equivalence and a congruence; let $s,\ s'$ be s.t. sRs'. Then let $\sigma,\ \sigma'$ be s.t. $\sigma R \sigma'$. If $(\sigma,\ s) \to (\sigma_1,\ s_1)$, then, since R is a F-bisimulation, $\exists \sigma_2,\ s_2.\ ((\sigma,\ s') \to (\sigma_2,\ s_2) \wedge\ \sigma_1 R \sigma_2 \wedge s_1 R s_2)$. We claim that, if $(\sigma,\ s') \to (\sigma_2,\ s_2)$, then $\exists \sigma_1',\ s_1'.\ ((\sigma',\ s') \to (\sigma_1',\ s_1') \ \wedge \ \sigma_2 R \sigma_1' \ \wedge \ s_2 R s_1')$. Hence by transitivity of R, we have the thesis. The claim is proved by induction on the length of the derivation of the judgement $(\sigma,\ s') \to (\sigma_2,\ s_2)$. The base case is immediate by definition (for *var* and $:=_1$) and using reflexivity of R (for $:=_2$). The induction step is dealt with by case analysis, according to the last rule applied in the derivation of $(\sigma,\ s') \to (\sigma_2,\ s_2)$, using the fact that R is a congruence.
ii) The proof follows from Lemma 3, Theorem 4, and from point i of this lemma. □

Corollary 3.2 *For all $s, s' \in Statement$ and for all $\sigma, \sigma' \in SynState$,*

$$(s \approx s' \ \wedge \ \sigma \approx \sigma') \implies \forall C[\ ].\mathcal{M}(C[s])(\sigma) = \mathcal{M}(C[s'])(\sigma')\ .$$

4 A Full Abstraction Result

In this section we consider the metric denotational semantics for $\mathcal{L}_{pas_2}$ *à la* de Bakker (see [2, 3]), and we compare the equivalence induced by this semantics, denoted by $\approx_{sem}$, and the hyperset equivalence $\approx$. We will show that $\approx_{sem}$ coincides with the equivalence $\approx$. It is interesting to notice that, as a consequence of this, we can characterize an equivalence defined using a contravariant functor (viz. $\approx_{sem}$) using solely "covariant" tools.

The equivalence $\approx_{sem}$ is induced by the metric denotational semantics defined using as domain of processes the unique solution P of the following recursive equation in the category CMS of Complete Metric Spaces:

$$P \ \cong \ (((IVar \to Val) \times (PVar \to P_{\frac{1}{2}})) \to^1 \mathcal{P}_{nco}((IVar \to Val) \times (PVar \to P_{\frac{1}{2}}) \times P_{\frac{1}{2}})) + \{E\},$$

where $P_{\frac{1}{2}}$ denotes the metric space obtained from the metric space P by contracting the distance of a factor $\frac{1}{2}$, $\to^1$ denotes the spaces of non distance increasing functions and $\mathcal{P}_{nco}$ denotes the spaces of non-empty compact subsets.

In order to show $\approx_{sem} = \approx$, we need some preliminary results on the metric denotational semantics. We proceed as in [2] for the language with second order communication $\mathcal{L}_{co_2}$. Namely, we prove that the equivalence $\approx_{sem}$ coincides with the equivalence induced on $Statement \times Statement$ by the extended operational semantics defined using Rutten's technique of "Processes as Terms" ([12]):

Definition 7. i) Let T be the set of terms defined by
$t ::= \ E \ | \ v := e \ | \ t;t \ | \ t+t \ | \ t \,||\, t \ | \ x \ | \ x := s \ | \ p \ ,$ where $p \in P$.
Let $SemState = (IVar \to Val) \times (PVar \to P)$ be the set of semantical states.
ii) Let $\mathcal{T}^{sem}_{pas_2}$ be the transition system defined as follows. The transition relation $\to_{sem}$ is the smallest subset of $(T \times SemState) \times (T \times SemState)$, satisfying

$$(\rho,\ v := e) \to_{sem} (\rho[\mathcal{V}_{sem}(e)(\rho)/v],\ E) \quad {}_{:=_1},$$

where $\mathcal{V}_{sem}$: $Expression \times Semstate \to Val$ is a function which associates values to expressions according to a semantical state;

$$\frac{(\rho,\ t) \to_{sem} (\rho_1,\ t_1)}{(\rho,\ t;t') \to_{sem} (\rho_1,\ t_1;t')} \ ;_1 \qquad \frac{t \not\to_{sem} \quad (\rho,\ t') \to_{sem} (\rho'_1,\ t'_1)}{(\rho,\ t;t') \to_{sem} (\rho'_1,\ t'_1)} \ ;_2$$

$$\frac{(\rho,\ t) \to_{sem} (\rho_1,\ t_1)}{(\rho,\ t+t') \to_{sem} (\rho_1,\ t_1)} \ +_1 \qquad \frac{(\rho,\ t') \to_{sem} (\rho'_1,\ t'_1)}{(\rho,\ t+t') \to_{sem} (\rho'_1,\ t'_1)} \ +_2$$

$$\frac{(\rho,\ t) \to_{sem} (\rho_1,\ t_1)}{(\rho,\ t \,||\, t') \to_{sem} (\rho_1,\ t_1 \,||\, t')} \ ||_1 \qquad \frac{(\rho,\ t') \to_{sem} (\rho'_1,\ t'_1)}{(\rho,\ t \,||\, t') \to_{sem} (\rho'_1,\ t \,||\, t'_1)} \ ||_2$$

$$(\rho,\ x) \to_{sem} (\rho,\ \rho(x)) \quad {}_{var}$$

$$(\rho,\ x := t) \to_{sem} (\rho[\mathcal{D}^{\#}(t)/x],\ E) \quad {}_{:=_2}$$

$$(\rho,\ p) \to_{sem} (\rho_1,\ p_1) \quad {}_{sem} \qquad \text{if } p \in P \ \wedge \ (\rho_1,\ p_1) \in p(\rho),$$

where $\mathcal{D}^{\#}$ denotes the extension of the metric denotational function *à la* de Bakker to all terms in T.

We will now define by finality an operational semantics $\mathcal{O}^{\#}$ on T.

Definition 8. Let $G : CMS \to CMS$ be the endofunctor defined by
$G(X) = (SemState \to^1 \mathcal{P}_{nco}(SemState \times X_{\frac{1}{2}})) + \{E\}$
$G(f) = (id \to^1 \mathcal{P}_{nco}(id \times f)) + id.$

Proposition 9. *The following G-coalgebra (T, β) is well defined:*

$$\beta(t) = \begin{cases} \{(\rho,\ \{(\rho_1,\ t_1) \mid (\rho,\ t) \to (\rho_1,\ t_1)\}) \mid \rho \in SemState\} & \textit{if } t \to_{sem} \\ E & \textit{if } t \not\to_{sem} . \end{cases}$$

Lemma 10. *(P, id) is the final G-coalgebra.*

Proof. The thesis follows from Theorem 5.5 of [13], since P is a fixed point of G and G is locally contracting.

□

Definition 9. i) Let $\mathcal{O}^{\#} : T \to P$ be the unique morphism from the G-coalgebra (T, β) to the final G-coalgebra (P, id), i.e.

$$\mathcal{O}^{\#}(t) = \begin{cases} \lambda\rho.\{(\rho_1,\ \mathcal{O}^{\#}(t_1)) \mid (\rho,\ t) \to_{sem} (\rho_1,\ t_1)\} & \text{if } t \to_{sem} \\ E & \text{if } t \not\to_{sem} \end{cases}$$

ii) Let $\approx^{\#}$ be the greatest G-bisimulation on the G-coalgebra (T, β).

As usual, a definition by finality induces a coinduction principle:

Definition 10. Let $\mathcal{T}_{\#} : \mathcal{P}(T \times T) \to \mathcal{P}(T \times T)$ be the operator defined as follows:
$\mathcal{T}_{\#}(R) = \{(t, t') \mid \forall\rho.(((\rho, t) \to (\rho_1, t_1) \ \Rightarrow\ \exists t'_1.((\rho, t') \to (\rho_1, t'_1) \ \wedge\ t_1 R t'_1)) \ \wedge$ $((\rho, t') \to (\rho_1, t'_1) \ \Rightarrow\ \exists t_1.((\rho, t) \to (\rho_1, t_1) \ \wedge\ t_1 R t'_1)))\}$.

Proposition 11. *The equivalence induced by $\mathcal{O}^{\#}$ coincides with $\approx^{\#}$, which is the greatest fixed point of $\mathcal{T}_{\#}$.*

Following [2], one can prove that $\mathcal{O}^{\#}$ coincides with $\mathcal{D}^{\#}$, by endowing the space T with a suitable distance induced by the distance on P, and by showing, using an inductive argument on T, that both $\mathcal{O}^{\#}$ and $\mathcal{D}^{\#}$ are fixed points of a suitable contractive higher order operator:

Theorem 12. $\qquad \mathcal{O}^{\#} = \mathcal{D}^{\#}$.

From Theorem 12 it follows immediately that the equivalence $\approx_{sem}$ induced by the metric denotational semantics coincides with the equivalence $\approx^{\#}$ restricted to *Statement* $\times$ *Statement*. Hence from now onwards we focus on the relation between $\approx$ and $\approx^{\#}$.

Proof of "$\approx_{sem} \subseteq \approx$".

Lemma 13. *i) Let $\sigma, \sigma_1 \in SynState$, $s, s_1 \in Statement$; if $(\sigma,\ s) \to (\sigma_1,\ s_1)$, then there exists $t_1 \in T$ s.t. $(\rho,\ s) \to_{sem} (\rho_1,\ t_1)$, where $\mathcal{O}^{\#}(\sigma) = \rho$, $\mathcal{O}^{\#}(\sigma_1) = \rho_1$ and $\mathcal{O}^{\#}(s_1) = \mathcal{O}^{\#}(t_1)$ (by abuse of notation, we use $\mathcal{O}^{\#}$ also for the natural extension to SynState).*
ii) Let $s \in SynState$, $t_1 \in T$, and let $\rho, \rho_1 \in SemState$ be s.t. there exists $\sigma \in Synstate$ s.t. $\mathcal{O}^{\#}(\sigma) = \rho$. If $(\rho,\ s) \to_{sem} (\rho_1,\ t_1)$, then there exist $\sigma_1 \in Synstate$, $s_1 \in Statement$ s.t. $(\sigma,\ s) \to (\sigma_1,\ s_1)$, $\mathcal{O}^{\#}(\sigma_1) = \rho_1$ and $\mathcal{O}^{\#}(s_1) = \mathcal{O}^{\#}(t_1)$.

Proof. i) Proceed by induction on the length of the derivation of the judgement $(\sigma,\ s) \to (\sigma_1,\ s_1)$ in the transition system.
ii) Proceed by induction on the length of the derivation of $(\rho,\ s) \to_{sem} (\rho_1,\ t_1)$ in the $\to_{sem}$-transition system.

□

Theorem 14. *The equivalence* $\approx_{sem}$ *induced by the metric denotational semantics is not coarser than the equivalence* $\approx$ *induced by the final hyperset semantics* $\mathcal{M}$, *i.e.* $\approx_{sem} \subseteq \approx$.

Proof. Using items i and ii of Lemma 13, one can easily show that $\{(s, s') \mid \mathcal{O}^{\#}(s) = \mathcal{O}^{\#}(s')\}$ is a F-bisimulation on the coalgebra $(Statement, \alpha)$.

□

Proof of "$\approx \subseteq \approx_{sem}$". This proof is rather complicated and it requires a number of results about $\mathcal{O}^{\#}$, some of which are interesting in their own right. First, we need another coinduction principle for $\approx^{\#}$.

Definition 11. Let $\mathcal{T}'_{\#} : \mathcal{P}(T \times T) \to \mathcal{P}(T \times T)$ be the operator defined as: $\mathcal{T}'_{\#}(R) = \{(t, t') \mid \forall\rho.(((\rho, t) \to (\rho_1, t_1) \Rightarrow \exists\rho'_1, t'_1.((\rho, t') \to (\rho'_1, t'_1) \wedge \rho_1 R \rho'_1 \wedge t_1 R t'_1)) \wedge ((\rho, t') \to (\rho'_1, t'_1) \Rightarrow \exists\rho_1, t_1.((\rho, t) \to (\rho_1, t_1) \wedge \rho_1 R \rho'_1 \wedge t_1 R t'_1)))\}$.

Lemma 15. $\approx^{\#}$ *is the greatest fixed point of* $\mathcal{T}'_{\#}$.

Proof. Since for all $p \in P$, $\mathcal{O}^{\#}(p) = p$, then $\mathcal{O}^{\#}$ is s.t.

$$\mathcal{O}^{\#}(t) = \begin{cases} \lambda\rho.\{(\mathcal{O}^{\#}(\rho_1),\ \mathcal{O}^{\#}(t_1)) \mid (\rho,\ t) \to_{sem} (\rho_1,\ t_1)\} & \text{if } t \to_{sem} \\ E & \text{if } t \not\to_{sem} \end{cases},$$

i.e. $\mathcal{O}^{\#}$ is the unique G'-coalgebra map from (T, β) to the final G'-coalgebra (P, id), where $G' : CMS \to CMS$ is defined as follows
$G'(X) = (SemState \to^1 \mathcal{P}_{nco}((IVar \to Val) \times (PVar \to X_{\frac{1}{2}}) \times X_{\frac{1}{2}})) + \{E\}$
$G'(f) = (id \to^1 \mathcal{P}_{nco}(id \times (id \to f) \times f)) + id.$

□

Definition 12 (Substitution). Let $s, r \in Statement$. Let $s[r/x] \in Statement$ be defined inductively as follows:
$s = E$ or $s = (v := e)$ or $s = (y := r') \implies s[r/x] = s;$
$s = x \implies s[r/x] = r;$
$s = s_1; s_2 \implies s[r/x] = s_1[r/x]; s_2[r/x];$
$s = s_1 + s_2 \implies s[r/x] = s_1[r/x] + s_2[r/x];$
$s = s_1 \mid\mid s_2 \implies s[r/x] = s_1[r/x] \mid\mid s_2[r/x].$

Lemma 16. *Let* $s, s', r, r' \in Statement$. *If* $s \approx s'$ *and* $r \approx r'$, *then* $s[r/x] \approx s'[r'/x]$.

Proof. Analysing the fine structure of derivations starting from s, one can show that the relation $R = \approx \cup \{(s[r/x], s'[r/x]) \mid s, s', r \in Statement,\ s \approx s'\}$ is an F-bisimulation on $(Statement, \alpha)$. Then, using Theorem 4 and transitivity of $\approx$, we conclude the proof.

□

A very detailed analysis of the fine structure of derivations in $\mathcal{T}^{sem}_{pas_2}$ is necessary to prove the next theorem.

Theorem 17. $\approx \subseteq \approx^{\#}$.

Proof. (Sketch) The proof uses the coinduction principle of Lemma 15. Define a relation $R \subseteq T \times T$, including $\approx$, as follows:

$$R = \approx \cup \{(\mathcal{O}^{\#}(s), \mathcal{O}^{\#}(s')) \mid s \approx s'\} \cup Id_{P\times P} \cup R^{\#} ,$$

where $R^{\#}$ is a relation on $(T \setminus Statement) \times (T \setminus Statement)$ defined as follows:
$R^{\#} = \{(t_1, t_1') \mid \exists s_1, s_1' \in Statement$ s.t.
$\exists\tau(\rho) : (\rho, s) \rightarrow_{sem} (\rho_1, t_1)$. $\exists\tau'(\rho) : (\rho, s') \rightarrow_{sem} (\rho_1', t_1')$. $\exists\tau(\sigma) : (\sigma, s) \rightarrow (\sigma_1, s_1)$. $\exists\tau'(\sigma) : (\sigma, s') \rightarrow_{sem} (\sigma_1, s_1')$. ($\tau(\sigma)$ and $\tau'(\sigma)$ are $\approx$-corresponding $\wedge$ $\tau(\rho)$ is strongly similar to $\tau(\sigma)$ $\wedge$ $\tau'(\rho)$ is strongly similar to $\tau'(\sigma))\}$.
(The notions of $\approx$-corresponding and strongly similar derivations are given in Appendix B.) One can show that R is a $\mathcal{T}'_{\#}$-bisimulation, and hence that $\approx \subseteq \approx^{\#}$ (the proof of $R \subseteq \mathcal{T}'_{\#}(R)$ uses Lemma 16 and the results of Appendix B).

□

Theorem 18 (Full Abstraction). $\approx = \approx_{sem}$.

5 Another Characterization of the Hyperset Equivalence

The basic idea of this section is to embed function spaces of the shape $A \rightarrow \mathcal{P}(B)$ into spaces of relations $\mathcal{P}(A \times B)$. Applying this idea to the mixed functor naturally induced by the functor F of Definition 4, we obtain a new functor H. We prove that the hyperset equivalence is the greatest H-bisimulation. Hence we obtain another coinduction principle for $\approx$, logically independent from those of the previous sections.

Definition 13. Let $H : Class^* \rightarrow Class^*$ be the endofunctor defined by
$H(X) = \mathcal{P}_{ne}(((IVar \rightarrow Val) \times (PVar \rightarrow X))^2 \times X) + \{E\}$.
$H(f) = \mathcal{P}_{ne}((id \times (id \rightarrow f))^2 \times f) + id$.

Theorem 19. *The unique map from the H-coalgebra $(Statement, \alpha)$ to the final H-coalgebra, induces the equivalence $\approx_H$. $\approx_H$ is the greatest fixed point of the operator $\mathcal{T}' : \mathcal{P}(Statement \times Statement) \rightarrow \mathcal{P}(Statement \times Statement)$ defined as follows:*
$\mathcal{T}'(R) = \{(s, s') \mid \forall\sigma.(((\sigma, s) \rightarrow (\sigma_1, s_1) \Rightarrow \exists\sigma', \sigma_1', s_1'.(\sigma R\sigma' \wedge (\sigma', s') \rightarrow (\sigma_1', s_1') \wedge \sigma_1 R\sigma_1' \wedge s_1 Rs_1')) \wedge ((\sigma, s') \rightarrow (\sigma_1', s_1') \Rightarrow \exists\sigma', \sigma_1, s_1.(\sigma' R\sigma \wedge (\sigma', s) \rightarrow (\sigma_1, s_1) \wedge \sigma_1 R\sigma_1' \wedge s_1 Rs_1')))\}$.

We now show that $\approx_H$ coincides with $\approx$.

Lemma 20. *i)* $r \approx_H r' \implies s[r/x] \approx_H s[r'/x]$.
ii) If there is a derivation $\tau(\sigma) : (\sigma, s) \rightarrow (\sigma_1, s_1)$, then, for all σ' s.t. $\sigma' \approx_H \sigma$, there exists $\tau(\sigma') : (\sigma', s) \rightarrow (\sigma_1', s_1')$ s.t. $\sigma_1' \approx_H \sigma_1 \wedge s_1' \approx_H s_1$.

Proof. i) Show that the following relation is an H-bisimulation:

$$R = \{(s[r/x], s[r'/x]) \mid s, r, r' \in Statement,\ r \approx_H r'\}$$

ii) Use item i of this lemma.

□

Theorem 21. $\approx\ =\ \approx_H$.

Proof. The inclusion $\approx\subseteq\approx_H$ follows from the fact that $\approx$ is an H-bisimulation. In order to show the other inclusion, i.e. $\approx_H\subseteq\approx$, prove that $\approx_H$ is an F-bisimulation, using item ii of Lemma 20.

□

6 Coinduction and Mixed Principles for $\mathcal{L}_{pas_2}$

In this section we give both syntactical and semantical coinduction and induction-coinduction principles for establishing the equivalences $\approx$, $\approx_{sem}$ and $\approx_{\#}$. In order to show the mixed induction-coinduction principles, we use techniques based on the notion of algebraically compact category introduced in [8].

In previous sections we have proved the following *pure* coinduction principles:

Theorem 22 (Syntactical Coinduction). *Let $R \subseteq Statement \times Statement$. The following principles hold:* $\dfrac{R \subseteq \mathcal{T}(R)}{R \subseteq\approx}$ $\dfrac{R \subseteq \mathcal{T}'(R)}{R \subseteq\approx}$.

Theorem 23 (Syntactical-semantical Coinduction). *Let $R \subseteq T \times T$, then the following principles hold:* $\dfrac{R \subseteq \mathcal{T}_{\#}(R)}{R \subseteq\approx^{\#}}$ $\dfrac{R \subseteq \mathcal{T}'_{\#}(R)}{R \subseteq\approx^{\#}}$.

An induction-coinduction principle for the identity relation on the metric domain of processes can be obtained by extending Pitts' categorical technique for $CPO_{\perp}$'s ([11]) to the algebraically compact category CMS:

Definition 14. Let $\mathcal{T}_{sem} : \mathcal{P}(P \times P) \times \mathcal{P}(P \times P) \to \mathcal{P}(P \times P)$ be the operator defined as follows:
$\mathcal{T}_{sem}(R^-, R^+) = \{(p, p') \mid \forall(\rho, \rho') \in R^-.((\rho, p) \to (\rho_1, p_1) \ \Rightarrow\ \exists \rho'_1, p'_1.((\rho', p') \to (\rho'_1, p'_1) \ \wedge\ \rho_1 R^+ \rho'_1 \ \wedge\ p_1 R^+ p'_1)) \ \wedge\ \forall(\rho', \rho) \in R^-.((\rho', p') \to (\rho'_1, p'_1) \ \Rightarrow\ \exists \rho_1, p_1.((\rho, p) \to (\rho_1, p_1) \ \wedge\ \rho'_1 R^+ \rho_1 \ \wedge\ p'_1 R^+ p_1))\}$.

Theorem 24 (Semantical Induction-coinduction). *Let $R^-, R^+ \subseteq P \times P$. The following principle holds*

$$\frac{R^- \subseteq \mathcal{T}_{sem}(R^+, R^-) \qquad R^+ \supseteq \mathcal{T}_{sem}(R^-, R^+)}{R^- \subseteq Id_P \subseteq R^+} \quad R^+ \text{ closed in } P \times P\ ,$$

where Id_P is the identity relation on P.

Using the notion of bisimulation "up to" and Theorem 24, we are able to prove a mixed induction-coinduction principle for the equivalence $\approx^{\#}$:

Definition 15. Let $\mathcal{T}_{\#,2} : \mathcal{P}(T \times T) \times \mathcal{P}(T \times T) \to \mathcal{P}(T \times T)$ be the operator defined as follows:
$\mathcal{T}_{\#,2}(R^-, R^+) = \{(t, t') \mid \forall(\rho, \rho') \in R^-.((\rho, t) \to (\rho_1, t_1) \ \Rightarrow\ \exists \rho'_1,\ t'_1.((\rho', t') \to (\rho'_1, t'_1) \ \wedge\ \rho_1 R^+ \rho'_1 \ \wedge\ t_1 R^+ t'_1)) \ \wedge\ \forall(\rho', \rho) \in R^-.((\rho', t') \to (\rho'_1, t'_1) \ \Rightarrow\ \exists \rho_1,\ t_1.((\rho, t) \to (\rho_1, t_1) \ \wedge\ \rho'_1 R^+ \rho_1 \ \wedge\ t'_1 R^+ t_1))\}$.

Definition 16. A relation $R \subseteq T \times T$ is *saturated* if and only if

$$R \ = \approx^{\#} \circ R \circ \approx^{\#} .$$

Theorem 25 (Syntactical-semantical Induction-coinduction). *Let $R^-, R^+ \subseteq T \times T$. The following principle holds*

$$\frac{R^- \subseteq \mathcal{T}_{\#,2}(R^+, R^-) \quad R^+ \supseteq \mathcal{T}_{\#,2}(R^-, R^+)}{R^- \subseteq \approx^{\#} \subseteq R^+} \ R^+ \ saturated \ ,$$

provided $(R^+/\approx^{\#})$ is closed w.r.t. the distance inherited from P.

Proof. Since $(T/\approx^{\#}) \simeq P$, from the semantical induction-coinduction principle (Theorem 24) it follows that, if $R^-, R^+ \subseteq T \times T$, then the following bisimulation "up to" principle holds: if $(R^+/\approx^{\#})$ is *closed* and R^+ is *saturated*, then

$$\frac{R^- \subseteq \mathcal{T}_{\#,2}(R^+, \approx^{\#} \circ R^- \circ \approx^{\#}) \quad R^+ \supseteq \mathcal{T}_{\#,2}(\approx^{\#} \circ R^- \circ \approx^{\#}, R^+)}{R^- \subseteq \approx^{\#} \subseteq R^+} \ .$$

Hence, since $R^- \subseteq \mathcal{T}_{\#,2}(R^+, R^-)$ implies $R^- \subseteq \mathcal{T}_{\#,2}(R^+, \approx^{\#} \circ R^- \circ \approx^{\#})$, and $R^+ \supseteq \mathcal{T}_{\#,2}(R^-, R^+)$ implies $R^+ \supseteq \mathcal{T}_{\#,2}(\approx^{\#} \circ R^- \circ \approx^{\#}, R^+)$, we get the thesis. □

We conclude this section by conjecturing the validity of a purely syntactical induction-coinduction principle:

Definition 17. Let $\mathcal{T}_2 : \mathcal{P}(Statement \times Statement) \times \mathcal{P}(Statement \times Statement) \to \mathcal{P}(Statement \times Statement)$ be the operator defined as follows:
$\mathcal{T}_2(R^-, R^+) = \{(s, s') \mid \forall \sigma, \sigma'. \sigma R^- \sigma'.((\sigma, s) \to (\sigma_1, s_1) \ \Rightarrow \ \exists \sigma'_1, s'_1.((\sigma', s') \to (\sigma'_1, s'_1) \ \wedge \ \sigma_1 R^+ \sigma'_1 \ \wedge \ s_1 R^+ s'_1)) \ \wedge \ \forall \sigma, \sigma'. \sigma' R^- \sigma.((\sigma', s') \to (\sigma'_1, s'_1) \ \Rightarrow \ \exists \sigma_1, s_1.((\sigma, s) \to (\sigma_1, s_1) \ \wedge \ \sigma'_1 R^+ \sigma_1 \ \wedge \ s'_1 R^+ s_1))\}$.

Conjecture. (Syntactical Induction-coinduction)
Let $R^-, R^+ \subseteq Statement \times Statement$. The following principle holds

$$\frac{R^- \subseteq \mathcal{T}_2(R^+, R^-) \quad R^+ \supseteq \mathcal{T}_2(R^-, R^+)}{R^- \subseteq \approx \subseteq R^+} \ R^+ closed \ ,$$

where R^+ *closed* means that R^+ must be closed w.r.t. the distance inherited from P.

7 Final Remarks and Directions for Future Work

The paper focuses on the language $\mathcal{L}_{pas_2}$ and the equivalence $\approx$. It is debatable as to whether such an equivalence is the "appropriate" one. The equivalence $\approx$ is probably too weak. For instance, by endowing $\mathcal{P}(Statement)$ with a suitable structure of F-coalgebra, one obtains an equivalence on *Statement* coarser than $\approx$, which is still a *congruence*. Furthermore, slight changes in the *var* and $:=_2$ rules would coarsen the $\approx$ equivalence itself. Experiments are called for in order to settle the issue. In any case, the same analysis carried out for the equivalence $\approx$ could be also done for other notions of equivalence on $\mathcal{L}_{pas_2}$.

In this paper we tried to hint at general methodologies. Generality should be achieved by introducing appropriate formats, in the style of tyft/tyxt, for dealing uniformly with a large class of higher order concurrent languages.

In Section 5 we derive a coinduction principle for $\approx$, by viewing the function space appearing in the mixed functor naturally induced by the functor F as a space of relations. By applying the same transformation to the *mixed* functor used for giving the metric denotational semantics, we get a purely *covariant* functor. It would be interesting to compare the equivalence $\approx_{sem}$ to the equivalence obtained using this covariant functor. General theorems are called for here.

It would be interesting to carry out more case-studies on final semantics. One could consider higher order languages with local stores instead of global store, e.g. in the style of CHOCS ([16]) or of Higher Order π-calculus ([15]).

Techniques based on algebraically compact categories should yield mixed induction-coinduction principles also for other languages; but especially the syntactical case deserves further study.

References

1. P.Aczel, *Non-wellfounded sets,* Number 14, Lecture Notes CSLI, 1988.
2. J.de Bakker, F.van Breugel, *Topological models for higher order control flow,* MF-PS'93 Conference Proc., Brookes et al. eds., Springer LNCS Vol.802, 1993.
3. J.de Bakker, F.van Breugel, *Metric Semantics for Second Order Communication,* Technical Report, McGill University, Montreal, July 1995.
4. M.Barr, *Terminal coalgebras in well-founded set theory,* TCS, 114:299–315, 1993.
5. Barwise, *Hypersets,* The Mathematical Intelligencer, Vol.13(4):31–41, 1991.
6. M.Forti, F.Honsell, *Set Theory with Free Construction Principles,* Annali Scuola Normale Sup. Pisa, Cl. Sci., (IV), 10:493–522, 1983.
7. M.Forti, F.Honsell, M.Lenisa, *Processes and Hyperuniverses,* MFCS'94 Conference Proceedings, I.Privara et al. eds., Springer LNCS Vol.841:352–363, Košice, 1994.
8. P.Freyd, *Algebraically complete categories,* A.Carboni et al. eds, Category Theory '90 Springer LNM, Vol.1144:95–104, Como, 1992.
9. J.F.Groote, F.Vaandrager, *Structured Operational Semantics and Bisimulation as a Congruence,* Information and Computation Vol.100:202–260, 1992.
10. F.Honsell, M.Lenisa, *Final Semantics for untyped λ-calculus,* M.Dezani et al. eds, TLCA'95 Springer LNCS Vol.902:249–265, Edinburgh, 1995.
11. A.M.Pitts, *Relational Properties of Recursively Defined Domains,* 8th LICS Conference Proceedings, IEEE Computer Society Press :86–97, 1993.
12. J.J.M.M.Rutten, *Processes as terms: non-wellfounded models for bisimulation,* Math.Struct.Comp.Sci., Vol.2(3):257–275, 1992.
13. J.J.M.M.Rutten, D.Turi, *On the Foundations of Final Semantics: Non-Standard Sets, Metric Spaces, Partial Orders,* REX Conference Proceedings, J.deBakker et al. eds., Springer LNCS Vol.666:477–530, 1993.
14. J.J.M.M.Rutten, D.Turi, *Initial algebra and final coalgebra semantics for concurrency,* Report CS-R9409, CWI, Amsterdam, January 1994.
15. D.Sangiorgi, *Expressing Mobility in Process Algebras: First-Oder and Higher-Order Paradigms,* Ph.D thesis CST-99-93, University of Edinburgh, 1992.

16. B.Thomsen, *A Calculus of Higher Order Communicating Systems,* Proc. 16th Annual ACM Symp. on Principles of Programming Languages :143–154, 1989.

Appendix A

We recall some categorical definitions (for more details see [13]).

Definition 18 (F-coalgebra). Let C be a category and $F : C \to C$ an endofunctor.
i) An *F-coalgebra* is a pair (A, α), where A is an object of C and $\alpha : C \to F(C)$ is a morphism of C.
ii) Let C_F be the category whose objects are F-coalgebras and whose morphisms are F-coalgebra morphisms. An F-coalgebra morphism $f : (A, \alpha) \to (B, \beta)$ is an arrow $f : A \to B$ in the category C such that the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & F(A) \arrow[d, "F(f)"] \\ B \arrow[r, "\beta"] & F(B) \end{tikzcd}$$

Definition 19 (F-bisimulation). Let C be a category with products. Let $F : C \to C$ be an endofunctor and (A, α) an F-coalgebra. An *F-bisimulation on the F-coalgebra* (A, α) is a relation $R \subseteq A \times A$ such that there exists an arrow $\gamma : R \to F(R)$ which makes the following diagram commute:

$$\begin{tikzcd} A \arrow[d, "\alpha"'] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\gamma"] & A \arrow[d, "\alpha"] \\ F(A) & F(R) \arrow[l, "F(\pi_1)"'] \arrow[r, "F(\pi_2)"] & F(A) \end{tikzcd}$$

In the following definition we recall the Antifoundation Axiom FCU (for more details see [6]):

Definition 20. Unique Free Construction w.r.t. a set U of atoms FCU:
Let X be a set such that $X \cap U = \emptyset$. For every function $f : X \to \mathcal{P}(X \cup U)$ there is a unique function $g : X \to V$ (V is the universe of sets) verifying

$$g(x) = (f(x) \cap U) \cup \{g(y) \mid y \in f(x) \cap X\}, \quad \forall x \in X.$$

Appendix B

In the proof of $R \subseteq \mathcal{T}'_{\#}(R)$, necessary to establish Theorem 17, one proceeds by an involved case analysis, making use of the following results.

Statements which are related by $\approx$ generate derivations in $\mathcal{T}_{pas_2}$, which are related in a precise sense:

Definition 21. Two derivations τ and τ' are $\approx$-*corresponding* if
- τ and τ' use the same axiom,
- the objects occurring in corresponding positions in the conclusions of τ and τ' are $\approx$-related, and
- if the axiom used in τ and τ' is *var*, then it is instantiated to the same variable both in τ and in τ'.

Lemma 26. *Let $s, s' \in Statement$ be such that $s \approx s'$, and let $\sigma \in SynState$. If there is a derivation $\tau(\sigma) : (\sigma, s) \to (\sigma_1, s_1)$, then there exists a $\approx$-corresponding derivation $\tau'(\sigma) : (\sigma, s') \to (\sigma'_1, s'_1)$.*

Proof. By case analysis on the axiom used in $\tau(\sigma)$ (if the axiom is *var*, use Lemma 16).

□

Syntactical derivations starting from a given statement are related to semantical derivations starting from the same statement in a precise sense:

Definition 22. Let $\tau(\sigma) : (\sigma, s) \to (\sigma_1, s_1)$ be a derivation in $\mathcal{T}_{pas_2}$. The derivation $\tau(\rho) : (\rho, s) \to (\rho_1, t_1)$ in $\mathcal{T}^{sem}_{pas_2}$ is strongly similar to $\tau(\sigma)$ if both are obtained instantiating the same sequence of rule schemata.

Lemma 27. *Let $s \in Statement$. Then, for all $\sigma \in SynState$ and $\rho \in SemState$, there exists a syntactical derivation $\tau(\sigma)$ of $(\sigma, s) \to (\sigma_1, s_1)$ if and only if there exists a semantical derivation $\tau(\rho)$ of $(\rho, s) \to (\rho_1, t_1)$ strongly similar to $\tau(\sigma)$.*

Finally we need a last technical lemma (omitted), which deals with semantical derivations starting from a given statement. It allows to postpone the introduction of a non purely syntactical term, by replacing a pair of semantical steps with a suitable sequence consisting of syntactical steps but the last one.

This article was processed using the LaTeX macro package with LLNCS style