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Polarized Multigames†

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In traditional game semantics, games are *sequential*, *i.e.*, at each step, either Player or Opponent moves (*global polarization*), moreover, only a *single* move can be performed at each step. More recently, *concurrent games* have been introduced, where global polarization is abandoned, and multiple moves are allowed. In this paper, we introduce *polarized multigames*, which are situated half-way between traditional sequential game semantics and concurrent game semantics: global polarization is still present, however *multiple* moves are possible at each step, *i.e.* a team of Players/Opponents moves in parallel. Usual game constructions can be naturally extended to multigames, which can be endowed with a structure of a monoidal closed category together with an exponential comonad. Multigames are useful to model languages with parallel features, *e.g.* they provide an *universal* model of unary PCF with *parallel or*. Interestingly, the category of polarized multigames turns out to be equivalent to a category of AJM-games with a new notion of tensor product, where at each step the current player performs a move in *at least* one component of the tensor game. This notion of *parallel tensor product* is inspired by Conway's *selective sum*.

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Introduction

In traditional game semantics (see *e.g.* (Abramsky *et al.* 2000; Hyland and Ong 2000)), games are *sequential*, *i.e.*, at each step, either Player or Opponent moves (*global polarization*), moreover, only a *single* move can be performed at each step. More recently, *concurrent games* (see *e.g.* (Abramsky and Melliès 1999; Ghica and Menaa 2010; Rideau and Winskel 2011; Clairambault *et al.* 2012)) have been introduced, where global polarization is abandoned, and multiple moves are allowed. In this paper, we introduce *polarized multigames*, which are situated half-way between traditional sequential game semantics and concurrent game semantics: global polarization is still present, however *multiple* moves are possible at each step, *i.e.* a team of Players/Opponents moves in parallel. More precisely, we define multigames as games where plays are sequences of *multimoves*, *i.e.* finite sets of (atomic) moves with the same polarity. The notion of strategy as well as the usual game constructions, such as tensor product, linear implication, and exponential, can be naturally extended to multigames. Moreover, we show that multigames and strategies can be endowed with a structure of a monoidal closed category together with an exponential comonad. The main difference between the category of multigames and categories of traditional games such as those in (Abramsky *et al.* 2000; Hyland and Ong 2000) lies in the fact that, in the *tensor product* of traditional games, at each step, the current player can move in exactly one component, while in the *multigame tensor*, by exploiting the parallel nature of multigames, in general the current player can perform a multimove consisting of atomic moves on *both* components. Similarly, in multigame strategies, at each step, the current player can possibly perform a multimove consisting of atomic moves on both components. As a consequence, the notion of strategy composition on multigames is not a straightforward adaptation of usual composition, but it requires a non-standard *parallel application* of strategies. Interestingly, the category of polarized multigames turns out to be *equivalent* to a category of games à la (Abramsky *et al.* 2000) (*AJM-games*) with a new notion of *parallel tensor product*, where at each step the current player performs a move in *at least* one component of the tensor game.

While traditional tensor on games is related to Conway's *disjunctive sum*, (Conway 2001), *parallel tensor product* is inspired by Conway's *selective sum*. Categories of coalgebraic games, *i.e.* possibly non-terminating games generalizing Conway's games and other notions of games, have been recently studied in (Honsell *et al.* 2012; Honsell *et al.* 2013), where disjunctive or selective sum is taken as tensor product. In particular, in (Honsell *et al.* 2012), categories of coalgebraic games with disjunctive sum are related to categories of traditional Game Semantics. On the other hand, the categories of AJM-games with parallel tensor product presented in this paper are related to the categories of coalgebraic games with selective sum studied in (Honsell *et al.* 2013).

Finally, a few words on the relationships between multigames and concurrent games, and on the interest of multigames.

Multigames, as they are defined in this paper, can be viewed as a special case of concurrent games of (Abramsky and Melliès 1999). Hence, one may ask what is their interest or to what extent they offer a model of parallelism. A first answer provided in the present paper is that they offer a sufficient level of parallelism for modeling (in a universal way) unary PCF with *parallel or*. We think that multigames can be also useful to model more complex languages with parallel features, so as to understand concurrency in proof theory (see *e.g.* (Abramsky 2003)). Further experiments are left for future work.

In conclusion, we feel that our investigation is worthwhile under various perspectives: multigames help in clarifying/factorizing the steps taking from sequential games (with global polarization and single moves) to concurrent games (no global polarization, multiple moves), offering a model of parallelism with a low level of complexity but still of a set-theoretic nature, compared to more complex concurrent games.

Synopsis. In Section 1, the notion of multigame is introduced together with strategies and multigame constructions. In Section 2, categories of multigames and strategies are defined and studied. In Section 3, categories of AJM-games with a new parallel tensor product are introduced and shown to be equivalent to multigame categories. In Section 4, an universal game model for unary PCF with parallel or is built in a category of AJM-games with parallel tensor. Conclusions and directions for future work appear in Section 5.

1. Multigames

In this section, we introduce the notion of *multigame*, namely a polarized game, where, at each step, the current player can perform a *multimove*, *i.e.* a non-empty *finite set* of (atomic) moves. Technically, a multigame is a game in the (Abramsky *et al.* 2000)-style, where plays are sequences of (finite) *sets* of (atomic) moves. The notion of strategy on games admits a natural extension on multigames. Similarly, the usual game constructors, such as tensor product, linear implication, and exponential, admit a natural definition on multigames, and, as we will see, they amount to the parallel counterparts of the traditional game constructors.

Definition 1.1 (Multigames). A *multigame* has two participants, *Player (P)* and *Opponent (O)*. A multigame A is a quadruple $(M_A, \lambda_A, P_A, \approx_A)$, where

— M_A is the set of *atomic moves* of the game.

— $\lambda_A : M_A \to \{O, P\}$ is the *labeling* function: it tells us if an atomic move is taken by the Opponent or by the Player. We denote by $^-$ the function which exchanges Player and Opponent, i.e. $\overline{O} = P$ and $\overline{P} = O$. We denote by $\overline{\lambda_A}$ the function which exchanges the polarity of moves, $\overline{\lambda_A}(a) = \overline{\lambda_A(a)}$.

— The set of atomic moves M_A determines the set $\mathcal{M}_A$ of *multimoves* of the game, which are non-empty finite sets of atomic moves with the same polarity. We denote by λ_A the obvious extension of λ_A to multimoves.

— P_A is the set of *plays*, *i.e.* a non-empty and prefix-closed subset of the set $\mathcal{M}_A^{\circledast}$, which satisfies the following conditions:
  - $s = \alpha t \ \Rightarrow\ \forall a \in \alpha.\ \lambda_A(a) = O$ (O starts)
  - $(\forall i : 1 \leq i \leq |s|)[\lambda_A(s_{i+1}) = \overline{\lambda_A(s_i)}]$ (alternating)

  where s_i denotes the i-th component of the sequence s.

— $\approx_A$ is an equivalence relation on P_A which satisfies the following properties:
  - $s \approx_A s' \Rightarrow |s| = |s'|$

- $s\alpha \approx_A s'\alpha' \Rightarrow s \approx_A s'$
- $s \approx_A s'$ & $s\alpha \in P_A \Rightarrow (\exists \alpha')[s\alpha \approx_A s'\alpha']$

The set of even-length (odd-length) plays will be denoted by P_A^{even} (P_A^{odd}). The empty sequence will be denoted by ϵ.

In what follows, we will often refer to multimoves simply as moves.

The difference between the above notion of multigame and the standard notion of AJM-game lies in the definition of plays, which are sequences of atomic moves on AJM-games, while on multigames they are sequences of multimoves.

Notice that every multigame can be viewed as a game whose moves are the multimoves, *i.e.* any multigame $A = (M_A, \lambda_A, P_A, \approx_A)$ induces a game $A_g = (\mathcal{M}_A, \lambda_A, P_A, \approx_A)$. However, as we will see, multigame tensor (as well as other game constructions) is *not* preserved under this mapping.

The notion of strategy naturally extends to multigames:

Definition 1.2 (Strategies). A *strategy* for the Player on a multigame A is a non-empty set $\sigma \subseteq P_A^{even}$ of plays of even length such that

— $\overline{\sigma} = \sigma \cup dom(\sigma)$ is prefix-closed, where $dom(\sigma) = \{t \in P_A^{odd} \mid (\exists \alpha)[t\alpha \in \sigma]\}$,
— $s\alpha, s\beta \in \sigma \;\Rightarrow\; \alpha = \beta$ (determinism).

A strategy σ for a multigame A is *history-free* if it satisfies the following property: $s\alpha\beta, t \in \sigma, t\alpha \in P_A \Rightarrow t\alpha\beta \in \sigma$.

The equivalence relation on plays $\approx_A$ can be naturally extended to strategies:

Definition 1.3. Let σ, τ be strategies, $\sigma \approx \tau$ if and only if

— $s\alpha\beta \in \sigma, s'\alpha'\beta' \in \tau, s\alpha \approx_A s'\alpha' \Rightarrow s\alpha\beta \approx_A s'\alpha'\beta'$
— $s \in \sigma, s' \in \tau, s\alpha \approx_A s'\alpha' \Rightarrow (\exists \beta)[s\alpha\beta \in \sigma]$ iff $(\exists \beta')[s'\alpha'\beta' \in \tau]$

Multigame Constructions. The usual definitions of game constructions yield on multigames the parallel counterparts of the standard ones.

Definition 1.4 (Tensor Product). Given multigames A and B, the tensor product $A \otimes B$ is the multigame defined as follows:

— $M_{A\otimes B} = M_A + M_B$
— $\lambda_{A\otimes B} = [\lambda_A, \lambda_B]$
— $P_{A\otimes B} \subseteq \mathcal{M}_{A\otimes B}^{\circledast}$ is the set of plays, s, which satisfy the following condition: the projections on each component (written as $s \upharpoonright A$ or $s \upharpoonright B$) are plays for the games A and B respectively[†]
— $s \approx_{A\otimes B} s' \iff s \upharpoonright A \approx_A s' \upharpoonright A, s \upharpoonright B \approx_B s' \upharpoonright B, (\forall i)[(s_i \in \mathcal{M}_A \Leftrightarrow s'_i \in \mathcal{M}_A) \wedge (s_i \in \mathcal{M}_B \Leftrightarrow s'_i \in \mathcal{M}_B)]$

[†] Formally $s \upharpoonright A$ is defined by: $\begin{cases} \epsilon \upharpoonright A = \epsilon \\ \alpha s' \upharpoonright A = \begin{cases} \{a \in \alpha \mid a \in M_A\}(s' \upharpoonright A) & \text{if } \{a \in \alpha \mid a \in M_A\} \neq \emptyset \\ s' \upharpoonright A \text{ otherwise} \end{cases} \end{cases}$.

Notice that, on the tensor product of multigames, $A \otimes B$, a multimove can contain atomic moves of both A and B. As a consequence, $(A \otimes B)_g$ is *not* isomorphic to $A_g \otimes B_g$. However, as we will see in Section 3, this isomorphism holds when a new notion of parallel tensor on standard AJM-games is considered.

Definition 1.5 (Linear Implication). Given multigames A and B, the linear implication multigame $A \multimap B$ is defined as follows:

— $M_{A \multimap B} = M_A + M_B$
— $\lambda_{A \multimap B} = [\overline{\lambda_A}, \lambda_B]$
— $P_{A \multimap B} \subseteq \mathcal{M}^{\circledast}_{A \multimap B}$ is the set of plays, s, which satisfy the following condition: $s \upharpoonright A$ and $s \upharpoonright B$ are plays for the multigames A and B respectively.
— $s \approx_{A \multimap B} s' \iff s \upharpoonright A \approx_A s' \upharpoonright A, s \upharpoonright B \approx_B s' \upharpoonright B, (\forall i)[(s_i \in \mathcal{M}_A \Leftrightarrow s'_i \in \mathcal{M}_A) \wedge (s_i \in \mathcal{M}_B \Leftrightarrow s'_i \in \mathcal{M}_B)]$

As on the tensor multigame, also on linear implication multimoves can include moves on both components. Similarly, on the exponential multigame, moves on finitely many components are possible at each step.

Definition 1.6 (Exponential). Given a multigame A, the multigame $!A$ is defined by:

— $M_{!A} = \omega \times M_A = \sum_{i \in \omega} M_A$
— $\lambda_{!A}(\langle i, \alpha \rangle) = \lambda_A(\alpha)$
— $P_{!A} \subseteq \mathcal{M}^{\circledast}_{!A}$ is the set of plays, s, which satisfy the following condition: $(\forall i \in \omega)[s \upharpoonright A_i \in P_{A_i}]$
— $s \approx_{!A} s' \iff \exists$ a permutation of indexes $\phi \in S(\omega)$ such that:
  - $\pi_1^*(s) = \phi^*(\pi_1^*(s'))$
  - $(\forall i \in \omega)[\pi_2^*(s \upharpoonright \phi(i)) \approx \pi_2^*(s' \upharpoonright i)]$

  where π_1 and π_2 are the projections of $\omega \times \mathcal{M}_A$ and $s \upharpoonright i$ is an abbreviation of $s \upharpoonright A_i$.

2. Categories of Multigames and Strategies

We start by defining a monoidal closed category $\mathcal{M}$, whose *objects* are multigames and whose *morphisms* are strategies. The main difficulty in defining this category is the definition of composition, which is based on a non-standard *parallel composition of strategies*. The difficulty arises from the fact that a multimove in a strategy between A and B can include atomic moves on both A and B.

A monoidal closed subcategory, $\mathcal{M}_{hf}$, can then be defined by restricting to history-free strategies. Finally, one can show that the exponential construction induces a comonad on both categories, which can be used to build cartesian closed categories via the usual co-Kleisli construction.

2.1. *Monoidal Closed Categories of Multigames*

Let $\mathcal{M}$ be the category defined by:
Objects: multigames.
Morphisms: a morphism between games A and B is an equivalence class of strategies

$\sigma : A \multimap B$ w.r.t. the relation $\approx_{A \multimap B}$. We denote the equivalence class of σ by $[\sigma]$. Notice that in a strategy on $A \multimap B$, the Opponent can only open in B, but then the Player can move in A or B or in both components, and so on.

Identity: the identity $id_A : A \multimap A$ is (the equivalence class of) the *copy-cat strategy*, defined by: $id_A = \{s \in P^{even}_{A \multimap A} \mid s \restriction 1 = s \restriction 2\}$. This definition works thanks to the fact that games are polarized. On the game $A \multimap A$, Opponent can only open on the righthand A component, then Player proceeds by copying the moves on the lefthand A component and so on, thus at each step Opponent has exactly one component to move in.

Composition: the composition is given by the extension on equivalence classes of the following composition of strategies.

Given strategies $\sigma : A \multimap B$ and $\tau : B \multimap C$, $\tau \circ \sigma : A \multimap C$ is obtained via the *swivel-chair strategy* and a non-standard *parallel application* of strategies as follows.

The opening (multi)move by O on $A \multimap C$ must be on C, since games are polarized. Then consider the P reply given by the strategy τ on $B \multimap C$, if it exists, otherwise the whole composition is undefined. If P moves in C, then we take this as the P (multi)move in the strategy $\tau \circ \sigma$. If the P move according to τ is in the B component of $B \multimap C$ or in both components B and C, then we use the swivel chair to view the P move in the B component as an O move in the B component of $A \multimap B$. Now, if P has a reply in $A \multimap B$ according to σ, then P moves in A or in B or in both A and B. In the first case, the P move in A together with the possible previous move by P in C form the P reply to the opening O move. In the latter two cases, using the swivel chair, the move in B can be viewed as an O move in the B component of $B \multimap C$, and we go on in this way: three cases can arise. Eventually, the P multimove is all in A or in C, or σ or τ is undefined, or the dialogue between the B components does not stop. In the first case, the last move on A or C, together with a possible previous move on C or A, form the answer to the opening O move, in the latter two cases the composition is undefined.

Now, in case of convergence, in order to understand how the strategy $\tau \circ \sigma$ behaves after the first pair of OP moves, it is convenient to list all situations which can arise after these initial moves, according to which player is next to move in each component. Namely, by case analysis, one can show that, after the first OP moves, the following four cases can arise:

1 $A^P \multimap B^O \quad B^P \multimap C^O$
2 $A^O \multimap B^P \quad B^O \multimap C^P$
3 $A^O \multimap B^P \quad B^O \multimap C^O$
4 $A^O \multimap B^O \quad B^P \multimap C^O$

where A^P (A^O) denotes that P (O) is next to move in that component. Notice that case 1 above corresponds to the initial situation. Thus we are left to discuss the behavior of $\tau \circ \sigma$ in the other cases. In case 2, O can only open in A; this case can be dealt with similarly as the initial case 1, and after a pair of OP moves, it takes again in one of the four situations above. Cases 3 and 4 are the interesting ones, where we need to apply the strategies σ and τ in *parallel*, by exploiting the parallelism of multigames. These two cases are dealt with similarly. Let us consider case 3. Then O can open in C or in A or in both components.

— If O opens in C, then the reply of P via τ must be in C, since, by definition of configuration 3, P cannot play in the B component of $B \multimap C$. This will be also the reply of P in the composition $\tau \circ \sigma$, and the final configuration coincides with configuration 3.

— If O opens in A, then the P reply given by σ can be in A, or in B, or both in A and B. In the first case, *i.e.* if the P reply is in A, this will be the reply of $\tau \circ \sigma$, and the new configuration coincides with configuration 3. In the latter two cases, the P move in the B component of $A \multimap B$ can be viewed, via the swivel chair, as an O move in $B \multimap C$. By definition of configuration 3, P can only reply in B via τ, and, either after finitely many applications of the swivel chair the P reply via σ ends up in A, or the dialogue between the B components goes on indefinitely, or σ or τ are undefined. In the latter two cases, the overall composition is undefined, while in the first case the P move in A will be the reply in $\tau \circ \sigma$ to the O move, and the final configuration still coincides with configuration 3.

— Finally, if O opens in A and C, we apply the two strategies σ and τ in parallel: by the form of configuration 3, the P answer of τ must be in C, while the P answer via σ can be either in A or in B or both in A and B. In this latter case, an infinite dialogue between the B components arises (or σ or τ is undefined at some point), and hence the overall composition is undefined.
If the P reply via σ is in A, then this, together with the P reply via τ in C, will form the P multimove in $\tau \circ \sigma$, and the final configuration coincides with configuration 3 itself. If the P reply via σ is in B, then again, either the dialogue between the B components goes on indefinitely, or σ or τ is undefined at some point, or, after finitely many applications of the swivel chair, σ will finally provide a P move in A. This, together with the P reply via τ in C, will form the P move in $\tau \circ \sigma$, and the final configuration coincides with configuration 3 again.

Similarly, one can deal with case 4. This proves closure under composition of the category $\mathcal{M}$.

Formally, strategy composition can be defined as follows.

First, we capture the parallel interaction between σ and τ, by defining:

$$\sigma || \tau = \{ s \in (\mathcal{M}_{A \multimap B} + \mathcal{M}_{B \multimap C})^* \mid s \upharpoonright (A, B) \in \overline{\sigma} \ \& \ s \upharpoonright (B, C) \in \overline{\tau} \ \& \ s \downarrow \} \ ,$$

where the predicate $s \downarrow$ expresses the fact that it is not the case that s goes on with an infinite sequence of moves on B, or σ or τ become undefined, before providing the complete reply of P to an O move. *I.e.*, $s \downarrow$ is defined as $\neg(s \uparrow)$, where $s \uparrow$ iff $\exists t \neq \epsilon.\ st \in \sigma || \tau \ \wedge \ \forall t \neq \epsilon.\ (st \in \sigma || \tau \ \Rightarrow \ t \upharpoonright B = t)$.

Now, we consider the set of sequences on $\mathcal{M}_A + \mathcal{M}_C$ obtained by restricting the sequences in $\sigma || \tau$ to the moves in $\mathcal{M}_A + \mathcal{M}_C$, and we denote this set by $(\sigma || \tau) \upharpoonright (A, C)$. In order to get the plays in the composition, we still need to merge pairs of subsequent P multimoves on A and C or on C and A, respectively, and consider only those sequences which give the complete P reply. Namely, for any $s \in (\sigma || \tau) \upharpoonright (A, C)$, we define a partial function $\widehat{s}$ by induction on s by:

$$\widehat{\alpha} = \alpha \qquad \widehat{s'\alpha_1\alpha_2} = \begin{cases} \widehat{s'}\alpha_1 \cup \alpha_2 & \text{if } \lambda(\alpha_1) = P \text{ and } \lambda(\alpha_2) = P \\ \widehat{s'\alpha_1}\alpha_2 & \text{if } \lambda(\alpha_1) = P \text{ and } \lambda(\alpha_2) = O \\ \widehat{s'}\alpha_1\alpha_2 & \text{if } \lambda(\alpha_1) = O \text{ and } \lambda(\alpha_2) = P \text{ and} \\ & \forall\alpha.\ \lambda(\alpha) = P \Rightarrow s'\alpha_1\alpha_2\alpha \notin (\sigma||\tau) \upharpoonright (A, C) \\ \text{undefined} & \text{otherwise .} \end{cases}$$

Let $\widehat{\sigma||\tau} = \{\widehat{s} \mid s \in (\sigma||\tau) \upharpoonright (A, C)\}$. Finally, we have: $\tau \circ \sigma = \widehat{\sigma||\tau}^{\,even}$.

Associativity of composition can also be proven by case analysis on the polarity of the current player in the various components.

Assume strategies $\sigma : A \multimap B$, $\tau : B \multimap C$, $\theta : C \multimap D$. We have to prove that $\theta \circ (\tau \circ \sigma) = (\theta \circ \tau) \circ \sigma$. Since games are alternating, in any of the two compositions, O can only open in D. Now, in any of the two compositions, one should consider the possible replies by P. We only discuss one case, the remaining being dealt with similarly. Assume the P reply via θ is in C. In both compositions, we proceed to apply the swivel chair, by viewing this latter move as an O move in the C component of $B \multimap C$. Then, in both compositions, we consider the P reply via τ. Assume P replies both in B and in C. At this point, the two compositions proceed differently, since in $\theta \circ (\tau \circ \sigma)$ we first apply the swivel chair to the move in C and we go on until we get a P answer in D, and then we apply the swivel chair to the P move in B. In $(\theta \circ \tau) \circ \sigma$, these two steps are reversed, first we apply the swivel chair to the P move in B until we get a P reply in A, then we apply the swivel chair to the P move in C until we get a P reply in D. The point is that these two steps, working on separate parts of the board (*i.e.* different components), are independent and can be exchanged. As a consequence, the behavior of $\theta \circ (\tau \circ \sigma)$ and $(\theta \circ \tau) \circ \sigma$ is the same.

The multigame constructions of tensor product and linear implication can be made functorial, determining a structure of a symmetric monoidal closed category on $\mathcal{M}$, with the empty multigame $I = (\emptyset, \emptyset, \{\epsilon\}, \{(\epsilon, \epsilon)\})$ as tensor unit.

Theorem 2.1. The category $\mathcal{M}$ is symmetric monoidal closed.

Strategy composition is closed under history-free strategies, and hence we can consider the corresponding subcategory $\mathcal{M}_{hf}$, getting:

Theorem 2.2. The category $\mathcal{M}_{hf}$ is symmetric monoidal closed.

2.2. *Cartesian Closed Categories of Multigames*

The standard construction of a cartesian closed category from the exponential easily extends to multigames. Namely, the exponential game construction of Definition 1.6 can be made functorial, by defining, for any strategy $\sigma : A \multimap B$, the strategy $!\sigma : !A \multimap !B$ by $!\sigma = \{s \in P_{!A \multimap !B} \mid \forall i\ \exists s' \in \sigma.\ (\forall s_1, s'_1$ prefixes of s, s' of the same even length.
$(s_1 \upharpoonright (A)_i = s'_1 \upharpoonright A \ \&\ s_1 \upharpoonright (B)_i = s'_1 \upharpoonright B))\}$.

Moreover, the exponential can be endowed with a comonad structure $(!, der,)$, where for each game A the morphisms $der_A : !A \multimap A$ and $_A : !A \multimap !!A$ are defined as follows:

- $der_A = [\{s \in P^{even}_{!A \multimap A} \mid \forall s'$ even length prefix of s. $(s' \upharpoonright (!A)_0 = s' \upharpoonright A \ \& \ \forall i \neq 0.\ s' \upharpoonright (!A)_i = \epsilon)\}]$
- $_A = [\{s \in P^{even}_{!A \multimap\ !!A} \mid \forall s'$ even length prefix of s. $(\forall i, j.\ s' \upharpoonright (!A)_{c(i,j)} = s' \upharpoonright (!(!A)_i)_j \ \& \ \forall k \notin codom(c).\ s' \upharpoonright (!A)_k = \epsilon)\}]$, where c is a *pairing function*, i.e. an injective map $c : \omega \times \omega \to \omega$.

Let $K_!(\mathcal{M})$ $(K_!(\mathcal{M}_{hf}))$ be the co-Kleisli category over the comonad $(!, der,)$, i.e.:

Objects of $K_!(\mathcal{M})$ $(K_!(\mathcal{M}_{hf}))$: multigames.

Morphisms of $K_!(\mathcal{M})$ $(K_!(\mathcal{M}_{hf}))$: a morphism between games A and B is an equivalence class of (history-free) strategies for the game $!A \multimap B$.

Composition on $K_!(\mathcal{M})$ $(K_!(\mathcal{M}_{hf}))$: given strategies $\sigma : !A \multimap B$ and $\tau : !B \multimap C$, the strategy $\tau \circ \sigma : A \to C$ is given by the composition in the category $\mathcal{M}$ $(\mathcal{M}_{hf})$ of the strategies $\sigma^\dagger : !A \multimap !B$ and $\tau : !B \multimap C$, where $\sigma^\dagger$ is defined by $(!\sigma)\circ_A$.

The following strategies give a commutative comonoid structure on $!A$:

- the empty strategy $weak_A : !A \multimap I$ (*weakening*),
- the *contraction* strategy $con_A : !A \multimap !A \otimes !A$,
  $con_A = [\{s \in P^{even}_{!A \multimap\ !A \otimes !A} \mid \forall s'$ even length prefix of s. $\forall i\ (s' \upharpoonright (!A)_{d(l,i)} = s' \upharpoonright ((!A)_l)_i \ \& \ s' \upharpoonright (!A)_{d(r,i)} = s' \upharpoonright ((!A)_r)_i) \ \& \ \forall j \notin codom(d).\ (s' \upharpoonright (A)_j = \epsilon)\}]$, where d is a *tagging function*, i.e. an injective map $d : \omega + \omega \to \omega$.

Identity on $K_!(\mathcal{M})$ $((K_!(\mathcal{M}_{hf})))$: the identity $id_A : !A \multimap A$ is der_A.

Using the above structure, one can define a cartesian product on $K_!(\mathcal{M})$ $((K_!(\mathcal{M}_{hf})))$ in a standard way, getting:

Theorem 2.3. The category $K_!(\mathcal{M})$ $(K_!(\mathcal{M}_{hf}))$ is cartesian closed.

3. Categories of Games with Parallel Tensor Product

In this section, we build a new category, where objects are AJM-games, but the usual tensor product is replaced by a new notion of parallel tensor.

In the standard tensor product of games, see *e.g.* (Abramsky *et al.* 2000), on the game $A \otimes B$, at each step, the player who has the turn can move exactly in one of the two components, A or B. In (Honsell *et al.* 2013), an alternative notion of tensor product, *i.e.* $A \vee B$, has been considered, where at each step the player who has the turn can either move in A, or in B, or in *both* components. In (Honsell *et al.* 2013), it has been shown that the game $A \vee B$, together with a non-standard definition of strategy composition, gives rise to a tensor product in a category of *coalgebraic games* and (total) strategies. This category turns out to be symmetric monoidal closed together with a monoidal comonad. Here we show that an analogous construction can be carried out on (Abramsky *et al.* 2000)-games.

Interestingly, the resulting category turns out to be *equivalent* to the category of multigames introduced in Section 2.

Let us consider AJM-games and (history-free) strategies defined in the usual way. On these games, we introduce the following *parallel tensor* product:

Definition 3.1 (Parallel Tensor Product). Let $A = (M_A, \lambda_A, P_A, \approx_A)$ and $B = (M_B, \lambda_B, P_B, \approx_B)$ be AJM-games, the tensor product $A \triangledown B$ is the game defined as follows:

— $M_{A\triangledown B} = M_A + M_B + (M_A^O \times M_B^O) + (M_A^P \times M_B^P)$

— $\lambda_{A\triangledown B} = [\lambda_A, \lambda_B, a \mapsto O, a \mapsto P]$, where $a \mapsto O$ ($a \mapsto P$) denotes the O-constant (P-constant) function.

— $P_{A\triangledown B} \subseteq M^{\circledast}_{A\triangledown B}$ is the set of plays, s, which satisfy the following condition: $s \upharpoonright A$ and $s \upharpoonright B$ are plays for the games A and B respectively.

— $s \approx_{A\triangledown B} s' \iff s \upharpoonright A \approx_A s' \upharpoonright A, s \upharpoonright B \approx_B s' \upharpoonright B, (\forall i)[(s_i \in M_A \Leftrightarrow s'_i \in M_A) \wedge (s_i \in M_B \Leftrightarrow s'_i \in M_B)]$

As for parallel tensor, also on parallel linear implication moving on both components at the same time is allowed.

Definition 3.2 (Parallel Linear Implication). Let $A = (M_A, \lambda_A, P_A, \approx_A)$ and $B = (M_B, \lambda_B, P_B, \approx_B)$ be AJM-games, the parallel linear implication game $A \multimap_\triangledown B$ is defined as follows:

— $M_{A\multimap_\triangledown B} = M_A + M_B + (M_A^O \times M_B^P) + (M_A^P \times M_B^O)$

— $\lambda_{A\multimap_\triangledown B} = [\overline{\lambda_A}, \lambda_B, \overline{\lambda_A} \times \lambda_B, \overline{\lambda_A} \times \lambda_B]$

— $P_{A\multimap_\triangledown B} \subseteq M^{\circledast}_{A\multimap_\triangledown B}$ is the set of plays, s, which satisfy the following condition: $s \upharpoonright A$ and $s \upharpoonright B$ are plays for the games A and B respectively.

— $s \approx_{A\multimap_\triangledown B} s' \iff s \upharpoonright A \approx_A s' \upharpoonright A, s \upharpoonright B \approx_B s' \upharpoonright B, (\forall i)[(s_i \in M_A \Leftrightarrow s'_i \in M_A) \wedge (s_i \in M_B \Leftrightarrow s'_i \in M_B)]$

Exponential amounts to infinite tensor product. On the parallel exponential game, at each step, the current player can move in finitely many components.

Definition 3.3 (Parallel Exponential). Given a game $A = (M_A, \lambda_A, P_A, \approx_A)$, the game $!_\triangledown A$ is defined by:

— $M_{!_\triangledown A} = \Sigma_{j\geq 1}\Sigma_{i_1,\ldots,i_j\in\omega}(\{i_1\} \times M_A^O) \times \ldots \times (\{i_j\} \times M_A^O) \ + \ \Sigma_{j\geq 1}\Sigma_{i_1,\ldots,i_j\in\omega}(\{i_1\} \times M_A^P) \times \ldots \times (\{i_j\} \times M_A^P)$

— $\lambda_{!_\triangledown}(\langle\langle i_1, a_1\rangle \ldots \langle i_j, a_j\rangle\rangle) = \lambda_A(a_1)$

— $P_{!_\triangledown A} \subseteq M^{\circledast}_{!_\triangledown A}$ is the set of plays, s, which satisfy the following condition: $(\forall i \in \omega)[s \upharpoonright A_i \in P_{A_i}]$.

— $s \approx_{!_\triangledown A} s' \iff \exists$ a permutation of indexes $\phi \in S(\omega)$ such that:
- $\pi_1^*(s) = \phi^*(\pi_1^*(s'))$
- $(\forall i \in \omega)[\pi_2^*(s \upharpoonright \phi(i)) \approx \pi_2^*(s' \upharpoonright i)]$.

3.1. *Monoidal Categories of AJM-games with Parallel Tensor*

The notions of parallel tensor and linear implication give rise to monoidal closed categories $\mathcal{G}^\triangledown$ and $\mathcal{G}^\triangledown_{hf}$ of AJM-games defined by:
Objects: AJM-games.
Morphisms: a morphism between games A and B is an equivalence class of (history-free) strategies $\sigma : A \multimap_\triangledown B$ w.r.t. the relation $\approx_{A\multimap_\triangledown B}$.
Identity: the identity $id_A : A \multimap_\triangledown A$ is (the equivalence class of) the *copy-cat strategy*.

Composition: the composition is given by the extension on equivalence classes of the composition of strategies. As for multigames, the difficulty in defining composition lies in the fact that the current player can play in both components of the linear game. Let $\sigma : A \multimap_{\triangledown} B$ and $\tau : B \multimap_{\triangledown} C$, $\tau \circ \sigma : A \multimap C$ is defined similarly as for multigames.

First, we capture the parallel interaction between σ and τ, by defining:

$$\sigma || \tau = \{ s \in (M_{A \multimap_{\triangledown} B} + M_{B \multimap_{\triangledown} C})^* \mid s \upharpoonright (A, B) \in \overline{\sigma} \ \& \ s \upharpoonright (B, C) \in \overline{\tau} \ \& \ s \downarrow \} \ ,$$

where the predicate $s \downarrow$ expresses the fact that it is not the case that s goes on with an infinite sequence of moves on B, or σ or τ become undefined, before providing the complete reply of P to an O move. *I.e.*, $s \downarrow$ is defined as $\neg(s \uparrow)$, where

$$s \uparrow \quad \text{iff } \exists t \neq \epsilon.\ st \in \sigma||\tau \ \wedge \ \forall t \neq \epsilon.\ (st \in \sigma||\tau \ \Rightarrow \ t \upharpoonright B = t) \ .$$

Now, we consider the set of sequences on $M_A + M_C$ obtained by restricting the sequences in $\sigma||\tau$ to the moves of $M_A + M_C$, and we denote this set by $(\sigma||\tau) \upharpoonright (A, C)$. In order to get the sequences of moves of the composition, we still need to merge pairs of subsequent P moves on A and C or on C and A, respectively, and consider only those sequences which give the complete P reply. Namely, for any $s \in (\sigma||\tau) \upharpoonright (A, C)$, we define a partial function $\widehat{s}$ by induction on s by:

$$\widehat{a} = a \qquad \widehat{s' a_1 a_2} = \begin{cases} \widehat{s'} \langle a_1, a_2 \rangle & \text{if } \lambda(a_1) = P \text{ and } \lambda(a_2) = P \\ \widehat{s' a_1} a_2 & \text{if } \lambda(a_1) = P \text{ and } \lambda(a_2) = O \\ \widehat{s'} a_1 a_2 & \text{if } \lambda(a_1) = O \text{ and } \lambda(a_2) = P \text{ and} \\ & \forall a. (\lambda(\alpha) = P \Rightarrow s' a_1 a_2 a \notin (\sigma||\tau) \upharpoonright (A, C) \\ \text{undefined} & \text{otherwise} \ . \end{cases}$$

Let $\widehat{\sigma||\tau} = \{ \widehat{s} \mid s \in (\sigma||\tau) \upharpoonright (A, C) \}$. Finally, we have:

$$\tau \circ \sigma = \widehat{\sigma||\tau}^{\,even} \ .$$

The fact that composition is well-defined is proved in a similarly way as for multigames.

The co-kleisli construction can be straightforwardly carried out on the categories $\mathcal{G}^{\triangledown}$ and $\mathcal{G}^{\triangledown}_{hf}$ getting CCC's $K_{!_{\triangledown}}(\mathcal{G}^{\triangledown})$ and $K_{!_{\triangledown}}(\mathcal{G}^{\triangledown}_{hf})$.

3.2. *Relating Games to Multigames*

The category $\mathcal{G}^{\triangledown}$ ($\mathcal{G}^{\triangledown}_{hf}$) of games with parallel tensor product turns out to be *equivalent* to the category $\mathcal{M}$ ($\mathcal{M}_{hf}$) of multigames, *i.e.* there exist functors $F : \mathcal{G}^{\triangledown} \to \mathcal{M}$ and $G : \mathcal{M} \to \mathcal{G}^{\triangledown}$, and natural isomorphisms $\eta : G \circ F \stackrel{\cdot}{\to} Id_{\mathcal{G}^{\triangledown}}$ and $\eta' : Id_{\mathcal{M}} \stackrel{\cdot}{\to} F \circ G$.

Namely, given a multigame $A = (M_A, \lambda_A, P_A, \approx_A)$, in Section 1, we have seen how this induces an AJM-game $A_g = (\mathcal{M}_A, \lambda_A, P_A, \approx_A)$, where the moves are the set of multimoves on the multigame A. Vice versa, given an AJM-game, $A = (M_A, \lambda_A, P_A, \approx_A)$, one build a multigame $A_m = (M_A, \lambda_A, \mathcal{P}_A, \approx_A)$, where $\mathcal{P}_A$ denotes the set of plays obtained from the plays in P_A by replacing each move instance a by the singleton multimove $\{a\}$. Clearly, for any game A, $(A_g)_m$ is isomorphic to A, and for any multigame A, $(A_m)_g$ is also isomorphic to A.

This allows us to define the object part of functors $F : \mathcal{G}^{\triangledown} \to \mathcal{M}$ and $G : \mathcal{M} \to \mathcal{G}^{\triangledown}$. Notice that F and G preserve tensor product on objects, up-to isomorphism.

Functors F and G can be extended to strategies as follows.

For any strategy on multigames $\sigma : A \multimap B$, we can associate a strategy $\sigma_g : A_g {\multimap_{\triangledown}} B_g$, where the plays of σ_g are obtained from the plays of σ by splitting each multimove of σ containing atomic moves both in A and in B into a pair of moves on A and B, respectively. Vice versa, any strategy on games $\sigma : A {\multimap_{\triangledown}} B$ induces a strategy on multigames $\sigma_m : A_m \multimap B_m$, whose plays are obtained from the plays of σ by transforming each move instance of A or B into a singleton multimove, and each pair of moves $\langle a, b\rangle$ as the multimove $\{a, b\}$.

Summarizing, we can define functors $F : \mathcal{G}^{\triangledown} \to \mathcal{M}$ and $G : \mathcal{M} \to \mathcal{G}^{\triangledown}$ by:
for any AJM-game A, $FA = A_m$, for any strategy $\sigma : A {\multimap_{\triangledown}} B$, $F([\sigma]) = [\sigma_m]$,
for any multigame A, $GA = A_g$, for any strategy $\sigma : A \multimap B$, $G([\sigma]) = [\sigma_g]$.

Then, we have:

Theorem 3.1. The functors $F : \mathcal{G}^{\triangledown} \to \mathcal{M}$ and $G : \mathcal{M} \to \mathcal{G}^{\triangledown}$ are monoidal, and they give an equivalence between the categories $\mathcal{G}^{\triangledown}$ and $\mathcal{M}$.

An analogous result can be proved for the categories $\mathcal{G}^{\triangledown}_{hf}$ and $\mathcal{M}_{hf}$.

4. A Game Model of Unary PCF with Parallel Or

In this section, we define a game model of unary PCF with parallel or. Models of unary PCF have been extensively studied in the literature, see *e.g.* (Laird 2003; Bucciarelli *et al.* 2003). In particular, in (Laird 2003) it is shown that any *standard order-extensional model* of unary PCF is universal either for unary PCF or for unary PCF extended with *parallel or*. More precisely, any standard order-extensional model of unary PCF which is *sequential*, is universal for unary PCF, while non-sequential models are universal for the extended language. *E.g.* the standard Scott model is universal for unary PCF with parallel or, while the bidomain model of (Laird 2003) is universal for unary PCF. In the context of games, we can recover both kinds of models.

Namely, the extensional quotient of the standard game model of unary PCF built over the Sierpinski game is sequential, and hence universal. On the other hand, one can build a non-sequential multigame game model in the category $K_!(\mathcal{M}_{hf})$, or equivalently, a non-sequential game model in the category $K_!(\mathcal{G}^{\bigtriangledown}_{hf})$ of AJM-games with parallel tensor product. In this way, the theory of standard Scott model is recovered in the context of games.

In this section, we describe the model in the category of games $K_!(\mathcal{G}^{\bigtriangledown}_{hf})$.

4.1. *Unary PCF*

We recall that unary PCF with parallel or is a typed λ-calculus with a ground type o, two ground constants, $\bot, \top$, a *sequential composition* constant $\wedge : o \to o \to o$, and a *parallel or* constant $\vee : o \to o \to o$. Sequential composition examines the two arguments sequentially: if the first argument is $\bot$, then it returns $\bot$, otherwise it returns the second argument. Parallel or examines its arguments in parallel, and it returns $\perp$ only if both are $\perp$. The formal definition of unary PCF is the following:

Definition 4.1 (Unary PCF).
The class *SimType* of *simple types* over a ground type o is defined by:

$$(\text{SimType} \ni)\ A \ ::=\ o \ \mid\ A \to A\ .$$

Raw terms are defined as follows:

$$\Lambda \ni\ M\ ::= \perp \ \mid\ \top\ \mid\ \wedge\ \mid\ \vee\ \mid\ x\ \mid\ \lambda x{:}A.M\ \mid\ MM\ ,$$

where $\perp, \top, \wedge, \vee$ are constants, and $x \in \text{Var}$.

Well-typed terms are terms typable in *typing judgements* of the form $\Gamma \vdash M : A$, where Γ is a *type environment*, *i.e.* a finite set $x_1 : A_1, \ldots, x_k : A_k$. The rules for deriving typing judgements are the following:

$$\frac{}{\Gamma \vdash \perp : o} \qquad \frac{}{\Gamma \vdash \top : o} \qquad \frac{}{\Gamma \vdash \wedge : o \to o \to o} \qquad \frac{}{\Gamma \vdash \vee : o \to o \to o}$$

$$\frac{}{\Gamma, x : A \vdash x : A} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : A \to B} \qquad \frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

The *conversion relation* between well-typed terms is the least relation generated by the following rules together with the rules for congruence closure (which we omit):
$\Gamma \vdash (\lambda x{:}A.M)N = M[N/x]\ :\ B$, where $\Gamma, x : A \vdash M : B$, and $\Gamma \vdash N : A$
$\Gamma \vdash \wedge\top M = M\ :\ o$ and $\Gamma \vdash \wedge\perp M = \perp\ :\ o$, where $\Gamma \vdash M\ :\ o$
$\Gamma \vdash \vee\top M = \top\ :\ o$ and $\Gamma \vdash \vee\perp M = M\ :\ o$, where $\Gamma \vdash M\ :\ o$.

4.2. *The Game Model*

In the cartesian closed category $K_{!_\triangledown}(\mathcal{G}^\triangledown)$, simple types are interpreted by the hierarchy of games over the following *Sierpinski game*:

Definition 4.2 (Sierpinski Game). The game $\mathcal{O}$ is defined as follows:

— $M_{\mathcal{O}} = \{q, a\}$
— $\lambda_{\mathcal{O}}(q) = OQ \quad \lambda_{\mathcal{O}}(a) = PA$
— $P_{\mathcal{O}} = \{\epsilon, q, qa\}$
— $\approx_{\mathcal{O}} = id_{P_{\mathcal{O}}}$

For any simple type A, we denote by $[\![A]\!]$ the interpretation of A in the game category $K_{!_\triangledown}(\mathcal{G}^\triangledown)$. Terms in contexts are interpreted as strategies in the usual way, *i.e.* $x_1 : A_1, \ldots, x_k : A_k \vdash M : A$ is interpreted as a strategy on the game $!_\triangledown[\![A_1]\!]\triangledown \ldots \triangledown !_\triangledown[\![A_k]\!] \multimap_\triangledown [\![A]\!]$, using standard categorical combinators, once the interpretation of constants has been fixed. Thus, we are left to specify this.

The interpretation of constants $\perp$, $\top$ are the only two strategies on the Sierpinski game: $[\![\perp]\!]$ is the empty strategy, while $[\![\top]\!] = \{qa\}$. The interpretation of the constant $\wedge$ is the strategy $\sigma_\wedge$ on the game $!_\triangledown\mathcal{O} \multimap_\triangledown !_\triangledown\mathcal{O} \multimap_\triangledown \mathcal{O}$, which interrogates its arguments sequentially. This is defined by the set of plays generated by the even-prefix closure of the play described in the picture. Parallel or $\vee$ is interpreted by the non-sequential strategy on the game $!_\triangledown\mathcal{O} \multimap_\triangledown !_\triangledown\mathcal{O} \multimap_\triangledown \mathcal{O}$, where Opponent opens in the right-hand $\mathcal{O}$-component, and Player answers with a *pair* of moves asking *both* arguments in parallel; then if Opponent answers in at least one argument (*i.e.* at least one argument is different from $\perp$), Player provides the final answer in the right-hand component. This strategy is obtained as the even-prefix closure of the plays in the picture.

$$
\begin{array}{lccccc}
\sigma_\wedge : & !_\triangledown\mathcal{O} & \multimap_\triangledown & !_\triangledown\mathcal{O} & \multimap_\triangledown \mathcal{O} & \\
 & & & & q & O \\
 & q & & & & P \\
 & a & & & & O \\
 & & & q & & P \\
 & & & a & & O \\
 & & & & a & P
\end{array}
$$

$$
\begin{array}{lccccc}
\sigma_\vee : & !_\triangledown\mathcal{O} & \multimap_\triangledown & !_\triangledown\mathcal{O} & \multimap_\triangledown \mathcal{O} & \\
 & & & & q & O \\
 & q & & q & & P \\
 & a & & & & O \\
 & & & & a & P \\
- & - & - & - & - & - \\
 & & & & q & O \\
 & q & & q & & P \\
 & & & a & & O \\
 & & & & a & P \\
- & - & - & - & - & - \\
 & & & & q & O \\
 & q & & q & & P \\
 & a & & a & & O \\
 & & & & a & P
\end{array}
$$

Using standard methods, one can prove that the theory induced by the game model is the theory of $\beta\eta$-normal forms. Moreover, in view of the results in (Laird 2003), the extensional quotient of the above game model is universal for the observational equivalence of unary PCF (see (Laird 2003) for more details).

5. Final Remarks

We have defined a new category of polarized multigames, where at each step the current player can perform a finite set of atomic moves. Multigames are situated half-way between traditional sequential game semantics and concurrent games, and they provide a sufficient level of parallelism for modeling *e.g.* unary PCF with parallel or. Interestingly, the category of multigames is equivalent to a category of AJM-games with a new notion of parallel tensor product.

Here is a list of comments and lines for future work.

— It would be interesting to understand the logical/proof-theoretic counterpart of multigame tensor, in the spirit of (Abramsky 2003).

— More experiments are called for in the context of the semantics of programming languages, in order to fully understand and exploit the expressivity of multigames, also vs. concurrent games. It is interesting that multigames capture parallel or, however notice that this operator is purely non-deterministic, while multigames (or parallel tensor on AJM-games) have a "truly concurrent" nature. We feel that this could be better exploited for modeling "truly concurrent" languages. Namely, in principle, for modeling non-deterministic operators such as parallel or, a setting where non-deterministic strategies are considered (see (Harmer 2000)) should be sufficient.

— A precise comparison of multigames with other notions of games, ranging from concurrent games of (Abramsky and Melliès 1999), (Ghica and Menaa 2010), and (Rideau and Winskel 2011; Clairambault *et al.* 2012) to asynchronous games of (Melliès 2006), but also to traditional Conway's games (Conway 2001) is called for. In particular, in the vein of (Honsell *et al.* 2012), it would be interesting to establish a connection between multigames and coalgebraic games.

— It would be also interesting to study the relationships between multigames and traditional denotational semantics based on domains.

— In our definition of multigames, we have omitted the question/answer machinery, which is often considered on games. The extension to multigames of this, especially the well-bracketing condition, requires further study.

— Finally, it would be interesting to provide a version of arena multigames in the style of (Hyland and Ong 2000). As usual, this should allow us to avoid explicit exponential construction and equivalence on plays. The difficulty in defining arena multigames lies in the definition of pointers.

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