Functors Determined by Values on Objects [4]

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Abstract

Functors which are determined, up to natural isomorphism, by their values on objects, are called *DVO* (*Defined by Values on Objects*). We focus on the collection of *polynomial functors* on a category of sets (classes), and we give a characterization theorem of the DVO functors over such collection of functors. Moreover, we show that the (κ-bounded) powerset functor is *not* DVO.

Key words: category of sets (classes), set functor, inclusion preserving functor, DVO functor.

1 Introduction

Set Theory and Category Theory are the main ambient theories for developing Semantics. However, for opposite ideological views, seldom the two are

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formally discussed one within the other. Our preformal intuition lives probably in a naive set theory, even when we work with categories. Nevertheless, we do not have yet a complete understanding of what are functors in a set theoretic universe. We do not claim, of course, that this is preliminary to their fruitful use, we claim only that it would enhance our feel for them and our understanding of the notion of naturality. This paper is a contribution to the set theoretic understanding of functors, especially in view of their use in defining inductive and coinductive data types.

In this paper, we focus on functors on *categories of sets (classes)*, i.e. categories where objects are sets (classes) of a set theoretical universe, and morphisms are set (class) functions.

A very natural question which arises on the boundary between Set Theory and Category Theory is the following: when can a *set operator* be extended to a *set functor*?
This problem has been investigated in the literature for various classes of set operators, [8,11,3,6]; it can be equivalently expressed, by purely categorical means, in terms of *solvability of functorial equations*, in the sense of [11]. An easy observation is that operators underlying functors are monotone w.r.t. object cardinalities, up to $\emptyset$ (i.e. for all $X \neq \emptyset$), since set functors preserve injective maps with non-empty domain. However, operators underlying set functors satisfy also various continuity properties, see e.g. [6]. Capitalizing on these, one can show that there exist various classes of (monotone) set operators which are not extensible to functors, [6].

The question of the *uniqueness* (up to natural isomorphism) of the functorial extension of a set operator, when it exists, is also quite natural, but also very difficult. This can be equivalently expressed by saying that the functor is *DVO* (*Defined by Value on Objects*), [3], i.e. its behaviour on objects determines uniquely the functor, up to natural isomorphism. In [11,2,3,4], various positive and negative results on DVO functors have been obtained. In particular, in [3], a characterization theorem is given for the DVO functors which are non-constant and do not contain a subfunctor naturally isomorphic to the Identity functor.

In this paper, we investigate the collection of *n-ary polynomial functors* on a category $\mathcal{C}$, where objects are sets (classes) of a universe of von Neumann-Bernays-Gödel, NBG, and where morphisms are set (class) theoretic functions. The main result that we prove is the following characterization of DVO *n-ary polynomial functors*:

Theorem 1.1 *A* strict polynomial *functor* $H^n : \mathcal{C} \to \mathcal{C}$ *is DVO if and only if*

- *either it is* affine polynomial *with* finite coefficients *or*
- *it is a* constant *functor to an object whose cardinality is less than Ord (the cardinality of classes).*

A *n-ary affine* polynomial functor is a polynomial functor in which each monomial is affine, i.e. each variable appears *at most* once in each monomial. A *strict* functor is a functor possibly redefined on the empty set, in such a way that it yields the empty set. The reason why we deal with strict functors is that, as we will see, the behaviour of functors on the empty set is somewhat irregular.

Theorem 1.1 above extends results in [11,2,3,5], where other classes of (unary) functors or some special cases of (unary) polynomial functors have been considered.

Some easy consequences of Theorem 1.1 are the following. The unary product functor $FX = X \times X$ is (the most elementary) non-DVO polynomial functor, while the product bifunctor $F(X_1, X_2) = X_1 \times X_2$ is DVO. The situation is different for the coproduct. Namely, the unary coproduct functor $FX = X + X$ is (naturally isomorphic to) the linear polynomial functor $FX = 2 \times X$, and hence it is DVO. The coproduct bifunctor $F(X_1, X_2) = X_1 + X_2$, which is affine (in each monomial) is also DVO. Another interesting easy observation is that the collection of DVO functors is *not* closed under composition: e.g. both *Id* and the binary product functor are DVO, but $Id \times Id$ is not DVO.

The proof of Theorem 1.1 above is quite difficult. In this paper, we provide details only for the unary case, i.e. $H : \mathcal{C} \to \mathcal{C}$, the proof in the n-ary case being a complex and lengthy generalization of the previous case. Non-constant polynomial functors and constant polynomial functors deserve separate treatments. In the non-constant case, the proof proceeds by defining a “candidate natural transformation”, and by proving that this is an isomorphism. To this aim, we also exploit a general κ-continuity result, for any infinite κ. In the constant case, a subtle cardinality reasoning is carried out.

Finally, we show that the whole family of κ-bounded powerset functors is *not* DVO, for any cardinal κ.

Summary

In Section 2, we provide set theory and categorical preliminaries. In Section 3, we collect some preliminary results about inclusion preserving functors and continuity properties of set functors. In Section 4, which is the main section, we carry out our investigation of DVO polynomial functors. In Section 5, we deal with the powerset. Final remarks and directions for future work appear in Section 6.

2 Set Theory and Set Category Preliminaries

We will work in a universe of sets and classes satisfying the theory NBG of von Neumann-Bernays-Gödel. The theory NBG is closely related to the more familiar set theory ZFC of Zermelo-Fraenkel with choice. The primary difference between ZFC and NBG is that NBG has proper classes among its objects. NBG and ZFC are actually equiconsistent, NBG being a conservative extension of ZFC. In NBG, the proper classes are differentiated from sets by the fact that they do not belong to other classes. Moreover, NBG classes satisfy the Axiom N of von Neumann, stating that all proper classes have the same cardinality of the set theoretic universe, which we denote by Ord.

In this paper, we will deal with a generic *category of sets*, possibly including also *classes*, i.e. a category whose objects are the sets (classes) of a NBG universe, and where morphisms are set (class) theoretic functions.

Notation. Throughout this paper, we denote by V the universe of sets. We use the symbol $\mathcal{C}$ to denote a generic category of sets (classes). Moreover, we use the following basic notation about functions. Let $f : X \to Y$ be any function on sets (or classes), and let $X' \subseteq X$, then: $gr(f)$ denotes the graph of f; $img\ f$ denotes the image of f; $f_{|img\ f} : X \to img\ f$ denotes the function obtained from f by restricting the codomain Y to the image of f; $f_{X'} : X' \to Y$ denotes the function obtained from f by restricting the domain of f to X'.

Remark. In this paper we will refer not only to *large objects*, such as proper classes, but also to *very large* objects, such as functors over categories whose objects are classes. A foundational formal theory which can accommodate naturally all our notions is not readily available. A substantial formalistic effort would be needed to "cross all our t's" properly. We shall therefore adopt a pragmatic attitude and freely assume that we have classes and functors over classes at hand. Worries concerning consistency can be eliminated by assuming that our ambient theory is a Set Theory with an inaccessible cardinal κ, and the model of our object theory consists of those sets whose hereditary cardinal is less than κ, V_κ say, the classes of our model are the subsets of V_κ, and functors live at the appropriate ranks of the ambient universe.

3 Inclusion Preserving Functors and Continuity Properties

In this section, we recall some properties of *inclusion preserving functors*, i.e. functors which preserve inclusion maps. For more details and proofs see [1,6]. The main result about inclusion preserving functors is the fact that *any* functor on a set category is naturally isomorphic, *up to* $\emptyset$, to a functor which is inclusion preserving. Two functors are *naturally isomorphic up to* $\emptyset$, if they are naturally isomorphic on the restriction of the category $\mathcal{C}$ to the category of non-empty sets. The above result allows us to restrict ourselves to inclusion preserving functors, whenever we reason about a property which is preserved under natural isomorphism, as it is the case for the DVO property.

In this section, we also recall a very general κ-continuity result of set functors, holding for *any infinite* cardinal κ.

In Section 4, we will use this continuity result in order to extend DVO properties of functors on categories of finite sets to categories including also infinite objects.

In the next definition, we recall the notion of inclusion preserving functor:

Definition 3.1 *$F : \mathcal{C} \to \mathcal{C}$ is* inclusion preserving *if*

$$\forall X, Y.\ X \subseteq Y \implies F(\iota_{X,Y}) = \iota_{F(X),F(Y)}\ ,$$

where $\iota_{X,Y} : X \to Y$ is the inclusion map from X to Y.

Inclusion preserving functors satisfy the following properties:

Proposition 3.2 *Let $F : \mathcal{C} \to \mathcal{C}$ be inclusion preserving. Then*

(i) *The operator underlying F is monotone, i.e. $X \subseteq Y \implies FX \subseteq FY$.*

(ii) *F preserves images of functions, i.e., for any $f : X \to Y$, $F(img\ f) = img\ F(f)$, [1].*

(iii) *The value of F on any morphism depends only on the graph of the morphism, i.e. for all $f : X \to Y$, $f' : X \to Y'$, $gr(f) = gr(f') \Rightarrow gr(F(f)) = gr(F(f'))$.*
Vice versa, if for all $f : X \to Y$, $f' : X \to Y'$, $gr(f) = gr(f') \Rightarrow gr(F(f)) = gr(F(f'))$, then F is inclusion preserving, [6].

(iv) *For all $X' \subseteq X$ and for all $f : X \to Y$, $gr(Ff_{X'}) = gr(Ff)_{FX'}$.*

(v) *F preserves non-empty* finite intersections, *i.e. for all X, Y such that $X \cap Y \neq \emptyset$, $F(X \cap Y) = FX \cap FY$, [10].*

Trivially, not every functor is inclusion preserving. Just consider any functor obtained by mapping isomorphically the value on a given set into a set which is disjoint from the value of the functor on a subset. However:

Theorem 3.3 *Any functor $F : \mathcal{C} \to \mathcal{C}$ is naturally isomorphic up to $\emptyset$ to a functor which is inclusion preserving.*

Theorem 3.3 above was originally proved by J. Adamek in his PhD thesis, see e.g. [1]; see also [6] for a simple proof.

The notion of κ-reachability below, essentially corresponds to Koubek's notion of "attainable cardinal", [7].

Definition 3.4 (κ-reachability) *Let $\kappa > 1$. Then FX is* κ-reachable *if $FX = \bigcup\{Ff \mid f : Y \to X\ \wedge\ |Y| < \kappa\}$.*

The following general continuity result has been originally proved in [7] under GCH. However this hypothesis can be eliminated, see [6] for a complete proof.

Theorem 3.5 (κ-continuity) *Let $F : \mathcal{C} \to \mathcal{C}$. For all $|X| = \kappa$ infinite,*

$$|FX| \leq \kappa \implies FX \text{ is } \kappa\text{-reachable}\ .$$

4 Polynomial Functors

In this section, we focus on the class of polynomial functors, and we prove the characterization Theorem 1.1 of Section 1 for the unary case. We recall that a functor is *DVO* (*Defined by Values on Objects*), [3], when its behaviour on objects determines uniquely, up to natural isomorphism, its behaviour on morphisms. Formally:

Definition 4.1 (DVO Functor) *A functor $F : \mathcal{C} \to \mathcal{C}$ is* DVO *iff for all $G : \mathcal{C} \to \mathcal{C}$ isomorphic on objects to F, G is naturally isomorphic to F.*

Notice that, in Definition 4.1 above, assuming the functor G to be isomorphic on objects to F is equivalent to assume that G coincides on objects to F.

We focus on the class of (unary) strict polynomial functors:

Definition 4.2 ((Strict) Polynomial Functor) • *A functor $H : \mathcal{C} \to \mathcal{C}$ is* polynomial *if it is defined by*

$$HX = \Sigma_{i \geq 0} K_i \times X^i \ ,$$

where $K_0, K_1, \ldots$ are objects of $\mathcal{C}$, called the coefficients *of the polynomial functor, and X^i denotes $\underbrace{X \times \ldots \times X}_{i}$. The definition of H on arrows is the standard one. We will often omit the symbol $\times$ in the definition of a polynomial functor, simply writing $\Sigma_{i \geq 0} K_i X^i$.*

- *A* strict polynomial *functor $H : \mathcal{C} \to \mathcal{C}$ is a functor which coincides with a polynomial functor on the restriction of the category $\mathcal{C}$ to the category of non-empty objects, and on $\emptyset$ it is defined as follows: $H\emptyset = \emptyset$, and for any $f : \emptyset \to X$, $Hf : \emptyset \to HX$ is the empty function. We will denote by $(\Sigma_{i \geq 0} K_i X^i)_\emptyset$ the strict polynomial functor corresponding to the polynomial $\Sigma_{i \geq 0} K_i X^i$.*

In this section, we will prove the following:

Theorem 4.3 *A* strict polynomial *functor $H : \mathcal{C} \to \mathcal{C}$ is DVO if and only if*

- *either it is* linear *with* finite coefficients *or*
- *it is a* constant *functor to an object whose cardinality is less than Ord.*

By a *linear* functor $H : \mathcal{C} \to \mathcal{C}$ we mean a functor such that the parameter X appears exactly once in HX.

The proof of Theorem 4.3 above is divided in two parts: in Section 4.1, we deal with the case of H non-constant, while in Section 4.2 we deal with the case of H constant.

4.1 Non-constant Polynomial Functors

The aim of this subsection is to prove the following:

Theorem 4.4 *Let $H : \mathcal{C} \to \mathcal{C}$ be a strict non-constant polynomial functor. Then H is DVO if and only if it is linear, with finite coefficients.*

The most difficult part to prove in Theorem 4.4 above is the implication ($\Leftarrow$). The proof of Theorem 4.4($\Rightarrow$) is inspired by the argument used in [3] for proving that the functor $X \times X$ is not DVO.

4.1.1 Proof of Theorem 4.4($\Leftarrow$): the general pattern

Let $H : \mathcal{C} \to \mathcal{C}$ be strict polynomial, non-constant, linear, with finite coefficients. Let $F : \mathcal{C} \to \mathcal{C}$ be isomorphic on objects to H, i.e., for all X, there exists $\sigma_X : HX \xrightarrow{\sim} FX$. By Theorem 3.3, we can assume F to be inclusion preserving.

The proof of Theorem 4.4($\Leftarrow$) proceeds by

(i) defining a "candidate" natural transformation $\tau : H \stackrel{\cdot}{\longrightarrow} F$;

(ii) proving that τ_X is injective for all X;

(iii) proving that τ_X is surjective for all X.

For the sake of clarity, we first carry out the proof of Theorem 4.4($\Leftarrow$) in the special case of the Identity functor. Then, we will give details also for the general case.

4.1.2 Proof of Theorem 4.4($\Leftarrow$): the case of the Identity

Let $F : \mathcal{C} \to \mathcal{C}$ be inclusion preserving and isomorphic on objects to Id, i.e., for all X, there exists $\sigma_X : X \xrightarrow{\sim} FX$.

In the next definition, we define our "candidate natural transformation" $\tau = \{\tau_X : X \to FX\}_X$.

Definition 4.5 *Let $x_0 \in V$ be any fixed element of the universe. For $X \neq \emptyset$, we define $\tau_X : X \to FX$ by*

$$\tau_X(x) = (F\delta^X_{x_0 x})(\sigma_{\{x_0\}}(x_0)) \ ,$$

where $\delta^X_{x_0 x} : \{x_0\} \to X$ is such that $\delta^X_{x_0 x}(x_0) = x$.

For $X = \emptyset$, we define $\tau_\emptyset = \varepsilon_\emptyset$, where $\varepsilon_\emptyset$ denotes the empty function into $\emptyset$.

The family τ is easily shown to be a natural transformation from Id to F.

Injectivity of τ: the case of the Identity.

We proceed by contradiction, i.e. let us assume that τ is not injective. We have the following lemmata:

Lemma 4.6 *If τ is not injective, then for all X, for all $x, x' \in X$, $\tau_X(x) = \tau_X(x')$.*

Proof. If τ is not injective for some X, then there exist x, x' such that $(F\delta^X_{x_0 x})(\sigma_{\{x_0\}}(x_0)) = (F\delta^X_{x_0 x'})(\sigma_{\{x_0\}}(x_0))$. Let Y be any object and let $y, y' \in$

Y. We show that $(F\delta^Y_{x_0y})(\sigma_{\{x_0\}}(x_0)) = (F\delta^Y_{x_0y'})(\sigma_{\{x_0\}}(x_0))$. Namely, let $f : X \to Y$ be such that $fx = y$ and $fx' = y'$. Then we have:

$$\begin{aligned} (F\delta^Y_{x_0y})(\sigma_{\{x_0\}}(x_0)) &= (Ff \circ F\delta^X_{x_0x})(\sigma_{\{x_0\}}(x_0)) && \text{by definition of } f \\ &= (Ff) \circ (F\delta^X_{x_0x'})(\sigma_{\{x_0\}}(x_0)) && \text{by hypothesis} \\ &= (F\delta^Y_{x_0y'})(\sigma_{\{x_0\}}(x_0)) && \text{by definition of } f. \end{aligned}$$

□

An immediate consequence of Lemma 4.6 above and of the fact that F is inclusion preserving, is that the images under F of singleton sets are all equated in a canonical point, i.e.:

Lemma 4.7 *If τ is not injective, then there exists c such that for all x, $F(\{x\}) = \{c\}$.*

More generally,

Lemma 4.8 *If τ is not injective, then there exists c such that for all $X \neq \emptyset$, $c \in FX$.*

Before proving that τ has to be injective, we still need a further ingredient, i.e. the fact that constant functions are mapped by F to constant functions. This is an easy consequence of Proposition 3.2(iii):

Lemma 4.9 *For any constant function $f : X \to Y$, Ff is constant.*

Finally, we are in the position of proving that:

Proposition 4.10 *τ is injective.*

Proof. Assume by contradiction that τ is not injective. Then let $X = \{x_1, x_2, x_3\}$. There are three different subsets of two elements of X. Since $|F(X)| = 3$ and $|FX'| = 2$ for $|X'| = 2$, using Lemma 4.8, one can easily show that the image of two 2-element subsets of X must coincide, e.g. let $F\{x_1, x_2\} = F\{x_2, x_3\}$. Now let $h : X \to X$ be defined by $h(x_1) = x_1$, $h(x_2) = x_2$, $h(x_3) = x_2$ (see Fig. 1). Since set functors preserve injective functions, using Proposition 3.2(iv) and Lemma 4.9, we have: $F(h)_{F\{x_1,x_2\}}$ is injective, while $F(h)_{F\{x_2,x_3\}}$ is constant, (see Fig. 1). This is a contradiction, since $F\{x_1, x_2\} = F\{x_2, x_3\}$, and hence $F(h)_{F\{x_1,x_2\}} = F(h)_{F\{x_2,x_3\}}$.

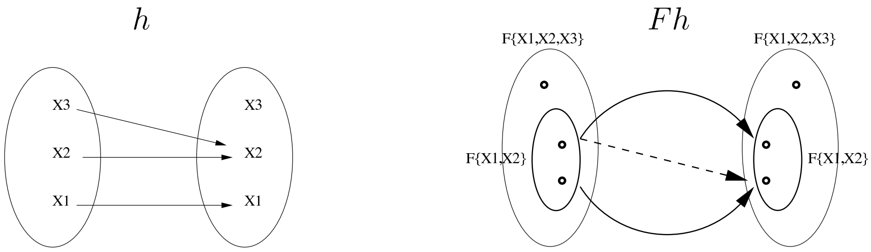

Fig. 1. The functions h and Fh.

□

Surjectivity of τ: the case of the Identity.

Since τ is injective, then it is also surjective on all finite objects. To extend the surjectivity of τ to infinite objects, we use the κ-continuity Theorem 3.5 of Section 3. More precisely, we prove the following lemma:

Lemma 4.11 *Let $F : \mathcal{C} \to \mathcal{C}$ be inclusion preserving and isomorphic on objects to Id. Then, for all X, FX is 2-reachable.*

Proof. By induction on $|X| = \kappa$. If $\kappa < \omega$, then the thesis is immediate, since, by Proposition 4.10, τ_X is injective, and hence surjective on finite sets. If $|X| = \kappa \geq \omega$, then, by Theorem 3.5 , since $|FX| = |X|$, FX is κ-reachable. Thus, for all $x \in FX$, there exist Y, $|Y| < \kappa$, and $f : Y \to X$ such that $x \in img(Ff)$. By induction hypothesis, FY is 2-reachable, and hence (by composition), also $x \in FX$ is 2-reachable. □

By Lemma 4.11 above and by definition of τ, we immediately have:

Theorem 4.12 *τ_X is surjective for all X.*

4.1.3 Proof of Theorem 4.4($\Leftarrow$): the general case

Let $H : \mathcal{C} \to \mathcal{C}$ be polynomial, non-constant, strict, linear, with finite coefficients, i.e. $HX = (K_0 + K_1 X)_\emptyset$, with $0 \leq |K_0| < \omega$, $0 < |K_1| < \omega$. Let $F : \mathcal{C} \to \mathcal{C}$ be inclusion preserving and isomorphic on objects to H, i.e., for all X, there exists $\sigma_X : HX \xrightarrow{\sim} FX$.

The definition of the candidate natural transformation τ in the general case requires some extra care, since it is not immediate how to extend the definition of τ (Definition 4.5) on the constant part K_0. To overcome this problem, we carry out an extra preliminary analysis on the behaviour of a functor F isomorphic on objects to a generic polynomial functor. The output of this analysis will be that for such a functor F we can single out special elements $c_1, \ldots, c_{|K_0|} \in FX$, for any $X \neq \emptyset$, "playing the role" of the constant elements in K_0. More precisely, we have:

Lemma 4.13 *Let F be inclusion preserving and isomorphic on objects to H. Then*

(i) *there exist elements $c_1, \ldots, c_{|K_0|}$ such that $c_1, \ldots, c_{|K_0|} \in FX$, for all X;*

(ii) *moreover, for all x_0, for all X, for all $x \in X$, $F\delta^X_{x_0 x}(c_i) = c_i$, for all $i = 1, \ldots, |K_0|$.*

Proof. We will prove the thesis for objects X of cardinality 1. Then the thesis in its full generality follows from the fact that F is inclusion preserving.
Let $\{x\}, \{y\}$ be two singleton sets. Since $|F\{x\}| = |F\{y\}| = |K_0| + |K_1|$, $|F\{x, y\}| = |K_0| + 2|K_1|$, and $F\{x\}, F\{y\} \subseteq F\{x, y\}$, then, for all x, y, there exist $c^{xy}_1, \ldots, c^{xy}_{|K_0|} \in F\{x\} \cap F\{y\}$. Now, notice that, given a set $F\{x\}$ of $|K_0| + |K_1|$ elements, there are $\begin{pmatrix} |K_0| + |K_1| \\ |K_0| \end{pmatrix}$ different subsets of $|K_0|$ ele-

ments. Thus, let $\{\{y_i\}\}_i$ be a family of $(|K_0|+|K_1|) \times \begin{pmatrix} |K_0|+|K_1| \\ |K_0| \end{pmatrix}$ distinct singleton sets different from $\{x\}$, and let us consider the intersection of $F\{x\}$ with each $F\{y_i\}$. Then at least $|K_0|+|K_1|$ sets in the family $\{F\{y_i\}\}_i$ have the same intersection with $F\{x\}$, say $\{c_1, \ldots, c_{|K_0|}\}$.
Now let x_0 be a fixed element. Then, for each c_i given above, there exist $y \in \{y_i\}_i$ such that, for some $a_i \in F\{x_0\}$, $F\delta_{x_0 x}(a_i) = F\delta_{x_0 y}(a_i) = c_i$. Now we show that, for any z, $F\delta_{x_0 z}(a_i) = c_i \in F\{z\}$. Let $X = \{x, y, z\}$ and let $f : X \to X$ be such that $f(x) = z$ and $f(y) = y$. Then we have

$$\begin{aligned}
&(F\delta^X_{x_0 z})(a_i) = (Ff \circ F\delta^X_{x_0 x})(a_i) && \text{by definition of } f\\
&= (Ff) \circ (F\delta^X_{x_0 y})(a_i) && \text{by hypothesis}\\
&= (F\delta^X_{x_0 y})(a_i) && \text{by definition of } f\\
&= c_i && \text{by definition}
\end{aligned}$$

Finally, we are left to show that $a_i = c_i$, for all i. Namely, for $x = x_0$, we have $F(\delta_{x_0 x_0})(a_i) = c_i$, but $F(\delta_{x_0 x_0})$ is the identity, thus $a_i = c_i$. □

Now we are in the position of introducing the candidate natural transformation $\tau = \{\tau_X : (K_0 + K_1 X)_\emptyset \to FX\}_X$. Let $K_0 = \{k_1, \ldots, k_{|K_0|}\}$ and $K_1 = \{k'_1, \ldots, k'_{|K_1|}\}$ be enumerations of K_0 and K_1, respectively. Let $x_0 \in V$ be fixed and let $F\{x_0\} = \{c_1, \ldots, c_{|K_0|}\} \cup \{b_1, \ldots, b_{|K_1|}\}$, where, the elements c_i are those given in Lemma 4.13(2) above. Using Lemma 4.13, one can easily check that the definition below gives a natural transformation:

Definition 4.14 *Let $\tau_X : (K_0 + K_1 X)_\emptyset \to FX$ be defined by:*
if $X = \emptyset$, then $\tau_X = \varepsilon_\emptyset$, otherwise if $X \neq \emptyset$, let $\overline{x} \in X$ be any element of X, then[5]

$$\tau_X(d) = \begin{cases} F\delta^X_{x_0 \overline{x}}(c_i) = c_i & \text{if } d = k_i \in K_0 \\ F\delta^X_{x_0 x}(b_i) & \text{if } d = (k'_i, x) \in K_1 \times X \ . \end{cases}$$

Injectivity of τ: the general case.

The proof of injectivity in the general case is a straightforward generalization of the case of the Identity. We proceed by contradiction, i.e. let us assume that τ is not injective. Then we have the following lemmata:

Lemma 4.15 *If τ is not injective, then there exists $k \in K_1$ such that for all X, for all $x, x' \in X$, $\tau_X(k, x) = \tau_X(k, x')$.*

[5] In the definition below, by abuse of notation, we denote by k_i both an element of K_0 and an element of the lefthand part of $K_0 + K_1 X$. And similarly, for the elements of $K_1 X$.

One can easily show that a consequence of Lemma 4.13 and Lemma 4.15 above is the following

Lemma 4.16 *If τ is not injective, then there exist $|K_0|+1$ canonical points $c_1, \ldots, c_{|K_0|}, c$, such that for all $X \neq \emptyset$, $c_1, \ldots, c_{|K_0|}, c \in FX$.*

An easy consequence of Proposition 3.2(iii) is the following:

Lemma 4.17 *If $f : X \to Y$ is constant, then $|imgFf| \leq |K_0| + |K_1|$.*

Finally, we have

Proposition 4.18 *τ is injective.*

Proof. Assume by contradiction that τ is not injective. Let $X = \{x_1, x_2, x_3\}$. Then $|FX| = |K_0| + |K_1| \times 3$. Let us consider the three subsets of X consisting of two elements. Since by Lemma 4.16, the images under F of all sets share $|K_0|+1$ canonical points, by a direct computation, one can show that there are two distinct sets, say $\{x_1, x_2\}$, $\{x_2, x_3\}$, such that $|F\{x_1, x_2\} \cap F\{x_2, x_3\}| \geq |K_0| + |K_1| + 1$ $(*)$. Now, let $h : X \to X$ be defined by $hx_1 = x_1$, $hx_2 = x_2$, $hx_3 = x_2$. We have: $|\mathrm{img}((Fh)_{F\{x_2,x_3\}})| \leq |K_0| + |K_1|$ by Lemma 4.17, while $(Fh)_{F\{x_1,x_2\}}$ is injective, and hence $|\mathrm{img}((Fh)_{F\{x_1,x_2\}})| = |K_0| + 2|K_1|$. Thus $|F\{x_1, x_2\} \cap F\{x_2, x_3\}|$ must be $< |K_0| + |K_1| + 1$, contradicting $(*)$. $\square$

Surjectivity of τ: the general case.

The proof of the surjectivity of τ_X follows exactly the same pattern used in Section 4.1.2 for the Identity case. Notice that, in the proof of Lemma 4.11 of Section 4.1.2, the only extra ingredient apart from the injectivity of τ_X and the κ-continuity property, is the property $|FX| = |X|$, for $|X|$ infinite. This holds also in the case of a general polynomial functor, provided the coefficients are all finite, which is our hypothesis.

This completes the proof of Theorem 4.4($\Leftarrow$).

4.1.4 Proof of Theorem 4.4($\Rightarrow$)

Let $H : \mathcal{C} \to \mathcal{C}$ be strict polynomial and non-constant. We prove that

(i) if H is non-linear, then H is not DVO;

(ii) if $H = (K_0 + K_1 X)_\emptyset$ and K_0 or K_1 are infinite, then H is not DVO.

Proof. (i) The argument is inspired by that used in [3] for proving that the family of functors $Q_n X = X^n$, with usual definition on arrows, are not DVO, for all $n > 1$. Let $HX = \Sigma_{i\geq 0} K_i X^i$ (or its strict version), with $|K_j| > 0$ for some $j \geq 2$. We will define a new functor H' isomorphic on objects to H but not naturally isomorphic to it. First of all, let $P_{\leq n}$ be the bounded powerset functor defined by $P_{\leq n}(X) = \{Y \subseteq X \mid 0 < |X| \leq n\}$, with the usual definition on arrows. Let us define the *quotient functor* of two functors F_1, F_2 with a common subfunctor G as the quotient of the disjoint union $F_1 + F_2$ that unifies G in F_1, F_2. Let us denote such quotient functor by

$(F_1 + F_2)/G$. [6] Now, let H' be obtained from H by substituting a non-linear occurrence of $X \times X$ by the quotient $(P_{\leq 2}(X) + P_{\leq 2}(X))/Id(X)$, i.e.: $H'X = \Sigma_{0 \leq i < j} K_i X^i + K_j \times ((P_{\leq 2}(X) + P_{\leq 2}(X))/Id(X)) \times X^{j-2} + \Sigma_{i>j} K_i X^i$. Then one can check that H' is isomorphic on objects to H but not naturally isomorphic to it, due to the presence of the disjoint sum in H' which induces a different behaviour on arrows (elements in the righthand components are always sent to righthand components, and similarly for lefthand components). (ii) Assume that K_0 or K_1 is infinite. Now, let $H'X = (K_0 + K_1 X^2)_\emptyset$, with standard definition on arrows. Then, for all X, $H'X \cong HX$, since $|K_0| \geq \omega$ or $|K_1| \geq \omega$. But, by item (i) above, H' is not DVO. □

Remark. Finally, one may ask what happens if we abandon the hypothesis H strict. One can prove that, for $|K_0| > 1$, $H = K_0 + K_1 X$ is not DVO. Namely, if $|K_0| > 1$, let $\bar{k} \in K_0$, and let $H' : \mathcal{C} \to \mathcal{C}$ be defined as H, apart from the value on empty functions $\varepsilon_Y : \emptyset \to Y$, for $Y \neq \emptyset$, where $H'(\varepsilon_Y) : K_0 \to K_0 + K_1 Y$ is the constant function $\lambda k.\bar{k}$. One can easily check that H' is a functor. Moreover, H' is trivially isomorphic on objects to H, but it is *not* naturally isomorphic to it.
On the other hand, if $|K_0| = 1$, one can prove that H is DVO, by exploiting the argument used in the previous subsections for the strict case, and by a direct analysis of the behaviour of the functor on $\emptyset$.

4.2 Constant Functors

Let K denote the functor constant to a class K, and let $K_\emptyset$ denote its strict version, i.e. the functor which is constant to K up to $\emptyset$.

If K is finite, then the pattern of Section 4.1 can be immediately extended to prove that the constant strict functor $K_\emptyset$ is DVO. In this section, we extend this result by proving that, on a generic set category $\mathcal{C}$, $K_\emptyset$ is DVO if and only if $|K| < Ord$.

Proving that $K_\emptyset$ is DVO for $|K| < Ord$ is quite difficult and it requires an "ad-hoc" argument, exploiting cardinality properties. The core of this subsection is to prove that if F is strict inclusion preserving and it has constant "small" cardinality on objects, then F is strict constant. To this end, we need the following Lemmata 4.19 and 4.20.

Lemma 4.19 *Let $F : \mathcal{C} \to \mathcal{C}$ be a (strict) inclusion preserving functor, which has constant cardinality $\kappa < Ord$ on objects. Then there exists a set $\bar{X}$ such that $F(Y) = F(\bar{X})$, for all $Y \supseteq \bar{X}$.*

Proof. Let X be a set. We define an increasing chain of sets as follows: $X_0 = X$, $X_{\alpha+1} = X_\alpha^+$ for any cardinality α, and $X_\lambda = \bigcup_{\gamma<\lambda} X_\gamma$ for any limit cardinality, where, for any α, X_α^+ is an arbitrary set such that $X_\alpha^+ \supset X_\alpha$ and

[6] This definition can be generalized to the case where G is not a subfunctor of F_1, F_2, but it has only an isomorphic copy in F_1 and F_2. In this case, a quotient functor is any pushout of F_1, F_2 with monotrasformations $\mu_1 : G \to F_1$, $\mu_2 : G \to F_2$.

$F(X^+_\alpha) \supset F(X_\alpha)$, if such a set exists, otherwise $X^+_\alpha = X_\alpha$. Since $|F(X)| = \kappa$ for all X, the chain $\{X_\alpha\}_{\alpha<2^\kappa}$ must be eventually constant. Let $\bar{X}$ be the least element of the chain after which the chain is eventually constant, then $F(\bar{X}) = F(Y)$ for all $Y \supseteq \bar{X}$. $\square$

Lemma 4.20 *Let $F : \mathcal{C} \to \mathcal{C}$ be a (strict) inclusion preserving functor, which has constant cardinality $\kappa <$ Ord on objects. Then, for any infinite set cardinality μ, F is constant on all non-empty objects of cardinality $\leq \mu$, i.e. there exists X_0 such that:*
for all $Z \neq \emptyset$ such that $|Z| \leq \mu$, $F(Z) = X_0$, moreover for all $f : X \to Y$ such that $0 < |X|, |Y| \leq \mu$, $F(f) = id_{X_0}$.

Proof. Let μ be an infinite set cardinal. The proof proceeds gradually by proving the following steps:

(i) there exists an infinite family $\mathcal{X}_0$ of disjoint sets of cardinality μ, such that F is constant on $\mathcal{X}_0$, i.e. there exists X_0 s.t. $F(X) = X_0$, for all $X \in \mathcal{X}_0$, moreover, for all $X, Y \in \mathcal{X}_0$ such that $X \neq Y$, for all $f : X \to Y$, $F(f) = id_{X_0}$;

(ii) for all $X \in \mathcal{X}_0$, for all $f : X \to X$, $F(f) = id_{X_0}$;

(iii) for all $X \in \mathcal{X}_0$, for all $Y \neq \emptyset$ such that $Y \subseteq X$, $F(X) = X_0$;

(iv) for all $X \in \mathcal{X}_0$ and for all Y such that $|Y| = |X|$ and $f : Y \to Y$, $F(Y) = X_0$ and $F(f) = id_{X_0}$;

(v) for all $Z \neq \emptyset$ such that $|Z| \leq \mu$, $F(Z) = X_0$, moreover for all $f : X \to Y$ such that $|X|, |Y| \leq \mu$, $F(f) = id_{X_0}$.

Proof of (i). If κ is finite, then let $\mathcal{X}$ be any countable family of disjoint sets of infinite cardinality μ. If κ is infinite, let $\mathcal{X}$ be any family of $2^{2^{k^+}}$ disjoint sets of infinite cardinality μ. By Lemma 4.19, there exists a set $\bar{X}$ such that $F(Y) = F(\bar{X})$, for all $Y \supseteq \bar{X}$. Since for any $X \in \mathcal{X}$, there exists $Y \supseteq \bar{X} \cup X$, then the value of F on elements of $\mathcal{X}$ is a subset of $F(\bar{X})$. Therefore since $|F(\bar{X})| = \kappa$, F is constant on infinitely many elements of $\mathcal{X}$ [7]. Let $\mathcal{X}_0$ be the family of sets of $\mathcal{X}$ on which F is constantly equal on objects to, say, X_0.
Now we show that, for all $X, Y \in \mathcal{X}_0$ such that $X \neq Y$, and $f : X \to Y$, $F(f) = id_{X_0}$. The following diagrams straightforwardly commute:

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\iota_{X,X\cup Y}"'] & & Y \arrow[d, phantom, "(2)"] & \\
& X \cup Y \arrow[ur, "f\cup id_Y"'] \arrow[u, phantom, "(1)"] & & Y \arrow[ul, "id_Y"'] \arrow[ll, "\iota_{Y,X\cup Y}"]
\end{tikzcd}$$

Hence, if we apply F to all diagrams, these still commute. Since F is inclusion preserving and the diagram (2) commutes, $F(f \cup id_Y)_{FY} = id_{FY} = id_{X_0}$. Hence since $F(X) = F(Y) = X_0$, also $F(f \cup id_Y)_{FX} = id_{X_0}$ and, by diagram (1), also $F(f) = id_{X_0}$.

Proof of (ii). Let $X \in \mathcal{X}_0$, $f : X \to X$. Since $\mathcal{X}_0$ has infinitely many elements,

[7] For κ infinite there are $2^{2^{\kappa^+}}$ elements of $\mathcal{X}$ on which F is constant.

and all elements of $\mathcal{X}_0$ are disjoint and have the same cardinality, there exists $Y \in \mathcal{X}_0$, such that $Y \cap X = \emptyset$ and an isomorphism $\tau : X \to Y$. The following diagram straightforwardly commutes

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\tau"'] & & X \\
& Y \arrow[ur, "f\circ\tau^{-1}"'] &
\end{tikzcd}$$

The diagram commutes also when we apply F. Hence, $F(f) = F(f \circ \tau^{-1}) \circ F(\tau)$. But by the proof of item (i), both $F(\tau) = id_{X_0}$ and $F(f \circ \tau^{-1}) = id_{X_0}$. Therefore, $F(f) = id_{X_0}$.

Proof of (iii). Let $X \in \mathcal{X}_0$, and $\emptyset \neq Y \subseteq X$. Let $\pi : X \to Y$ be such that $\pi_{|Y} = id_Y$. Then, by item (ii), $\iota_{Y,X} \circ \pi : X \to X$ is such that $F(\iota_{Y,X} \circ \pi) = id_{X_0}$. Moreover, since $gr(\pi) = gr(\iota_{Y,X} \circ \pi)$, by Proposition 3.2(iii), also $gr(F(\pi)) = id_{X_0}$. Therefore, $F(X) = F(Y)$.

Proof of (iv). Let Y be such that $|Y| = |X|$ and $X \in \mathcal{X}_0$. We prove that $F(Y) = X_0$. Since X and Y have the same infinite cardinality, also $|Y \cup X| = |X|$. Hence, there exists an isomorphism $\tau : Y \cup X \to X$. The following diagram straightforwardly commutes.

$$\begin{tikzcd}
& Y \cup X \arrow[dr, "\tau"] & \\
X \arrow[ur, "\iota_{X,Y\cup X}"] \arrow[rr, "\tau_{|X}"'] & & X
\end{tikzcd}$$

Then, we apply F to the diagram above. By item (ii), $F(\tau_{|X}) = id_{X_0}$. Moreover F is inclusion preserving, and, therefore, $F(\iota_{X,Y\cup X}) = \iota_{FX,F(Y\cup X)}$. Hence, since $F(\tau)$ is bijective, $F(Y \cup X) = F(X)$ and $F(\tau) = id_{X_0}$.
In order to conclude, we still need to show that $F(Y \cup X) = F(Y)$.
Let $\pi : Y \cup X \to Y$ be a function such that $\pi_{|Y} = id_Y$. The following diagram trivially commutes.

$$\begin{tikzcd}
& Y \arrow[dr, "\iota_{Y,Y\cup X}"] & \\
Y \cup X \arrow[ur, "\pi"] & & Y \cup X \arrow[d, "\tau"] \\
X \arrow[u, "\iota_{X,Y\cup X}"] \arrow[rr, "\tau\circ\iota_{Y,Y\cup X}\circ\pi\circ\iota_{X,Y\cup X}"'] & & X
\end{tikzcd}$$

We apply F to the diagram above. Since F is inclusion preserving, and $F(X) = F(Y \cup X)$, then $F(\iota_{X,Y\cup X}) = id_{X_0}$. Moreover, by item (ii), $F(\tau \circ \iota_{Y,Y\cup X} \circ \pi \circ \iota_{X,Y\cup X}) = id_{X_0}$ and, by above, $F(\tau) = id_{X_0}$. Hence, $F(\iota_{Y,Y\cup X}) \circ F(\pi) = id_{X_0}$. As a result, $F(\pi)$ needs to be injective and not only surjective. Moreover, since F is inclusion preserving, $F(\pi) = id_{X_0}$. Therefore, $F(Y \cup X) = F(Y)$.
We are left to show that, if $|X| = |Y|$ and $f : Y \to Y$, then $F(f) = id_{X_0}$. The

following diagram straightforwardly commutes.

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "\tau"'] & Y \arrow[d, "\tau"] \\
X \arrow[r, "\tau \circ f \circ \tau^{-1}"'] & X
\end{tikzcd}$$

We apply F to the diagram above. By item (ii), $F(\tau \circ f \circ \tau^{-1}) = id_{X_0}$. Since the diagram commutes, $F(f) = F(\tau^{-1}) \circ id_{X_0} \circ F(\tau)$. Hence, $F(f) = id_{X_0}$.

Proof of (v). Let $Z \neq \emptyset$ be such that $|Z| \leq \mu$. Then there exists Z_1 such that $Z \subseteq Z_1$ and $|Z_1| = \mu$. By the item (iv), $F(Z_1) = X_0$, then using (iv), with an argument similar to the one used for proving item (iii), one can show that $F(Z) = X_0$. Here, we only prove that, for all $f : X \to Y$, $0 < |X|, |Y| \leq \mu$, $F(f) = id_{X_0}$.

There are two cases: either `(a)` $X \cap Y = \emptyset$ or `(b)` $X \cap Y \neq \emptyset$.

`(a)`

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\iota_{X,X\cup Y}"'] & \arrow[d, phantom, "(1)"] & Y & \\
& X \cup Y \arrow[ur, "f \cup id_Y"] & \arrow[u, phantom, "(2)"] & Y \arrow[ll, "\iota_{Y,X\cup Y}"] \arrow[ul, bend right, "id_Y"']
\end{tikzcd}$$

Since $X \cap Y = \emptyset$, both diagrams, (1) and (2), commute. We apply the functor F to both diagrams. Since by above, $F(X) = F(X \cup Y) = F(Y)$, then $F(\iota_{X,X\cup Y}) = id_{X_0}$. Analogously, $F(\iota_{Y,X\cup Y}) = id_{X_0}$. By commutativity of diagram (2), $F(id_Y) = F(f \cup id_Y) \circ F(\iota_{Y,X\cup Y})$. Hence, $F(f \cup id_Y) = id_{X_0}$. Therefore, by commutativity of diagram (1), $Ff = id_{X_0}$.

`(b)` Let X' and Y' be such that $X' \cap X = \emptyset$, $Y' \cap Y = \emptyset$, $X' \cap Y' = \emptyset$, and $X \cong X'$, $Y \cong Y'$, that is, there exist two isomorphisms $\tau_X : X \to X'$ and $\tau_Y : Y \to Y'$. By `(a)`, $F(\tau_X) = id_{X_0}$, $F(\tau_Y) = id_{X_0}$ and $F(\tau_Y \circ f \circ \tau_X^{-1}) = id_{X_0}$. Therefore, $F(\tau_Y) \circ F(f) \circ F(\tau_X^{-1}) = id_{X_0}$ and hence $F(f) = id_{X_0}$.

□

Proposition 4.21 *Any endofunctor F on $\mathcal{C}$, which is strict inclusion preserving and has constant cardinality on objects $< Ord$, is a strict constant functor.*

Proof. We proceed by contradiction. We assume that F is not constant everywhere on objects. Then there exist $X, Y \neq \emptyset$ such that $F(X) \neq F(Y)$. Let $|X| \geq |Y|$. If $|X|$ is infinite, then we immediately have a contradiction by Lemma 4.20. Otherwise, if $|X|$ is finite, then we consider a set X_0 of infinite cardinality μ, such that $X_0 \supseteq X, Y$. Then by Lemma 4.20 we have a contradiction. Therefore, F must be constant on objects. Moreover, using again Lemma 4.20, one can easily check that F must give the identity on every morphism, i.e. F is a constant functor. □

Finally, we have:

Theorem 4.22 *A constant functor K on $\mathcal{C}$ is DVO if and only if either $|K| = 1$ or it is strict and constant to an object of cardinality $\kappa < Ord$.*

Proof. ($\Leftarrow$) If $|K| = 1$, then the thesis follows immediately by a direct reasoning. Otherwise, the thesis follows by Proposition 4.21 and Theorem 3.3.
($\Rightarrow$) Assume that F is a constant non-strict functor K, with $|K| > 1$. Then it is not DVO, since it can be redefined on morphisms $f : \emptyset \to X$, where $X \neq \emptyset$, in such a way that $Ff : F\emptyset \to FX$ is a constant function to a given $\overline{\kappa} \in K$. Then we still get a functor which is isomorphic on objects to K, but it is not naturally isomorphic to K.
Finally, we are left to show that if F is strict constant to an object of cardinality Ord, then F is not DVO. Namely, let $G : \mathcal{C} \to \mathcal{C}$ be defined as follows. For all $X \neq \emptyset$, let $G(X) = \coprod_{k \in Ord} X^\kappa$, where X^κ denotes the function space $[\kappa \to X]$, i.e. all sequences of elements of X of length κ. For all $f : X \to Y$, let $G(f) : \coprod_{\kappa \in Ord} X^k \to \coprod_{\kappa \in Ord} Y^k$ be such that $G(f)(\boldsymbol{x}) = \boldsymbol{y}$ iff $\forall i \; f(x_i) = y_i$. Then, for all X, $FX \simeq GX$. By Theorem 3.3, there exists an inclusion preserving functor $G' : \mathcal{C} \to \mathcal{C}$, which is naturally isomorphic to G. But G cannot be constantly equal to the identity on functions, and hence it is not naturally isomorphic to F. □

5 The Powerset Functor

In this section, we prove that the whole family of *κ-bounded powerset* functors is not DVO.

Definition 5.1 (κ-Powerset) *Let $\kappa > 1$ be a cardinal. Let $P_\kappa : \mathcal{C} \to \mathcal{C}$ be defined by: for all X, $P_\kappa(X) = \{Y \mid Y \subseteq X \;\wedge\; |Y| < \kappa\}$, and for any $f : X \to Y$, $P_\kappa(f) = f^+$, where $f^+(u) = \{f(x) \mid x \in u\}$.*

Notice that, for $\kappa = Ord$, P_κ coincides with the standard (unbounded) powerset functor P.

Theorem 5.2 *For all $\kappa > 1$, the powerset functor $P_\kappa : \mathcal{C} \to \mathcal{C}$ is not DVO.*

Proof. Let P^*_κ be the functor defined by: for all X, $P^*_\kappa(X) = P_\kappa(X)$, and for all $f : X \to Y$, for all $u \in P_\kappa(X)$, $P^*_\kappa(f) = \begin{cases} f^+(u) & \text{if } f \text{ is injective on } u \\ \emptyset & \text{otherwise .} \end{cases}$
Then, it is easy to check that P^*_κ is not naturally isomorphic to P_κ. □

Remark. Notice that the functor P^*_κ defined in the proof of Theorem 5.2 above coincides with the functor $\bigcup_{\alpha<\kappa} B_\alpha$, where B_α is the functor defined by: $B_\alpha X = \{u \subseteq X \mid |u| = \alpha\} \cup \{\emptyset\}$, and for all $f : X \to Y$,

$$B_\alpha f(u) = \begin{cases} f^+(u) = \{fx \mid x \in u\} & \text{if } f \text{ is injective on } u \\ \emptyset & \text{otherwise .} \end{cases}$$

The family of functors B_α has been studied in [11,3], where B_α, for $\alpha > 2$, has been proved to be DVO if and only if $\alpha < \omega$. Notice that, by Theorem 4.3, using the remark at the end of Section 4.1.4, also B_0 and B_1 are DVO.
Finally, notice that by slightly modifying the definition of the κ-powerset functor P_κ by not including the empty set in the definition of the object part, we get the family of functors R_κ, which have been investigated in [3]. The functors R_κ have an odd behaviour: R_κ is DVO if and only if κ is finite and $\kappa \neq 3$.

6 Final Remarks and Directions For Future Work

We conclude the paper with a list of final comments and lines for future work.

- In [3], a characterization theorem has been given for the (unary) DVO functors which are non-constant and do not contain a subfunctor naturally isomorphic to the Identity. In [2], a special class of functors containing a subfunctor naturally isomorphic to the Identity has been studied. This class includes the polynomial functors X^n, for $n > 0$. In this paper, we have extended previous results in two ways. First of all, we have considered n-ary functors. Moreover, we have studied polynomial functors in full generality, thus covering the case of (possibly constant) polynomial functors with arbitrary coefficients. This extends, in particular, the study about the class of DVO functors which contain a subfunctor naturally isomorphic to the Identity. However, a general theorem covering such class in the whole is still missing.
- In this paper, we have studied how constraints on the object part determine the behaviour of a functor on a category of sets (classes). A natural question which arises is what happens for functors on other categories, e.g. categories of domains. Moreover, it would be also interesting to explore how constraints on the morphism part determine the behaviour of a functor on the object part.
- Finally, both DVO functors and functors uniform on maps in the sense of Aczel are functors whose behaviour on objects determines in some way the behaviour on morphisms. However, these two notions are apparently orthogonal, since there are functors which are DVO and not uniform on maps, and vice versa, [6].

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