# Categories of Coalgebraic Games with Selective Sum 

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#### Abstract

Joyal's categorical construction on (well-founded) Conway games and winning strategies provides a compact closed category, where tensor and linear implication are defined via Conway disjunctive sum (in combination with negation for linear implication). The equivalence induced on games by the morphisms coincides with the contextual closure of the equideterminacy relation w.r.t. the disjunctive sum. Recently, the above categorical construction has been generalized to non-wellfounded games. Here we investigate Joyal's construction for a different notion of sum, i.e. selective sum. While disjunctive sum reflects the interleaving semantics, selective sum accommodates a form of parallelism, by allowing the current player to move in different parts of the board simultaneously. We show that Joyal's categorical construction can be successfully extended to selective sum, when we consider alternating games, i.e. games where each position is marked as Left player $(\mathrm{L})$ or Right player $(\mathrm{R})$, that is only L or R can move from that position, R starts, and $\mathrm{L} / \mathrm{R}$ positions strictly alternate. Alternating games typically arise in the context of Game Semantics. This category of well-founded games with selective sum is symmetric monoidal closed, and it induces exactly the equideterminacy relation. Generalizations to non-wellfounded games give linear categories, i.e. models of Linear Logic. Our game models, providing a certain level of parallelism, may be situated halfway between traditional sequential alternating game models and the concurrent game models by Abramsky and Mellies. We work in a context of coalgebraic games, whereby games are viewed as elements of a final coalgebra, and game operations are defined as final morphisms.


Keywords: Games, Strategies, Conway Games, Coalgebras, Categories of Games and Strategies

## 1. Introduction

In [21], Joyal showed how to endow (well-founded) Conway games and winning strategies [12] with a structure of compact closed category. This construction is based on the disjunctive sum of games, which induces a tensor product, and, in combination with negation, yields linear implication. The equivalence determined on games by the morphisms of the above category coincides with the contextual closure of the equideterminacy relation w.r.t. disjunctive sum. Recently, the above categorical construction and equivalences have been generalized to non-wellfounded games [14, 15, 16], while, in the context of Linear Logic, various categories of possibly non-wellfounded Conway games have been introduced and studied in [22, 23, 26].

In this paper, we investigate Joyal's construction for a different, less-known, but nonetheless interesting notion of sum, i.e. selective sum. This has been introduced by Conway in [12], Chapter 14 "How to Play Several Games at Once in a Dozen Different Ways". In the disjoint sum of two games, at each step, the player who moves chooses one component and performs a move in that component, leaving unchanged the other; in the selective sum instead, at each step, the player who moves can choose to move in one or in both components. Interestingly from the perspective of Semantics of Concurrency, disjoint sum reflects an interleaving semantics, while selective sum accommodates a form of (true) parallelism, by allowing the current player to move in different parts of the board simultaneously.

In [12], Conway defines and studies selective sum only for a special class of games, i.e. the impartial ones, where, at each position, the two players, Left $(\mathrm{L})$ and Right $(\mathrm{R})$, have the same moves. In this paper, we first provide a definition of selective sum on general Conway games, then we show that Joyal's categorical paradigm can be successfully extended to selective sum, when we restrict the class of Conway games to alternating games. These are games where each position is marked as L or R , that is only L or $R$ can move from that position, R starts, and $\mathrm{L} / \mathrm{R}$ positions strictly alternate. Alternating games typically arise in the context of traditional Game Semantics [3, 18] (see [16] for a precise analysis and comparison). Our category of alternating well-founded Conway games with selective sum can be endowed with a structure of symmetric monoidal closed category (not compact closed, since alternating games are not closed under negation). This category characterizes the equideterminacy relation, which coincides, in this setting, with its contextual closure. In the present paper, we also generalize the above construction to (alternating) non-wellfounded games, both in the case of fixed games, i.e. games where each play is winning for one of the players, and in the case of mixed games, i.e. games where each play can be either winning or a draw. Fixed games arise in the context of semantics of Linear Logic, see e.g. [2, 17], while the more general mixed games are those considered in [8]. The categories of games that we obtain are linear in the sense of [9], i.e. models of intuitionistic Linear Logic. In particular, the categorical construction in the case of mixed games is based on an analysis of mixed games in terms of pairs of fixed games. The difficulty in defining a category of mixed games arises from the lack of compositionality of strategies on mixed games.

The interest of our investigation on selective sum is manifold.
To our knowledge, the extension of Joyal's paradigm to selective sum and the categories of games studied in the present paper are the first results in the literature in this direction. Selective sum introduces a level of parallelism in game models: models based on selective sum may be situated halfway between the traditional sequential alternating models of Linear Logic [2,17] and the concurrent models introduced by Abramsky and Mellies [4]. Our model, compared with [4] and more recent works on concurrent and distributed games [28, 11], where the departure from traditional Game Semantics is radical, is more
conservative and less complex, still providing, up to a certain extent, an account of parallelism. It is interesting to notice that the level of concurrency accounted for in our game model is sufficient to recover e.g. parallel or in the setting of semantics of functional languages. It will be worth to explore to what extent our model is useful for the semantics of concurrent and distributed languages.

Moreover, it is well-known that it is not so easy to define good semantic models where strategies are allowed to play several moves in a row. As we will see, building a good category where selective sum plays the rôle of tensor is not trivial, in particular the definition of strategy composition is subtle, and it requires a non standard parallel application of strategies. Under this perspective, our investigation is valuable also because it contributes to the overall understanding of games.

The equivalences on games that we study, i.e. equideterminacy, its contextual closure, and the categorical equivalence induced by the existence of morphisms, originate from Conway's and Joyal's fundamental works [12, 21]. These equivalences are interesting in themselves. In the context of Game Semantics, they capture game invariants, such as type inhabitability.

In the present paper, we work in a context of coalgebraic games, whereby possibly non-wellfounded Conway games are viewed as elements of a final coalgebra, and game operations, including sum, negation, exponential, are defined as final morphisms. Conway games have been first represented as elements of a final coalgebra in [14], while in [16] a quite general notion of coalgebraic game has been introduced and shown to subsume various notions of games, including games arising in Game Semantics à la [3]. Coalgebraic methods appear very natural and useful in this context, since they allow to abstract away superficial features of positions in games, and to smoothly define game operations as final morphisms. Moreover, representing games as points of a final coalgebra is in the spirit of the original Conway presentation, where games are defined as sets rather than as graphs, as in many other approaches in the literature. The coalgebraic/set-theoretical representation is motivated by the fact that the existence of winning/non-losing strategies is invariant w.r.t. graph bisimilarity.

We formalize the notion of play as a sequence of pairs move-position, and, on top of it, we define a strategy as a function on plays. Again in the spirit of [12], we focus on total strategies for a given player, i.e. strategies that must provide an answer, if any, for the player. These differ from partial strategies, in which the player can refuse an answer and give up the game. Interesting equivalences on games are induced when the strategies are total. Total strategies are formalized via the notion of winning/non-losing strategy.
Related work. Generalizations of Joyal's category to non-wellfounded games have been introduced in [22, 23, 26], in the context of Semantics of Linear Logic. Generalizations to non-wellfounded games and total strategies have been studied in $[15,16]$ in a coalgebraic context, and categorical equivalences have been investigated and related to contextual equivalences.

Coalgebraic methods for modeling games have been used also in [6], where the notion of membership game has been introduced. This corresponds to a subclass of our coalgebraic games, where at any position $L$ and $R$ have the same moves, and all infinite plays are deemed winning for player II (the player who does not start). However, no operations on games are considered in that setting. In the literature, various notions of bisimilarity equivalences have been considered on games, see e.g. [27, 7]. But, contrary to our approach, such games are defined as graphs of positions, and equivalences on graphs, such as trace equivalences or various bisimilarities are considered. By defining games as the elements of a final coalgebra, we directly work up-to bisimilarity of game graphs.
Summary. In Section 2, we recall the framework of coalgebraic games and strategies. In Section 3, we introduce game operations, including selective sum, as final morphisms into the final coalgebra of
games. In Section 4, we extend Joyal's paradigm to the case of selective sum, by building a symmetric monoidal closed category of alternating well-founded games, and linear categories of fixed alternating non-wellfounded games, and mixed alternating non-wellfounded games. We also study the corresponding categorical equivalences. Conclusions and directions for future work appear in Section 5.

We assume the reader familiar with (sequential) Game Semantics, see e.g. [1], with categorical semantics of Linear Logic, see e.g. [25], with basic notions of coalgebras, see [20], and with the theory of non-wellfounded sets, see [13,5] for more details.

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## 2. Coalgebraic Games and Strategies

In this section, we recall the definitions of coalgebraic game, play, and strategy, as they have been introduced in [14, 15].

We consider a general notion of 2-player game of perfect information, where the two players are called Left $(\mathrm{L})$ and Right $(\mathrm{R})$. A game $x$ is identified with its initial position; at any position, there are moves for L and R , taking to new positions of the game. By abstracting superficial features of positions, games can be viewed as elements of the final coalgebra for the functor $F(X)=\mathcal{P}_{<\kappa}(\mathcal{A} \times X)$, where $\mathcal{A}=\{L, R\}$ is a set of two atoms marking each move with the name of the player who performs the move, and $\mathcal{P}_{<\kappa}$ is the set of all subsets of cardinality $<\kappa$, where $\kappa$ can be $\omega$ if only games with finitely many moves are considered, or it can be an inaccessible cardinal if we are interested in more general games. The coalgebra structure captures, for any position, the moves of the players and the corresponding next positions.

We work in the category Set* of sets belonging to a universe satisfying the Antifoundation Axiom, see [13, 5], where the objects are the sets with hereditary cardinal less than $\kappa$, and whose morphisms are the functions with hereditary cardinal less than $\kappa^{1}$. Of course, we could work in the category Set of well-founded sets, but we prefer to use $S e t^{*}$ so as to be able to use identities, i.e. extentional equalites in formal set theory, rather than isomorphisms in some naive set theory. Formally, we define:

## Definition 2.1. (Coalgebraic Games)

Let $\mathcal{A}=\{L, R\}$, let $F: \operatorname{Set}^{*} \rightarrow \operatorname{Set}^{*}$ be the functor defined by $F(X)=\mathcal{P}_{<\kappa}(\mathcal{A} \times X)$ (with usual definition on morphisms), and let ( $\mathcal{G}, i d$ ) be the final $F$-coalgebra ${ }^{2}$. A coalgebraic game is an element $x$ of the carrier $\mathcal{G}$ of the final coalgebra.

The elements of the final coalgebra $\mathcal{G}$ are the minimal graphs up-to bisimilarity, and they extend Conway original games [12] to non-wellfounded ones ${ }^{3}$. We call player I the player who starts the game

[^0](who can be L or R in general), and player II the other. Once a player has moved on a game $x$, this leads to a new game/position $x^{\prime}$. We define the plays on $x$ as the sequences of pairs move-position from $x$; moves in a play are not necessarily alternating (this generality will be useful in the sequel, in defining operations on games):

## Definition 2.2. (Plays)

A play on a coalgebraic game $x_{0}$ is a possibly empty finite or infinite sequence of pairs in $\mathcal{A} \times \mathcal{G}$, $s=\left\langle a_{1}, x_{1}\right\rangle \ldots$ such that $\forall n \geq 0 .\left\langle a_{n+1}, x_{n+1}\right\rangle \in x_{n}$.
We denote by Play $x$ the set of plays on $x$ and by FPlay $x$ the set of finite plays.
We focus on strategies that always provide an answer, if any, of the player to the moves of the opponent player. In this sense, these strategies are "total", opposite to "partial strategies", where the player can refuse an answer and give up the game. Formally, strategies in our framework are partial functions on finite plays ending with a position where the player is next to move, and yielding (if any) a pair in $\mathcal{A} \times \mathcal{G}$, consisting of "a move of the given player together with a next position" on the game $x$. In what follows, we denote by

- FPlay ${ }_{x}^{L I}\left(\right.$ FPlay $\left._{x}^{R I}\right)$ the set of possibly empty finite plays on the game $x$ on which $\mathrm{L}(\mathrm{R})$ acts as player I , and ending with a position where $\mathrm{R}(\mathrm{L})$ was last to move, i.e. $s=\left\langle a_{1}, x_{1}\right\rangle \ldots\left\langle a_{n}, x_{n}\right\rangle$, $a_{1}=L\left(a_{1}=R\right)$ and $a_{n}=R\left(a_{n}=L\right)$.
- $\operatorname{FPlay} y_{x}^{L I I}\left(\right.$ FPlay $\left._{x}^{R I I}\right)$ the set of finite plays on the game $x$ on which $\mathrm{L}(\mathrm{R})$ acts as player II, and ending with a position where $\mathrm{R}(\mathrm{L})$ was last to move, i.e. $s=\left\langle a_{1}, x_{1}\right\rangle \ldots\left\langle a_{n}, x_{n}\right\rangle, a_{1}=R$ $\left(a_{1}=L\right)$ and $a_{n}=R\left(a_{n}=L\right)$.

Formally, we define:

## Definition 2.3. (Strategies)

Let $x$ be a coalgebraic game. A strategy $\sigma$ for LI (i.e. L acting as player I) is a partial function $\sigma$ : FPlay ${ }_{x}^{L I} \rightarrow \mathcal{A} \times \mathcal{G}$ such that, for any $s \in$ FPlay $_{x}^{L I}$,
$-\sigma(s)=\left\langle a, x^{\prime}\right\rangle \Longrightarrow a=L \wedge s\left\langle a, x^{\prime}\right\rangle \in$ FPlay $_{x}$
$-\exists\left\langle a, x^{\prime}\right\rangle .\left(s\left\langle a, x^{\prime}\right\rangle \in\right.$ FPlay $\left._{x} \wedge a=L\right) \Longrightarrow s \in \operatorname{dom}(\sigma)$.
Similarly, one can define strategies for players LII, RI, RII.
For any player LI, LII, RI, RII, we define the opponent player as RII, RI, LII, LI, respectively. A counterstrategy of a strategy for a player in $\{\mathrm{LI}, \mathrm{LII}, \mathrm{RI}, \mathrm{RII}\}$ is a strategy for the opponent player.

We are interested in studying the interactions of a strategy for a given player with the (counter)strategies of the opponent player. When a player plays on a game according to a strategy $\sigma$, against an opponent player who follows a (counter)strategy $\sigma^{\prime}$, a play arises. Formally, we define:

## Definition 2.4. (Product of Strategies)

Let $x$ be a coalgebraic game.
(i) Let $s$ be a play on $x$, and $\sigma$ a strategy for a player in $\{$ LI,LII,RI,RII $\}$. Then $s$ is coherent with $\sigma$ if, for any proper prefix $s^{\prime}$ of $s$, ending with a position where the player is next to move, $\sigma\left(s^{\prime}\right)=\left\langle a, x^{\prime}\right\rangle \Longrightarrow$ $s^{\prime}\left\langle a, x^{\prime}\right\rangle$ is a prefix of $s$.
(ii) Given a strategy $\sigma$ on $x$ and a counterstrategy $\sigma^{\prime}$, we define the product of $\sigma$ and $\sigma^{\prime}, \sigma * \sigma^{\prime}$, as the unique play coherent with both $\sigma$ and $\sigma^{\prime}$.

Notice that a play arising from the product of strategies is alternating.
We distinguish between well-founded games, i.e. well-founded sets as elements of the final coalgebra $\mathcal{G}$ (corresponding to original Conway games), and non-wellfounded games, i.e. non-wellfounded sets in $\mathcal{G}$. Clearly, strategies on well-founded games generate only finite plays, while strategies on nonwellfounded games can generate infinite plays.

Strategies for a given player, as we have defined so far, simply provide an answer (if any) of the player to all possible moves of the opponent. Intuitively, a strategy is winning/non-losing for a player, if it generates winning/non-losing plays against any possible counterstrategy. We take a finite play to be winning for the player who performs the last move, while infinite plays are taken to be winning for $\mathrm{L} / \mathrm{R}$ or draws. In order to formalize winning/non-losing strategies, we need to introduce a payoff function on plays, and to enrich the notion of game with a payoff function. Hence we define:

## Definition 2.5. (Payoff function)

Let $x$ be a coalgebraic game. A payoff function $\nu:$ Play $_{x} \rightarrow\{0,1,-1\}$ is a function such that, for any finite play $s, \nu(s)= \begin{cases}1 & \text { if the last move is by } \mathrm{L} \\ -1 & \text { if the last move is by } \mathrm{R} .\end{cases}$

Note that the action of a payoff function is uniquely determined on finite plays, while it is not constrained a priori on infinite plays.

## Definition 2.6. (Winning/non-losing Plays)

Let $x$ be a coalgebraic game, $s$ a play on $x$, and $\nu$ a payoff function defined on $x$. Then
(i) A play $s$ is winning for player $\mathrm{L}(\mathrm{R})$ if $\nu(s)=1(\nu(s)=-1)$.
(ii) A play $s$ is a draw if $\nu(s)=0$.
(iii) A play $s$ is non-losing for player $\mathrm{L}(\mathrm{R})$ if $\nu(s) \in\{0,1\}(\nu(s) \in\{0,-1\})$.

Games come endowed with a payoff function, i.e.:

## Definition 2.7. (Games)

A game $\left\langle x, \nu_{x}\right\rangle$ is a coalgebraic game $x$ together with a payoff function on $x$.
In what follows, we will denote games $\left\langle x, \nu_{x}\right\rangle$ simply by $x$.

## Definition 2.8. (Winning/non-losing Strategies)

Let $\nu:$ Play $_{x} \rightarrow\{0,1,-1\}$ be a payoff function on $x$.
(i) A strategy $\sigma$ on $x$ for LI (LII) is winning (non-losing) for LI (LII) if for any strategy $\sigma^{\prime}$ for RII (RI), $\nu\left(\sigma * \sigma^{\prime}\right)=1\left(\nu\left(\sigma * \sigma^{\prime}\right) \in\{0,1\}\right)$.
(ii) A strategy $\sigma$ on $x$ for RI (RII) is winning (non-losing) if for any strategy $\sigma^{\prime}$ for LII (LI), $\nu\left(\sigma * \sigma^{\prime}\right)=-1$ $\left(\nu\left(\sigma * \sigma^{\prime}\right) \in\{0,-1\}\right)$.

Winning strategies on well-founded games correspond to Conway winning strategies.

## Special classes of games.

- Mixed games. The class of all games, where plays can be winning or draws. These correspond to the loopy or mixed games considered in [8].
- Fixed games. The subclass of games where all plays are winning for one of the players, i.e. they include all well-founded games, but only the subclass of non-wellfounded games where draws are not admitted.
- Alternating games. The subclass of games with the following special structure: R starts, at any non-ending position only moves for R or L are available, along any path in the game graph R/L moves strictly alternate. Alternating games play a central rôle in the construction of our categories based on selective sum. Fixed alternating games arise in traditional Game Semantics of Linear Logic, see e.g. [2, 17], see also [16].

The following determinacy results hold (see [14] for more details):

## Theorem 2.9. (Determinacy for Fixed Games)

Any fixed game has a winning strategy precisely for one of the following players: L (independently if she/he plays as I or II ${ }^{4}$ ), R (independently if she/he plays as I or II), I (independently if she/he plays as L or $\mathrm{R}^{5}$ ), II (independently if she/he plays as L or R ).

The above theorem generalizes to mixed games as follows:

## Theorem 2.10. (Determinacy for Mixed Games)

Any mixed game has a non-losing strategy at least for one of the following players: L, R, I, II.

## 3. Game Operations

In this section, we show how to define various operations on games, including selective sum, negation, linear implication, and exponential. In our framework, game operations can be conveniently defined via final morphisms, using (some generalizations of) the standard coiteration schema. These coalgebraic definitions capture the structure of compound games; the extra structure of the payoff function on infinite plays of the compound game is obtained from the payoff of the components. In particular, we define two notions of sum generalizing Conway selective sum, one on mixed and the other on fixed games. These two sums have the same coalgebraic structure, and only differ by the definition of the payoff on infinite plays.

Selective sum. The coalgebraic structure of the sum of two games is given as follows. On the sum game, at each step, the next player selects either one (non-ended) or both component games, and makes a legal move in each of the selected components, while the component which has not been chosen (if any) remains unchanged. Notice that, in this way, even if the play on the sum game agrees with turns of L and R , the subplays in the single components may not agree with turns, in general.

Definition 3.1. (Selective Sum, coalgebraic structure)
The selective sum of two games $\vee: \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$ is defined by:
$x \vee y=\left\{\left\langle a, x^{\prime} \vee y\right\rangle \mid\left\langle a, x^{\prime}\right\rangle \in x\right\} \cup\left\{\left\langle a, x \vee y^{\prime}\right\rangle \mid\left\langle a, y^{\prime}\right\rangle \in y\right\} \cup\left\{\left\langle a, x^{\prime} \vee y^{\prime}\right\rangle \mid\left\langle a, x^{\prime}\right\rangle \in x \&\left\langle a, y^{\prime}\right\rangle \in y\right\}$.

[^1]The above definition corresponds to a standard coiteration schema, namely the function $\vee$ is the final morphism from the coalgebra ( $\mathcal{G} \times \mathcal{G}, \alpha_{\vee}$ ) to the final coalgebra ( $\mathcal{G}$, id), where the coalgebra morphism $\alpha_{\vee}: \mathcal{G} \times \mathcal{G} \longrightarrow F(\mathcal{G} \times \mathcal{G})$ is defined by: $\alpha_{\vee}(x, y)=\left\{\left\langle a,\left\langle x^{\prime}, y\right\rangle\right\rangle \mid\left\langle a, x^{\prime}\right\rangle \in x\right\} \cup\left\{\left\langle a,\left\langle x, y^{\prime}\right\rangle\right\rangle \mid\left\langle a, y^{\prime}\right\rangle \in\right.$ $y\} \cup\left\{\left\langle a,\left\langle x^{\prime}, y^{\prime}\right\rangle\right\rangle \mid\left\langle a, x^{\prime}\right\rangle \in x \&\left\langle a, y^{\prime}\right\rangle \in y\right\}$.

In what follows, we denote by $\vee$ the selective sum on well-founded games. Two kinds of sum arise on non-wellfounded games from the above coalgebraic definition, by suitably defining the payoff function on infinite plays:

- Mixed sum $\vee_{m}$. This is defined on mixed games. The payoff of an infinite play will be $1(-1)$ if all infinite plays in the components have payoff $1(-1)$ and 0 otherwise. This definition is inspired by the extension of disjoint sum to mixed games of [8].
- Fixed sum $\vee_{f}$. This is defined on fixed games. The payoff of an infinite play is 1 (i.e. winning for L ) iff all infinite subplays in the components have payoff 1 and -1 otherwise. This definition is inspired by the definition of tensor product in the context of Game Semantics for Linear Logic, see e.g. [1].

Remark. Notice that, on both sums, since plays which agree with turns do not necessarily induce subplays on the components which agree with turns, in order to define the payoff on infinite plays, we need the payoff on all plays of the components, also those non conformed to turns. This is the reason for such a liberal definition of plays in Section 2. But, if we restrict ourselves to alternating games, then any play on the sum game which agrees with turns induces subplays with the same property in the components.

Negation. The negation is a unary game operation, which allows us to build a new game, where the rôles of L and R are exchanged. For $a \in \mathcal{A}$, we define $\bar{a}= \begin{cases}L & \text { if } a=R \\ R & \text { if } a=L .\end{cases}$

The definition of game negation is as follows:

## Definition 3.2. (Negation)

The negation ${ }^{-}: \mathcal{G} \longrightarrow \mathcal{G}$ is defined by: $\bar{x}=\left\{\left\langle\bar{a}, \overline{x^{\prime}}\right\rangle \mid\left\langle a, x^{\prime}\right\rangle \in x\right\}$.
The payoff on infinite plays of $\bar{x}$ is taken to be opposite to the payoff on $x$.
Also negation is an instance of the coiteration schema. It is the final morphism from the coalgebra $\left(\mathcal{G}, \alpha_{-}\right)$to the final coalgebra $(\mathcal{G}$, id $)$, where the coalgebra morphism $\alpha_{-}: \mathcal{G} \longrightarrow F(\mathcal{G})$ is defined by: $\alpha_{-}(x)=\left\{\left\langle\bar{a}, x^{\prime}\right\rangle \mid\left\langle a, x^{\prime}\right\rangle \in x\right\}$.

Clearly, winning/non-losing strategies for a given player on $x$ become winning/non-losing strategies for the opponent player on $\bar{x}$, and $\overline{\bar{x}}=x$, i.e. negation is involutive.
Linear implication. Using the two notions of sum, and negation, we can now define the following linear implications:

## Definition 3.3. (Linear Implications)

We define:

- on well-founded games: the linear implication $x \multimap y$ as the game $\overline{x \vee \bar{y}}$;
- on mixed games: the linear implication $x \multimap_{m} y$ as the game $\overline{x \vee_{m} \bar{y}}$;
- on fixed games: the linear implication $x \multimap_{f} y$ as the game $\overline{x \vee_{f} \bar{y}}$.

Of course both $\multimap_{m}$ and $\multimap_{f}$ extend $\multimap$ to non-wellfounded games, but in two different ways. Notice that mixed sum satisfies the equality $\overline{x \vee_{m} y}=\bar{x} \vee_{m} \bar{y}$, hence the linear implication $x \multimap_{m} y$ amounts to $\bar{x} \vee_{m} y$, while the corresponding equality for fixed sum does not hold. More precisely, the coalgebraic structure of the game $x \multimap_{f} y$ coincides with the coalgebraic structure of $\bar{x} \vee_{m} y$, but the winning condition on infinite plays is different, namely an infinite play is winning for L on $x \multimap_{f} y$ if and only if the subplay on $\bar{x}$ or that on $y$ is infinite and winning for L , while an infinite play is winning for L on $x \multimap_{m} y$ if and only if all infinite subplays on $\bar{x}$ and $y$ are winning for L .
Infinite sum. We can enrich games with a further interesting unary coalgebraic operation, $\mathrm{V}^{\infty}$, an infinite selective sum: on the game $\bigvee^{\infty} x$, at each step, the current player can perform a move in finitely many of the infinite components of $x$. Our infinite selective sum is related to the exponential modality defined on a category of non-wellfounded games in [26], and it will induce a comonad on the categories of non-wellfounded games that we will consider in Section 4.

## Definition 3.4. (Infinite Sum)

We define the infinite selective sum $\bigvee^{\infty}: \mathcal{G} \longrightarrow \mathcal{G}$ by:

$$
\bigvee^{\infty} x=\left\{\left\langle a, x_{1}^{\prime} \vee \ldots \vee x_{n}^{\prime} \vee \bigvee^{\infty} x\right\rangle \mid n \geq 1 \&\left\langle a, x_{1}^{\prime}\right\rangle, \ldots,\left\langle a, x_{n}^{\prime}\right\rangle \in x\right\}
$$

The above definition defines a unique function $\bigvee^{\infty}$, being an instance of the guarded coiteration schema of [10], where the guard $g: G(\mathcal{G}) \longrightarrow \mathcal{G}$, for $G$ the functor $G(X)=\coprod_{n \geq 1}\left(\mathcal{G}^{n} \times X\right)$, is defined by $g\left(x_{1}, \ldots, x_{n+1}\right)=x_{1} \vee \ldots \vee x_{n+1}$.
Operations on alternating games. Notice that alternating games are not closed under the operations defined above. However, one can define corresponding sums and linear implications on alternating games, simply by "pruning" the graph of the resulting games. A similar construction has been considered also in [22], in the case of disjoint sum. In our setting, the pruning operation of a game into an alternating one can be defined as a coalgebraic operation, using a definition by mutual recursion:

## Definition 3.5. (Pruning)

Let ()$_{a_{L}},()_{a_{R}}: \mathcal{G} \longrightarrow \mathcal{G}$ be the mutually recursive functions defined as:

$$
\left\{\begin{array}{l}
(x)_{a_{L}}=\left\{\left\langle R,\left(x^{\prime}\right) a_{a_{R}}\right\rangle \mid\left\langle R, x^{\prime}\right\rangle \in x\right\} \\
(x)_{a_{R}}=\left\{\left\langle L,\left(x^{\prime}\right) a_{L}\right\rangle \mid\left\langle L, x^{\prime}\right\rangle \in x\right\}
\end{array}\right.
$$

We define the pruning operation ()$_{a}$ as ()$_{a_{L}}$.
The payoff on infinite plays of $(x)_{a}$ is induced by the payoff of $x$.
The above schema of mutual recursion defines a unique pair of functions ()$_{a_{L}},()_{a_{R}}$, since it is an instance of the T-coiteration schema, a schema which generalizes standard coiteration, see [10] for more details.

Once we have the pruning operation, we can define alternating sums and linear implications:

## Definition 3.6. (Alternating Operations)

Let $x, y$ be alternating games. We define:

- the alternating selective sum as the game $(x \vee y)_{a}$;
- the alternating linear implication as the game $(x \multimap y)_{a}$;
- the alternating infinite sum as the game $\left(\bigvee^{\infty} x\right)_{a}$.

Alternating operations on mixed and fixed non-wellfounded games are defined by suitably determining the payoff of the compound games as above. In the following, by abuse of notation, we will use the same symbols for alternating operations and the corresponding operations on all games.

## 4. Game Categories based on Selective Sum

In this section, we extend Joyal's paradigm [21] to the case of selective sum. We work with alternating games, since the whole class of games fails to give a category when selective sum is considered, because of lack of identities, as we will see. Our category of alternating well-founded games turns out to be symmetric monoidal closed. In particular, the proof of closure under composition requires a non standard parallel application of strategies. Moreover, we show that this category induces on games exactly the equideterminacy relation, which, on alternating games, coincides with its contextual closure under selective contexts.

Then, we investigate generalizations of the above construction to non-wellfounded games. First, we build a linear category [9, 25], i.e. a model of intuitionistic Linear Logic, of fixed alternating games, still inducing the equideterminacy relation, then, following the approach of [16], where non-wellfounded games with disjoint sum are considered, we build a linear category which includes mixed alternating games as objects, where mixed games are characterized in terms of pairs of fixed games. Equideterminacy and contextual equivalence still coincide on mixed alternating games, but the categorical equivalence turns out to be only strictly included in the equideterminacy relation.

Finally, a technical issue. In defining categories of games, we need to extend game operations to functors. But this requires to move from the purely set-theoretical extensional definition of game operations of Section 3 to a more intensional one. The problem arises because, on the sum game $x \vee y$ as it is defined in Definition 3.1, we are not able, in general, to distinguish between moves "coming from" $x$ or $y$. This distinction will be necessary e.g. in extending sum to a bifunctor. A possible way out is to extend the set of atoms $\mathcal{A}=\{L, R\}$ in order to be able to keep track, on the sum game, of the component from which each move comes from. That is, we can define:

$$
(\widehat{\mathcal{A}} \ni) a::=L|R|\langle a,-\rangle|\langle-, a\rangle|\langle a, a\rangle .
$$

Now, let $\lambda: \widehat{\mathcal{A}} \rightarrow\{L, R, \perp\}$ be the following function, which gives the name of the player, when this is unique, i.e.:

$$
\lambda(a)= \begin{cases}a & \text { if } a=L \text { or } a=R \\ \lambda a^{\prime} & \text { if } a=\left\langle a^{\prime},-\right\rangle \text { or } a=\left\langle-, a^{\prime}\right\rangle \text { or }\left(a=\left\langle a^{\prime}, a^{\prime \prime}\right\rangle \text { and } \lambda a^{\prime}=\lambda a^{\prime \prime}\right) \\ \perp & \text { otherwise }\end{cases}
$$

Then we define $\mathcal{A}=\{a \in \widehat{\mathcal{A}} \mid \lambda a \in\{L, R\}\}$.

Finally, the definition of sum will be:

$$
\begin{aligned}
& x \vee y=\left\{\left\langle\langle a,-\rangle, x^{\prime} \vee y\right\rangle \mid\left\langle a, x^{\prime}\right\rangle \in x\right\} \cup\left\{\left\langle\langle-, a\rangle, x \vee y^{\prime}\right\rangle \mid\left\langle a, y^{\prime}\right\rangle \in y\right\} \cup \\
& \\
& \left\{\left\langle\left\langle a, a^{\prime}\right\rangle, x^{\prime} \vee y^{\prime}\right\rangle \mid\left\langle a, x^{\prime}\right\rangle \in x \&\left\langle a^{\prime}, y^{\prime}\right\rangle \in y \& \lambda\left\langle a, a^{\prime}\right\rangle \in\{L, R\}\right\} .
\end{aligned}
$$

The other operations are defined accordingly.
In what follows, we tacitly assume to work with the above definitions.

### 4.1. Well-founded Games

Definition 4.1. (The Category $\mathcal{Y}_{\mathrm{V}}$ )
Objects: alternating well-founded games.
Morphisms: $\quad \sigma: x \rightarrow y$ winning strategy for LII on $x \multimap y=(\bar{x} \vee y)_{a}$.
Identities on $\mathcal{Y}_{\vee}$ are the copy-cat strategies. These work thanks to the fact that games are alternating, so as, on the game $x \multimap x=(\bar{x} \vee x)_{a}$, R can only open on $x$, then L proceeds by copying the move on $\bar{x}$ and so on, at each step R has exactly one component to move in. Notice that, if the games were not alternating, then R could play on both components $x$ and $\bar{x}$, preventing L to apply the copy-cat strategy.

The proof that the above category is well-defined follows.
Theorem 4.2. The category $\mathcal{Y}_{\vee}$ is well-defined.

## Proof:

Copy-cat strategies behave as identities w.r.t. composition as defined below.
Composition in the category $\mathcal{Y}_{V}$ is obtained via the swivel-chair strategy and a non standard parallel application of strategies. I.e., given winning strategies for LII, $\sigma$ on $x \multimap y$ and $\tau$ on $y \multimap z$, the composition, $\tau \circ \sigma$ on $x \multimap z$, is obtained as follows. R opening move on $x \multimap z$ must be on $z$, since games are alternating. Then consider the L answer given by the strategy $\tau$ on $y \multimap z$. If L moves in $z$ only, then we take this as the L answer in the strategy $\tau \circ \sigma$. If the L move according to $\tau$ is in the $y$ component of $y \multimap z$ or in both components $y$ and $z$, then we use the swivel chair to view the L move in the $y$ component as a R move in the $y$ component of $x \multimap y$. Now L has an answer in $x \multimap y$, according to $\sigma$. If this move is in $y$ or in both $x$ and $y$, then, using the swivel chair, the move in $y$ can be viewed as a R move in the $y$ component of $y \multimap z$, and so on. Since games are well-founded, we are guaranteed that the dialogue between the $y$ components does not go on forever, and eventually an L move according to $\sigma$ or $\tau$ will be only on $x$ or $z$, respectively. At this point, the procedure for computing the L answer to the R opening move stops. The L answer will consists of this latter move in $x$ or $z$ together with another possible L move in $z$ or $x$ obtained during the above procedure. The fact that games are alternating guarantees that we obtain exactly one L move in $x$ or $z$ or two L moves, one in $x$ and one in $z$.

Now, in order to understand how the strategy $\tau \circ \sigma$ behaves after the first pair of RL moves, it is convenient to list all situations which can arise after these moves, according to which player is next to move in each component. Namely, by case analysis, one can show that, after the first RL moves, the following four cases can arise:

$$
\text { 1. } x^{+} \multimap y^{-} y^{+} \multimap z^{-}
$$

2. $x^{-} \multimap y^{+} y^{-} \multimap z^{+}$
3. $x^{-} \multimap y^{+} y^{-} \multimap z^{-}$
4. $x^{-} \multimap y^{-} y^{+} \multimap z^{-}$
where $x^{+}\left(x^{-}\right)$denotes that $\mathrm{L}(\mathrm{R})$ is next to move in that component. Notice that case 1 above corresponds to the initial situation. Thus we are left to discuss the behavior of $\tau \circ \sigma$ in the other cases. In case 2 , R can only open in $x$; this case can be dealt with similarly as the initial case 1 , and after a pair of RL moves, it brings again in one of the four situations above. Cases 3 and 4 are the interesting ones, where we need to apply the strategies $\sigma$ and $\tau$ in parallel, by exploiting the parallelism of $\vee$. These two cases are dealt with similarly. Let us consider case 3 . Then R can open in $z$ or in $x$ or in both components.

- If R opens in $z$, then the answer of L via $\tau$ must be in $z$, since, by definition of configuration 3, L cannot play in the $y$ component of $y \multimap z$. This will be also the answer of L in the composition $\tau \circ \sigma$, and the final configuration coincides with configuration 3 .
- If R opens in $x$, then the L answer given by $\sigma$ can be in $x$ or in $y$, but not both in $x$ and $y$, as we show below. If the L answer is in $x$, this will be the answer of $\tau \circ \sigma$, and the new configuration coincides with configuration 3. If the L move is in the $y$ component of $x \multimap y$, then this can be viewed, via the swivel chair, as a R move in $y \multimap z$. By definition of configuration 3 , L can only answer in $y$ via $\tau$, and, after finitely many applications of the swivel chair, the L answer via $\sigma$ end up in $x$. The final configuration still coincides with configuration 3. Then, we are left to show that, after the R opening move in $x$, L cannot answer both in $x$ and $y$ via $\sigma$. Namely, the move in $y$ can be viewed, via the swivel chair, as a R move in the $y$ component of $y \multimap z$. Then, by definition of configuration 3 , the L answer via $\tau$ cannot be in $z$ but only in the $y$ component of $y \multimap z$. As a consequence, an infinite dialogue between the $y$ components arises, contradicting the fact that games are well-founded.
- Finally, if R opens in $x$ and $z$, we apply the two strategies $\sigma$ and $\tau$ in parallel: by the form of configuration 3, the L answer of $\tau$ must be in $z$, while the L answer via $\sigma$ can be either in $x$ or in $y$, but not both in $x$ and $y$, since, otherwise, by the swivel chair, the L move in $y$ could be viewed as a R move in the $y$ component of $y \multimap z$, but L can only answer in the same component via $\tau$, thus leading to an infinite dialogue between the $y$ components of $x \multimap y$ and $y \multimap z$, contradicting the fact that games are well-founded. Therefore, if the L answer via $\sigma$ is in $x$, then this, together with the L answer via $\tau$ in $z$, will be the L answer of $\tau \circ \sigma$, and the final configuration coincides with configuration 3 itself. If the L answer via $\sigma$ is in $y$, then, after finitely many applications of the swivel chair, $\sigma$ will finally provide a L move in $x$. This, together with the L answer via $\tau$ in $z$, will be the L answer of $\tau \circ \sigma$, and the final configuration coincides with configuration 3 again.

Similarly, one can deal with case 4 . This proves closure under composition of the category $\mathcal{Y}_{V}$.
Associativity of composition can also be proven by case analysis on the polarity of the current player in the various components.

Assume strategies $\sigma: x \multimap y, \tau: y \multimap z, \theta: z \multimap w$. We have to prove that $\theta \circ(\tau \circ \sigma)=(\theta \circ \tau) \circ \sigma$. Since games are alternating, in any of the two compositions, R can only open in $w$. Now, in any of the two compositions, one should consider the possible answers by L. We only discuss one case, the remaining
being dealt with similarly. Assume the L answer via $\theta$ is in $z$. In both compositions, we proceed to apply the swivel chair, by viewing this latter move as a R move in the $z$ component of $y \multimap z$. Then, in both compositions, we consider the L answer via $\tau$. Assume L answers both in $y$ and in $z$. At this point, the two compositions proceed differently, since in $\theta \circ(\tau \circ \sigma)$ we first apply the swivel chair to the move in $z$ and we go on until we get a L answer in $w$, and then we apply the swivel chair to the L move in $y$. In $(\theta \circ \tau) \circ \sigma$, these two steps are reversed, first we apply the swivel chair to the L move in $y$ until we get a L answer in $x$, then we apply the swivel chair to the L move in $z$ until we get a L answer in $w$. The point is that these two steps, working on separate parts of the board (i.e. different components), are independent and can be exchanged. As a consequence, the behavior of $\theta \circ(\tau \circ \sigma)$ and $(\theta \circ \tau) \circ \sigma$ is the same.

This concludes the proof that Definition 4.1 is well-given.
Notice that the above proof relies on the possibility of keeping track, on the linear game, of the component(s) where each player moves. This is also evident in extending sum to a bifunctor, where from strategies $\sigma: x \multimap y, \tau: x^{\prime} \multimap y^{\prime}$, we define a strategy $\sigma \vee \tau: x \vee x^{\prime} \multimap y \vee y^{\prime}$ by letting $\sigma$ and $\tau$ play in parallel.

Selective sum gives rise to a tensor product on $\mathcal{Y}_{V}$, which determines a structure of a symmetric monoidal closed category. The proof of this fact is standard, in particular the identity $(x \vee y) \multimap z=$ $y \multimap(x \multimap z)$ holds, this latter following from the definition of the $\multimap$ game and from the fact that negation is involutive.

Theorem 4.3. The category $\mathcal{Y}_{V}$ is symmetric monoidal closed.
Notice that $\mathcal{Y}_{V}$ is not compact closed neither $*$-autonomous, since alternating games are not closed under negation.

### 4.1.1. Equivalences on well-founded games.

Following [12], we consider equideterminacy, a natural equivalence on games induced by the existence of winning strategies. In the following, we denote by $x \Downarrow_{L I I}$ the fact that the game $x$ has a winning strategy for LII.

## Definition 4.4. (Equideterminacy)

Let $x, y$ be well-founded games. We define the equideterminacy relation $\mathbb{\Downarrow}$ by: $x \mathbb{\Downarrow} y$ iff $\left(x \Downarrow_{L I I} \Longleftrightarrow\right.$ $\left.y \Downarrow_{L I I}\right)$.

Following the approach in $[14,15]$ for the case of disjunctive sum, we consider the contextual closure of equideterminacy w.r.t. selective sum:

Definition 4.5. (Contextual Equivalence)
Let $x, y$ be well-founded games. We define the contextual equivalence $\approx_{\vee}$ by:

$$
x \approx \vee y \text { iff } \forall C[] .(C[x] \Uparrow \overparen{\Downarrow} C[y]),
$$

where $C[]$ is a selective context, i.e. a context of the shape [ ] $\vee z$, for $z$ well-founded game.

On alternating well-founded games, equideterminacy is a congruence w.r.t. selective contexts, and hence it coincides with the contextual equivalence. This is a consequence of the following interesting property characterizing winning strategies on the selective sum of alternating games:

## Lemma 4.6. (Main)

Let $x, y$ be alternating well-founded games. Then

$$
x \vee y \Downarrow_{L I I} \Longleftrightarrow\left(x \Downarrow_{L I I} \& y \Downarrow_{L I I}\right) .
$$

## Proof:

The implication $(\Leftarrow)$ is immediate to prove. Namely, for any given move of R in $x \vee y$, L answers using his winning strategy in $x$ or in $y$ or both winning strategies in $x$ and $y$. For the converse implication $(\Rightarrow)$, assume by contradiction that e.g. $x \vee y \Downarrow_{L I I}$ but $x \Downarrow_{L I I}$. Then, by the Determinacy Theorem 2.9, $x \Downarrow_{R I}$ (i.e. RI has a winning strategy on $x$ ), but then it is easy to check that also $x \vee y \Downarrow_{R I}$, contradicting $x \vee y \Downarrow_{L I I}$. Namely, if $x \Downarrow_{R I}$, then, thanks to the fact that games are alternating, R can use his winning strategy to play and win on $x \vee y$.

Notice that the above property does not hold on general well-founded games (a simple example: for $x=\{\langle L, \emptyset\rangle,\langle R,\{\langle L, \emptyset\rangle\}\rangle\}$ and $y=\{\langle L, \emptyset\rangle,\langle R, \emptyset\rangle\}$, we have that $x \vee y \Downarrow_{L I I}$, while $y \psi_{L I I}$. However, it is interesting to notice that Lemma 4.6 also holds on the special class of impartial games, i.e. games where at each position $L$ and $R$ have exactly the same moves, see [12], Chapter 14.

As a consequence of Lemma 4.6, we have:

## Lemma 4.7. (Equideterminacy is a Congruence)

Let $x, y$ be alternating well-founded games. Then

$$
x \Uparrow y \Longrightarrow \forall z \cdot(x \vee z) \Uparrow(y \vee z)
$$

## Proof:

Let $x \Uparrow y$ and $x \vee z \Downarrow_{L I I}$. By Lemma 4.6, $x \Downarrow_{L I I}$ and $z \Downarrow_{L I I}$. Moreover, since $x \Uparrow y$, then also $y \Downarrow_{L I I}$. Hence $y \vee z \Downarrow_{L I I}$.

An immediate consequence of Lemma 4.7 is:

## Theorem 4.8. $\hat{\mathbb{V}}=\approx \vee$.

Following [21], we can consider the categorical equivalence, i.e. the equivalence induced on games by the existence of morphisms between them, namely:

## Definition 4.9. (Categorical Equivalence)

The category $\mathcal{Y}_{\checkmark}$ induces the following pre-equivalence on alternating well-founded games:

$$
x \leq_{\vee} y \text { iff there exists a winning strategy for LII on } x \multimap y .
$$

The categorical equivalence $\sim_{v}$ is defined as $\leq_{v} \cap\left(\leq_{v}\right)^{-1}$.

The categorical properties of $\mathcal{Y}_{V}$ ensure that $\sim_{V}$ is an equivalence and a congruence w.r.t. selective sum. Namely, identities on the category $\mathcal{Y}_{\mathcal{A}}$ correspond to reflexivity of $\sim_{V}$, closure under composition corresponds to transitivity, while functoriality of tensor ensures congruence w.r.t. selective sum.

Interestingly, all the equivalences, equideterminacy, contextual equivalence, and categorical equivalence coincide, thus providing different characterizations of the same equivalence:

## Theorem 4.10. (Characterization of the Equivalence)

$$
\sim_{v}=\approx_{\mathrm{v}}=\mathbb{\Downarrow} .
$$

## Proof:

It is sufficient to show that $\sim_{\vee}=\Uparrow$. $(\subseteq)$ Assume $x \leq_{\vee} y$, i.e. $(\bar{x} \vee y)_{a} \Downarrow_{L I I}$, and $x \Downarrow_{L I I}$. We show that also $y \Downarrow_{L I I}$. Namely, if by contradiction $y \Downarrow_{L I I}$, then, by Theorem 2.9, $y \Downarrow_{R I}$. Moreover, $\bar{x} \Downarrow_{R I I}$. Hence $(\bar{x} \vee y)_{a} \Downarrow_{R I}$, since R can play on $y$ as player I , according to his winning strategy, and she/he can answer to any move of L on $\bar{x}$ using his winning strategy on $\bar{x}$ as player II. Notice that here the hypothesis that games are alternating is fundamental. Thus we get a contradiction. ( $\supseteq$ ) Assume $x \Downarrow_{L I I} \Rightarrow y \Downarrow_{L I I}$, and assume by contradiction that $(\bar{x} \vee y)_{a} \psi_{L I I}$, i.e. $(\bar{x} \vee y)_{a} \Downarrow_{R I}$. Then $\bar{x} \psi_{L I}$, otherwise RI would not have a winning strategy on $(\bar{x} \vee y)_{a}$. Hence $x \not_{R I}$, i.e., by Theorem 2.9, $x \Downarrow_{L I I}$. Thus, by hypothesis, also $y \Downarrow_{L I I}$, hence $(\bar{x} \vee y)_{a} \Downarrow_{L I I}$, contradicting the hypothesis $(\bar{x} \vee y)_{a} \Downarrow_{R I}$.

### 4.2. Non-wellfounded Games

Following [16], where Joyal's construction has been extended to non-wellfounded games, we first build a linear category, in the sense of [9], of fixed alternating games with selective sum, and then a linear category of mixed alternating games, where each game is characterized via a pair of fixed games, and we study the corresponding equivalences.

### 4.2.1. Fixed games.

Definition 4.11. (The Category $\mathcal{Y}_{f}$ )
Objects: fixed alternating games.
Morphisms: $\quad \sigma: x \rightarrow y$ winning strategy for LII on $x \longrightarrow_{f} y$.
Identities on $\mathcal{Y}_{f}$ are the copy-cat strategies. Closure under composition derives from the use of the swivel-chair strategy and the parallel application of strategies, in a similar way as for $\mathcal{Y}_{\mathrm{V}}$; notice that, in the case of $\mathcal{Y}_{f}$, the dialogue between the $y$ components is guaranteed not to go on forever by the winning condition on infinite plays on the game $x \rightarrow_{f} y$, spelled out in Section 3. Namely, let $\sigma$ be a winning strategy for LII on $x \multimap_{f} y$ and $\tau$ a winning strategy for LII on $y \multimap_{f} z$. The proof that there is a winning strategy for LII $\tau \circ \sigma$ on $x \multimap_{f} z$ proceeds as for $\mathcal{Y}_{\vee}$, we only have to rule out the case that the play goes on forever because there are infinite subplays on the $y$ components of $x \rightarrow_{f} y$ and $y \multimap_{f} z$. Namely, if this would be the case, since the subplay on $x \multimap_{f} y$ must be winning for L , then the infinite subplay on the $y$ component of $x \multimap_{f} y$ must be winning for L . But then, since the roles of $\mathrm{L} / \mathrm{R}$ are reversed on the $y$ component of $y \multimap_{f} z$, we have an infinite subplay on the $y$ component of $y \multimap_{f} z$, which is winning for R , contradicting the fact that the play on $y \multimap_{f} z$ is winning for L .

Theorem 4.12. The category $\mathcal{Y}_{f}$ is a linear category.

## Proof:

(Sketch) Fixed selective sum gives rise to a tensor product on $\mathcal{Y}_{f}$, which determines a structure of a symmetric monoidal closed category; in particular, the identity $\left(x \vee_{f} y\right) \multimap_{f} z=y \multimap_{f}\left(x \multimap_{f} z\right)$ holds, this latter following from the definition of the $\omega_{f}$ game and from the fact that negation is involutive. Moreover, one can show that the infinite sum operation $\bigvee^{\infty}$ induces a symmetric monoidal comonad, determining on $\mathcal{Y}_{f}$ a structure of linear category in the sense of [9]. We skip the lengthy (but direct) proof ${ }^{6}$.

Equivalences on fixed games. The notions of equideterminacy (Definition 4.4) and contextual equivalence (Definition 4.5) directly extend to fixed games, so as the Main Lemma 4.6 characterizing winning strategies on selective sum still holds, hence $\mathbb{\rrbracket}_{f}=\approx_{f}$, where $\mathbb{\rrbracket}_{f}$ and $\approx_{f}$ denote equideterminacy and contextual equivalence on fixed games. Also the Characterization Theorem 4.10 extends to fixed games. Summarizing, denoting by $\sim_{f}$ the categorical equivalence induced by $\mathcal{Y}_{f}$, we have:

Theorem 4.13. $\sim_{f}=\approx_{f}=\Uparrow_{f}$.

### 4.2.2. Mixed games.

Defining a category of mixed games and non-losing strategies is not straightforward, the reason being that non-losing strategies are not closed under composition. The situation has been analyzed in [15, 16] for the disjoint sum, and various solutions have been proposed. Similar problems arise for selective sum. Here we present a solution in the line of [16], based on the analysis of mixed games as pairs of fixed games. The idea, which is inspired by [8], is to represent a mixed game $x$ as a pair of fixed games $\left\langle x^{-}, x^{+}\right\rangle$, obtained by considering all draws to be winning for R or for L respectively. These are objects in a category of pairs of fixed alternating games, where tensor product and linear implication are defined as fixed tensor and fixed implication in the single components. Carrying out this construction for selective sum, we get a linear category $\mathcal{Y}_{p}$.

## Definition 4.14. (The Category $\mathcal{Y}_{p}$ )

Objects: $\quad$ pairs of fixed alternating games $x=\left\langle x_{1}, x_{2}\right\rangle$.
Morphisms: $\quad$ pairs of winning strategies for LII on fixed games, $\left\langle\sigma_{1}, \sigma_{2}\right\rangle:\left\langle x_{1}, x_{2}\right\rangle \rightarrow\left\langle y_{1}, y_{2}\right\rangle$.
Mixed games $x$ are objects of the category $\mathcal{Y}_{p}$, when $x$ is represented as $\left\langle x^{-}, x^{+}\right\rangle$, where $x^{-}$is obtained from $x$ by taking all infinite plays which are draws as winning for R , and $x^{+}$is obtained by taking all infinite plays which are draws on $x$ as winning for L. Notice that each mixed game is uniquely determined by its corresponding pair of fixed games.

The category $\mathcal{Y}_{p}$, being the cartesian product of $\mathcal{Y}_{f}$ with itself, inherits from $\mathcal{Y}_{f}$ all constructions, in particular:

[^2]Theorem 4.15. $\mathcal{Y}_{p}$ is a linear category.
Equivalences on mixed games. The notions of equideterminacy (Definition 4.4) and contextual equivalence (Definition 4.5) can be naturally extended to the case of mixed games, by replacing the notion of "winning strategy" by that of "non-losing strategy", getting relations $\imath_{m}$ and $\approx_{m}$. The Main Lemma 4.6 and Theorem 4.8 also extend to mixed alternating games. However, the Characterization Theorem 4.10 does not hold in its full form. A more precise analysis follows.

The equivalence $\sim_{m}$ on mixed games, induced by $\mathcal{Y}_{p}$, is defined by:
$x \sim_{m} y$ iff $\left(x^{-} \multimap_{f} y^{-}\right) \Downarrow_{L I I}^{f}$ and $\left(x^{+} \multimap_{f} y^{+}\right) \Downarrow_{L I I}^{f}$,
where $\Downarrow_{L I I}^{f}$ denotes the existence of a winning strategy on fixed games. Now, one can prove that
Lemma 4.16. For $x, y$ mixed games,
(i) $\left(\overline{x^{-} \vee_{f} \overline{y^{-}}}\right)_{a} \Downarrow_{L I I}^{f} \Longleftrightarrow\left(\overline{x^{-} \vee_{m} \overline{y^{-}}}\right)_{a} \Downarrow_{L I I}^{m}$ and
(ii) $\left(\overline{x^{+} \vee_{f} \overline{y^{+}}}\right)_{a} \Downarrow_{L I I}^{f} \Longleftrightarrow\left(\overline{x^{+} \vee_{m} \overline{y^{+}}}\right)_{a} \Downarrow_{L I I}^{m}$,
where $\Downarrow_{L I I}^{m}$ denotes the existence of a non-losing strategy on mixed games.

## Proof:

We proceeds by proving (i), the proof of (ii) being similar. Item (i) is equivalent to $\left(x^{-} \vee_{f} \overline{y^{-}}\right)_{a} \Downarrow_{R I I}^{f}$ $\Longleftrightarrow\left(x^{-} \vee_{m} \overline{y^{-}}\right)_{a} \Downarrow_{R I I}^{m}$. This latter result follows from the fact that, for fixed games $x, y$, the game $x \vee_{f} y$ has a winning strategy for R (independently whether she/he plays I or II) if and only if $x \vee_{m} y$ has a non-losing strategy for R (independently whether she/he plays I or II). This follows from the fact that any infinite play on $x \vee_{f} y$ is winning for R if and only if all infinite subplays are winning for R ; but since $x, y$ are fixed, all infinite subplays of the given play must be in the components where infinite plays are fixed to be winning for R . Hence this play is winning for R also on $x \vee_{m} y$.

As a consequence, the following characterization of $\sim_{m}$ directly in terms of operations on mixed games and non-losing strategies holds:

Theorem 4.17. Let $x, y$ be mixed alternating games. Then

$$
x \sim_{m} y \Longleftrightarrow\left(x^{-} \multimap_{m} y^{-}\right) \Downarrow_{L I I}^{m} \&\left(x^{+} \multimap_{m} y^{+}\right) \Downarrow_{L I I}^{m} .
$$

Finally, we have:
Theorem 4.18. $\sim_{m} \subsetneq \imath_{m}$.

## Proof:

$(\subseteq)$ Assume $x \sim_{m} y$ and $x \Downarrow_{L I I}^{m}$. Then $x^{+} \Downarrow_{L I I}^{m}$ and $\overline{x^{+}} \Downarrow_{R I I}^{m}$. If by contradiction $y \psi_{L I I}^{m}$, then also $y^{+} \psi_{L I I}^{m}$, and hence $y^{+} \Downarrow_{R I}^{m}$, but then $\left(\overline{x^{+}} \vee_{m} y^{+}\right)_{a} \psi_{L I I}^{m}$. Contradiction. ( $\left.\subsetneq\right)$ The following counterexample shows that the inclusion is only strict. Let $x=\emptyset, y=\{\langle R,\{\langle L, y\rangle\}\rangle\}$, where the only infinite play on $y$ is a draw. Then $x \mathbb{\Downarrow}_{m} y$, but $\left(\overline{x^{-}} \vee_{m} y^{-}\right)_{a}=y^{-} \psi_{L I I}^{m}$.

Summarizing, the difficulty in providing a categorical characterization of equideterminacy stems from the fact that non-losing strategies are not closed under composition, which, in terms of equivalences, can be expressed by saying that the relation induced on games by the existence of a non-losing strategy is not transitive.

## 5. Final Remarks and Directions for Future Work

We have explored Joyal's categorical paradigm in the case of selective sum. We have shown that this paradigm can be carried out in the setting of alternating games, producing a categorical characterization of the equideterminacy relation. This requires a non trivial categorical construction, where composition is defined via a non-standard parallel application of strategies. We have also considered generalizations to the case of non-wellfounded games, both fixed and mixed. For fixed games, the generalization goes through smoothly, for mixed games a different categorical equivalence emerges.

Here is a list of further comments and directions for future work.

- Alternating games. The problem of finding a categorical characterization of the contextual equivalence remains open on mixed games, even if they are alternating. Notice that also on alternating games with disjunctive sum the equideterminacy relation coincides with the contextual equivalence.
- Beyond alternating games. The issue of extending our categorical constructions for alternating games to the whole class of general games, including impartial ones, remains open. Characterizing the contextual equivalence induced by the selective sum on general games appears rather difficult. Consider for example the games $x=\{\langle L, \emptyset\rangle,\langle R, \emptyset\rangle\}$ and $y=\{\langle L, \emptyset\rangle,\langle R,\{\langle R, \emptyset\rangle\}\rangle\}$. Both are winning for I , but they have rather different behaviors. For instance, if $z=\{\langle L, \emptyset\rangle,\langle R,\{\langle L, \emptyset\rangle\}\rangle\}$, then $x \vee z$ is winning for L (if L plays I then she/he wins by moving on both $x$ and $z$, if L plays II, then he/she wins after any opening R move), while $y \vee z$ is winning for I (if L plays I, she/he wins by moving on both $y$ and $z$, if R plays I , then she/he wins by opening on $y$ ). Notice that $x$ and $y$ are equivalent w.r.t. the contextual equivalence of the disjunctive sum.
If we restrict to impartial games, the contextual equivalence coincides with equideterminacy, in the case of selective sum, as Conway pointed out. However, it is still open how to characterize it categorically, using a monoidal operation.
- Other notions of sum. In [12], Chapter 14 "How to Play Several Games at Once in a Dozen Different Ways", Conway introduces a number of different ways in which (impartial) games can be played. Apart from disjunctive and selective sum, Conway defines the conjunctive sum, where at each step the current player makes a move in each (non-ended) component. A first attempt to extend Joyal's paradigm to conjunctive sum fails, even in the case of alternating games, since trivially copy-cat strategies do not work. Alternative approaches are called for.
- Semantics of concurrency. In the literature, notions of concurrent games [4], asynchronous games [24], and distributed games [28, 11] have been introduced as concurrent extensions of traditional games. Our categories based on selective sum are more in the traditional line, but nonetheless, they reflect a form of parallelism. E.g. in the context of functional languages, our categories based on selective sum accommodate parallel or. It would be interesting to explore to what extent they can be used for modeling concurrent and distributed languages, possibly featuring true concurrency. This would require an extension of our approach in order to account also for interference between moves/events.
- Mixed games. Typically, in categories for Game Semantics, fixed games are considered. In this paper, we have shown how to build a well-behaved category including the more general mixed
games. In principle, by accommodating also draws, mixed games appear to model the interactions between non-terminating processes in a more general and natural way. Applications in the context of semantics are called for.
- Significance of game congruences. The notions of congruences that we have studied originate in the setting of Conway games, and they have not received direct attention in Game Semantics so far. Clearly game congruences induce game invariants, such as inhabitability. A more detail investigation is called for.


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[^0]:    ${ }^{1}$ We recall that the hereditary cardinal of a set is the cardinality of its transitive closure, namely the cardinality of the downward membership tree which has the given set as its root.
    ${ }^{2}$ The final coalgebra of the powerset functor exists since the powerset functor is bounded by $\kappa$.
    ${ }^{3}$ Here we consider a notion of coalgebraic game tuned for Conway games, where moves only carry the information about the player who performs the move. By enriching the set $\mathcal{A}$, one can capture other kinds of games. E.g., by considering also move names, one can recover the games used in Game Semantics à la [3], see [16] for more details.

[^1]:    ${ }^{4}$ I.e. there are winning strategies for both LI and LII.
    ${ }^{5}$ I.e. there are winning strategies for both LI and RI.

[^2]:    ${ }^{6}$ Following [26], one could also prove that $\bigvee^{\infty} x$ associates a free commutative comonoid to each $x$, thus providing a Lafont category.

