# Multigames and strategies, coalgebraically 

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#### Abstract

Coalgebraic games have been recently introduced as a generalization of Conway games and other notions of games arising in different contexts. Using coalgebraic methods, games can be viewed as elements of a final coalgebra for a suitable functor, and operations on games can be analyzed in terms of (generalized) coiteration schemata. Coalgebraic games are sequential in nature, i.e., at each step either the Left $(\mathrm{L})$ or the Right $(\mathrm{R})$ player moves (global polarization); moreover, only a single move can be performed at each step. Recently, in the context of Game Semantics, concurrent games have been introduced, where global polarization is abandoned, and multiple moves are allowed. In this paper, we introduce coalgebraic multigames, which are situated half-way between traditional sequential games and concurrent games: global polarization is still present, however multiple moves are possible at each step, i.e., a team of $\mathrm{L} / \mathrm{R}$ players moves in parallel. Coalgebraic operations, such as sum and negation, can be naturally defined on multigames. Interestingly, sum on coalgebraic multigames turns out to be related to Conway's selective sum on games, rather than the usual (sequential) disjoint sum. Selective sum has a parallel nature, in that at each step the current player performs a move in at least one component of the sum game, while on disjoint sum the current player performs a move in exactly one component at each step. A presentation of strategies on coalgebraic games is given via a final coalgebra of a pair of mutually recursive functors, and a suitable notion of simulation. A monoidal closed category of coalgebraic multigames in the vein of a Joyal category of Conway games is then built. The relationship between coalgebraic multigames and games is then formalized via an equivalence of the multigame category and a monoidal closed category of coalgebraic games where tensor is selective sum.


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## 1. Introduction

In [15,16], coalgebraic games have been introduced as a generalization of Conway games [11] to possibly non-terminating games. Coalgebras offer an elementary but sufficiently abstract framework, where games are represented as elements of a final coalgebra for a suitable functor, by abstracting away superficial features of positions, and operations are smoothly defined as final morphisms via (generalized) coiteration schemata. Coalgebraic games have been further studied and generalized in $[17,18]$. In particular, in [18], coalgebraic games have been shown to subsume also games arising in the context of Game Semantics. In [3] a similar notion of coalgebraic game has been used to model games arising in Economics (see also [25,26]).

Coalgebraic games are 2-player games of perfect information, the two players being Left ( L ) and Right ( R ). A game is identified with its initial position. At any position, there are moves for $L$ and $R$ taking to new positions of the game. Contrary

[^0]to other approaches in the literature, where games are defined as graphs, we view possibly non-wellfounded games as points of a final coalgebra of graphs, i.e., minimal graphs w.r.t. bisimilarity. This coalgebraic representation is more in the spirit of Conway's original presentation, and it is motivated by the fact that the existence of winning/non-losing strategies is invariant w.r.t. graph bisimilarity.

Coalgebraic games are sequential in nature, i.e., at each step either L or R moves (global polarization), moreover only a single move can be performed at each step. Recently, in the context of Game Semantics, concurrent games have been introduced, see e.g. [2,13,33,10], where global polarization is abandoned, and multiple moves are allowed.

In this paper, we introduce coalgebraic multigames, which are situated half-way between traditional sequential games and concurrent games: global polarization is still present, however multiple moves are possible at each step, i.e., a team of $L / R$ players moves in parallel. Coalgebraic operations, such as sum and negation, can be naturally defined on multigames via (generalized) coiteration schemata.

The main difference between coalgebraic multigames and games lies in the fact that in the sum of games, at each step, the current player can move in exactly one component, while in the multigame sum, by exploiting the parallel nature of multigames, the current player can perform a multimove consisting of atomic moves on both components. Sum on games amounts to Conway's disjoint sum, which corresponds to interleaving semantics and standard tensor product in Game Semantics (see e.g. [1,20]), while multigame sum is related to Conway's selective sum, a form of parallel sum on games, where the current player can possibly move in both components.

We define the strategies on coalgebraic multigames as a final coalgebra of a pair of mutually recursive functors. The notion of being a strategy on a specific multigame is then captured via a suitable notion of simulation between the strategy and the multigame. To author's knowledge, this coalgebraic characterization of strategies is new.

We formalize the relationship between coalgebraic games and multigames in categorical terms. In particular, inspired by Joyal's categorical construction of Conway games [22], we build a symmetric monoidal closed category of coalgebraic multigames, where tensor is multigame sum. Namely, in [22], Joyal showed how to endow (well-founded) Conway games and winning strategies with a structure of compact closed category. This construction is based on the disjunctive sum of games, which induces a tensor product, and, in combination with negation, yields linear implication. Recently, the above categorical construction has been generalized to non-wellfounded games, see e.g. [17,18], while, in the context of Linear Logic and Semantics, various categories of possibly non-wellfounded Conway games have been introduced and studied in [27,28,31,19].

In the present paper, we build a category of coalgebraic multigames and strategies in the line of the above constructions. Moreover, somehow surprisingly, we show that this is equivalent to a category of coalgebraic games with a parallel tensor product inspired by Conway's selective sum. In particular, we carry out these categorical constructions in the context of polarized (multi)games, i.e., (multi)games where each position is marked as L or R, that is only L or R can move from that position, R starts, and $\mathrm{L} / \mathrm{R}$ positions strictly alternate. Polarized games typically arise in Game Semantics, see e.g. [1,20,19].

Technically, the main difficulty in defining the above categories of (multi)games with parallel tensor product lies in the definition of strategy composition, which is not a straightforward adaptation of usual composition, but it requires a non-standard parallel application of strategies.

The interest of coalgebraic (multi)games is manifold. Multigames help in clarifying/factorizing the steps taking from sequential games (with global polarization and single moves) to concurrent games (no global polarization, multiple moves), offering a model of parallelism with a low level of complexity but still of a set-theoretic nature, compared to more complex concurrent games. Notably, the coalgebraic approach, both in the game and in the multigame versions, appears significantly simpler than the traditional Game Semantics approach, where definitions of games and strategies require complex additional structures, such as equivalences on plays and strategies in the style of [1], or pointers in the arena style [20]. Such extra structure is not needed in the coalgebraic framework: both basic definitions of games and strategies, and constructions of categories of coalgebraic (multi)games appear much more natural and transparent in the coalgebraic context.

The present paper is an extended version of [24]. The two main novelties w.r.t. the previous version are: a coalgebraic presentation of strategies via a final coalgebra of a pair of mutually recursive functors and a suitable notion of simulation, and a presentation of coalgebraic game operations in terms of SOS specifications, in the style of [23]. The coalgebraic presentation of strategies allows in particular for smoother definitions of functors on game categories.

Related work. Coalgebraic methods for modeling games have been used also in [5], where the notion of membership game has been introduced. This corresponds to a subclass of our coalgebraic games, where at any position $L$ and $R$ have the same moves, and all infinite plays are deemed winning for player II (the player who does not start). However, no operations on games are considered in that setting. In the literature, various notions of bisimilarity equivalences have been considered on games, see e.g. [32,6]. But, contrary to our approach, such games are defined as graphs of positions, and equivalences on graphs, such as trace equivalences or various bisimilarities are considered. By defining games as the elements of a final coalgebra, we directly work up to bisimilarity of game graphs.

Summary. In Section 2, we introduce the notions of coalgebraic (multi)game, strategy, and play. In Section 3, we define operations on the final coalgebra of (multi)games via suitable (generalized) coiteration schemata. In Section 4, we build a monoidal closed category of polarized coalgebraic multigames and strategies, where tensor is sum. In Section 5, the relationship between coalgebraic multigames and games is expressed in categorical terms via an equivalence between the category of multigames and a monoidal closed category of coalgebraic games where tensor is selective sum. Conclusions and
directions for future work appear in Section 6. In Appendix A, we recall the guarded coiteration schema, which generalizes standard coiteration.

We assume the reader familiar with basic notions of coalgebras, see e.g. [21], and monoidal closed categories, see [30] for a complete overview of monoidal categories, in connection to the semantics of Linear Logic.

## 2. Coalgebraic (multi)games and strategies

In this section, we introduce the definitions of coalgebraic (multi)game, strategy, and play. First, we give the notion of game, then multigames are viewed as an instance of games, where moves have the structure of sets of atomic moves.

We consider a general notion of 2-player game of perfect information, where the two players are called Left (L) and Right (R). On a game, at each step, one of the two players can perform a move. A game $x$ is identified with its initial position; at any position, there are moves for $L$ and $R$, taking to new positions of the game. By abstracting superficial features of positions, games can be viewed as elements of the final coalgebra for the functor $F_{\mathcal{A}}(X)=\mathcal{P}_{<\kappa}(\mathcal{A} \times X)$, where $\mathcal{A}$ is a (non-empty) set of moves, each move is marked with the name of the player who performs it (polarity of the move), and $\mathcal{P}_{<\kappa}$ is the set of all subsets of cardinality $<\kappa$, where $\kappa$ can be $\omega$, if only finitely branching games are considered, or it can be an inaccessible cardinal, if we are interested in more general games. The coalgebra structure captures, for any position, the moves of the players and the corresponding next positions.

We work in the category Set* of sets belonging to a universe satisfying the Antifoundation Axiom, see [12,4], where the objects are the sets with hereditary cardinal less than $\kappa$, and whose morphisms are the functions with hereditary cardinal less than $\kappa .{ }^{1}$ Of course, we could work in the category Set of well-founded sets, but we prefer to use Set* so as to be able to use identities, i.e., extensional equalities in formal set theory, rather than isomorphisms in some naive set theory. As a consequence, final coalgebras are endowed with identities, rather than isomorphisms. Formally, we define:

## Definition 2.1 (Coalgebraic games).

- Let $\mathcal{A}$ be a non-empty set of atoms with functions:
(i) $\mu: \mathcal{A} \rightarrow \mathcal{N}$ yielding the name of the move (for a set $\mathcal{N}$ of names),
(ii) $\lambda: \mathcal{A} \rightarrow\{L, R\}$ yielding the player who has moved.

We denote by $\mathcal{A}_{L}\left(\mathcal{A}_{R}\right)$ the set of moves $a$ in $\mathcal{A}$ such that $\lambda a=L(\lambda a=R)$. We assume $\mathcal{A}_{L}, \mathcal{A}_{R} \neq \emptyset$.

- Let $F_{\mathcal{A}}:$ Set ${ }^{*} \rightarrow$ Set ${ }^{*}$ be the functor defined by

$$
F_{\mathcal{A}}(X)=\mathcal{P}_{<\kappa}(\mathcal{A} \times X)
$$

with the usual definition on morphisms, and let $\left(\mathbf{G}_{\mathcal{A}}\right.$, id $)$ be the final $F_{\mathcal{A}}$-coalgebra. ${ }^{2}$

- A coalgebraic game is an element $x$ of the carrier $\mathbf{G}_{\mathcal{A}}$ of the final coalgebra.

The elements of the final coalgebra $\mathbf{G}_{\mathcal{A}}$ are the minimal graphs up to bisimilarity.
We call player I the player who starts the game (who can be L or R in general), and player II the other. Once a player has moved on a game $x$, this leads to a new game/position $x^{\prime}$.

We consider strategies for player LI, i.e. L acting as player first, or LII, i.e. L acting as player second, or RI or RII. Intuitively, a strategy for player L (LI or LII) provides, at each step, for any possible move of $R$, an answer for $L$, or no answer, in the case there are no L-choices or when L gives up the game. More precisely, a strategy $\sigma$ for LI can be represented as a pair ( $a^{L}, \tau$ ), where $a^{L}$ is a move for $L$ and $\tau$ is a strategy for LII, or as $\perp$, if there are no $L$-moves or $L$ gives up the game. A strategy $\tau$ for LII can be viewed as a function with graph $\left\{\left(a_{i}^{R}, \sigma_{i}\right) \mid i \in I\right\}$, where $\left\{a_{i}^{R} \mid i \in I\right\} \subseteq \mathcal{A}_{R}$ represents the set of all possible R-moves at that stage, and $\left\{\sigma_{i}\right\}_{i \in I}$ are strategies for LI.

Coalgebraically, the pair $\left(\mathcal{S}^{L I}, \mathcal{S}^{L I I}\right)$, consisting of the set of strategies for LI, and the set of strategies for LII, respectively, can be defined as a final coalgebra for the following functor on Set* $\times$ Set $^{*}$ :

## Definition 2.2 (Coalgebraic strategies).

- Let $H_{\mathcal{A}}^{L}:$ Set $^{*} \times$ Set $^{*} \rightarrow$ Set $^{*} \times$ Set $^{*}$ be the functor defined by

$$
\begin{aligned}
H_{\mathcal{A}}^{L}(X, Y) & =\left(\left(\mathcal{A}_{L} \times Y\right)+\mathbf{1}, \coprod_{U \subseteq \mathcal{A}_{R}}[U \rightarrow X]\right) \\
H_{\mathcal{A}}^{L}(f, g) & =\left(\left(i d_{\mathcal{A}_{L}} \times g\right)+i d_{\mathbf{1}}, \coprod_{U \subseteq \mathcal{A}_{R}}\left[i d_{U} \rightarrow f\right]\right),
\end{aligned}
$$

where $\mathbf{1}$ is a one-element set.

[^1]- Let $\left(\left(\mathcal{S}^{L I}, \mathcal{S}^{L I I}\right), i d\right)$ be the final $H_{\mathcal{A}}^{L}$-coalgebra.
- A strategy for LI (LII) is an element $\sigma$ of the set $\mathcal{S}^{L I}\left(\mathcal{S}^{L I I}\right)$. Similarly, one defines strategies for RI and RII, $\left(\mathcal{S}^{R I}, \mathcal{S}^{R I I}\right)$.

The final coalgebras $\left(\mathcal{S}^{L I}, \mathcal{S}^{L I I}\right)$ and $\left(\mathcal{S}^{R I}, \mathcal{S}^{R I I}\right)$ capture the strategies on all coalgebraic games. Strategies on a specific game $x$ are characterized via a suitable notion of simulation, which we call saturated. First of all, notice that any strategy can be injectively mapped into the coalgebra $\mathbf{G}_{\mathcal{A}}$ of games. Then the strategies for L can be characterized via the notion of $R$-saturated simulation on $\mathbf{G}_{\mathcal{A}}$. Namely, a strategy for L on the game $x$ is a strategy in $\mathcal{S}^{L I}$ or $\mathcal{S}^{L I I}$, which is $R$-saturated simulated by the game $x$, according to the following definition, whereby the notions of being a strategy for LI or LII on a game $x$ are defined in a mutual recursive way:

Definition 2.3. The strategy $\sigma$ in $\mathcal{S}^{L I}\left(\mathcal{S}^{L I I}\right)$ is a strategy for $L I(L I I)$ on the game $x$ if $\sigma \leq^{L I} x\left(\sigma \leq{ }^{L I I} x\right)$, where $\leq \leq^{L I}$ and $\leq^{L I I}$ are defined by

$$
\begin{aligned}
& \sigma \leq^{L I} x \Longleftrightarrow\left(\sigma \xrightarrow{a^{L}} \sigma^{\prime} \Longrightarrow x \xrightarrow{a^{L}} x^{\prime} \& \sigma^{\prime} \leq^{L I I} x^{\prime}\right) \text { and } \\
& \sigma \leq^{L I I} x \Longleftrightarrow\left(\operatorname{dom}(\sigma)=\pi_{1}[x] \&\left(x \xrightarrow{a^{R}} x^{\prime} \Rightarrow \sigma\left(a^{R}\right) \leq^{L I} x^{\prime}\right)\right),
\end{aligned}
$$

where

- $\operatorname{dom}(\sigma)$ denotes the domain of $\sigma$
- $\pi_{1}[x]=\left\{\pi_{1}\left(a, x^{\prime}\right) \mid\left(a, x^{\prime}\right) \in x\right\}$
- $\sigma \xrightarrow{a} \sigma^{\prime}, x \xrightarrow{a} x^{\prime}$ denote $\left(a, \sigma^{\prime}\right) \in \sigma,\left(a, x^{\prime}\right) \in x$, respectively.

A similar definition can be given for strategies of RI and RII.
We have called the above notion of simulation "R-saturated", since it behaves as a simulation w.r.t. L-moves, while w.r.t. R-moves the strategy is required to be defined on all R-moves of the game. That is, if $\sigma$ is a strategy for L on the game $x$, then player L must provide an answer to any R-move on the game $x$.

We are interested in studying the interactions of a strategy for a given player with the (counter)strategies of the opponent player. When a player plays on a game $x$ according to a strategy $\sigma$, against an opponent player who follows a (counter)strategy $\sigma^{\prime}$, a play arises, i.e. an alternating (possibly infinite) sequence of pairs in $\mathcal{A} \times \mathbf{G}_{\mathcal{A}}$. Formally, we define:

Definition 2.4 (Plays).

- A play is a (possibly infinite) stream of pairs on $\mathcal{A} \times \mathbf{G}_{\mathcal{A}}, s=\left(a_{1}, x_{1}\right) \ldots$, such that $\forall n \geq 1 . \lambda a_{n+1}=\overline{\lambda a_{n}}$, where $\bar{R}=L$ and $\bar{L}=R$.
- A play $s$ is a play on a game $x$ if and only if $s \leq x$, where $\leq$ denotes simulation, i.e.:

$$
s \leq x \Longleftrightarrow\left(s=\left(a, x^{\prime}\right) s^{\prime} \Longrightarrow x \xrightarrow{a} x^{\prime} \& s^{\prime} \leq x^{\prime}\right) .
$$

Plays on $\mathcal{A} \times \mathbf{G}_{\mathcal{A}}$ can be injectively mapped into the final coalgebra of strategies/games. This justifies the above and the following definition:

Definition 2.5 (Product of strategies). Let $x$ be a coalgebraic game, let $\sigma$ be a strategy on $x$, and $\sigma^{\prime}$ a counterstrategy on $x$ (i.e. a strategy for the opponent player). We define the product of $\sigma$ and $\sigma^{\prime}, \sigma * \sigma^{\prime}$, as the unique play simulated both by $\sigma$ and $\sigma^{\prime}$.

We call well-founded games those games which correspond to well-founded sets as elements of the final coalgebra $\mathbf{G}_{\mathcal{A}}$, and non-wellfounded games the non-wellfounded sets in $\mathbf{G}_{\mathcal{A}}$. Clearly, strategies on well-founded games generate only finite plays, while strategies on non-wellfounded games can generate infinite plays.

A special subclass of games on which we focus on in the sequel is that of polarized games. On such games, at any non-ending position, only moves either for R or for L are available, and along any path in the game graph moves by $\mathrm{R} / \mathrm{L}$ strictly alternate. Polarized games play a central rôle in the construction of our categories of multigames in Section 4. Such games arise in traditional Game Semantics of Linear Logic and Programming Languages, see e.g. [1,20]. Polarized games admit a direct coalgebraic definition:

Definition 2.6 (Polarized games).

- Let $F_{\mathcal{A}_{L}}, F_{\mathcal{A}_{R}}:$ Set $^{*} \rightarrow$ Set ${ }^{*}$ be the functors defined as $F_{\mathcal{A}}$ (see Definition 2.1 above), when $\mathcal{A}$ is taken to be $\mathcal{A}_{L}, \mathcal{A}_{R}$, respectively. Then we define the functor $F_{\mathcal{A}}^{P}: \operatorname{Set}^{*} \times$ Set $^{*} \rightarrow$ Set ${ }^{*} \times$ Set $^{*}$ by

$$
\begin{aligned}
& F_{\mathcal{A}}^{P}(X, Y)=\left(F_{\mathcal{A}_{L}}(Y), F_{\mathcal{A}_{R}}(X)\right) \\
& F_{\mathcal{A}}^{P}(f, g)=\left(F_{\mathcal{A}_{L}}(g), F_{\mathcal{A}_{R}}(f)\right)
\end{aligned}
$$

- Let $\left(\left(\mathbf{G}_{\mathcal{A}_{L}}, \mathbf{G}_{\mathcal{A}_{R}}\right), i d\right)$ be the final $F_{\mathcal{A}}^{P}$-coalgebra.
- A polarized game is an element $x$ of $\mathbf{G}_{\mathcal{A}_{L}}$ (L starts the game) or $\mathbf{G}_{\mathcal{A}_{R}}$ (R starts the game).


### 2.1. Coalgebraic multigames

Multigames can be viewed as an instance of coalgebraic games, when moves are multimoves, i.e. sets of atomic moves. More precisely, we define:

Definition 2.7 (Coalgebraic multigames).

- Let $\mathcal{A}$ be a non-empty set of atoms with functions:
(i) $\mu: \mathcal{A} \rightarrow \mathcal{N}$ yielding the name of the atomic move (for a set $\mathcal{N}$ of names),
(ii) $\lambda: \mathcal{A} \rightarrow\{L, R\}$ yielding the player who has moved.
- Let $\mathcal{M}_{\mathcal{A}}$ be the set of multimoves, i.e. the powerset of all finite sets of atomic moves with the same polarity:

$$
\mathcal{M}_{\mathcal{A}}=\left\{\alpha \in \mathcal{P}_{f}(\mathcal{A}) \mid \forall a, a^{\prime} \in \alpha . \lambda a=\lambda a^{\prime}\right\}
$$

We extend functions $\mu$ and $\lambda$ on $\mathcal{M}_{\mathcal{A}}$ in the standard way.

- Multigames are defined as the coalgebraic games obtained by considering multimoves $\mathcal{M}_{\mathcal{A}}$ as the set $\mathcal{A}$ of moves in Definition 2.1 (by abuse of notation). We denote the set of multigames by $\mathbf{M}_{\mathcal{A}}$.

We will use capital letters $X, Y, \ldots$ to denote multigames.
Games and multigames differentiate when operations are defined (see Section 3 below). Namely, the structure of multimoves is exploited to define a notion of parallel sum on multigames, which does not correspond to the standard disjoint sum on games and to usual tensor in game categories. However, as we will see, multigame sum corresponds to an alternative sum that can be defined on games, i.e. selective sum, which has a parallel flavor.

## 3. (Multi)game operations

In this section, we show how to define various operations on (multi)games, including sum, negation, linear implication, and infinite sum. The crucial operation on (multi)games is sum. On games we will define two kinds of sum: disjoint sum, which is the standard one, and selective sum, which is related to the parallel sum defined on multigames. In our coalgebraic framework, operations can be conveniently defined via final morphisms, using (some generalizations of) the standard coiteration schema. We will present final morphisms corresponding to (multi)game operations using SOS specifications, in the style of [23].

### 3.1. Operations on games

In order to define negation and sum, we need to assume the set of actions $\mathcal{A}$ to be closed under:

- complementation, i.e. $a \in \mathcal{A} \Rightarrow \bar{a} \in \mathcal{A}$, where $\mu \bar{a}=\mu a$ and $\lambda \bar{a}=\overline{\lambda a}$;
- tagging, i.e. $a \in \mathcal{A} \Rightarrow(1, a),(2, a) \in \mathcal{A}$, where $\mu(1, a)=\mu(2, a)=\mu a$ and $\lambda(1, a)=\lambda(2, a)=\lambda a$;
- pairing, i.e. $(a, b \in \mathcal{A} \& \lambda a=\lambda b) \Rightarrow(a, b) \in \mathcal{A}$, where $\mu(a, b)=(\mu a, \mu b)$ and $\lambda(a, b)=\lambda a=\lambda b$.

Sums. Following [11], coalgebraic games can be endowed with various notions of sum, the most studied being disjoint sum: at each step, the current player performs a move either in the first or in the second component, while the component which has not been chosen remains unchanged.

Definition 3.1 (Disjoint sum). The disjoint sum of two games $\oplus: \mathbf{G}_{\mathcal{A}} \times \mathbf{G}_{\mathcal{A}} \longrightarrow \mathbf{G}_{\mathcal{A}}$ is defined by coiteration via the following set of SOS rules:

$$
\frac{x \xrightarrow{a} x^{\prime}}{x \oplus y \xrightarrow{(1, a)} x^{\prime} \oplus y} \quad \frac{y \xrightarrow{b} y^{\prime}}{x \oplus y \xrightarrow{(2, b)} x \oplus y^{\prime}}
$$

The above SOS specification corresponds to a coiteration schema, namely the function $\oplus$ is obtained as the final morphism from the coalgebra $\left(\mathbf{G}_{\mathcal{A}}, f_{\oplus}\right)$ to the final coalgebra $\left(\mathbf{G}_{\mathcal{A}}\right.$, id), where the coalgebra morphism $f_{\oplus}: \mathbf{G}_{\mathcal{A}} \times \mathbf{G}_{\mathcal{A}} \longrightarrow F_{\mathcal{A}}\left(\mathbf{G}_{\mathcal{A}} \times \mathbf{G}_{\mathcal{A}}\right)$ is defined by:

$$
f_{\oplus}(x, y)=\left\{\left((1, a),\left(x^{\prime}, y\right)\right) \mid\left(a, x^{\prime}\right) \in x\right\} \cup\left\{\left((2, b),\left(x, y^{\prime}\right)\right) \mid\left(b, y^{\prime}\right) \in y\right\}
$$

Remark. Notice that, in composing games via the sum, we keep track of the moves coming from the two different components by using tags. This definition is different from original Conway's sum on games [11], which is a purely set-theoretic extensional operation, possibly allowing for identifications between the two components. Our sum definition, where we keep track of the component in which each move has been performed, is necessary e.g. for extending sum to a bifunctor in categories of games and strategies, or even to define strategy composition in these categories. Nonetheless, notice that, from a determinacy point of view, this sum and Conway's original one behave in the same way, i.e. they are equivalent w.r.t. the existence of (winning) strategies.

An alternative notion of sum on games is Conway's selective sum, where at each step the current player can perform a move either in the first or in the second component or in both components. This notion of sum admits a definition by coiteration similar to the one for disjoint sum.

Definition 3.2 (Selective sum). The selective sum of two games $\vee: \mathbf{G}_{\mathcal{A}} \times \mathbf{G}_{\mathcal{A}} \longrightarrow \mathbf{G}_{\mathcal{A}}$ is defined by the following rules:

$$
\frac{x \xrightarrow{a} x^{\prime}}{x \vee y \xrightarrow{(1, a)} x^{\prime} \vee y} \quad \frac{y \xrightarrow{b} y^{\prime}}{x \vee y \xrightarrow{(2, b)} x \vee y^{\prime}} \quad \frac{x \xrightarrow{a} x^{\prime} y \xrightarrow{b} y^{\prime}}{x \vee y \xrightarrow{(a, b)} x^{\prime} \vee y^{\prime}}
$$

On the game $x \vee y$, the first player can choose to move either sequentially in one of the two components or in parallel in both subgames. Thinking of $x$ and $y$ as two game boards, the two players can play truly in parallel on the two boards. While on $x \oplus y$, the subgames $x$ and $y$ can only be played in an interleaved fashion.
Negation. The negation is a unary game operation, which allows us to build a new game, where the rôles of $L$ and $R$ are exchanged.

The definition of negation is as follows:

Definition 3.3 (Negation). The negation ${ }^{-}: \mathbf{G}_{\mathcal{A}} \longrightarrow \mathbf{G}_{\mathcal{A}}$ is defined by the following SOS rule:

$$
\frac{x^{\alpha} x^{\prime}}{\bar{x} \xrightarrow{\alpha} \overline{x^{\prime}}} .
$$

Also negation is an instance of the coiteration schema. It is the final morphism from the coalgebra ( $\mathbf{G}_{\mathcal{A}}, f_{-}$) to the final coalgebra $\left(\mathbf{G}_{\mathcal{A}}\right.$, id), where the coalgebra morphism $f_{-}: \mathbf{G}_{\mathcal{A}} \longrightarrow F_{\mathcal{A}}\left(\mathbf{G}_{\mathcal{A}}\right)$ is defined by:

$$
f_{-}(x)=\left\{\left(\bar{a}, x^{\prime}\right) \mid\left(a, x^{\prime}\right) \in x\right\}
$$

Linear arrows. Using the above notions of sum and negation, we can now define the following linear arrows, which correspond to the notion of linear implication in Linear Logic:

Definition 3.4 (Linear arrows). The linear arrows of the games $x, y$ induced by the disjoint and selective sum respectively, are defined by

$$
x \multimap_{\oplus} y=\bar{x} \oplus y \quad x \multimap_{\vee} y=\bar{x} \vee y
$$

### 3.2. Operations on polarized games

Notice that the above notions of sums and linear arrows are not closed under polarized games. On such games, we define corresponding notions of polarized sums and implications as follows:

Definition 3.5 (Polarized disjoint sum). The polarized disjoint sums of two games with the same polarity $\mathrm{P}, \oplus_{P}: \mathbf{G}_{\mathcal{A}_{P}} \times \mathbf{G}_{\mathcal{A}_{P}} \longrightarrow$ $\mathbf{G}_{\mathcal{A}_{P}}$, for $P \in\{L, R\}$, are mutually defined by the following set of SOS rules:

$$
\frac{x \xrightarrow{a} x^{\prime} \quad \lambda a=P}{x \oplus_{P} y \xrightarrow{(1, a)} x^{\prime} \oplus_{\bar{P}} y} \quad \frac{y \xrightarrow{b} y^{\prime} \quad \lambda b=P}{x \oplus_{P} y \xrightarrow{(2, b)} x \oplus_{\bar{P}} y^{\prime}}
$$

Similarly, one can define polarized selective sums $\vee_{L}, \vee_{R}$. Then, we can define polarized linear arrows as follows:
Definition 3.6 (Polarized linear arrows). For $x, y$ polarized games with the same polarity P, we define:

$$
x \multimap_{\oplus P} y=\bar{x} \oplus_{P} y \quad x \multimap_{\vee_{P}} y=\bar{x} \vee_{P} y
$$

By way of example, let us consider the polarized game $O=\left\{\left(q^{R},\left\{\left(a^{L}, \emptyset\right)\right\}\right)\right\}$. On the game $O \vee_{P} O \multimap_{\vee_{P}} O$, we can consider the following strategy implementing a sort of parallel or: R can only open in the righthand $O$-component with $q^{R}$, then L asks in parallel both arguments by performing the move ( $q^{L}, q^{L}$ ). Hence, independently whether R answers in one of the arguments or in both, L "copies" the result in the righthand O-component. The above parallel strategy exploits the parallel nature of selective sum.

When we work on polarized games of a given polarity P , let say $P=R$, these can be endowed with a structure of symmetric monoidal category, by considering as tensor either disjoint sum, see [18], or selective sum, and the corresponding linear arrows as spaces of morphisms. As we will see in Section 5, this construction, carried out in the case of games with selective sum, gives rise to a category equivalent to the category of multigames defined in Section 4.

Finally, we point out that one can define notions of infinite sums on games (and polarized games), corresponding to disjoint and selective sums, respectively, whereby infinitely many components of the game are available in the sum. These are related to the exponential modality defined on game categories, and they induce a comonad structure on the corresponding categories of games. We will provide details on the infinite sum in the case of multigames.

### 3.3. Operations on multigames

In order to define negation and sum, we assume the set of atomic actions $\mathcal{A}$ to be closed under complementation and tagging.

Notice that, contrary to the case of games, here the closure is required on the atomic actions and not on the (multi)moves. Namely, in defining multigame operations, we exploit the fine structure of multimoves.

Sum. In the context of multigames, the following notion of sum arises naturally. On the sum multigame, at each step, the next player selects either one (non-ended) or both component multigames, and makes a legal move in each of the selected components, while the component which has not been chosen (if any) remains unchanged. Formally:

Definition 3.7 (Sum). The sum of two multigames $\nabla: \mathbf{M}_{\mathcal{A}} \times \mathbf{M}_{\mathcal{A}} \longrightarrow \mathbf{M}_{\mathcal{A}}$ is defined by the following set of SOS rules:

$$
\frac{X \xrightarrow{\alpha} X^{\prime}}{X \nabla Y \xrightarrow{\alpha^{\prime}} X^{\prime} \nabla Y} \quad \frac{Y \xrightarrow{\beta} Y^{\prime}}{X \nabla Y \xrightarrow{\beta^{\prime}} X \nabla Y^{\prime}} \quad \frac{X \xrightarrow{\alpha} X^{\prime} Y \xrightarrow{\beta} Y^{\prime}}{X \nabla Y \xrightarrow{\alpha+\beta} X^{\prime} \nabla Y^{\prime}}
$$

where $\alpha^{\prime}=\{(1, a) \mid a \in \alpha\}, \beta^{\prime}=\{(2, b) \mid b \in \beta\}$, and $\alpha+\beta=\{(1, a) \mid a \in \alpha\} \cup\{(2, b) \mid b \in \beta\}$.
Negation. The negation is a unary multigame operation, which allows us to build a new game, where the rôles of $L$ and $R$ are exchanged. For $\alpha \in \mathcal{M}_{\mathcal{A}}$, we define

$$
\bar{\alpha}=\{\bar{a} \mid a \in \alpha\} .
$$

The definition of multigame negation is as follows:

Definition 3.8 (Negation). The negation ${ }^{-}: \mathbf{M}_{\mathcal{A}} \longrightarrow \mathbf{M}_{\mathcal{A}}$ is defined by the following SOS rule:

$$
\frac{X \xrightarrow{\alpha} X^{\prime}}{\bar{X} \xrightarrow{\bar{\alpha}} \overline{X^{\prime}}} .
$$

Linear arrow. Using the above notions of sum and negation, we can now define the following linear arrow:
Definition 3.9 (Linear arrow). The linear arrow of the multigames $X, Y, X \multimap Y$, is defined by

$$
X \multimap Y=\bar{X} \nabla Y
$$

Infinite sum. We can enrich multigames with a further interesting unary coalgebraic operation, $\nabla^{\infty}$, an infinite sum: on the multigame $\nabla^{\infty} X$, at each step, the current player can perform a move in finitely many of the infinite components of $X$. Our infinite sum is related to the exponential modality defined on a category of games in [31], and its polarized version will induce a comonad on the categories of multigames that we will consider in Section 4.

Definition 3.10 (Infinite sum). The system of SOS rules defining the infinite sum $\nabla^{\infty}: \mathbf{M}_{\mathcal{A}} \longrightarrow \mathbf{M}_{\mathcal{A}}$ includes the following family of SOS rules:

$$
\begin{equation*}
\left.\left\{\xrightarrow{\nabla^{\alpha_{1}} X_{1}^{\prime} \ldots X \xrightarrow{\alpha_{n}} X_{n}^{\prime} \quad \lambda \alpha_{1}=\ldots=\lambda \alpha_{n}}\right\}^{\alpha_{i}} \nabla^{n+1}\left(X_{1}^{\prime}, \ldots, X_{n}^{\prime}, \nabla^{\infty} X\right) \quad\right\}_{n \geq 1} \tag{1}
\end{equation*}
$$

together with rules for operations $\nabla^{n}$ of arity $n$, for any $n \geq 2$, generalizing $\nabla$ :

$$
\begin{equation*}
\left\{\xrightarrow{\left\{X_{i} \xrightarrow{\alpha_{i}} X_{i}^{\prime}\right\}_{i \in I} \quad \emptyset \neq I \subseteq\{1, \ldots, n\} \quad \forall i \in I . \lambda \alpha_{i}=P} \nabla^{n}\left(X_{1}, \ldots, X_{n}\right) \xrightarrow{\Sigma_{i \in \in} \alpha_{i}} \nabla^{n}\left(X_{1}^{\prime \prime}, \ldots, X_{n}^{\prime \prime}\right) \quad\right\}_{n \geq 2} \tag{2}
\end{equation*}
$$

where, for any $k \in\{1, \ldots, n\}, \quad X_{k}^{\prime \prime}= \begin{cases}X_{k}^{\prime} & \text { if } k \in I \\ X_{k} & \text { if } k \notin I .\end{cases}$

Some comments are in order about the above specification for the infinite sum. This is not captured by standard coiteration, but it is an instance of a more general schema, guarded coiteration, introduced and studied in [9]. The essence of this schema is that, in the specification, an extra operation appears, which "guards" the application of the operation being defined in the "continuation term". E.g., the specification of infinite sum is guarded by the family of operations $\left\{\nabla^{n}\right\}_{n \geq 2}$. The interested reader can see Appendix A or directly refer to [9] for the formal definition of the guarded coiteration schema, and sufficient conditions for it to define a unique function.

Operations on polarized multigames. Similarly as in the case of games, one can define sum and linear arrow operations on polarized multigames with the same polarity. That is, we denote by $\nabla_{P}$ multigame polarized sum, by $\nabla_{P}^{\infty}$ the corresponding notion of infinite sum, and, for $X, Y$ polarized multigames with the same polarity P , we define multigame polarized linear arrow by $\bar{X} \nabla_{p} Y$. In what follows, we will omit the subscript $P$ in denoting operations on polarized multigames, and we will denote $\nabla_{P}, \nabla_{P}^{\infty}, \multimap_{P}$ simply by $\nabla_{,} \nabla^{\infty}, \multimap_{\text {, respectively. }}$

## 4. Categories of multigames and strategies

We define a monoidal closed category $\mathcal{Y}_{\mathbf{M}_{\mathcal{A}}}$, whose objects are polarized multigames with R acting as first player, and whose morphisms are strategies on linear arrow multigames. We work with polarized multigames, since the whole class of multigames fails to give a category, because of lack of identities, as we will see.

The main difficulty in defining this category is the definition of composition, which is based on a non-standard parallel composition of strategies. The difficulty arises from the fact that a move in a strategy between $X$ and $Y$ can include atomic moves on both $X$ and $Y$.

Definition 4.1 (The category $\mathcal{Y}_{\mathbf{M}_{\mathcal{A}}}$ ). Let $\mathcal{Y}_{\mathbf{M}_{\mathcal{A}}}$ be the category defined by:
Objects: polarized multigames in $\mathbf{G}_{\mathcal{A}_{R}}$.
Morphisms: a morphism between multigames $X$ and $Y, \sigma: X \multimap Y$, is a strategy for LII on $X \multimap Y$.

In Theorem 4.1 below, we provide a proof that $\mathcal{Y}_{\mathbf{M}_{\mathcal{A}}}$ is a monoidal closed category. But before giving formal definitions and arguments, we provide an informal presentation of the categorical construction.

Usual copy-cat strategies play the rôle of identities. Intuitively, this work thanks to the fact that multigames are polarized, so as, on the multigame $X \multimap X=(\bar{X} \nabla X)$, R can only open on $X$ choosing a move $\alpha_{1}^{R}$ such that $\exists X_{1} .\left(\alpha_{1}^{R}, X_{1}\right) \in X$, but then L proceeds by copying the move on $\bar{X}$, and we reduce to the game $X_{1} \nabla \bar{X}_{1}$ (notice that polarities are now reversed on the components). Again R has only one component to move in ( $X_{1}$ ) and L copies the move in the other component, and so on, at each step R has exactly one component to move in, and L copies the move in the other component. The situation can be represented as follows.

$$
\begin{array}{cc}
\bar{X} & \nabla \\
& X_{1}^{R} \\
& \alpha_{1}^{R} \\
\alpha_{1}^{L} & \\
\hline X_{1} & \nabla \bar{X}_{1} \\
\alpha_{2}^{R} & \\
& \alpha_{2}^{L} \\
\hline \bar{X}_{2} & \nabla X_{2}
\end{array}
$$

A formal conductive definition of identities is given in the proof of Theorem 4.1.
Notice that, if the games were not polarized, then R could play on both components $X$ and $\bar{X}$, preventing L to apply the copy-cat strategy.

For general strategies on $X \multimap Y$, notice that Player R can only open in $Y$, but then Player L can move in $X$ or $Y$ or in both components; in this latter case, R has now the possibility of moving in both components. As a consequence, a non-standard parallel definition of composition is required. Namely, given strategies $\sigma: X \multimap Y$ and $\tau: Y \multimap Z$ for LII, the composition $\tau \circ \sigma: X \multimap Z$ is obtained via the swivel-chair strategy, using the terminology of [7] (or the copy-cat strategy, in Game Semantics terminology), and a non-standard parallel application of strategies as follows.

The opening move by R on $X \multimap Z$ must be on $Z$, since multigames are polarized. Then consider the L reply given by the strategy $\tau$ on $Y \multimap Z$, if it exists, otherwise the whole composition is undefined. If $L$ moves in $Z$, then we take this as the L move in the strategy $\tau \circ \sigma$. If the L move according to $\tau$ is in the $Y$ component of $Y \multimap Z$ or in both components $Y$ and $Z$, then we use the swivel chair to view the L move in the $Y$ component as a R move in the $Y$ component of $X \multimap Y$. Now, if L has a reply in $X \multimap Y$ according to $\sigma$, then L moves in $X$ or in $Y$ or in both $X$ and $Y$. In the first case, the move by L in $X$ together with the possible previous move by L in $Z$ form the reply by L to the opening R move. In the latter two cases, using the swivel chair, the move in $Y$ can be viewed as a $R$ move in the $Y$ component of $Y \multimap Z$, and we go on in this way: three cases can arise. Eventually, the multimove by L is all in $X$ or in $Z$, or $\sigma$ or $\tau$ is undefined, or the dialogue between the $Y$ components does not stop. In the first case, the last move on $X$ or $Z$, together with a possible previous move on $Z$ or $X$, form the answer to the opening move by R , in the latter two cases the composition is undefined.

Summarizing, in order to understand how the strategy $\tau \circ \sigma$ behaves, it is convenient to list all possible situations, according to the first pair of moves by $R$ and $L$. The following four cases can arise:
1.

$$
\begin{array}{ccccccc}
\sigma: \bar{X} \nabla & Y & \tau: & \bar{Y} & \nabla & Z \\
& & & & & \alpha_{Z}^{R} \\
& \gamma_{1 Y}^{R}< & - & -\gamma_{1 Y}^{L} & & \\
& \gamma_{2 Y}^{L} & - & -> & \gamma_{2 Y}^{R} & & \\
& \vdots & & \vdots & & \\
& & \gamma_{n Y}^{L}- & -> & \gamma_{n Y}^{R} & & \\
& & & & \beta_{Z}^{L} \\
\hline \sigma^{\prime}: \bar{X} \nabla & Y^{\prime} & \tau^{\prime}: \bar{Y}^{\prime} & \nabla & Z^{\prime}
\end{array}
$$

with $n \geq 0$.
The answer by L to the initial move $\alpha_{Z}^{R}$ in the composition $\tau \circ \sigma$ is $\beta_{Z}^{L}$. After this pair of moves, we reduce to the new pair of strategies $\sigma^{\prime}$ and $\tau^{\prime}$.
2.

$$
\begin{aligned}
& \sigma: \bar{X} \nabla Y \quad \tau: \bar{Y} \nabla Z \\
& \gamma_{1 Y}^{R}<--\gamma_{1 Y}^{L} \\
& \begin{array}{cc}
\gamma_{2 Y}^{L}-->\gamma_{2 Y}^{R} \\
\vdots & \vdots
\end{array} \\
& \gamma_{n Y}^{R} \leqslant--\gamma_{n Y}^{L} \\
& \frac{\beta_{X}^{L}}{\sigma^{\prime}: X^{\prime} \nabla \bar{Y}^{\prime} \quad \tau^{\prime}: Y^{\prime} \nabla \bar{Z}^{\prime}}
\end{aligned}
$$

with $n \geq 1$.
The answer by L to the initial move $\alpha_{Z}^{R}$ in the composition $\tau \circ \sigma$ is $\beta_{X}^{L}$. After this pair of moves, we reduce to the new pair of strategies $\sigma^{\prime}$ and $\tau^{\prime}$, defined on multigames whose shape is a variant of the shape of the multigames on which $\sigma$ and $\tau$ are defined (the polarities of the single components are reversed).
3.

$$
\left.\begin{array}{cccccc}
\sigma: \bar{X} \nabla & Y & \tau: & \bar{Y} & \nabla & Z \\
& & & & & \alpha_{Z}^{R}
\end{array}\right]
$$

with $n \geq 0, m \geq 1$.
The answer by $L$ to the initial move $\alpha_{Z}^{R}$ in the composition $\tau \circ \sigma$ is $\beta_{X}^{L}+\beta_{Z}^{L}$. After this pair of moves, we reduce to the new pair of strategies $\sigma^{\prime}$ and $\tau^{\prime}$, defined on multigames whose shape is a variant of the shape of the multigames on which $\sigma$ and $\tau$ are defined.
4.

with $n \geq 1, m \geq 1$.
The answer by L to the initial move $\alpha_{Z}^{R}$ in the composition $\tau \circ \sigma$ is $\beta_{X}^{L}+\beta_{Z}^{L}$. After this pair of moves, we reduce to the new pair of strategies $\sigma^{\prime}$ and $\tau^{\prime}$, defined on multigames whose shape is a variant of the shape of the multigames on which $\sigma$ and $\tau$ are defined.

In case 1 above, the new pair of strategies, $\sigma^{\prime}, \tau^{\prime}$, to which we reduce after the first pair of moves by R and L is defined on multigames with the same shape of the initial ones, while in the remaining cases $\sigma^{\prime}$ and $\tau^{\prime}$ are defined on multigames whose shapes are variants of the initial ones.

Thus we are left to discuss how the composition goes on in the latter three cases. Notice that in case 2 we reach a situation symmetric w.r.t. the initial one (polarities of components are reversed). Here R can only move in $X^{\prime}$, and after a pair of moves by $R$ and $L$, we reach again one of the four situations above. Cases 3 and 4 are the interesting ones, where we need to apply the strategies $\sigma$ and $\tau$ in parallel, by exploiting the parallelism of multigames. These two cases are dealt with similarly. Let us consider case 3. Here R can open either in $Z^{\prime}$ or in $X^{\prime}$ or in both components.

If R opens in $Z^{\prime}$, then the reply of $L$ via $\tau^{\prime}$ must be in $Z^{\prime}$, since $L$ cannot play in the $Y^{\prime}$ component of $Y^{\prime} \nabla Z^{\prime}$. This will be also the reply of L in the composition $\tau \circ \sigma$, and the next configuration coincides with configuration 3 . That is, we have:

$$
\begin{array}{lrl}
\sigma^{\prime}: X^{\prime} \nabla \bar{Y}^{\prime} \quad \tau^{\prime}: Y^{\prime} \nabla Z^{\prime} \\
& & \alpha_{Z^{\prime}}^{R} \\
& & \beta_{Z^{\prime}}^{L} \\
& & \tau^{\prime \prime}: Y^{\prime} \nabla Z^{\prime \prime}
\end{array}
$$

If R opens in $X^{\prime}$, then the reply by L given by $\sigma^{\prime}$ can be directly in $X^{\prime}$, or in $Y^{\prime}$ (but in this latter case, after finitely many dialogue moves between the $Y^{\prime}$ components, Player L must end up in $X^{\prime}$, otherwise the composition is undefined), or L replies both in $X^{\prime}$ and $Y^{\prime}$.

More in detail, assume R opens in $X^{\prime}$ and the reply by L is directly in $X^{\prime}$ or in $Y^{\prime}$. In this latter case, we apply the swivel chair and, either after finitely many applications of it the reply by L via $\sigma^{\prime}$ ends up in $X^{\prime}$, or the dialogue between the $Y$ components goes on indefinitely, or $\sigma$ or $\tau$ are undefined. In the latter two cases, the overall composition is undefined, while in the first case the move by L in $X^{\prime}$ will be the reply in $\tau \circ \sigma$ to the R move, and the final configuration still coincides with configuration 3 . This situation is represented as follows:

$$
\begin{aligned}
& \sigma^{\prime}: X^{\prime} \nabla \quad \bar{Y}^{\prime} \quad \tau^{\prime}: Y^{\prime} \nabla Z^{\prime} \\
& \alpha_{X^{\prime}}^{R} \\
& \begin{array}{cc}
\gamma_{1 Y^{\prime}}^{L}-->\gamma_{1 Y^{\prime}}^{R} \\
\vdots & \vdots
\end{array} \\
& \gamma_{n Y^{\prime}}^{R}<--\gamma_{n Y^{\prime}}^{L} \\
& \frac{\beta_{X^{\prime}}^{L}}{\sigma^{\prime \prime}: X^{\prime \prime} \nabla \bar{Y}^{\prime \prime} \quad \tau^{\prime \prime}: Y^{\prime \prime} \nabla Z^{\prime \prime}}
\end{aligned}
$$

with $n \geq 0$.
Notice that, if at some point in the case above the reply by L is both in $X^{\prime}$ and $Y^{\prime}$, then the composition is undefined, because the move by $L$ in the $Y^{\prime}$ component can be viewed, via the swivel chair, as a move by $R$ in $Y^{\prime} \nabla Z^{\prime}$, but then $L$ can only answer in the $Y^{\prime}$ component via $\tau^{\prime}$, and again this move can be viewed, via the swivel chair, as a R move in the $Y^{\prime}$ component of $X^{\prime} \nabla Y^{\prime}$, and so on, but L can never end up neither in $X^{\prime}$ nor in $Z^{\prime}$. Thus, either at some point $\sigma^{\prime}$ or $\tau^{\prime}$ is undefined or the dialogue between the $Y^{\prime}$ components goes on indefinitely. In any way, the overall composition is undefined.

Finally, if R moves in $X^{\prime}$ and $Z^{\prime}$, we apply the two strategies $\sigma^{\prime}$ and $\tau^{\prime}$ in parallel: by the form of configuration 3, the answer by $L$ in $\tau^{\prime}$ must be in $Z^{\prime}$, while the answer by $L$ via $\sigma^{\prime}$ can be either in $X^{\prime}$ or in $Y^{\prime}$ or both in $X^{\prime}$ and $Y^{\prime}$. In this latter case, an infinite dialogue between the $Y^{\prime}$ components arises (or $\sigma^{\prime}$ or $\tau^{\prime}$ is undefined at some point), and hence the overall composition is undefined.

If the $L$ reply via $\sigma^{\prime}$ is directly in $X^{\prime}$, then this, together with the L reply via $\tau^{\prime}$ in $Z^{\prime}$, will form the L move in $\tau \circ \sigma$. Otherwise, if the reply by $L$ via $\sigma^{\prime}$ is in $Y^{\prime}$, then again, either the dialogue between the $Y^{\prime}$ components goes on indefinitely, or $\sigma$ or $\tau$ is undefined at some point, or, after finitely many applications of the swivel chair, $\sigma$ will finally provide a move by L in $X^{\prime}$. This, together with the reply by L via $\tau$ in $Z^{\prime}$, will form the move by L in $\tau \circ \sigma$, and the final configuration coincides with configuration 3 again:

$$
\begin{aligned}
& \sigma^{\prime}: X^{\prime} \nabla \bar{Y}^{\prime} \quad \tau^{\prime}: \quad Y^{\prime} \quad \nabla Z^{\prime} \\
& \alpha_{X^{\prime}}^{R} \quad \alpha_{Z^{\prime}}^{R} \\
& \gamma_{1 Y^{\prime}}^{L}-->\gamma_{1 Y^{\prime}}^{R} \quad \beta_{Z^{\prime}}^{L} \\
& \gamma_{n Y^{\prime}}^{R}<--\gamma_{n Y^{\prime}}^{L} \\
& \frac{\beta_{X^{\prime}}^{L}}{\sigma^{\prime \prime}: X^{\prime \prime} \nabla \bar{Y}^{\prime \prime} \quad \tau^{\prime \prime}: Y^{\prime \prime} \nabla Z^{\prime \prime}}
\end{aligned}
$$

with $n \geq 0$.
Similarly, one can deal with case 4 . This justifies closure under composition of the category $\mathcal{Y}_{\mathbf{m}_{\mathcal{A}}}$. Formal definitions are given in the proof of Theorem 4.1 below.

Associativity of composition can also be proven by case analysis on the polarities of the various components.
Assume strategies $\sigma: X \multimap Y, \tau: Y \multimap Z, \theta: Z \multimap W$. We have to prove that $\theta \circ(\tau \circ \sigma)=(\theta \circ \tau) \circ \sigma$. Since multigames are polarized, in any of the two compositions, R can only open in $W$. Now, in any of the two compositions, one should consider
the possible replies by L . We only discuss one case, the remaining being dealt with similarly. Assume the reply by L via $\theta$ is in $Z$. In both compositions, we proceed to apply the swivel chair, by viewing this latter move as a move by R in the $Z$ component of $Y \multimap Z$. Then, in both compositions, we consider the reply by $L$ via $\tau$. Assume $L$ replies both in $Y$ and in $Z$. At this point, the two compositions proceed differently, since in $(\theta \circ \tau) \circ \sigma$ we first apply the swivel chair to the move in $Z$ and we go on until we get an answer by L in $W$, and then we apply the swivel chair to the move by L in $Y$. In $\theta \circ(\tau \circ \sigma)$, these two steps are reversed, first we apply the swivel chair to the move by L in $Y$ until we get a reply by L in $X$, then we apply the swivel chair to the move by $L$ in $Z$ until we get a reply by L in $W$. The point is that these two steps, working on separate parts of the board (i.e. different components), are independent and can be exchanged. As a consequence, the behavior of $\theta \circ(\tau \circ \sigma)$ and $(\theta \circ \tau) \circ \sigma$ is the same. The behavior of $(\theta \circ \tau) \circ \sigma$ is represented below (the order of steps 1 and 2 is reversed in the other composition).

$$
\begin{array}{ccccccc}
\sigma: \bar{X} \nabla & Y & (\tau: \bar{Y} \nabla & Z & \theta: \bar{Z} & \nabla W) \\
& & & & & \\
& & \gamma_{1}^{R}<--\gamma_{1}^{L} & \\
& \delta_{1}^{R}<--^{2}--\delta_{1}^{L} & \gamma_{2}^{L}-\frac{1}{2}>\gamma_{2}^{R} & \\
& \vdots & \vdots & \vdots & \vdots & \\
& \delta_{n}^{R}<---\delta_{n}^{L} & \gamma_{n}^{L}-->\gamma_{n}^{R} & \\
\beta_{1}^{L} & & & & \beta_{2}^{L}
\end{array}
$$

The multigame constructions of tensor product and linear implication can be made functorial, determining a structure of a symmetric monoidal closed category on $\mathcal{Y}_{\mathbf{m}_{\mathcal{A}}}$, with the empty multigame as tensor unit. In particular, in defining tensor bifunctor $\nabla$, we proceed as follows. Let $\sigma_{1}: X_{1} \multimap Y_{1}$ and $\sigma_{2}: X_{2} \multimap Y_{2}$ be strategies. In order to define the strategy $\sigma_{1} \nabla \sigma_{2}: X_{1} \nabla X_{2} \multimap Y_{1} \nabla Y_{2}$, we let the two strategies $\sigma_{1}$ and $\sigma_{2}$ play in parallel. I.e. we consider the opening move by R : if it is in the $Y_{1}\left(Y_{2}\right)$ component, then the L answer will be given by the strategy $\sigma_{1}\left(\sigma_{2}\right)$; if the R move is both in $Y_{1}$ and $Y_{2}$, then we apply the two strategies in parallel. We proceed in a similar way for the next moves. Clearly, for the above definition of $\sigma_{1} \nabla \sigma_{2}$, it is essential that in the sum multigame we keep track of the components from which each move comes from. Otherwise, without tags in the moves of the sum, when e.g. $Y_{1}=Y_{2}$, we would not know on which component the R opening move on $Y_{1} \nabla Y_{2}$ comes, and then we do not know whether applying $\sigma_{1}$ or $\sigma_{2}$. This justifies our definition of sum using tags in Section 3.3. Formal definitions are given in the proof of Theorem 4.1 below.

Summarizing, we have:
Theorem 4.1. The category $\mathcal{Y}_{\mathbf{M}_{\mathcal{A}}}$ is symmetric monoidal closed.
Proof. Identities: We define families of copy-cat strategies $\left\{\sigma_{X}^{c c}: \bar{X} \nabla X\right\}_{X}$ and $\left\{\bar{\sigma}_{X}^{c c}: X \nabla \bar{X}\right\}_{X}$ by mutual conduction as follows:

$$
\sigma_{X}^{c c}=\left\{\left(\alpha^{R},\left(\alpha^{L}, \bar{\sigma}_{X^{\prime}}^{c c}\right)\right) \mid\left(\alpha, X^{\prime}\right) \in X\right\} \quad \bar{\sigma}_{X}^{c c}=\left\{\left(\alpha^{R},\left(\alpha^{L}, \sigma_{X^{\prime}}^{c c}\right)\right) \mid\left(\alpha, X^{\prime}\right) \in X\right\},
$$

where $\alpha^{R}$ denotes the move $\alpha$ by R on the second (first) component of $X$ in $\bar{X} \nabla X(X \nabla \bar{X})$.
The identity morphism $i d_{X}: X \multimap X$ is the copy-cat strategy $\sigma_{X}^{c c}$.
Composition: In view of the above discussion, in order to formalize the definition of strategy composition, we need to consider a generalized notion of composition for strategies $\sigma, \tau$ ranging on variants of linear arrow games obtained by changing polarities of the components, as follows:

1. $\sigma: \bar{X} \nabla Y, \tau: \bar{Y} \nabla Z$
2. $\sigma: X \nabla \bar{Y}, \tau: Y \nabla \bar{Z}$
3. $\sigma: X \nabla \bar{Y}, \tau: Y \nabla Z$
4. $\sigma: X \nabla Y, \tau: \bar{Y} \nabla Z$.

Composition in the various cases above is defined by mutual conduction as follows ${ }^{3}$ :

1. For $\sigma: \bar{X} \nabla Y, \tau: \bar{Y} \nabla Z$, we define the strategy $\tau \circ \sigma: \bar{X} \nabla Z$ by

$$
\begin{aligned}
\tau \circ \sigma= & \left\{\left(\alpha_{Z}^{R},\left(\beta_{Z}^{L}, \tau^{\prime} \circ \sigma^{\prime}\right)\right) \mid \exists n \geq 0 . \exists \gamma_{1 Y}^{R}, \gamma_{2 Y}^{L}, \ldots, \gamma_{n Y}^{L} \cdot \exists Y^{\prime}, Z^{\prime} \cdot \exists \sigma^{\prime}: \bar{X} \nabla Y^{\prime},\right. \\
& \tau^{\prime}: \bar{Y}^{\prime} \nabla Z^{\prime} \cdot\left(\alpha_{Z}^{R},\left(\beta_{Z}^{L}, Z^{\prime}\right)\right) \in Z \&\left(\gamma_{1 Y}^{R}, \gamma_{2 Y}^{L}, \ldots, \gamma_{n Y}^{L}, Y^{\prime}\right) \in Y \& \\
& \left.\left(\alpha_{Z}^{R}, \gamma_{1 Y}^{L}, \gamma_{2 Y}^{R}, \ldots, \gamma_{n Y}^{R}, \beta_{Z}^{L}, \tau^{\prime}\right) \in \tau \&\left(\gamma_{1 Y}^{R}, \gamma_{2 Y}^{L}, \ldots, \gamma_{n Y}^{L}, \sigma^{\prime}\right) \in \sigma\right\} \cup
\end{aligned}
$$

[^2]\[

$$
\begin{aligned}
& \left\{\left(\alpha_{Z}^{R},\left(\beta_{X}^{L}, \tau^{\prime} \circ \sigma^{\prime}\right)\right) \mid \exists n \geq 1 . \exists \gamma_{1 Y}^{R}, \gamma_{2 Y}^{L}, \ldots, \gamma_{n Y}^{R} \cdot \exists X^{\prime}, Y^{\prime}, Z^{\prime} \cdot \exists \sigma^{\prime}: X^{\prime} \nabla \bar{Y}^{\prime},\right. \\
& \tau^{\prime}: Y^{\prime} \nabla \bar{Z}^{\prime} .\left(\alpha_{Z}^{R}, \bar{Z}^{\prime}\right) \in Z \&\left(\gamma_{1 Y}^{R}, \gamma_{2 Y}^{L}, \ldots, \gamma_{n Y}^{R}, \bar{Y}^{\prime}\right) \in Y \&\left(\beta_{X}^{L}, X^{\prime}\right) \in X \& \\
& \left.\left(\alpha_{Z}^{R}, \gamma_{1 Y}^{L}, \gamma_{2 Y}^{R}, \ldots, \gamma_{n Y}^{L}, \tau^{\prime}\right) \in \tau \&\left(\gamma_{1 Y}^{R}, \gamma_{2 Y}^{L}, \ldots, \gamma_{n Y}^{R}, \beta_{X}^{L}, \sigma^{\prime}\right) \in \sigma\right\} \cup \\
& \left\{\left(\alpha_{Z}^{R},\left(\beta_{X}^{L}+\beta_{Z}^{L}, \tau^{\prime} \circ \sigma^{\prime}\right)\right) \mid \exists n \geq 0, m \geq 1 . \exists \gamma_{1 Y}^{R}, \gamma_{2 Y}^{L}, \ldots, \gamma_{n+m Y}^{R} \cdot \exists X^{\prime}, Y^{\prime}, Z^{\prime} .\right. \\
& \exists \sigma^{\prime}: X^{\prime} \nabla \bar{Y}^{\prime}, \tau^{\prime}: Y^{\prime} \nabla Z^{\prime} .\left(\alpha_{Z}^{R},\left(\beta_{Z}^{L}, Z^{\prime}\right)\right) \in Z \&\left(\beta_{X}^{L}, X^{\prime}\right) \in \bar{X} \& \\
& \left(\gamma_{1 Y}^{R}, \gamma_{2 Y}^{L}, \ldots, \gamma_{n+m Y}^{R}, \bar{Y}^{\prime}\right) \in Y \& \\
& \left(\alpha_{Z}^{R}, \gamma_{1 Y}^{L}, \gamma_{2 Y}^{R}, \ldots, \gamma_{n Y}^{R}, \gamma_{n+1 Y}^{L}+\beta_{Z}^{L}, \gamma_{n+2 Y}^{R}, \ldots, \gamma_{n+m Y}^{L}, \tau^{\prime}\right) \in \tau \& \\
& \left.\left(\gamma_{1 Y}^{R}, \gamma_{2 Y}^{L}, \ldots, \gamma_{n+m Y}^{R}, \beta_{X}^{L}, \sigma^{\prime}\right) \in \sigma\right\} \cup \\
& \left\{\left(\alpha_{Z}^{R},\left(\beta_{X}^{L}+\beta_{Z}^{L}, \tau^{\prime} \circ \sigma^{\prime}\right)\right) \mid \exists n \geq 1, m \geq 1 . \exists \gamma_{1 Y}^{R}, \gamma_{2 Y}^{L}, \ldots, \gamma_{n+m Y}^{L} \cdot \exists X^{\prime}, Y^{\prime}, Z^{\prime} .\right. \\
& \exists \sigma^{\prime}: X^{\prime} \nabla Y^{\prime}, \tau^{\prime}: \bar{Y}^{\prime} \nabla Z^{\prime} .\left(\alpha_{Z}^{R},\left(\beta_{Z}^{L}, Z^{\prime}\right)\right) \in Z \&\left(\beta_{X}^{L}, X^{\prime}\right) \in \bar{X} \& \\
& \left(\gamma_{1 Y}^{R}, \gamma_{2 Y}^{L}, \ldots, \gamma_{n+m Y}^{L}, Y^{\prime}\right) \in Y \&\left(\alpha_{Z}^{R}, \gamma_{1 Y}^{L}, \gamma_{2 Y}^{R}, \ldots, \gamma_{n+m Y}^{R}, \beta_{Z}^{L}, \tau^{\prime}\right) \in \tau \& \\
& \left.\left(\gamma_{1 Y}^{R}, \gamma_{2 Y}^{L}, \ldots, \gamma_{n Y}^{R}, \beta_{X}^{L}+\gamma_{n+1 Y}^{L}, \gamma_{n+2 Y}^{R}, \ldots, \gamma_{n+m Y}^{L}, \sigma^{\prime}\right) \in \sigma\right\} .
\end{aligned}
$$
\]

2. This case is symmetric to case 1 w.r.t. the polarities of component games. We omit the complete definition.
3. For $\sigma: X \nabla \bar{Y}, \tau: Y \nabla Z$, we define the strategy $\tau \circ \sigma: X \nabla Z$ by

$$
\begin{aligned}
\tau \circ \sigma=\{ & \left.\left(\alpha_{Z}^{R},\left(\beta_{Z}^{L}, \tau^{\prime} \circ \sigma\right)\right) \mid \exists Z^{\prime} \cdot \exists \tau^{\prime}: Y \nabla Z^{\prime} .\left(\alpha_{Z}^{R},\left(\beta_{Z}^{L}, \tau^{\prime}\right)\right) \in \tau\right\} \cup \\
& \left\{\left(\alpha_{X}^{R},\left(\beta_{X}^{L}, \tau^{\prime} \circ \sigma^{\prime}\right)\right) \mid \exists n \geq 0 . \exists \gamma_{1 Y}^{R}, \gamma_{2 Y}^{L}, \ldots, \gamma_{n Y}^{L} \cdot \exists X^{\prime}, Y^{\prime} \cdot \exists \sigma^{\prime}: X^{\prime} \nabla \bar{Y}^{\prime},\right. \\
& \tau^{\prime}: Y^{\prime} \nabla Z^{\prime} \cdot\left(\alpha_{X}^{R},\left(\beta_{X}^{L}, X^{\prime}\right)\right) \in X \&\left(\gamma_{1 Y}^{R}, \gamma_{2 Y}^{L}, \ldots, \gamma_{n Y}^{L}, Y^{\prime}\right) \in Y \& \\
& \left.\left(\alpha_{X}^{R}, \gamma_{1 Y}^{L}, \gamma_{2 Y}^{R}, \ldots, \gamma_{n Y}^{R}, \beta_{X}^{L}, \sigma^{\prime}\right) \in \sigma \&\left(\gamma_{1 Y}^{R}, \gamma_{2 Y}^{L}, \ldots, \gamma_{n Y}^{L}, \tau^{\prime}\right) \in \tau\right\} \cup \\
& \left\{\left(\alpha_{X}^{R}+\alpha_{Z}^{R},\left(\beta_{X}^{L}+\beta_{Z}^{L}, \tau^{\prime} \circ \sigma^{\prime}\right)\right) \mid \exists n \geq 0 . \exists \gamma_{1 Y}^{R}, \gamma_{2 Y}^{L}, \ldots, \gamma_{n Y}^{L} \cdot \exists X^{\prime}, Y^{\prime}, Z^{\prime} .\right. \\
& \exists \sigma^{\prime}: X^{\prime} \nabla \bar{Y}^{\prime}, \tau^{\prime}: Y^{\prime} \nabla Z^{\prime} .\left(\alpha_{X}^{R}, \beta_{X}^{L}, X^{\prime}\right) \in X \&\left(\gamma_{1 Y}^{R}, \gamma_{2 Y}^{L}, \ldots, \gamma_{n Y}^{L}, Y^{\prime}\right) \in Y \& \\
& \left.\left(\alpha_{Z}^{R}, \beta_{Z}^{L}, Z^{\prime}\right) \in Z \&\left(\alpha_{X}^{R}, \gamma_{1 Y}^{L}, \gamma_{2 Y}^{R}, \ldots, \gamma_{n Y}^{R}, \beta_{X}^{L}, \sigma^{\prime}\right) \in \sigma \&\left(\alpha_{Z}^{R}, \beta_{Z}^{L}, \tau^{\prime}\right) \in \tau\right\} .
\end{aligned}
$$

4. This case can be dealt with similarly to case 3 . We omit the complete definition.

Now one should check that composition behaves correctly w.r.t. identities and that it is associative.
One can prove by mutual coinduction that copy-cat strategies behave as left (and right) identities. Namely, for the case of left identities, let $\sigma_{X}^{c c}: \bar{X} \nabla X$ and $\tau: \bar{X} \nabla Y$, then by applying the coinductive definition of composition, we obtain $\tau \circ \sigma_{X}^{c c}: X \multimap Y=\left\{\left(\alpha_{Y}^{R},\left(\beta_{Y}^{L}, \tau^{\prime} \circ \sigma_{X}^{c c}\right)\right) \mid \exists \tau^{\prime}: \bar{X} \nabla Y^{\prime} .\left(\alpha_{Y}^{R},\left(\beta_{Y}^{L}, Y^{\prime}\right)\right) \in Y \wedge\left(\alpha_{Y}^{R},\left(\beta_{Y}^{L}, \tau^{\prime}\right)\right) \in \tau\right\} \cup\left\{\left(\alpha_{Y}^{R},\left(\beta_{X}^{L}, \tau^{\prime} \circ \sigma_{X^{\prime}}^{c c}\right)\right) \mid \exists \sigma_{X^{\prime}}^{c c}:\right.$ $\left.X^{\prime} \nabla \bar{X}^{\prime} . \exists \tau^{\prime}: X^{\prime} \nabla \bar{Y}^{\prime} .\left(\alpha_{Y}^{R}, \bar{Y}^{\prime}\right) \in Y \wedge\left(\beta_{X}^{L}, X^{\prime}\right) \in \bar{X} \wedge\left(\alpha_{Y}^{R},\left(\beta_{X}^{L}, \tau^{\prime}\right)\right) \in \tau \wedge\left(\beta_{X}^{R}, \beta_{X}^{L}, \sigma_{X^{\prime}}^{c c}\right) \in \sigma_{X}^{c c}\right\}$. By mutual coinduction, the above coincides with $\tau$.

In order to prove that composition is associative, one can proceed by applying the coinductive definition of composition to $\theta \circ(\tau \circ \sigma)$ and $(\theta \circ \tau) \circ \sigma$, for $\sigma: \bar{X} \nabla Y, \tau: \bar{Y} \nabla Z, \theta: \bar{Z} \nabla W$. Intuitively, if R opens in $W$ with $\alpha_{W}^{R}$, then the dialogue between L and R goes on in the same way in the two compositions until L answers in both components $Y$ and $Z$ via $\tau$. If this happens, then the two compositions differ in the order in which the two moves by L are dealt with (see the discussion preceding Theorem 4.1), but the final behavior is the same, namely we get (by omitting details) $\theta \circ(\tau \circ \sigma)=\left\{\left(\alpha_{Z}^{R},\left(\beta_{Z}^{L}, \theta^{\prime} \circ\right.\right.\right.$ $\left.\left.\left.\left(\tau^{\prime} \circ \sigma^{\prime}\right)\right)\right) \mid \ldots\right\} \cup\left\{\left(\alpha_{Z}^{R},\left(\beta_{X}^{L}, \theta^{\prime} \circ\left(\tau^{\prime} \circ \sigma^{\prime}\right)\right)\right) \mid \ldots\right\} \cup\left\{\left(\alpha_{Z}^{R},\left(\beta_{X}^{L}+\beta_{Z}^{L}, \theta^{\prime} \circ\left(\tau^{\prime} \circ \sigma^{\prime}\right)\right)\right) \mid \ldots\right\}$ and $(\theta \circ \tau) \circ \sigma=\left\{\left(\alpha_{Z}^{R},\left(\beta_{Z}^{L},\left(\theta^{\prime} \circ \tau^{\prime}\right) \circ\right.\right.\right.$ $\left.\left.\left.\sigma^{\prime}\right)\right) \mid \ldots\right\} \cup\left\{\left(\alpha_{Z}^{R},\left(\beta_{X}^{L},\left(\theta^{\prime} \circ \tau^{\prime}\right) \circ \sigma^{\prime}\right)\right) \mid \ldots\right\} \cup\left\{\left(\alpha_{Z}^{R},\left(\beta_{X}^{L}+\beta_{Z}^{L},\left(\theta^{\prime} \circ \tau^{\prime}\right) \circ \sigma^{\prime}\right)\right) \mid \ldots\right\}$. Thus, by coinduction $\theta \circ(\tau \circ \sigma)=(\theta \circ \tau) \circ \sigma$.
Tensor functor: The tensor functor, called by abuse of notation $\nabla: \mathcal{Y}_{\mathbf{m}_{\mathcal{A}}} \times \mathcal{Y}_{\mathbf{M}_{\mathcal{A}}} \longrightarrow \mathcal{Y}_{\mathbf{M}_{\mathcal{A}}}$, is the extension of $\nabla$ multigame operation to strategies, defined as follows. Let $\sigma_{1}: X_{1} \multimap Y_{1}, \sigma_{2}: X_{2} \multimap Y_{2}$, then $\sigma_{1} \nabla \sigma_{2}: X_{1} \nabla X_{2} \multimap Y_{1} \nabla Y_{2}$ is the strategy coinductively defined by

$$
\begin{aligned}
\sigma_{1} \nabla \sigma_{2}=\{ & \left.\left(\alpha_{1}^{R},\left(\beta_{1}^{L}, \sigma_{1}^{\prime} \nabla \sigma_{2}\right)\right) \mid\left(\alpha_{1}^{R},\left(\beta_{1}^{L}, \sigma_{1}^{\prime}\right)\right) \in \sigma_{1}\right\} \cup \\
& \left\{\left(\alpha_{2}^{R},\left(\beta_{2}^{L}, \sigma_{1} \nabla \sigma_{2}^{\prime}\right)\right) \mid\left(\alpha_{2}^{R},\left(\beta_{2}^{L}, \sigma_{2}^{\prime}\right)\right) \in \sigma_{2}\right\} \cup \\
& \left\{\left(\alpha_{1}^{R}+\alpha_{2}^{R},\left(\beta_{1}^{L}+\beta_{2}^{L}, \sigma_{1}^{\prime} \nabla \sigma_{2}^{\prime}\right)\right) \mid\left(\alpha_{1}^{R},\left(\beta_{1}^{L}, \sigma_{1}^{\prime}\right)\right) \in \sigma_{1} \wedge\left(\alpha_{2}^{R},\left(\beta_{2}^{L}, \sigma_{2}^{\prime}\right)\right) \in \sigma_{2}\right\} .
\end{aligned}
$$

Linear arrow functor: The linear arrow functor, called by abuse of notation $-: \mathcal{Y}_{\mathbf{M}_{\mathcal{A}}}^{o p} \times \mathcal{Y}_{\mathbf{M}_{\mathcal{A}}} \longrightarrow \mathcal{Y}_{\mathbf{M}_{\mathcal{A}}}$, is the extension of the $\multimap$ multigame operation to strategies defined as follows. Let $\sigma: X_{2} \multimap X_{1}, \tau: Y_{1} \multimap Y_{2}$, i.e. $\sigma: \overline{X_{2}} \nabla X_{1}, \tau: \overline{Y_{1}} \nabla Y_{2}$, then $\sigma \multimap \tau:\left(X_{1} \multimap X_{2}\right) \multimap\left(Y_{1} \multimap Y_{2}\right)$, i.e., $\sigma \multimap \tau: \overline{X_{2}} \nabla \overline{Y_{1}} \nabla X_{1} \nabla Y_{2}$ is the strategy coinductively defined by

$$
\begin{aligned}
\sigma \multimap \tau= & \left\{\left(\alpha_{1}^{R},\left(\beta_{1}^{L}, \sigma_{1}^{\prime} \nabla \sigma_{2}\right)\right) \mid\left(\alpha_{1}^{R},\left(\beta_{1}^{L}, \sigma_{1}^{\prime}\right)\right) \in \sigma_{1}\right\} \cup \\
& \left\{\left(\alpha_{2}^{R},\left(\beta_{2}^{L}, \sigma_{1} \nabla \sigma_{2}^{\prime}\right)\right) \mid\left(\alpha_{2}^{R},\left(\beta_{2}^{L}, \sigma_{2}^{\prime}\right)\right) \in \sigma_{2}\right\} \cup \\
& \left\{\left(\alpha_{1}^{R}+\alpha_{2}^{R},\left(\beta_{1}^{L}+\beta_{2}^{L}, \sigma_{1}^{\prime} \nabla \sigma_{2}^{\prime}\right)\right) \mid\left(\alpha_{1}^{R},\left(\beta_{1}^{L}, \sigma_{1}^{\prime}\right)\right) \in \sigma_{1} \wedge\left(\alpha_{2}^{R},\left(\beta_{2}^{L}, \sigma_{2}^{\prime}\right)\right) \in \sigma_{2}\right\} .
\end{aligned}
$$

Finally, one could show that the infinite sum operation $\nabla^{\infty}$ induces a symmetric monoidal comonad, determining on $\mathcal{Y}_{\mathbf{M}_{\mathcal{A}}}$ a structure of linear category in the sense of [8]. Notice that, contrary to traditional Game Semantics, in our coalgebraic framework this construction does not require the definition of an equivalence on strategies and a quotient operation on them in the style of [1]. Using a coalgebraic approach, it makes categorical constructions much more natural and clear.

## 5. Relating multigames to games

In this section, we show that coalgebraic multigames are related to coalgebraic games, when selective sum is considered in place of disjoint sum.

Coalgebraic games together with partial or total strategies can be endowed with a structure of symmetric monoidal closed category, where tensor is disjoint sum (see [18] for the case of total strategies). This construction is in the spirit of traditional categorical constructions of Game Semantics, see e.g. [1]. Here we consider the case where tensor is selective sum, and we build the category $\mathcal{Y}_{\mathbf{G}_{\mathcal{A}}}$ of polarized games with R acting as first player, and strategies on $\bar{x} \vee y$ as morphisms from the game $x$ to the game $y$. We omit the details of the construction, which can be carried out in a similar way as for the category of multigames $\mathcal{Y}_{\mathbf{M}_{\mathcal{A}}}$, the main difficulty being the definition of composition.

### 5.1. Categorical correspondence

The category $\mathcal{Y}_{\mathbf{G}_{\mathcal{A}}}$ of games turns out to be equivalent to the category $\mathcal{Y}_{\mathbf{M}_{\mathcal{A}}}$ of multigames, i.e., there exist functors $S: \mathcal{Y}_{\mathbf{G}_{\mathcal{A}}} \rightarrow \mathcal{Y}_{\mathbf{M}_{\mathcal{A}}}$ and $T: \mathcal{Y}_{\mathbf{M}_{\mathcal{A}}} \rightarrow \mathcal{Y}_{\mathbf{G}_{\mathcal{A}}}$, and natural isomorphisms $\eta: T \circ S \rightarrow \operatorname{Id}_{\mathcal{Y}_{\mathcal{A}}}$ and $\eta^{\prime}: \operatorname{Id}_{\mathcal{Y}_{\mathbf{M}_{\mathcal{A}}}} \rightarrow S \circ T$.

Namely, given a multigame $X$ in $\mathbf{M}_{\mathcal{A}}$, this induces a game $X_{g}$, where the atomic moves are the sets of (multi)moves on the multigame $X$, i.e.:

$$
\begin{equation*}
X_{g}=\left\{\left(\alpha, X_{g}^{\prime}\right) \mid\left(\alpha, X^{\prime}\right) \in X\right\} \tag{3}
\end{equation*}
$$

Vice versa, given a game $x$, one builds a multigame $x_{m}$, where each atomic move $a$ is replaced by the singleton multimove \{a\}, i.e.:

$$
\begin{equation*}
x_{m}=\left\{\left(\{a\}, x_{m}^{\prime}\right) \mid\left(a, x^{\prime}\right) \in x\right\} . \tag{4}
\end{equation*}
$$

For any multigame $X,\left(X_{g}\right)_{m}$ is isomorphic to $X$, and for any game $x,\left(x_{m}\right)_{g}$ is also isomorphic to $x$, with a sort of copy-cat strategies as isomorphisms. Namely, the isomorphism $\iota_{X}: X \multimap\left(X_{g}\right)_{m}$ is the strategy conductively defined by

$$
\iota_{X}=\left\{\left(\left\{\alpha^{R}\right\},\left(\alpha^{L}, \iota_{X^{\prime}}\right)\right) \mid\left(\alpha, X^{\prime}\right) \in X\right\}
$$

The other isomorphism is defined similarly.
At the first sight, the existence of the above isomorphisms could be surprising, since, in mapping multigames to games, we lose the structure of multimoves. However, notice that, in moving on multigames, each player can only consider a multimove "in toto", but she/he cannot work at the level of atomic moves, e.g. by choosing a subset of a multimove. This is the reason why the above isomorphisms work.

Equations (1) and (4) above allow us to define the object part of functors $T: \mathcal{Y}_{\mathbf{M}_{\mathcal{A}}} \rightarrow \mathcal{Y}_{\mathbf{G}_{\mathcal{A}}}$ and $S: \mathcal{Y}_{\mathbf{G}_{\mathcal{A}}} \rightarrow \mathcal{Y}_{\mathbf{M}_{\mathcal{A}}}$, respectively. Notice that $S$ and $T$ preserve tensor product on objects, up to isomorphism, i.e., for all games $x, y$ and for all multigames $X, Y$,

$$
(x \vee y)_{m} \simeq x_{m} \nabla y_{m} \quad(X \nabla Y)_{g} \simeq X_{g} \vee Y_{g} .
$$

Again, the isomorphisms are sorts of copy-cat strategies, e.g. the isomorphism $\eta_{X Y}:(X \nabla Y)_{g} \multimap X_{g} \vee Y_{g}$ is the strategy coinductively defined by

$$
\begin{aligned}
\eta_{X Y}= & \left\{\left(\alpha^{R},\left(\alpha^{L}, \eta_{X^{\prime} Y}\right)\right) \mid\left(\alpha, X^{\prime}\right) \in X\right\} \cup\left\{\left(\beta^{R},\left(\beta^{L}, \eta_{X Y^{\prime}}\right)\right) \mid\left(\beta, Y^{\prime}\right) \in Y\right\} \cup \\
& \left\{\left(\left(\alpha^{R}, \beta^{R}\right),\left(\alpha^{L}+\beta^{L}, \eta_{X^{\prime} Y^{\prime}}\right)\right) \mid\left(\alpha, X^{\prime}\right) \in X \&\left(\beta, Y^{\prime}\right) \in Y\right\}
\end{aligned}
$$

Essentially, selective sum $\vee$ on games and multigame sum $\nabla$ use two different (but isomorphic) ways of codifying a move in two components.

Functors $S$ and $T$ can be extended to strategies as follows.
For any strategy on multigames $\sigma: X \multimap Y$, we can associate a strategy $\sigma_{g}: X_{g} \multimap Y_{g}$, coinductively defined by:

$$
\begin{aligned}
\sigma_{g}= & \left\{\left(\alpha_{Y}^{R},\left(\beta_{X}^{L}, \sigma_{g}^{\prime}\right)\right) \mid\left(\alpha_{Y}^{R},\left(\beta_{X}^{L}, \sigma^{\prime}\right)\right) \in \sigma\right\} \cup \\
& \left\{\left(\alpha_{Y}^{R},\left(\beta_{Y}^{L}, \sigma_{g}^{\prime}\right)\right) \mid\left(\alpha_{Y}^{R},\left(\beta_{Y}^{L}, \sigma^{\prime}\right)\right) \in \sigma\right\} \cup \\
& \left\{\left(\alpha_{Y}^{R},\left(\left(\beta_{X}^{R}, \beta_{Y}^{L}\right), \sigma_{g}^{\prime}\right)\right) \mid\left(\alpha_{Y}^{R},\left(\beta_{X}^{L}+\beta_{Y}^{L}, \sigma^{\prime}\right)\right) \in \sigma\right\}
\end{aligned}
$$

Vice versa, any strategy on games $\sigma: x \multimap y$ induces a strategy on multigames $\sigma_{m}: x_{m} \multimap y_{m}$ defined by:

$$
\begin{aligned}
\sigma_{m}= & \left\{\left(\left\{\alpha_{Y}^{R}\right\},\left(\left\{\beta_{X}^{L}\right\}, \sigma_{m}^{\prime}\right)\right) \mid\left(\alpha_{Y}^{R},\left(\beta_{X}^{L}, \sigma^{\prime}\right)\right) \in \sigma\right\} \cup \\
& \left\{\left(\left\{\alpha_{Y}^{R}\right\},\left(\left\{\beta_{Y}^{L}\right\}, \sigma_{m}^{\prime}\right)\right) \mid\left(\alpha_{Y}^{R},\left(\beta_{Y}^{L}, \sigma^{\prime}\right)\right) \in \sigma\right\} \cup \\
& \left\{\left(\left\{\alpha_{Y}^{R}\right\},\left(\left(\left\{\beta_{X}^{R}\right\}+\left\{\beta_{Y}^{L}\right\}\right), \sigma_{m}^{\prime}\right)\right) \mid\left(\alpha_{Y}^{R},\left(\left(\beta_{X}^{L}, \beta_{Y}^{L}\right), \sigma^{\prime}\right)\right) \in \sigma\right\}
\end{aligned}
$$

Summarizing, we can define the following functors:

Definition 5.1. Let $S: \mathcal{Y}_{\mathbf{G}_{\mathcal{A}}} \rightarrow \mathcal{Y}_{\mathbf{M}_{\mathcal{A}}}$ and $T: \mathcal{Y}_{\mathbf{M}_{\mathcal{A}}} \rightarrow \mathcal{Y}_{\mathbf{G}_{\mathcal{A}}}$ be the functors defined by:
for any game $x, S x=x_{m}$, for any strategy $\sigma: X \multimap Y, \mathcal{S} \sigma=\sigma_{m}$,
for any multigame $X, T X=X_{g}$, for any strategy $\sigma: X \multimap Y, T \sigma=\sigma_{g}$.

Then, we have:

Theorem 5.1. The functors $S: \mathcal{Y}_{\mathbf{G}_{\mathcal{A}}} \rightarrow \mathcal{Y}_{\mathbf{M}_{\mathcal{A}}}$ and $T: \mathcal{Y}_{\mathbf{M}_{\mathcal{A}}} \rightarrow \mathcal{Y}_{\mathbf{G}_{\mathcal{A}}}$ are monoidal, and they give an equivalence between the categories $\mathcal{Y}_{\mathbf{G}_{\mathcal{A}}}$ and $\mathcal{Y}_{\mathbf{M}_{\mathcal{A}}}$.

## 6. Final remarks and directions for future work

We have introduced coalgebraic multigames, where at each step the current player performs a multimove, i.e., a (finite) set of atomic moves. Coalgebraic multigames introduce a certain level of parallelism, and they are situated half-way between traditional sequential games and concurrent games. Multigame operations are smoothly defined in our coalgebraic framework as final morphisms via (generalized) coiteration schemata. A monoidal closed category of multigames and strategies is built, where tensor is sum. The relationship between coalgebraic multigames and games is expressed in categorical terms via an equivalence between the category of multigames and a monoidal closed category of coalgebraic games where tensor is selective sum.

Here is a list of comments and directions for future work.

- Total strategies. In this paper, coalgebraic (multi)games are endowed with a notion of partial strategy, whereby the given player can possibly refuse to provide an answer and give up the game. Alternatively, one could consider notions of total strategies, in the line of $[16,18]$, where the player is forced to give an answer, if there exists any. On total strategies one can then define the notion of winning/non-losing strategy for a player, if it generates winning/non-losing plays against any possible counterstrategy. A finite play is taken to be winning for the player who performs the last move, while infinite plays are taken to be winning for L/R or draws. In order to formalize winning/non-losing strategies, one shall introduce a payoff function on plays, and enrich the notion of multigame with a payoff. We claim that the results of the present paper can be rephrased also in the context of total strategies.
- Semantics of concurrency. In the literature, notions of concurrent games [2], asynchronous games [29], and distributed games $[13,33,10]$ have been introduced as concurrent extensions of traditional games. Our categories of coalgebraic multigames and coalgebraic games with selective sum are more in the traditional line, but nonetheless, they reflect a form of parallelism. Namely, in [19] it has been shown that, in the context of functional languages, categories of multigames in the style of [1] accommodate parallel or. It would be interesting to explore to what extent multigames can be used for modeling concurrent and distributed languages, possibly featuring true concurrency. This would require an extension of our approach in order to account also for interference between moves/events.
- Relating multigames to domains. Since multigames introduce a level of parallelism, and as it is shown in [19], they accommodate for e.g. parallel or in functional programming, one may expect a tighter connection between multigames and traditional denotational semantics based on domains.
- Relating (multi)actions and (multi)games structure. It would be interesting to explore the connection between the (multi)action structure and the monoidal structure of (multi)games and strategies. This could be possibly viewed as an instance of the microcosm principle, see [14].
- Other notions of sum. In [11], Chapter 14 "How to Play Several Games at Once in a Dozen Different Ways", Conway introduces a number of different ways in which games can be played. Apart from disjunctive and selective sum, Conway defines conjunctive sum, where at each step the current player makes a move in each (non-ended) component. A first attempt to extend Joyal's categorical construction to conjunctive sum fails, even in the case of polarized games, since trivially copy-cat strategies do not work. Alternative approaches are called for.


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## Appendix A. The guarded coiteration schema

Here we recall a generalized coiteration schema introduced in [9], and used in Section 3.3 of the present paper to define infinite sum on games. This schema generalizes standard coiteration:

Definition (Guarded coiteration). Let $\left(\Omega, \alpha_{\Omega}\right)$ be a final coalgebra for a functor $F:$ Set* $^{*} \rightarrow$ Set*, let $G:$ Set $^{*} \rightarrow$ Set* $^{*}$ be a functor, and let $A$ be a set. A guarded specification for a morphism $h: A \rightarrow \Omega$ is of the form:

$$
\alpha_{\Omega} \circ h=F(g \circ G(h)) \circ \delta_{A},
$$

where $\delta_{A}: A \rightarrow F(G A)$ and $g: G(\Omega) \rightarrow \Omega$ are given, $g$ is the guard.
I.e. $h$ makes the following diagram commute:


Under suitable conditions, the above schema admits a unique solution. For more details and complete theorems, see [9]. In particular, a generalized distributive law $\lambda: G F \rightarrow F G$ is required for which $\left(\Omega, g, \alpha_{\Omega}\right)$ is a $\lambda$-bialgebra.

The specification for infinite sum given in Definition 3.10 of the present paper defines a unique function $\nabla^{\infty}$, since it is an instance of the guarded coiteration above, where

- the functor $G:$ Set $^{*} \rightarrow$ Set* is defined by

$$
G(A)=\coprod_{n \geq 1}\left(\mathbf{M}_{\mathcal{A}}^{n} \times A\right)
$$

- the guard $g: G\left(\mathbf{M}_{\mathcal{A}}\right) \longrightarrow \mathbf{M}_{\mathcal{A}}$ is defined by

$$
g\left(X_{1}, \ldots, X_{n+1}\right)=X_{1} \nabla \ldots \nabla X_{n+1}
$$

- the function $\delta_{\mathbf{M}_{\mathcal{A}}}: \mathbf{M}_{\mathcal{A}} \rightarrow \mathcal{P}_{<\kappa}\left(\mathcal{M}_{\mathcal{A}} \times \coprod_{n \geq 1}\left(\mathbf{M}_{\mathcal{A}}^{n} \times \mathbf{M}_{\mathcal{A}}\right)\right)$ is defined by

$$
\begin{aligned}
\delta_{\mathbf{M}_{\mathcal{A}}}(X)= & \left\{\left(\alpha_{1}+\ldots+\alpha_{n}, X_{1}^{\prime}, \ldots, X_{n}^{\prime}, X\right) \mid n \geq 1 \& \lambda \alpha_{1}=\ldots=\lambda \alpha_{n} \&\right. \\
& \left.\left(\alpha_{1}, X_{1}^{\prime}\right), \ldots,\left(\alpha_{n}, X_{n}^{\prime}\right) \in X\right\}
\end{aligned}
$$

- $\left(\mathbf{M}_{\mathcal{A}}, g, i d\right)$ is a $\lambda$-bialgebra for $\lambda: G F \rightarrow F G$ distributive law defined by

$$
\begin{aligned}
& \lambda_{A}: \coprod_{n \geq 1}\left(\mathbf{M}_{\mathcal{A}}^{n} \times \mathcal{P}_{<\kappa}\left(\mathcal{M}_{\mathcal{A}} \times A\right)\right) \rightarrow \mathcal{P}_{<\kappa}\left(\mathcal{M}_{\mathcal{A}} \times \coprod_{n \geq 1}\left(\mathbf{M}_{\mathcal{A}}^{n} \times A\right)\right) \\
& \lambda_{A}\left(X_{1}, \ldots, X_{n+1}\right)=\left\{\left(\alpha_{1}+\ldots+\alpha_{k}, X_{1}^{\prime}, \ldots, X_{n+1}^{\prime}\right) \mid k \geq 1 \&\right. \\
&\left(\alpha_{1}, X_{i_{1}}^{\prime}\right) \in X_{i_{1}}, \ldots,\left(\alpha_{k}, X_{i_{k}}^{\prime}\right) \in X_{i_{k}} \& 1 \leq i_{1}, \ldots, i_{k} \leq n \& \\
&\left.\forall j \in\{1, \ldots, n\} \backslash\left\{i_{1}, \ldots, i_{k}\right\} . X_{j}^{\prime}=X_{j}\right\} .
\end{aligned}
$$

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[^1]:    ${ }^{1}$ We recall that the hereditary cardinal of a set is the cardinality of its transitive closure, namely the cardinality of the downward membership tree which has the given set as its root.
    2 The final coalgebra of the powerset functor exists since the powerset functor is bounded by $\kappa$.

[^2]:    ${ }^{3}$ In what follows, we use the notation $\left(\gamma_{1}, \gamma_{2}, \ldots, \gamma_{n}, Y^{\prime}\right)$ in place of $\left(\gamma_{1},\left\{\left(\gamma_{2},\left\{\ldots\left(\gamma_{n}, Y^{\prime}\right) \ldots\right\}\right), \ldots\right\}\right)$.

