Categories of Coalgebraic Games

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Abstract. We consider a general notion of *coalgebraic game*, whereby games are viewed as elements of a *final coalgebra*. This allows for a smooth definition of *game operations* (*e.g.* sum, negation, and linear implication) as *final morphisms*. The notion of coalgebraic game subsumes different notions of games, *e.g.* possibly non-wellfounded Conway games and games arising in Game Semantics à la [AJM00]. We define various categories of coalgebraic games and (total) strategies, where the above operations become functorial, and induce a structure of $*$-autonomous category. In particular, we define a category of coalgebraic games corresponding to AJM-games and winning strategies, and a generalization to non-wellfounded games of Joyal's category of Conway games. This latter construction provides a categorical characterization of the equivalence by Berlekamp, Conway, Guy on *loopy games*.
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Introduction

In this paper, we consider a general notion of *coalgebraic game*, whereby games are viewed as elements of a *final coalgebra*. This notion of coalgebraic game is general enough to subsume various notions of games, *e.g.* possibly non-wellfounded Conway games [Con01], and games arising in Game Semantics à la [AJM00]. Coalgebraic methods appear very natural and useful in this context, since they allow to abstract away superficial features of positions in games, and to smoothly define *game operations* as *final morphisms*.

The kind of games that we consider are 2-player games of perfect information, the two players being *Left* (L) and *Right* (R). A game is identified with its *initial* position. At any position, there are *moves* for L and R taking to new positions of the game. Contrary to other approaches in the literature, where games are defined as graphs, we view possibly non-wellfounded games as points of a *final coalgebra* of graphs, *i.e. minimal* graphs w.r.t. bisimilarity. This coalgebraic representation is motivated by the fact that the existence of winning/non-losing strategies is invariant w.r.t. graph bisimilarity. We formalize the notion of *play* as a sequence of pairs move-position, and, on top of it, we define a *strategy* as a function on plays. We focus on *total* strategies for a given player, *i.e.* strategies that must provide an answer, if any, for the player. These differ from *partial strategies*, in which the player can refuse an answer and give up the game. In particular, we introduce and study *winning/non-losing* strategies, which provide winning/non-losing plays when played against any counterstrategy.

In our general coalgebraic framework, we define and discuss various *game operations* arising in the literature, *i.e. sum* and *negation* introduced by Conway [Con01] to analyze games such as Go, or Nim, and *linear logic connectives* of Abramsky et al., see *e.g.* [Abr96,AJM00]. Coalgebraically, such operations can be naturally defined as *final morphisms*, and they uniformly and naturally subsume the corresponding original operations, allowing in particular for a comparison of operations arising in different contexts, such as Conway disjunctive sum and tensor sum on AJM-games. Then, on the basis of these operations, we discuss various categorical constructions, in the spirit of [Joy77], which generalize categories of AJM-games as well as Joyal's original category of Conway games. In particular, we provide a general construction of a $*$-autonomous category of possibly non-wellfounded games and (total) strategies, which subsumes Joyal's compact closed category as a *full* subcategory. Interestingly, our category characterizes the equivalence on *loopy games* defined in [BCG82].

Constructions generalizing Joyal's category to non-wellfounded games have been previously considered in [Mel09,MTT09], but in the context of partial strategies; hence they subsume Joyal's category as a subcategory, but *not* as a full subcategory, and the equivalence on games induced by the existence of partial strategies becomes trivial. Solutions to the problem of defining a well-behaved category of non-wellfounded games and total strategies have been presented in [HLR11], for the class of non-wellfounded Conway games where all infinite plays are draws. The solution in the present paper is based on a different and more general construction, and it applies to the class of *mixed games* (infinite plays can be either winning for any of the players or draws). To our knowledge, this is the first category of mixed games subsuming Joyal's construction as a full subcategory and capturing the original loopy equivalence of [BCG82].

The coalgebraic notion of game in this paper generalizes the one introduced in [HL11] for characterizing non-wellfounded Conway games. Coalgebraic methods for modeling games have been used also in [BM96], where the notion of *membership game* has been introduced. This corresponds to a subclass of our coalgebraic games, where at any position L and R have the same moves, and all infinite plays are deemed winning for player II (the player who does not start). However, no operations on games are considered in that setting. In the literature, various notions of bisimilarity equivalences have been considered on games, see *e.g.* [Pau00,Ben02]. But, contrary to our approach, such games are defined as *graphs* of positions, and equivalences on graphs, such as trace equivalences or various bisimilarities are considered. Differently, defining games as the elements of a final coalgebra, we directly work up-to bisimilarity of game graphs.

Summary. In Section 1, we introduce our framework of coalgebraic games and strategies, and we instantiate it to Conway games and AJM-games. In Section 2, we introduce and study general game operations, and in Section 3 we present two parametric categories of games and strategies, subsuming as special instances categories arising in Game Semantics as well as Joyal's category of Conway games. Conclusions and directions for future work appear in Section 4.

1 Coalgebraic Games and Strategies

We consider a general notion of 2-player game of perfect information, where the two players are called *Left* (L) and *Right* (R). A game x is identified with its *initial position*; at any position, there are *moves* for L and R, taking to new *positions* of the game. By abstracting from superficial features of positions, games can be viewed as elements of the final coalgebra for the functor $F_{\mathcal{A}}(X) = \mathcal{P}_{<\kappa}(\mathcal{A} \times X)$, where $\mathcal{A}$ is a parametric set of *atoms* which encode information on moves and positions, *i.e.* move names, and the player who has moved, and $\mathcal{P}_{<\kappa}$ is the set of all subsets of cardinality $< \kappa$. The coalgebra structure captures, for any position, the moves of the players and the corresponding next positions.

We work in the category Set^* of sets belonging to a universe satisfying the Antifoundation Axiom, see [FH83,Acz88]. Of course, we could work in the category Set of well-founded sets, but we prefer to use Set^* so as to be able to use identities rather than isomorphisms. Formally, we define:

Definition 1 (Coalgebraic Games). *Let $\mathcal{A}$ be a set of atoms with functions:*
(i) $\mu : \mathcal{A} \to \mathcal{M}$ yielding the name of the move (for a set $\mathcal{M}$ of names),
(ii) $\lambda : \mathcal{A} \to \{L, R\}$ yielding the player who has moved.
Let $F_{\mathcal{A}} : Set^* \to Set^*$ be the functor defined by $F_{\mathcal{A}}(X) = \mathcal{P}_{<\kappa}(\mathcal{A} \times X)$ (with usual definition on morphisms), and let $(\mathcal{G}_{\mathcal{A}}, id)$ be the final $F_{\mathcal{A}}$-coalgebra.
A coalgebraic game *is an element x of the carrier $\mathcal{G}_{\mathcal{A}}$ of the final coalgebra.*

The elements of the final coalgebra $\mathcal{G}_{\mathcal{A}}$ are the *minimal* graphs up-to bisimilarity. In the following, we often refer to *coalgebraic games* simply as *games*. We call *player I* the player who starts the game (who can be L or R in general), and *player II* the other. Once a player has moved on a game x, this brings to a new game/position x'. We define the *plays* on x as the sequences of pairs, move-position, from x; moves in a play are *not* necessarily alternating (this generality will be useful in the sequel, in defining operations on games):

Definition 2 (Plays). *A* play *on a game x_0 is a possibly empty finite or infinite sequence of pairs in $\mathcal{A} \times \mathcal{G}_{\mathcal{A}}$, $s = \langle a_1, x_1 \rangle \ldots$ such that $\forall n \geq 0.\ \langle a_{n+1}, x_{n+1} \rangle \in x_n$. We denote by $Play_x$ the set of plays on x and by $FPlay_x$ the set of finite plays.*

The kind of strategies for a given player on which we focus are those that *always* provide an answer, *if any*, of the player to the moves of the opponent player. In this sense, such strategies are "total", opposite to "partial strategies", where the player can possibly refuse an answer and give up the game. Formally, strategies in our framework are partial functions on finite plays ending with a position where the player is next to move, and yielding (if any) a pair in $\mathcal{A} \times \mathcal{G}_{\mathcal{A}}$, consisting of "a move of the given player together with a next position" on the game x. In what follows, we denote by

– $FPlay_x^{LI}$ ($FPlay_x^{RI}$) the set of possibly empty finite plays on which L (R) acts as player I, and ending with a position where R (L) was last to move, *i.e.* $s = \langle a_1, x_1 \rangle \ldots \langle a_n, x_n \rangle$, $\lambda a_1 = L$ and $\lambda a_n = R$ ($\lambda a_1 = R$ and $\lambda a_n = L$).
– $FPlay_x^{LII}$ ($FPlay_x^{RII}$) the set of finite plays on which L (R) acts as player II, and

ending with a position where R (L) was last to move, *i.e.* $s = \langle a_1, x_1\rangle \dots \langle a_n, x_n\rangle$, $\lambda a_1 = R$ ($\lambda a_1 = L$) and $\lambda a_n = R$ ($\lambda a_n = L$).

Formally, we define:

Definition 3 (Strategies). *Let x be a game. A strategy σ for LI (*i.e. *L acting as player I) is a partial function $\sigma : FPlay_x^{LI} \to \mathcal{A} \times \mathcal{G}_{\mathcal{A}}$ such that, for any $s \in FPlay_x^{LI}$,*
– $\sigma(s) = \langle a, x'\rangle \implies \lambda a = L \ \wedge\ s\langle a, x'\rangle \in FPlay_x$
– $\exists \langle a, x'\rangle.\ (s\langle a, x'\rangle \in FPlay_x \ \wedge\ \lambda a = L) \implies s \in dom(\sigma)$.
Similarly, one can define strategies for players LII, RI, RII.

We are interested in studying the interactions of a strategy for a given player with the (counter)strategies of the opponent player. When a player plays on a game according to a strategy σ, against an opponent player who follows a (counter)strategy σ', a play arises. Formally, we define:

Definition 4 (Product of Strategies). *Let x be a game.*
(i) Let s be a play on x, and σ a strategy for a player in $\{LI,LII,RI,RII\}$. Then s is coherent with σ *if, for any proper prefix s' of s, ending with a position where the player is next to move, $\sigma(s') = \langle a, x'\rangle \implies s'\langle a, x'\rangle$ is a prefix of s.*
(ii) Given a strategy σ on x and a counterstrategy σ', we define the product *of σ and σ', $\sigma * \sigma'$, as the unique play coherent with both σ and σ'.*

Notice that a play arising from the product of strategies is alternating.

We distinguish between *well-founded games*, *i.e.* well-founded sets as elements of the final coalgebra $\mathcal{G}_{\mathcal{A}}$, and *non-wellfounded games*, *i.e.* non-wellfounded sets in $\mathcal{G}_{\mathcal{A}}$. Clearly, strategies on well-founded games generate only finite plays, while strategies on non-wellfounded games can generate infinite plays.

Strategies for a given player, as we have defined so far, simply provide an answer (if any) of the player to all possible moves of the opponent. Intuitively, a strategy is *winning/non-losing* for a player, if it generates winning/non-losing plays against any possible counterstrategy. We take a finite play to be winning for the player who performs the last move. While infinite plays are taken to be winning for L/R or draws. Formally, we define:

Definition 5 (Winning/non-losing Play). *Let $\nu : Play_x \to \{0, 1, -1\}$ be a payoff function defined on plays of a game x.*
(i) A play s is winning *for player L (R) if $\nu(s) = 1$ ($\nu(s) = -1$).*
(ii) A play s is a draw *if $\nu(s) = 0$.*
(iii) A play s is non-losing *for player L (R) if $\nu(s) \in \{0, 1\}$ ($\nu(s) \in \{0, -1\}$).*

Definition 6 (Winning/non-losing Strategy). *Let $\nu : Play_x \to \{0, 1, -1\}$ be a payoff function on x.*
(i) A strategy σ on x for LI (LII) is winning (non-losing) *for LI (LII) if for any strategy σ' for RII (RI), $\nu(\sigma * \sigma') = 1$ ($\nu(\sigma * \sigma') \in \{0, 1\}$).*
(ii) A strategy σ on x for RI (RII) is winning (non-losing) *if for any strategy σ' for LII (LI), $\nu(\sigma * \sigma') = -1$ ($\nu(\sigma * \sigma') \in \{0, -1\}$).*

We will refer to the whole class of coalgebraic games, where plays can be winning or draws, as *mixed games*; and we will call *fixed games* the subclass of games where all plays are winning for one of the players.

The notion of strategy of Definition 3 is quite general, being defined on plays which carry the information on moves and positions. Often, we are interested in considering special classes of strategies, depending either on moves or on positions (or even only on the last move/position). Here we collect the relevant definitions. For any play s, we denote by $s_{|\mathcal{A}}$ the sequence obtained by erasing all positions, and by $s_{|\mathcal{P}}$ the sequence obtained by erasing all moves from s.

Definition 7. *Let σ be a strategy on a game x.*
(i) σ is pos-independent *if* $\forall s, s' \in dom(\sigma).\ (s_{|\mathcal{A}} = s'_{|\mathcal{A}} \implies \sigma(s) = \sigma(s'))$.
(ii) σ is move-independent *if* $\forall s, s' \in dom(\sigma).\ (s_{|\mathcal{P}} = s'_{|\mathcal{P}} \implies \sigma(s) = \sigma(s'))$.

1.1 Conway Games. Conway (wellfounded) games are inductively defined in [Con01] as pairs of sets $x = (X^L, X^R)$, where X^L (X^R) is the set of next positions to which L (R) can move. Such games are purely positional, no move names are considered. In [BCG82], non-wellfounded games are considered, called *loopy* or *mixed* games, but these are defined as graphs of positions, rather than sets, *i.e.* graphs up-to bisimilarity. Here we extend the original set-theoretical definition of [Con01], by representing possibly non-wellfounded Conway games as coalgebraic games for $\mathcal{A}$ the two-element set $\{a^L, a^R\}$, where $\mu a^L = \mu a^R = a$, $\lambda a^L = L$ and $\lambda a^R = R$. These correspond to loopy games taken up-to graph bisimilarity. Our coalgebraic approach is motivated by the fact that the existence of winning/non-losing strategies is preserved under graph bisimilarity of loopy games. Winning/non-losing strategies on Conway games correspond to (move-independent) winning/non-losing strategies of Definition 6.

1.2 Game Semantics. In Game Semantics various notions of games are used, here we focus on the basic games à la [Abr96,AJM00], called *AJM-games*. We define an *AJM-game* as a tuple $G = (M_G, \lambda_G, P_G, W_G)$, where M_G is the set of moves, the function $\lambda_G : M_G \to \{O, P\}$ specifies for each move if it is an O (Opponent) or a P (Player) move; O and P move in strict alternation, O starts the game; the set P_G is a non-empty prefix-closed set of finite alternating sequences of moves starting with an O-move, which represents the set of *legal positions*. These correspond to the finite plays in our setting when positions are omitted. We define P_G^∞ as the set of infinite plays, *i.e.* infinite sequences whose finite prefixes are legal positions. The winning condition for a player on a finite play corresponds to the absence of moves for the other player, while any infinite play is fixed to be winning either for O or for P via the predicate W_G, which holds on an infinite play s ($W_G(s)\downarrow$) iff s is winning for P.

The underlying structure of any such game can be represented in our framework by considering the tree of legal positions (plays). This can be viewed as an element of our final coalgebra $\mathcal{G}_\mathcal{A}$, provided we perform a bisimilarity quotient on nodes (since the tree of plays is not necessarily minimal w.r.t. bisimilarity), thus getting the graph of positions. Formally, we represent such games as follows:

– P is player L and O player R, R starts the game;
– the set $\mathcal{A}$ includes atoms a_m for any $m \in M_G$ s.t. $\mu a_m = m$, $\lambda a_m = \lambda_G m$;
– nodes $\{x_p\}_{p \in P_G}$, $x_p = \{(a_m, x_{p'}) \mid p' = pa \in P_G\}$, are taken up-to bisimilarity;
– the initial position is x_ϵ;
– the payoff function ν is defined on infinite plays by $\nu(s) = \begin{cases} 1 & \text{if } W_G(s) \downarrow \\ -1 & \text{otherwise.} \end{cases}$

Coalgebraic games representing AJM-games are fixed and have a special structure: R starts, at any non-ending position only moves for R or L are available, for any move there is at most one arc labeled by that move, and along any path in the game graph R/L moves strictly alternate. We call *strict games* such subclass of coalgebraic games. They form a subcoalgebra of our final coalgebra.

Winning strategies on AJM-games are defined as suitable subsets of the legal positions, see [Abr96] for more details, and hence they only depends on the sequence of moves (pos-independent strategies in our setting). AJM-games together with winning strategies form a $*$-autonomous category $\mathcal{C}$, see [Abr96]. The precise relationship between $\mathcal{C}$ and the corresponding category of coalgebraic games is formalized in Section 3 via an *equivalence of categories*.

2 Game Operations

In this section, we show how to define various operations on coalgebraic games, including *sum*, *negation*, and *linear implication*. In our framework, game operations can be conveniently defined via final morphisms. These capture the structure of compound games; the extra structure of the payoff function on infinite plays of the compound game is obtained inductively from the payoff of the components.

On mixed games, we define a notion of sum, inspired by Conway *disjunctive sum*; while, on fixed games, we define a notion of sum subsuming the *tensor product* of Game Semantics. The two notions of sum have the same coalgebraic structure, and only differ by the definition of the payoff on infinite plays. This neatedly emerges from the analysis carried out in our coalgebraic framework.

Sum. We start by defining the coalgebraic structure of the sum of two games. On the sum game, at each step, the next player selects any of the component games and makes a legal move in that component, the other component remaining unchanged. The other player can either choose to move in the same component or in a different one. Notice that in this way, even if the play on the sum game agrees with turns of L and R, the subplays in the single components may not agree with turns, in general.

Definition 8 (Sum, coalgebraic structure). *The* sum *of two games is given by the final morphism* $+ : (\mathcal{G}_\mathcal{A} \times \mathcal{G}_\mathcal{A}, \alpha_+) \longrightarrow (\mathcal{G}_\mathcal{A}, id)$*, where the coalgebra morphism* $\alpha_+ : \mathcal{G}_\mathcal{A} \times \mathcal{G}_\mathcal{A} \longrightarrow F_\mathcal{A}(\mathcal{G}_\mathcal{A} \times \mathcal{G}_\mathcal{A})$ *is defined by:*
$\alpha_+(x, y) = \{\langle a, \langle x', y\rangle\rangle \mid \langle a, x'\rangle \in x\} \cup \{\langle a, \langle x, y'\rangle\rangle \mid \langle a, y'\rangle \in y\}$.
That is: $x + y = \{\langle a, x' + y\rangle \mid \langle a, x'\rangle \in x\} \cup \{\langle a, x + y'\rangle \mid \langle a, y'\rangle \in y\}$.

Two kinds of sum arise from the above coalgebraic definition, by suitably defining the payoff function on infinite plays:
(i) *Mixed sum* $\oplus$. This is defined on mixed games, and it is inspired by Conway disjunctive sum. The payoff of an infinite play will be 1 (-1) if all infinite plays in the components have payoff 1 (-1), it will be 0 otherwise.
(ii) *Fixed sum* $\otimes$. This is defined on fixed games and it generalizes the tensor product of Game Semantics. The payoff of an infinite play is 1 (winning for L) iff all infinite subplays in the components have payoff 1, it will be -1 otherwise. This "asymmetric" definition is motivated by the interpretation of the linear logic tensor connective.

Notice that, on both sums, since plays which agree with turns do not necessarily induce subplays on the components which agree with turns, in order to define the payoff on infinite plays, we need the payoff on *all* plays of the components, also those non conformed to turns. This is the reason for such a liberal definition of plays in Section 1. But, if we restrict ourselves to coalgebraic *strict* games, which correspond to games of Game Semantics, then any play on the sum game which agrees with turns induces subplays with the same property in the components.

Negation. The *negation* is a unary game operation, which allows us to build a new game, where the roles of L and R are exchanged. Let us assume that the set of atoms $\mathcal{A}$ is closed under an involution operation, *i.e.*, for any $a \in \mathcal{A}$, let $\overline{a} \in \mathcal{A}$ be such that $\lambda\overline{a} = \overline{\lambda a}$, $\nu\overline{a} = -\nu(a)$, $\mu\overline{a} = \mu a$, where $\overline{L} = R$ and $\overline{L} = R$. The coalgebraic definition of game negation is as follows:

Definition 9 (Negation). *The* negation *of a game is given by the final morphism* $\,^{-} : (\mathcal{G}_{\mathcal{A}}, \alpha_{-}) \longrightarrow (\mathcal{G}_{\mathcal{A}}, id)$*, where the coalgebra morphism* $\alpha_{-} : \mathcal{G}_{\mathcal{A}} \longrightarrow F_{\mathcal{A}}(\mathcal{G}_{\mathcal{A}})$ *is:* $\alpha_{-}(x) = \{\langle \overline{a}, x' \rangle \mid \langle a, x' \rangle \in x\}$*. That is:* $\overline{x} = \{\langle \overline{a}, \overline{x'} \rangle \mid \langle a, x' \rangle \in x\}$*. The payoff on infinite plays of* $\overline{x}$ *is taken to be opposite to the payoff on* x*.*

Clearly, winning/non-losing strategies for a given player on x become winning/non-losing strategies for the opponent player on $\overline{x}$, and $\overline{\overline{x}} = x$, *i.e.* negation is involutive. Notice that both mixed and fixed games are closed under negation. But strict games are not, of course.

Linear implication. Using the two notions of sum, and negation, we can now define two *linear implications*:

Definition 10 (Linear Implications). *We define*
(i) on mixed games: the linear implication $x \to y$ *as the game* $\overline{x \oplus \overline{y}}$*;*
(ii) on fixed games: the linear implication $x \multimap y$ *as the game* $\overline{x \otimes \overline{y}}$*.*

Notice that mixed sum satisfies the equality $\overline{x \oplus y} = \overline{x} \oplus \overline{y}$, hence the linear implication $x \to y$ amounts to $\overline{x} \oplus y$, while the corresponding equality for fixed sum does not hold. More precisely, the coalgebraic structure of the game $x \multimap y$ coincides with the coalgebraic structure of $\overline{x} \otimes y$, but the winning condition on infinite plays is different, namely an infinite play is winning for L on $x \multimap y$ iff the subplay on $\overline{x}$ or that on y is infinite and winning for L, while an infinite play is winning for L on $\overline{x} \otimes y$ iff all infinite subplays on $\overline{x}$ and y are winning for L.

3 Game Categories

The fine analysis of game operations carried out in Section 2 allows us to provide two very general categorical constructions arising from such operations on games. In particular, we provide a category $\mathcal{X}_{\mathcal{A}}$ of *fixed games* and *winning strategies*, parametric w.r.t. the set of atoms $\mathcal{A}$, which is $*$-autonomous, and a symmetric monoidal closed category $\mathcal{Y}_{\mathcal{A}}$ of *mixed games*, also parametric w.r.t. $\mathcal{A}$, obtained by analyzing mixed games via pairs of fixed games. A special instance of $\mathcal{X}_{\mathcal{A}}$ is obtained by instantiating $\mathcal{A}$ as shown in Section 1.2, in order to recover (up-to bisimilarity) AJM-games. A significant result that we obtain, which clarifies the relationships between the original AJM-games and their representation in our framework, is an equivalence between the category of games and winning strategies of [Abr96] and our category of strict coalgebraic games and pos-independent strategies. On the other hand, the category $\mathcal{Y}_{\mathcal{A}}$ of mixed games is related to Conway games. Namely, by suitably instantiating $\mathcal{A}$, we get a category whose objects correspond to the (non-wellfounded) mixed Conway games of [BCG82], a full subcategory of which is Joyal's compact closed category. Remarkably, the equivalence on mixed games induced by the morphisms of our category coincides with the equivalence defined in [BCG82] on loopy games. To our knowledge, this is the first category of mixed games subsuming Joyal's construction as a full subcategory and capturing the original loopy equivalence.

The category $\mathcal{X}_{\mathcal{A}}$ of fixed games. The notions of sum and linear implication on fixed games give rise to a $*$-autonomous category $\mathcal{X}_{\mathcal{A}}$ that generalizes categories of AJM-games and winning strategies.

Definition 11 (The Category $\mathcal{X}_{\mathcal{A}}$).
Objects*:* *fixed games.*
Morphisms*:* $\sigma : x \to y$ *winning strategy for LII on* $x \multimap y$*.*

Identities on $\mathcal{X}_{\mathcal{A}}$ are the *copy-cat strategies*, and closure under composition is obtained via the *swivel-chair strategy. I.e.*, given winning strategies for LII, σ on $x \multimap y$, τ on $y \multimap z$, the composition strategy, $\tau \circ \sigma$ on $x \multimap z$, is obtained by using the "swivel chair", as follows. Assume R opens on $x \multimap z$, playing either in z or in x, *e.g.* assume R opens in z. Then consider the L move given by the strategy τ on $y \multimap z$: if such L move is in z, then we take this as the L answer in the strategy on $\tau \circ \sigma$; if the L move according to σ is in the y component of $y \multimap z$, then, using the "swivel chair", we can view this move as an R move in the y component of $x \multimap y$. Now L has a next move in $x \multimap y$, according to τ. If this move is in the x component, then we take this as the L answer in $\tau \circ \sigma$; otherwise, if the L move is in y, then we use our swivel chair, viewing this as a move of R in the y component on $y \multimap z$, and so on. Since both σ and τ are winning strategies, by the winning condition on infinite plays on the $\multimap$ game, spelled out at the end of Section 2, we are guaranteed that the dialogue between the y components does not go on forever, and eventually the L move according to σ or τ will be on z or x. This is the L answer to the starting R move in the strategy $\tau \circ \sigma$. Then, we go on in the same way, for any possible next R move. Associativity of composition is also proved by a standard argument.

Fixed sum gives rise to a tensor product on $\mathcal{X}_{\mathcal{A}}$, which determines a structure of a symmetric monoidal closed category; in particular the identity $x \otimes y \multimap z = y \multimap (x \multimap z)$ holds, this latter following from the definition of the $\multimap$ game and from the fact that negation is involutive. Negation is also functorial, and, together with tensor, provides a $*$-autonomous structure on $\mathcal{X}_{\mathcal{A}}$, namely we have in particular the identity $x \otimes y \multimap \overline{z} = x \multimap (\overline{y \otimes z})$. Summarizing:

Theorem 1. *The category $\mathcal{X}_{\mathcal{A}}$ is $*$-autonomous.*

The above construction encompasses categories used in Game Semantics. Namely, let $\mathcal{C}$ be the category of AJM-games and winning strategies of [Abr96], and let us instantiate the parameter $\mathcal{A}$ of $\mathcal{X}_{\mathcal{A}}$ with the set of moves M for such games, getting the category $\mathcal{X}_M$. If we consider the subcategory $\mathcal{SX}_M$ of strict games and pos-independent winning strategies, then we obtain the following equivalence of categories:

Theorem 2. *The category $\mathcal{SX}_M$ of strict games and pos-independent winning strategies is equivalent to the category $\mathcal{C}$ of AJM-games and winning strategies.*

Proof. The equivalence between the categories $\mathcal{SX}_M$ and $\mathcal{C}$ is given by the functor $H : \mathcal{C} \to \mathcal{SX}_M$, which, for a game in $\mathcal{C}$, yields the coalgebraic game obtained by performing a bisimilarity quotient on the tree of legal positions. Winning strategies of $\mathcal{C}$, which are defined on legal positions (*i.e.* plays where positions are omitted, in our setting) are naturally mapped to pos-independent winning strategies in $\mathcal{SX}_M$, providing a one-to-one correspondence. Moreover, each strict coalgebraic game is the image of an AJM-game via H. □

The category $\mathcal{Y}_{\mathcal{A}}$ of mixed games. Defining a category of mixed games and non-losing strategies is not straightforward, the reason being that non-losing strategies are *not* closed under composition. The situation has been analyzed in [HLR11] for hypergames, *i.e.* non-wellfounded Conway games where all infinite plays are draws. The solution proposed there is to restrict the class of morphisms to non-losing *fair* strategies. This category is symmetric monoidal with the mixed sum $\oplus$ as tensor product, but it is *not* monoidal closed; moreover the categorical construction does not immediately extend to the whole class of mixed games. Here, by exploiting the investigation on operations carried out in Section 2, we propose a different solution, which is inspired by the analysis of mixed Conway games x in terms of pairs of fixed games, $\langle x^-, x^+\rangle$, [BCG82]. The idea is to represent mixed games as pairs of fixed games obtained by considering all draws to be winning for R or for L respectively, and to work with fixed tensor product and the corresponding linear implication in the single components. We carry out this construction in the full generality offered by our framework, building a $*$-autonomous category $\mathcal{Y}_{\mathcal{A}}$, parametric w.r.t. $\mathcal{A}$.

Definition 12. *Let x be a mixed game. We define the pair $\langle x^-, x^+\rangle$ of fixed games as follows: x^- is obtained from x by taking all infinite plays which are draws on x to be winning for R; x^+ is obtained from x by taking all infinite plays which are draws on x to be winning for L.*

Notice that each mixed game is uniquely determined by its corresponding pair of fixed games. In particular, for fixed games x, we have $x = x^- = x^+$.

Definition 13 (The Category $\mathcal{Y}_{\mathcal{A}}$).
Objects: *pars of fixed games $x = \langle x_1, x_2 \rangle$.*
Morphisms: *pairs of winning strategies for LII, $\langle \sigma_1, \sigma_2 \rangle : \langle x_1, x_2 \rangle \to \langle y_1, y_2 \rangle$.*

Mixed games are objects in the above category $\mathcal{Y}_{\mathcal{A}}$. This inherits from $\mathcal{X}_{\mathcal{A}}$ all constructions, hence:

Theorem 3. *The category $\mathcal{Y}_{\mathcal{A}}$ is $*$-autonomous.*

Moreover, $\mathcal{Y}_{\mathcal{A}}$ restricted to well-founded games is compact closed. Namely, the copy-cat strategy induces natural winning strategies for LII on $x \otimes \overline{x} \multimap 0$ and $0 \multimap x \otimes \overline{x}$, where 0 denotes the empty game, for any well-founded game x. Thus we have:

Theorem 4. *The full subcategory of $\mathcal{Y}_{\mathcal{A}}$, consisting of well-founded games and winning strategies, is compact closed.*

As shown in Section 1.1, non-wellfounded Conway games can be represented in our framework by instantiating $\mathcal{A}$ to an appropriate 2-element set. Let us denote by $\mathcal{Y}_2$ the corresponding category. As a corollary of Theorem 3 we get:

Corollary 1. *The category $\mathcal{Y}_2$ of non-wellfounded Conway games is $*$-autonomous.*

Moreover, by restricting to well-founded games, we obtain Joyal's category as a full subcategory, and from Theorem 4 we have:

Corollary 2 ([Joy77]). *The category of Conway games and winning strategies is compact closed.*

Game Equivalences. Having defined games as a final coalgebra, games are already taken up-to bisimilarity, thus abstracting from superficial features of positions. Bisimilarity is a first structural equivalence on game graphs, but on top of this one can define various equivalences and congruences, by looking at strategies. Such equivalences arise in many conceptually different ways, and they have been studied for Conway games and hypergames in [HL11,HLR11].

Our categorical constructions give rise to interesting notions of (pre)equivalences on games induced by the morphisms, and defined by:

$x \leq_{\mathcal{Y}_{\mathcal{A}}} y$ iff there exists a winning strategy for LII on $x \multimap y$.

Notice that, since $\mathcal{X}_{\mathcal{A}}$ is a full subcategory of $\mathcal{Y}_{\mathcal{A}}$, the (pre)equivalence induced by $\mathcal{X}_{\mathcal{A}}$ coincides with the restriction on fixed games of $\leq_{\mathcal{Y}_{\mathcal{A}}}$.

Identities on the category $\mathcal{Y}_{\mathcal{A}}$ correspond to reflexivity of $\leq_{\mathcal{Y}_{\mathcal{A}}}$, closure under composition corresponds to transitivity, while functoriality of tensor and negation ensures congruence of $\leq_{\mathcal{Y}_{\mathcal{A}}}$ w.r.t. the corresponding operations.

Interestingly, the equivalence $\leq_{\mathcal{Y}_2}$ captured by our category $\mathcal{Y}_2$ of mixed Conway games coincides with the loopy game equivalence of [BCG82].

Definition 14 (Loopy Equivalence). *For x, y mixed games, we define:*
$x \leq_l y$ *iff there are non-losing strategies for LII on* $\overline{x^-} \oplus y^-$ *and* $\overline{x^+} \oplus y^+$.

Then we have:

Theorem 5. *For mixed games x, y,* $\quad x \leq_l y \iff x \leq_{\mathcal{Y}_2} y$.

Proof. We prove that $\overline{x^- \otimes \overline{y^-}}$ $(\overline{x^+ \otimes \overline{y^+}})$ has a *winning strategy* for LII iff $\overline{x^- \oplus \overline{y^-}}$ $(\overline{x^+ \oplus \overline{y^+}})$ has a *non-losing strategy* for LII. Equivalently, $x^- \otimes \overline{y^-}$ $(x^+ \otimes \overline{y^+})$ has a *winning strategy* for RII iff $x^- \oplus \overline{y^-}$ $(x^+ \oplus \overline{y^+})$ has a *non-losing strategy* for RII. This follows since, for fixed games, $x \otimes y$ has a winning strategy for R (I or II) iff $x \oplus y$ has a non-losing strategy for R (I or II). $\square$

4 Final Remarks and Directions for Future Work

We have considered a general notion of coalgebraic game, whereby non-wellfounded games are viewed as elements of a final coalgebra. This allows for a unified treatment of games arising in different settings, in particular Conway games and AJM-games, and it helps in shedding light on the relationships between them. We have introduced and studied general notions of categories of games and strategies, subsuming Joyal's category of Conway games as well as categories used in Game Semantics. Categorical equivalences have been defined, providing in particular a characterization of the equivalence on loopy games of [BCG82].

Here is a list of further comments and directions for future work.

Partial strategies. In this paper, we have considered *total* strategies, but in contexts such as Game Semantics, often *partial* strategies are considered. Categories of games and partial strategies in the spirit of Joyal's category have been studied *e.g.* in [HS02,Mel09,MTT09]. Partial strategies could be naturally modeled in our framework. It would be interesting to investigate the relationships between partial and total strategies in generality. Intuitively, partial strategies should allow to approximate total strategies up to plays of certain length.

Exponential. The general categorical constructions carried out in the present paper provide symmetric monoidal closed and $*$-autonomous categories, which allow to model fragments of Linear Logics. We claim that mixed games can be endowed with an exponential operation, endowing $\mathcal{Y}_\mathcal{A}$ with a structure of linear category. The exponential operation can be defined by $!x = \Sigma^\infty x^R$, where x^R is obtained from x by erasing all L-opening moves, and Σ^∞ is an "infinite sum" operation, which can be defined via a suitable *generalized coiteration schema.*

Games with payoff on a partially ordered set. In [San02], partial infinite games are introduced. Various operations are defined, including a kind of sum operation. A precise comparison with our categorical sums has still to be investigated.

Generalizing the coalgebraic framework. Coalgebraic games can be further generalized, by encoding more information in the parameter set $\mathcal{A}$, *e.g.* the payoff or the turn of the players. These allow us to model a wider range of games, including automata games and games arising in Economics. It would be also interesting

to investigate a generalization for *non-perfect information games*. An approach could be that of explaining them using the notion of coalgebra morphism.
Coinductive specification of strategies. One can give a *coinductive* definition of the set of strategies for a player, via corecursive equations. Intuitively, if we denote by $\mathcal{S}_L(s)$ the set of strategies for L starting on the play s, a strategy in $\mathcal{S}_L(s)$, where s ends with the current position $\langle a, x\rangle$, amounts to: either the empty strategy, if L has no move in the position x; or a strategy for L in $\mathcal{S}_L(s\langle a', x'\rangle)$, where $\langle a', x'\rangle \in x$ and a' is a L move, if L is next to move in the current position $\langle a, x\rangle$, *i.e.* a strategy in $\Sigma_{\langle a',x'\rangle \in x . \lambda a' = L} \mathcal{S}_L(s\langle a', x'\rangle)$; or a collection of strategies for L, for any possible R move from the current position $\langle a, x\rangle$, if R is next to move, *i.e.* a collection of strategies in $\Pi_{\langle a',x'\rangle \in x . \lambda a' = R} \mathcal{S}_L(s\langle a', x'\rangle)$.
It would be interesting to formalize the above idea.

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